Celestial Mechanics.

Newton wrote down differential equations governing the motions of celestial bodies, that is, stars, planets, comets, and satellites. When only two bodies are involved, say the sun and a planet, Newton could solve his equations. In so doing he recovered Kepler's laws of planetary motion and became famous.

Add a third body to the mix. We get the famous three-body problem, not simply an unsolved problem, but an unsolvable problem. Poincaré and Bruns proved that the three-body problem is "unsolvable" in the precise sense that It does not admit 'enough' analytic integrals to integrate it. Like all the great impossibility proofs – the impossibility of trisecting an angle, of solving the general quintic, or Gödel's incompleteness theorem, this impossibility proof adds much more to our knowledge than it subtracts. In developing his impossibilty proof he discovered what we call "chaos", and built the foundations for a whole new field of inquiry – the qualitative theory of dynamical systems.

In the qualitative theory we do not focus on a single solution, but rather consider the family of all possible solutions and how they fit together. Each solution forms a curve within phase space. A single point in phase space represents the positions and velocities of all N bodies. Phase space has an even dimension $2n$ where n is called "the number of degrees of freedom" and is of the order of the number N of bodies. The motion of all solutions taken together forms a kind of complicated fluid flow in phase space, so that the motion of a particular solution is like the trace made by a single water molecule.

The dominant features of phase space are collisions, central configurations, and periodic orbits. Associated with each of these features is a theorem. These theorems represent the main theoretical advances in celestial mechanics since the time of Poincaré.

Collision and Blow-up. When two or more bodies collide, their accelerations and velocities tend to infinity. Collision solutions are examples of blow-up: solutions that cannot be analytically extended past some finite time. If a solution blows up, some of its velocities must tend to infinity. Is collision the only kind of blow-up? Jeff Xia and Joseph Gerver (separately) have established that for 5 or greater bodies non-collision blow-up solutions exist. As the blow-up time is approached in their solutions, the distances between some of the bodies behave roughly like the function $(1/t)\sin(1/t)$ as $t \to 0$. These bodies alternate between getting very close, and then flying very far away. Their positions have no well-defined limit.

One of the main new tools used by Xia in obtaining his solution was a different kind of blow-up, Dick McGehee's blow-up of the collision singularity. McGehee found a way to slow down time and rescale positions and velocities so that it takes infinite time to reach collision, while the single singular point of N-tuple collision in phase space is "blown-up" into an entire manifold, called the collision manifold (roughly the tangent bundle of a sphere) of dimension proportional to N. The rest points on the collision manifolds and their stable and unstable manifolds poking out into normal phase space organize the near-collision flow and enabled Xia to construct his solution.

Central Configurations. According to the qualitative theory the most important feature of a differential equation (a vector field) is its zeros, or rest points. The N-body problem has no rest points. The bodies cannot just sit in space without moving, since they all attract each other. The next best thing to a rest point is a rest point modulo symmetries. These are solutions which evolve purely by symmetry – by rotation or scaling – so that the ratios of the interbody distances all remain constant. They are called central configurations. For the three-body problem there are exactly five up to symmetry. Three were discovered by Euler. They are collinear. Two were found by Lagrange. They have the shape of equilateral triangles. It was not until the 20th century that these central configurations shown theoretically to exist by Lagrange were discovered to actually exist in our solar system. In the plane of the orbit of Jupiter around the sun, there are two equilateral triangles with the sun and Jupiter at two of the vertices; one triangle has a third vertex, generally denoted $L4$, sixty degrees in advance of Jupiter, and the other, $L5$, sixty degrees behind. In 1906, astronomers discovered "Achilles" – the first of what are now over 1,000 small bodies called Jupiter's Trojan satellites clustered around the Lagrange point $L4$. while the same year, "Patroclus" became the first of over 600 (as of November 2004) further Trojan satellites clustered around $L5$.

How many central configurations are there for four bodies? Until 2003, we did not even know if this number was finite. In a tour-de-force Hampton and Moeckel using a clever way to write the equations of central configurations due to Albouy and Chenciner, and the "GGK" theory from algebraic geometry, have established finiteness of the number of central configurations for 4 bodies. The question remains open for 5 bodies.

Periodic Solutions and Stability. A solution is periodic if, after some period, all of the bodies return to their original positions, along with their velocities. All the elliptic orbits of Kepler are periodic. We tend to think of the motion of the solar system as periodic, with the planets marching around the sun in an orderly fashion, returning to where they began after some period .

Is the solar system stable? Suppose that the planets and Sun, left to themselves with no other forces acting on them but their mutual Newtonian gravitational attractions, remain on their current essentially elliptical orbits for all time. Introduce a dust particle, coming in to the solar system from far beyond Pluto. Could the dust's influence disrupt the solar system, sending the Earth spinning off towards Alpha Centauri? To formulate the question in mathematical terms, imagine the solution of the N-body problem represented by the solar system, and assume that it is a known periodic solution. Perturb the initial conditions (the starting point of the solution in phase space) by slightly increasing the speed of Mercury, or by pushing the Earth an angstrom closer to the sun. Does the perturbed solution stay in a neighborhood of our known periodic solution? By general theory, it must stay close for a long time. But does it stay close for all time? And does it stay close for all time and for all sufficiently small perturbations? If the answer is yes the known solution is called "stable". For the two-body system every solution with negative energy (the Kepler ellipses) is periodic and stable. For the three-body problem we do not know if a single stable periodic solution exists! (For the restricted three body problem in which one of the masses is zero we do have stability. See below.) This problem is one of the great problems in celestial mechanics.

Despite our ignorance, we have something nearly as good as stability, a weaker type of stability called KAM stability. KAM stands for Kolmogorov, Arnold, Moser. Kolmogorov formulated the theorem and gave the original proof in a seminar. Arnold, his student, wrote out the proof and extended it to a degenerate case necessary for applications to the reduced three-body problem (described below). Moser further extended the proof. The KAM theorem says that if a certain necessary condition (the KAM twist condition) is satisfied along a periodic orbit then that orbit is "KAM stable". The necessary condition can sometimes be checked in practice and if it holds the theorem asserts that a great many orbits starting near our known orbit will stay near to it for all time.

KAM ideas can be tested on our solar system. Through a combination of careful numerical methods and new theoretical tools, Jacques Laskar has found evidence that the motion of our solar system occurs in a region of phase space where there are few if any KAM torii. The outer (Jupiter and beyond) and inner solar system behave quite differently and the outer one is more stable. Diffusion occurring within the inner one leads to a high probability that Mercury and Venus will collide within 5 billion years.

Scads of Periodic Orbits. How does one find an individual periodic orbit to begin with? Until recently the main method has been perturbation theory. Thanks to Newton, we know the exact solutions of the 2-body system. We can view the N-body problem as a perturbation of a number of uncoupled 2-body systems by making one mass, the Sun, very big in relation to all the other planets. In this uncoupled system we have a great many periodic orbits, where each planet moves around the Sun in its own Keplerian ellipse without affecting the other planets. If we normalize the mass of the Sun to be 1, then the planet masses are 0 in the uncoupled limit. If the true planet masses are m_i set $m_i = \epsilon\mu_i$ and call ϵ the "perturbation parameter". Turn up ϵ from $\epsilon = 0$ (uncoupled) and see if you can "follow" the uncoupled periodic orbits to the real system $\epsilon = 1$, as a function of ϵ. Methods of advanced calculus often allow us to succeed with this "following" at least partway into the realm of nonzero ϵ.

But if all the masses are equal, as in perhaps a triple star system, perturbation theory fails. A new method has been developed over the last decade to deal with this case, and has resulted in the discovery of a horde of new periodic orbits. As with most "new ideas" it it is really an old idea. Its main outline appears in an 1896 paper of Poincaré, and is as follows. A solution to the N-body equation is a curve in phase space, and if the solution is periodic, then the curve is closed. Instead of focusing on solutions, focus on periodicity by considering the space of all closed paths in phase space, perhaps imposing extra conditions if necessary. In order to single out the periodic solutions from among all the periodic paths, we use the "action function". This is a function which takes a path and produces a number by integrating the difference of the instantaneous kinetic energies and the potential energies over the path. Its critical points – the points where its derivative is zero – are precisely the solutions to Newton's equations. This critical point property, called "the action principle," was discovered by Euler and Lagrange, and is central to Feynman's approach to quantum mechanics.

The main technical difficulty in using the action principle to find periodic solutions is collisions. The action function is not differentiable along paths with collision and the principle breaks down.

In the past 5 years a number of researchers, including Chenciner and Montgomery, have been able to deal with this by a combination of symmetry constraints and detailed understanding of the central configurations. The results have been a horde of new orbits, many with beautiful floret type patterns. But so far, only one of these new orbits has been found to be KAM stable, the "figure eight orbit" in which all three bodies move around a single figure eight in the plane.

Extrasolar planets

One of the most dramatic recent applications of classical celestial mechanics has been the series of discoveries, starting in the 1990's, of planets orbiting other stars. Despite the usual formulation that "Newton's Laws imply Kepler's Laws," there is the crucial difference that according to Kepler, the sun is fixed, whereas Newton tells us that the sun also moves about the common center of gravity. In the case of other stars, it is precisely the motion of the star induced by an orbiting planet that is detected, allowing us to deduce not only the existence of the planet, but also key orbital information, such as the period, semimajor axis and eccentricity of its orbit. These are all obtained by an analysis of the precise motion of the star over a period of time, or more specifically, of the component of its motion along our line of sight.

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