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AN ESTIMATE FOR THE VOLUME ENTROPY OF NONPOSITIVELY CURVED GRAPH-MANIFOLDS

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ABSTRACT. Let M be a closed 3-dimensional graph-manifold. It is proved that $h(g) > 1$ for every geometrization g of M, where $h(g)$ is the topological entropy of the geodesic flow of g.

§1. INTRODUCTION

Asymptotic geometry of a nonpositively curved (for brevity, NPC) graph-manifold is a complicated mixture of flat and hyperbolic parts, which both contribute nontrivially to the general picture. We recall that an NPC-metric on a closed 3-dimensional graph-manifold M recovers the JSJ-decomposition of M in the following sense. There is a unique (up to isotopy) minimal finite collection E of pairwise disjoint flat geodesically embedded tori and Klein bottles such that the metric completion of each connected component of the complement of E is a Seifert space called a block of M. Each block M_v is fibered over a 2-orbifold S_v with negative Euler characteristic, $\chi(S_v) < 0$. Furthermore, along the interior of each block the metric locally splits as $U \times (-\varepsilon, \varepsilon)$, where U is an NPC-surface, this splitting is compatible with the fibration, the fibers are closed geodesics, and the regular fibers have one and the same length $l_v > 0$ depending only on the block.

Since we are interested in asymptotic properties, which are certainly the same for any finite covering of M, we may assume for simplicity that M is orientable, the collection E consists of tori, and each block is a trivial S^1-bundle over a compact surface S_v with boundary, $M_v = S_v \times S^1$. We also assume that the graph-manifold structure of M is nontrivial, i.e., that M itself is not a Seifert fibered space (though it may consist of one block).

The flat part of the asymptotic geometry of M was studied in [BS], [CK]; see also [HS]. Roughly speaking, this part can be described by fairly special geodesic rays $[0, \infty) \to M$, which leave any block, passing through separating tori $e \in E$ almost tangentially and spending most of the time near tori; moreover, this time rapidly increases at each step. Though the set of such rays is a negligible part of all rays, it keeps an important information about the geometry of M: in [CK] it was shown how this information allows one to recover (up to scaling) the marked length spectrum of the closed geodesics on S_v, as well as the fiber length for each block M_v.

In this paper we study the hyperbolic part of the asymptotic geometry of M, assuming that the surface U occurring in the local splitting $U \times (-\varepsilon, \varepsilon)$ as above has constant curvature $K = -1$. In other words, each block fibers over a *hyperbolic* orbifold (surface) S_v. An NPC-metric on M satisfying this condition is called a *geometrization* of M

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(because the metric of each block is modeled on $\mathrm{H}^2 \times \mathbb{R}$). We note that any geometrization of M is only $C^{1,1}$-smooth, being analytic along the interior of each block. It is known [L] that M admits an NPC-metric if and only if M admits a geometrization. In topological terms, necessary and sufficient conditions for M to carry an NPC-metric were found in [BK].

A relevant metric invariant that measures hyperbolicity of a space is the volume entropy h. Let $\pi : X \to M$ be the universal covering, and let $x_0 \in X$. We recall that $h = h(X)$ is defined by the formula

$$h = \lim_{R\to\infty} \frac{1}{R} \ln \operatorname{vol} B_R(x_0),$$

where $B_R(x)$ is the ball in X of radius R and centered at x. It is well known (see [M]) that the limit above exists, the quantity h is independent of the choice of x_0, and that if M is NPC, then h coincides with the topological entropy of the geodesic flow of M. The volume entropy scales as l^{-1}, where l is the length. Thus, the choice of a geometrization g of M also serves as a normalization. For the sectional curvatures of g we have $-1 \le K \le 0$. Consequently, $h(g) \le 2$, by comparison with H^3. Moreover, for the Ricci curvatures of g we have $-1 \le \mathrm{Ric}_g \le 0$ (whereas $\mathrm{Ric}_{\mathrm{H}^3} = -2$). The space $\sqrt{2}\,\mathrm{H}^3$, where the distances are those of H^3 multiplied by $\sqrt{2}$, has the constant Ricci curvature $\mathrm{Ric}_{\sqrt{2}\,\mathrm{H}^3} = -1$ and the volume entropy $h(\sqrt{2}\,\mathrm{H}^3) = \frac{1}{\sqrt{2}} h(\mathrm{H}^3) = \sqrt{2}$. The Bishop comparison theorem yields

$$\operatorname{vol} B_R^g \le \operatorname{vol} B_R^{\sqrt{2}\,\mathrm{H}^3}$$

for the volumes of the balls of one and the same radius R in the metric g and in $\sqrt{2}\,\mathrm{H}^3$. Thus, $h(g) \le \sqrt{2}$ for every geometrization g.

Our main result is as follows.

Theorem 1.1. *For any geometrization g of a graph-manifold M we have $h(g) > 1$.*

Remark 1.2. Though the universal covering X of M looks much more complicated than the model space $\mathrm{H}^2 \times \mathbb{R}$, even the estimate $h(g) \ge 1 = h(\mathrm{H}^2 \times \mathbb{R})$ is not obvious and nontrivial: X contains no isometrically and geodesically embedded H^2, which would lead to $h(g) \ge 1$; on the other hand, the attempt to compare X and $\mathrm{H}^2 \times \mathbb{R}$ via exponential maps identifying some tangent spaces fails, because the Jacobian of such a map is greater than 1 at some points. Finally, the estimate in [BW] for the measure-theoretic entropy of the geodesic flow, which never exceeds $h(g)$, gives only $\pi/4 < 1$ as a lower bound for any geometrization g of M (even if we ignore the fact that the $C^{1,1}$-smoothness of g is not sufficient for the application of that estimate).

In the proof of Theorem 1.1 we use the well-known fact that $h(g)$ coincides with the critical exponent of the Poincaré series

$$\mathcal{P}(t) = \sum_{\gamma\in\Gamma} e^{-t|x_0 - \gamma x_0|},$$

where the fundamental group $\Gamma = \pi_1(M)$ acts on X isometrically as the deck transformation group. Actually, instead of $\mathcal{P}$ we use a modified Poincaré series $\mathcal{P}_W$ in which summation is taken over some set W of walls in X. Our proof involves three ingredients:

(i) A local estimate, which is technical and used in (ii); this estimate is obtained in §2.

(ii) An accumulating procedure, which consists in the inductive construction of appropriate broken geodesics in X between the base point x_0 and the walls in W; the choice of these paths is the key point of the proof. The outcome of the accumulating procedure is the generating set for $\mathcal{P}_W$ to be used in (iii); the procedure is described in §3.

(iii) A self-similarity type argument. In this part of the proof we use a standard idea from self-similarity theory and the generating set obtained in (ii) to show that $\mathcal{P}_W(h)$ diverges for some $h > 1$. This is done in §4.

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§2. Local estimate

Let F be the universal covering of a compact hyperbolic surface S with geodesic boundary. We identify F with a convex subset $F \subset \mathrm{H}^2$ bounded by countably many disjoint geodesic lines and fix a point $o \in \mathrm{H}^2 \setminus F$. Let w_0 be the boundary line of F closest to o, and let $o_0 \in w_0$ be the point on w_0 closest to o, so that $|o - o_0| = \operatorname{dist}(o, F) =: l > 0$.

We denote by A the set of the boundary lines of F different from w_0. For $w \in A$, let $o_w \in w$ be the point closest to o. Then the geodesic segment oo_w intersects w_0 at some point t_w, and for $\widetilde{l}_w = |o - o_w|$ we have

$$\widetilde{l}_w = \widetilde{l}'_w + l''_w,$$

where $\widetilde{l}'_w = |o - t_w|$, $l''_w = |t_w - o_w|$ (all distances are taken in H^2).

Next, we identify H^2 with $\mathrm{H}^2 \times 0 \subset \mathrm{H}^2 \times \mathbb{R}$, so that F becomes a subset of $\mathrm{H}^2 \times \mathbb{R}$; we keep the notation introduced above. Observe that the point o_0 is the closest to o among the points of the wall $w_0 \times \mathbb{R}$. We take a nonhorizontal geodesic line $\sigma \subset w_0 \times \mathbb{R}$ through o_0, i.e., $\sigma \neq w_0 \times 0$, and take $s_w \in \sigma$ with $|s_w - o_0| = |t_w - o_0|$. Now, we put $l'_w := |o - s_w|$ (the distance is taken in $\mathrm{H}^2 \times \mathbb{R}$), and $\Delta_w := \widetilde{l}'_w - l'_w$.

In other words, we replace the distance $\widetilde{l}'_w$ between o and t_w in the hyperbolic plane H^2 by the distance l'_w in $\mathrm{H}^2 \times \mathbb{R}$, which is shorter by comparison: the triangles $oo_0t_w \subset \mathrm{H}^2 \times 0$, $oo_0s_w \subset \mathrm{H}^2 \times \mathbb{R}$ have right angles at o_0 ($\angle(oo_0t_w) = \pi/2 = \angle(oo_0s_w)$), have a common side oo_0, and have equal sides $|o_0 - t_w| = |o_0 - s_w|$. Since oo_0t_w lies in the hyperbolic plane $\mathrm{H}^2 \times 0$, but oo_0s_w does not, we have $\Delta_w > 0$ except in the case where $t_w = o_0 = s_w$. Now, we want to estimate the accumulation of the differences Δ_w from below. The precise statement is as follows.

Lemma 2.1. *For any $l_0 > 0$, and any $\alpha_0 \in (0, \pi/2]$, there exists $\lambda_0 > 1$, which depends only on l_0, α_0, and the compact surface S, so that*

$$\lambda(F, l, \alpha) := e^l \sum_{w \in A} e^{\Delta_w} e^{-\widetilde{l}_w} \geq \lambda_0$$

whenever $l = \operatorname{dist}(o, w_0) \geq l_0$ and the angle α between the lines $w_0 \times 0$ and σ is at least α_0, i.e., $\alpha_0 \leq \alpha \leq \pi/2$.

Proof. By a well-known formula of hyperbolic geometry, we have $e^l = (\tan \frac{\psi}{4})^{-1}$, $e^{-\widetilde{l}_w} = \tan \frac{\psi_w}{4}$, where ψ and ψ_w are the angles under which w_0 and $w \in A$ are observed in H^2 from o, respectively. The boundary at infinity $\partial_\infty F \subset \partial_\infty \mathrm{H}^2 = S^1$ coincides with the limit set of $\pi_1(S)$ represented in $\operatorname{Iso}(\mathrm{H}^2)$ as a Fuchsian group of the second kind. It is well known that the Hausdorff dimension of $\partial_\infty F$ (with respect to the angle metric) is less than 1; in particular, the Lebesgue measure of $\partial_\infty F$ is zero. Thus, $\psi = \sum_{w \in A} \psi_w$. Therefore, $\tan \psi/4 \leq \sum_{w \in A} \tan \frac{\psi_w}{4}$, and

$$e^l \sum_{w \in A} e^{-\widetilde{l}_w} = \sum_{w \in A} \tau_w \geq 1,$$

where $\tau_w = e^{l-\widetilde{l}_w}$. However, the sum $\sum_{w\in A}\tau_w$ can be made as close to 1 as we like (e.g., we can let $l\to\infty$). As was mentioned above, $\Delta_w > 0$ unless $s_w = t_w$. Thus, we always have $\lambda(F,l,\alpha) > 1$. The point is that $\lambda(F,l,\alpha)$ is separated away from 1 uniformly over all $l \ge l_0$, $\alpha \ge \alpha_0$.

Consider the subset $A_0 \subset A$ that consists of all $w \in A$ with $|t_w - o_0| \ge 1$. Then $\Delta_w \ge \delta_0 > 0$ for all $w \in A_0$, where δ_0 depends only on l_0 and α_0. We claim that

$$\sum_{w\in A_0}\tau_w \ge m_0 > 0, \tag{1}$$

where $m_0 = m_0(\widetilde{F}, l_0)$ is independent of l.

Assuming (1), we obtain

$$\begin{aligned}\lambda(F,l,\alpha) &= \sum_{w\in A} e^{\Delta_w}\tau_w \ge \sum_{w\in A_0} e^{\Delta_w}\tau_w + \sum_{w\in A\setminus A_0}\tau_w\\ &\ge e^{\delta_0}\sum_{w\in A_0}\tau_w + \sum_{w\in A\setminus A_0}\tau_w\\ &= (e^{\delta_0}-1)\sum_{w\in A_0}\tau_w + \sum_{w\in A}\tau_w \ge (e^{\delta_0}-1)m_0 + 1 =: \lambda_0 > 1.\end{aligned}$$

It remains to prove (1). Let $o'_w \in w$ be the point closest to $o_0 \in w_0$. Then $\widetilde{l}_w \le |o - o'_w| \le l + |o_0 - o'_w|$, whence $\widetilde{l}_w - l \le \operatorname{dist}(o_0, w)$. Consequently, $\tau_w \ge e^{-\operatorname{dist}(o_0,w)} \ge \widetilde{\psi}_w/4$, where w is observed from o_0 under the angle $\widetilde{\psi}_w$. Since the Lebesgue measure class on $\partial_\infty \mathrm{H}^2$ is independent of the choice of a marked point, we have $\sum_{w\in A}\widetilde{\psi}_w = \pi$.

Since $l \ge l_0$, for a sufficiently small $m_0 = m_0(S, l_0) > 0$ the sectors $S^+(m_0)$ and $S^-(m_0)$ to be defined below intersect no line $w \in A \setminus A_0$. The definition of the $S^\pm(m_0)$ is as follows. The common vertex o_0 of $S^\pm(m_0)$ divides the line w_0 into two opposite rays $w_0^\pm$. The sectors $S^\pm(m_0) \subset \mathrm{H}^2$ are bounded by the rays $w_0^\pm$, $s^\pm(m_0)$, where $\angle_{o_0}(s^\pm(m_0), w_0^\pm) = 2m_0$, and $s^\pm(m_0) \cap F \ne \emptyset$.

Therefore, from the relation $\sum_{w\in A}\widetilde{\psi}_w = \pi$ it follows that $\sum_{w\in A_0}\tau_w \ge m_0$, which completes the proof. □

§3. Accumulating procedure

In order to describe the accumulating procedure, we need some information about the metric structure of the universal covering X of (M, g), where g is a geometrization.

3.1. Metric structure of the universal covering. We recall (see, e.g., [BS], [CK]) that X can be represented as a countable union $X = \bigcup_v X_v$ of blocks, where each X_v is a closed convex subset in X isometric to the metric product $F_v \times \mathbb{R}$, and F_v is the universal covering of a compact hyperbolic surface S_v with geodesic boundary. Any two blocks are either disjoint, or intersect each other along a boundary component that is 2-flat in X and separates them; consequently, no three blocks have a point in common. The 2-flats in X that separate blocks are called *walls*. A common wall w of two blocks X_v and $X_{v'}$ covers a 2-torus $e \subset M$, which separates (possibly, only locally) the blocks $M_v = \pi(X_v)$, $M_{v'} = \pi(X_{v'})$ of M. The metric decompositions $X_v = F_v \times \mathbb{R}$ and $X_{v'} = F_{v'} \times \mathbb{R}$ do not agree on w, and their $\mathbb{R}$-factors induce two fibrations of w by parallel geodesics. We denote by α_w the angle between these fibrations, $0 < \alpha_w \le \pi/2$. Since M is compact and the set E of separating tori in M is finite, we have $\alpha_0 := \inf_w \alpha_w > 0$, where the infimum is taken over all walls in X.

3.2. Modified Poincaré series. We fix a wall $w^* \subset X$, a block $X_{v^*} \subset X$ for which w^* is a boundary wall, and a base point $x_0 \in w^*$. Let W_0 be the set of all boundary walls of X_{v^*} different from w^*; we denote by W_n, $n \geq 1$, the set of all walls in X that lie at the combinatorial distance $n+1$ from w^*, i.e., $w \in W_n$ if and only if the interior of any geodesic segment in X between x_0 and w intersects n walls, including a wall from W_0. Observe that $W = \bigcup_{n \geq 0} W_n$ consists of all walls in X that lie on the same side from w^* as X_{v^*}. Now, we define a modified Poincaré series by the formula

$$\mathcal{P}_W(t) = \sum_{w \in W} e^{-t \operatorname{dist}(x_0, w)}.$$

Comparing P_W with $\mathcal{P}(t) = \sum_{\gamma \in \Gamma} e^{-t|x_0 - \gamma x_0|}$, and using the triangle inequality, we easily see that $\mathcal{P}(t) \geq e^{-D} \mathcal{P}_W(t)$, where $D > 0$ is the maximum diameter of the tori $e \in E$. Recall that the critical exponent of $\mathcal{P}$ is defined as the infimum of all $t \in \mathbb{R}$ for which $\mathcal{P}(t) < \infty$. Therefore, the critical exponent $\overline{h}$ of $\mathcal{P}_W$ satisfies $\overline{h} \leq h(g)$, and Theorem 1.1 will be proved if we show that $\overline{h} > 1$.

3.3. Special broken-geodesic paths.

Proposition 3.1. *For each $n \geq 0$, we have*

$$\mathcal{P}_n(1) := \sum_{w \in W_n} e^{-\operatorname{dist}(x_0, w)} \geq \frac{\pi}{4} \lambda_0^n,$$

where $\lambda_0 > 1$ is the same constant as in Lemma 2.1.

Proof. We use induction on n. For each $n \geq 0$ and each wall $w \in W_n$, we construct a broken geodesic ξ_w in X between x_0 and w as follows. For $w \in W_0$ we put $\xi_w = x_0 x_w$, where $x_w \in w$ is the point closest to x_0. The segment $x_0 x_w$ lies in a horizontal slice $F_{v^*} \times \{r_0\}$ of the block $X_{v^*} = F_{v^*} \times \mathbb{R}$.

Suppose that the broken geodesic ξ_w is already defined for all k with $0 \leq k \leq n-1$ and all $w \in W_k$, and that ξ_w is a juxtaposition $\eta_{w_0} \eta_{w_1} \cdots \eta'_{w_k}$ of geodesic segments, where the sequence $w_i \in W_i$ of walls leads to $w = w_k$. Moreover, we assume that each segment of ξ_w lies in a block and connects its different boundary components, the last segment η'_{w_k} lies in a horizontal slice of the block that contains it, and η'_{w_k} is orthogonal to the wall w.

Let $w \in W_n$. Then there is a unique $\overline{w} \in W_{n-1}$ that precedes w. By assumption, the last edge $\eta'_{\overline{w}} = s_{\overline{w}} x_{\overline{w}}$ of $\xi_{\overline{w}}$ lies in a horizontal slice $F_{v'} \times \{r_{v'}\}$ of its block $X_{v'} = F_{v'} \times \mathbb{R}$ and is orthogonal to the wall $\overline{w}$ (at the endpoint $x_{\overline{w}}$). It is important that ξ_w contains all segments of $\xi_{\overline{w}}$ except for the last segment $\eta'_{\overline{w}}$.

Let $X_v \subset X$ be the other block adjacent to $\overline{w}$; in particular, the walls w and $\overline{w}$ are boundary components of X_v. We recall that the metric splitting $X_v = F_v \times \mathbb{R}$ does not agree with the metric splitting $X_{v'}$ along $\overline{w}$. Let $F_v \times \{r_v\}$ be the horizontal slice of X_v that contains the point $x_{\overline{w}}$ on the corresponding boundary component. The boundary lines of the slices $F_{v'} \times \{r_{v'}\}$ and $F_v \times \{r_v\}$, which lie in $\overline{w}$, contain $x_{\overline{w}}$ and form an angle of $\alpha_{\overline{w}} \in (\alpha_0, \pi/2]$.

To construct ξ_w, we act as follows. We take an isometric copy $F_v \subset \mathrm{H}^2 \times \{r_{v'}\}$ of $F_v \times \{r_v\}$ (by rotating the latter through the angle $\alpha_{\overline{w}}$), where $F_{v'} \times \{r_{v'}\} \subset \mathrm{H}^2 \times \{r_{v'}\}$, so as to ensure that $F_{v'} \times \{r_{v'}\}$ and F_v are sitting in the hyperbolic plane $\mathrm{H}^2 \times \{r_{v'}\}$ and are adjacent along their common boundary component, for which we use the same notation $\overline{w}$. Now, we connect the initial point $s_{\overline{w}}$ of the last edge $\eta'_{\overline{w}} \subset \xi_{\overline{w}}$ with the boundary component of F_v corresponding to w by the shortest geodesic segment $s_{\overline{w}} x'_w \subset \mathrm{H}^2 \times \{r_{v'}\}$ and take $t_w = s_{\overline{w}} x'_w \cap \overline{w}$. The segment $t_w x'_w$ turned back to $F_v \times \{r_v\}$ gives the last segment $\eta'_w = s_w x_w$ of ξ_w. Thus, $s_w \in \overline{w} \cap F_v \times \{r_v\}$, $|s_w - x_{\overline{w}}| = |t_w - x_{\overline{w}}|$, and the segment $\eta'_w \subset F_v \times \{r_v\}$ is orthogonal to w at x_w. To complete the construction of ξ_w, we delete the last segment $\eta'_{\overline{w}} \subset \xi_{\overline{w}}$, replacing it with $\eta_{\overline{w}}\eta'_w$, where $\eta_{\overline{w}} = s_{\overline{w}}s_w \subset X_{v'}$. Clearly, the resulting ξ_w has all properties advertised above.

Let $l = |s_{\overline{w}} - x_{\overline{w}}| = L(\eta'_{\overline{w}})$ be the length of the last segment of $\xi_{\overline{w}}$, and let $l'_w = L(\eta_{\overline{w}})$, $l''_w = L(\eta'_w)$. Then

$$L(\xi_w) = L(\xi_{\overline{w}}) - l + l'_w + l''_w = L(\xi_{\overline{w}}) - l + \widetilde{l}_w - \Delta_w,$$

where $\widetilde{l}_w = L(s_{\overline{w}}x'_w)$ and $\widetilde{l}'_w = L(s_{\overline{w}}t_w)$, so that $\widetilde{l}_w = \widetilde{l}'_w + l''_w$, and $\Delta_w = \widetilde{l}'_w - l'_w$. We arrive at the same configuration that was studied in §2, and we are going to apply Lemma 2.1 to estimate $\mathcal{P}_n(1)$ from below. Since l is the length of a segment in X that connects different boundary components of a block, l is separated away from 0 by some positive constant l_0 depending only on M, $l \geq l_0 > 0$.

For $n = 0$ we have

$$\mathcal{P}_0(1) = \sum_{w \in W_0} e^{-|x_0 - x_w|} = \sum_{w \in W_0} \tan\frac{\psi_w}{4} \geq \frac{\pi}{4},$$

where ψ_w is the angle under which the boundary component w of X_{v^*} is observed from x_0 (in the horizontal direction).

By the induction assumption, we have

$$\mathcal{P}_{n-1}(1) \geq \sum_{w \in W_{n-1}} e^{-L(\xi_w)} \geq \frac{\pi}{4}\lambda_0^{n-1}.$$

We write $W_n = \bigcup_{\overline{w}} W_{n,\overline{w}}$, where the union is taken over all $\overline{w} \in W_{n-1}$ and each $w \in W_{n,\overline{w}}$ follows $\overline{w}$. Then

$$\mathcal{P}_n(1) \geq \sum_{w \in W_n} e^{-L(\xi_w)} = \sum_{\overline{w} \in W_{n-1}} e^{-L(\xi_{\overline{w}})} e^l \sum_{w \in W_{n,\overline{w}}} e^{\Delta_w} e^{-\widetilde{l}_w}.$$

Applying Lemma 2.1 with $A = W_{n,\overline{w}}$, we obtain

$$\mathcal{P}_n(1) \geq \lambda_0 \sum_{\overline{w} \in W_{n-1}} e^{-L(\xi_{\overline{w}})} \geq \frac{\pi}{4}\lambda_0^n,$$

which completes the proof. □

§4. Self-similarity argument

The constant $\lambda_0 > 1$ occurring in Proposition 3.1 depends only on some metric data of M. Therefore, for some $n \in \mathbb{N}$, $n = n(M)$, we have $\frac{\pi}{4}\lambda_0^n > 1$. Proposition 3.1 implies that

$$\mathcal{P}_n(\overline{h}) = \sum_{w \in W_n} e^{-\overline{h}\,\mathrm{dist}(x_0, w)} \geq 1$$

for some $\overline{h} > 1$. Furthermore, taking n sufficiently large, we can find $\overline{h} > 1$ such that $\mathcal{P}_n(\overline{h}) \geq 1$ for any choice of the initial block X_{v^*}, of its wall w^*, and of the base point $x_0 \in w^*$, because the set of such choices (up to isometries of X) is compact.

We fix $n \in \mathbb{N}$ with the above property and select a subset $W^* = \bigcup_{k \geq 1} W_k^* \subset W$, where $W_k^* = W_{kn}$. The set W_1^* serves as the generating set for W^*. Connecting x_0 with each wall $w \in W_1^*$ by the shortest geodesic segment x_0x_w, we obtain new base points $x_w \in w$ (these x_w may differ from the x_w constructed in the proof of Proposition 3.1). By induction, we find a base point $x_w \in w$ for each $w \in W_k^*$, $k \geq 1$, with the property that $\operatorname{dist}(x_{\overline{w}}, w) = |x_{\overline{w}} - x_w|$, where $\overline{w} \in W^*_{k-1}$ precedes w. Furthermore, by the choice of n and $\overline{h}$, for each $\overline{w} \in W^*_{k-1}$ we have

$$\sum_w e^{-\overline{h}|x_{\overline{w}} - x_w|} \geq 1,$$

where summation is over all $w \in W^*_k$ that follow $\overline{w}$. Since $\operatorname{dist}(x_0, w) \leq \operatorname{dist}(x_0, x_w) \leq \operatorname{dist}(x_0, x_{\overline{w}}) + |x_{\overline{w}} - x_w|$, we see that

$$\begin{aligned}\mathcal{P}_{W^*_k}(\overline{h}) &= \sum_{w \in W^*_k} e^{-\overline{h}\operatorname{dist}(x_0, w)} \\ &\geq \sum_{w \in W^*_{k-1}} e^{-\overline{h}\operatorname{dist}(x_0, x_w)} \geq \sum_{w \in W_n} e^{-\overline{h}\operatorname{dist}(x_0, x_w)} \geq 1\end{aligned}$$

for all $k \geq 1$. Therefore, the modified Poincaré series

$$\mathcal{P}_W(\overline{h}) \geq \mathcal{P}_{W^*}(\overline{h}) = \sum_{k \geq 1} \mathcal{P}_{W^*_k}(\overline{h})$$

diverges at $\overline{h}$, whence $h(g) \geq \overline{h} > 1$. This completes the proof of Theorem 1.1.

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