This is called the Langlands decomposition of $q_1$ [GaVa, Knap2, Wall2]. It is easy to see that $m_1$ is a reductive Lie subalgebra of $g$. We can define a corresponding subgroup $M_1$ of $G$ as follows. The group $Q_1 \cap \theta(Q_1)$ has a Cartan decomposition (cf. (A.2.3.1))

$$Q_1 \cap \theta(Q_1) = (Q_1 \cap \theta(Q_1) \cap K) \exp(q_1 \cap p).$$

Set

(A.2.4.5) \hspace{1cm} M_1 = (Q_1 \cap \theta(Q_1) \cap K) \exp(m_1 \cap p).

Then the Cartan decomposition plus decompositions (A.2.4.4) tell us that

(A.2.4.6) \hspace{1cm} P_1 = M_1 A_1 N_1^+

in the strong sense that the map from $M_1 \times A_1 \times N_1^+$ defined by multiplication to $P_1$ is a diffeomorphism. The factorization (A.2.4.6) is called the Langlands decomposition of $P_1$.

The procedure sketched above constructs $2^r$, where $r = \#(\mathcal{F})$, parabolic subgroups of $\tilde{G}$ containing $P_0$. These are all possible parabolics containing $P_0$. To show this requires a more detailed study of root systems than we wish to give here. Instead we will finish as we started, by looking at $GL_n$. We will sketch how to see that possibilities for subgroups of $GL_n$ containing the Borel subgroup of upper triangular matrices are the groups of block upper triangular matrices defined by various partial flags (cf. §1.4). Consider the basis $\{E_{jk}\}_{j,k=1}^n$ of standard matrix units for $gl_n$. These satisfy the commutation relations

$$[E_{jk}, E_{lm}] = \delta_{kl}E_{jm} - \delta_{jm}E_{lk}.$$ 

The upper triangular matrices $b^+$ are the span of the $E_{jk}$ with $j \leq k$. Suppose we add to this another element $x = \sum c_{lm}E_{lm}$. Since the $E_{lm}$’s are eigenvectors for the $adE_{jj}$, with distinct eigenvalues, we find that if $c_{lm} \neq 0$, then $E_{lm}$ is in the algebra generated by $b^+$ and $x$. So take $x = E_{lm}$ for some $l > m$. Taking commutators with $E_{jl}$, $j \leq l$, shows us $E_{jm}$ belongs to the algebra generated by $E_{lm}$ and $b^+$. Similarly, we must have $E_{lk}$, $k \geq m$, in this algebra. Repeating this process, we find that all $E_{jk}$, $j \leq l$, $k \geq m$, are in the algebra. These span the whole block to the upper right of $E_{lm}$. Next suppose we have two elements $E_{lm}, E_{rs}$ which generate overlapping blocks, in the sense that $m < s \leq l < r$. Then from the argument above, we can find $E_{rl}$ in the algebra generated by $b^+$ and $E_{rs}$. Hence $[E_{rl}, E_{lm}] = E_{rm}$ is in our algebra, and therefore so is the smallest diagonal block containing both $E_{lm}$ and $E_{rs}$. Thus we get the general parabolic containing $b^+$ by adding disjoint diagonal blocks. We remark that the calculations sketched above are similar to those used in the context of general root systems.

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