Алгебра и анализ
Том. 15 (2003), вып. 1

St. Petersburg Math. J.
Vol. 15 (2004), No. 1, Pages 127–137
S 1061-0022(03)00805-7
Article electronically published on December 31, 2003

EXPONENTIAL GROWTH OF SPACES WITHOUT CONJUGATE POINTS

N. D. LEBEDEVA

§1. Introduction

An *n-dimensional polyhedral space* is a length space $\mathfrak{M}$ (with intrinsic metric) triangulated into n-simplexes with smooth Riemannian metrics. In the definitions below, we assume that the triangulation is fixed.

The *boundary* of $\mathfrak{M}$ is the union of the $(n-1)$-simplexes of the triangulation that are adjacent to only one $(n-1)$-simplex.

As usual, a *geodesic* in $\mathfrak{M}$ is a naturally parametrized locally shortest curve defined on an interval. We say that $\mathfrak{M}$ *has no conjugate points* if any two points in the universal covering space $\widetilde{\mathfrak{M}}$ of $\mathfrak{M}$ are joined by a unique geodesic.

We say that *the volume entropy of* $\widetilde{\mathfrak{M}}$ *is positive* if the volume of metric balls in $\widetilde{\mathfrak{M}}$ has at least exponential growth.

Now, we state the main result of this paper.

Theorem 1. *Let $\mathfrak{M}$ be an n-dimensional compact polyhedral space without boundary and without conjugate points. If the triangulation of $\mathfrak{M}$ contains three n-simplexes with a common $(n-1)$-face, then the volume entropy of $\widetilde{\mathfrak{M}}$ is positive.*

Corollary 1. *Under the assumptions of Theorem* 1*, the fundamental group $\pi_1(\mathfrak{M})$ of $\mathfrak{M}$ has at least exponential growth.*

§2. Geodesics in $\mathfrak{M}$: $\mathbf{G}$, $\mathbb{S}\mathfrak{M}$, etc.

A geodesic in $\mathfrak{M}$ is *complete* if it is defined on the entire real line $\mathbb{R}$. A geodesic is *generic* if it intersects no $(n-2)$-simplexes and intersects $(n-1)$-simplexes transversally. We denote by $\mathbf{G}$ the set of complete generic geodesics in $\mathfrak{M}$ and consider the geodesic flow transformation (GFT)

$$\varphi_t : \mathbf{G} \to \mathbf{G}, \quad \varphi_t\gamma(s) = \gamma(t+s).$$

We observe that a generic geodesic $\gamma : [a,b] \to \mathfrak{M}$ with $\gamma(b) \notin \mathfrak{M}^{n-1}$ is uniquely continued beyond b. If $\gamma(b)$ belongs to a common $(n-1)$-face F of n-simplexes $\Delta_1, \dots, \Delta_l$, then γ is continued beyond b in $l-1$ different ways (uniquely into each of the remaining $l-1$ simplexes by the rule "the angle of incidence is equal to the angle of reflection").

The *tangent space* $T_x\mathfrak{M}$ of $\mathfrak{M}$ at a point $x \in \mathfrak{M}$ is the tangent cone of $\mathfrak{M}$ at x. If $x \in \mathfrak{M} \setminus \mathfrak{M}^{n-1}$, then $T_x\mathfrak{M}$ is isometric to $\mathbb{R}^n$.

2000 *Mathematics Subject Classification.* Primary 52B70.

Key words and phrases. Polyhedral space, volume entropy, conjugate points.

Partially supported by RFBR (grants nos. 02-01-00090 and 00-15-96024) and by CRDF (grant no. RM1-2381-ST-02).

If x belongs to an $(n-1)$-simplex F that is a common $(n-1)$-face of n-simplexes $\Delta_1, \dots, \Delta_l$, then $T_x\mathfrak{M}$ is the union of the half-spaces $T_x\Delta_i$ with common boundary hyperplane T_xF. We say that the vectors in $T_x\Delta_i \setminus T_xF$ *go in the direction of* Δ_i.

For each unit vector $\mathbf{e} \in T_x\mathfrak{M} \setminus T_xF$, there exists a geodesic γ with $\gamma'(0) = \mathbf{e}$. Observe that if we "glue together" two geodesic segments with initial velocity vectors making an angle of π, then we obtain a geodesic segment.

In what follows, we consider only tangent vectors at points in $\mathfrak{M} \setminus \mathfrak{M}^{n-2}$. For $x \in \mathfrak{M}$, we let $\mathbb{S}_x \subset T_x\mathfrak{M}$ be the set of unit tangent vectors in $T_x\mathfrak{M}$. For any set $K \subset \mathfrak{M}$, we define $\mathbb{S}K := \bigcup_{x\in K} \mathbb{S}_x$. Thus, $\mathbb{S}\mathfrak{M}$ is the space of all unit tangent vectors of $\mathfrak{M}$.

§3. The Liouville measure μ_L

Let $\mathfrak{M}$ be a polyhedral space. A canonical measure μ_L on the space $\mathbb{S}\mathfrak{M}$ is defined in a standard way as the product of two measures: the Riemannian volume on $\mathfrak{M}$ and Lebesgue measure λ_x on the unit $(n-1)$-sphere $\mathbb{S}_x$, $x \in \mathfrak{M}$. This measure is called *Liouville measure*.

Let $A = \{\gamma : [a,b] \to \mathfrak{M}\}$ be a set of generic geodesics. We define

$$A'(t) := \{\gamma'(t) \mid \gamma \in A\} \subset \mathbb{S}\mathfrak{M}, \quad t \in [a,b].$$

The *multiset* $A^\dagger(t)$ is the pair $(A'(t), \mathbf{1}_{A^\dagger(t)})$, where

$$\mathbf{1}_{A^\dagger(t)} : \mathbb{S}\mathfrak{M} \to \{0\} \cup \mathbb{N}$$

is the "indicator function" acting by the rule

$$\mathbf{e} \mapsto \#\{\gamma \in A \mid \gamma'(t) = \mathbf{e}\}.$$

The *measure* $\mu_L(A^\dagger(t))$ *of* $A^\dagger(t)$ is the integral of $\mathbf{1}_{A^\dagger(t)}$:

$$\mu_L(A^\dagger(t)) := \int_{\mathbb{S}\mathfrak{M}} \mathbf{1}_{A^\dagger(t)} d\mu_L.$$

If for any two geodesics $\gamma_1, \gamma_2 \in A$ we have $\gamma_1'(t) \neq \gamma_2'(t)$, then $A^\dagger(t)$ may be regarded as the usual set $A'(t)$, and $\mathbf{1}_{A^\dagger(t)}$ is the usual indicator function of $A'(t)$ (equal to 1 on $A'(t)$ and vanishing outside $A'(t)$): $\mathbf{1}_{A^\dagger(t)} = \mathbf{1}_{A'(t)}$. Obviously, in this case, we have

$$\mu_L(A^\dagger(t)) = \mu_L(A'(t)).$$

We say that two generic geodesics defined on the segment $[a,b]$ have *one combinatorial type* if they traverse the simplexes in the same succession. (In particular, they pass the branchings in the same way.)

Claim. *Let $A = \{\gamma : [a,b] \to \mathfrak{M}\}$ be a set of generic geodesics of one combinatorial type. Then*

$$\mu_L(A'(a)) = \mu_L(A'(b)), \tag{3.1}$$

i.e., the "transformation of the geodesic flow along A" preserves Liouville measure.

To see this, it suffices to prove that Liouville measure is preserved in a neighborhood of a point of any $(n-1)$-simplex adjacent exactly to two n-simplexes. We give a precise statement.

Lemma 1. *Let F be a common $(n-1)$-face of two n-simplexes Δ_1 and Δ_2. Let $U \subset \Delta_1 \cup \Delta_2$ be an open ball with center in F, and let $B = \{\gamma : [0,c] \to U\}$ be a set of generic geodesics. Then*

$$\mu_L(B'(0)) = \mu_L(B'(c)).$$

Proof. The set B is a countable union of sets B_k such that each geodesic in B_k intersects F at an angle greater than $1/k$. It suffices to prove that $\mu_L(B'_k(0)) = \mu_L(B'_k(c))$ for each k. Therefore, we may assume that each geodesic in B intersects F once.

Let dx be the volume element on F. We consider the measure μ_F on $\mathbb{S}F$ with the density

$$d\mu_F(\mathbf{v}) = |\cos\alpha(\mathbf{v})| d\lambda_x(\mathbf{v}) dx,$$

where $\alpha(\mathbf{v})$ is the angle between $\mathbf{v}$ and the normal to F (see [1] and [4, Chapter 6]). For $C \subset \mathbb{S}F$, $\mu_F(C)$ is equal to the flux across C of the vector field generating the geodesic flow on U.

Let t_γ be the value of the parameter for which $\gamma \in B$ intersects F: $\{t_\gamma\} = \gamma^{-1}(F)$. The mapping

$$B'(0) \to \mathbb{S}F \times [0, c], \quad \gamma'(0) \mapsto (\gamma'(t_\gamma), t_\gamma),$$

determines coordinates $(\mathbf{v}, t)$ on $B'(0)$.

Since the Liouville measure μ_L is preserved within one simplex, the density of μ_L in the coordinates $(\mathbf{v}, t)$ has the form $d\mu_L(\mathbf{v}, t) = d\mu_F(\mathbf{v}) dt$ (see [4, Chapter 6]). Under the passage from $B'(0)$ to $B'(c)$, the vector $\mathbf{v} \in \mathbb{S}F$ changes to the opposite one, and the parameter along the flow changes by a constant. Hence, the Liouville measure μ_L is preserved. $\square$

Proposition 1. *Let $A = \{\gamma : [a, b] \to \mathfrak{M}\}$ be a set of generic geodesics. Then*

$$\mu_L(A^\dagger(a)) = \mu_L(A^\dagger(b)). \tag{3.2}$$

Proof. 1) First, suppose that the geodesics in A have one combinatorial type. Then the velocity vector at t uniquely determines a geodesic in A, so that $\mathbf{1}_{A^\dagger(t)} = \mathbf{1}_{A'(t)}$ for each $t \in [a, b]$, whence

$$\mu_L(A^\dagger(a)) = \mu_L(A'(a)) \overset{(*)}{=} \mu_L(A'(b)) = \mu_L(A^\dagger(b)).$$

2) In the general case, A splits into countably many subsets A_k in each of which the geodesics have one combinatorial type. For each k, we have

$$\mu_L(A_k^\dagger(a)) = \mu_L(A_k^\dagger(b)).$$

Summing these relations, we obtain (3.2). $\square$

§4. Proof of Theorem 1

Let X be an $(n-1)$-simplex that is a common hyperface of n-simplexes $\Upsilon_1, \ldots, \Upsilon_d$, where $d \geq 3$.

Notation. Let $\gamma : (a, b) \to \mathfrak{M}$ be a generic geodesic. Suppose that γ passes from an n-simplex Δ_1 to an n-simplex Δ_2 and intersects their common $(n-1)$-face at a point $\gamma(c)$, $c \in (a, b)$. We define

$$\gamma'_+(c) := \gamma'(c) \in T_{\gamma(c)}\Delta_2 \subset T_{\gamma(c)}\mathfrak{M} \quad \text{and} \quad \gamma'_-(c) := \bar{\gamma}'(-c) \in T_{\gamma(c)}\Delta_1 \subset T_{\gamma(c)}\mathfrak{M},$$

where $\bar{\gamma}(t) = \gamma(-t)$.

A tangent vector $\mathbf{v}$ at a point of an $(n-1)$-simplex F is said to be *almost orthogonal* to F if $\mathbf{v}$ makes an angle less than $\pi/10$ with one of the normals of F.

We denote by $(\widetilde{\mathfrak{M}}, \widetilde{\rho})$ a universal cover of $(\mathfrak{M}, \rho)$, where $\widetilde{\rho}$ is the lifting of the metric ρ.

Let $\Omega \subset X$ be a (sufficiently small) region. We denote by $\widetilde{\Omega}$ the preimage of Ω in $\widetilde{\mathfrak{M}}$.

We recall that $\mathfrak{M}$ is isometric to the quotient space $\widetilde{\mathfrak{M}}/\Gamma$, where Γ is a subgroup of the group of isometries of $\widetilde{\mathfrak{M}}$ isomorphic to $\pi_1(\mathfrak{M})$. We denote by $\mathfrak{M}_0$ a fundamental domain in $\widetilde{\mathfrak{M}}$.

4.1. A special set. For convenience, we introduce a certain subset $\mathbf{G}_{\mathrm{reg}} \subset \mathbf{G}$ within which all geodesics extend uniquely in both directions, except the branching on Ω.

For this, we introduce the following structure. For each $(n-1)$-simplex F, we cyclically order the n-simplexes adjacent to F. We say that a generic geodesic γ is *regular* if for every $x \in \gamma^{-1}(\mathfrak{M}^{n-1})$ one of the following two conditions is fulfilled:

1) $\gamma(x) \in \Omega$, and the vector $\gamma'_-(x)$ is almost orthogonal to X;

2) the point $\gamma(x)$ belongs to a common $(n-1)$-face F of n-simplexes $\Delta_1, \dots, \Delta_l$, and

$$\gamma'_-(x) \in T_{\gamma(x)}\Delta_i, \quad \gamma'_+(x) \in T_{\gamma(x)}\Delta_{i+1}.$$

(As usual, we set $\Delta_{l+1} := \Delta_1$.)

We denote by $\mathbf{G}_{\mathrm{reg}}$ the set of complete regular geodesics.

Remark 1. 1) For "almost every" unit tangent vector $v \in \mathbb{S}\mathfrak{M}$, each regular geodesic with initial velocity vector v can be continued to a generic regular complete geodesic. (Cf. the Appendix.)

2) The set $\mathbf{G}_{\mathrm{reg}}$ is GFT-invariant.

For $V \subset \mathbb{S}\mathfrak{M}$, we define

$$\mathbf{G}(V) := \{\gamma \in \mathbf{G} \mid \gamma'(0) \in V\}, \quad \mathbf{G}_{\mathrm{reg}}(V) := \{\gamma \in \mathbf{G}_{\mathrm{reg}} \mid \gamma'(0) \in V\}.$$

4.2. A special measure on $\mathbf{G}_{\mathrm{reg}}$. In the Appendix it is proved that there is a measure $\mathbf{m}$ on $\mathbf{G}$ such that $\mathbf{G}_{\mathrm{reg}}$ has full measure in $\mathbf{G}$ and the following properties 1)–3) are fulfilled (an invariant measure on $\mathbf{G}$ satisfying property 2) is described in detail in [1]):

1) the measure $\mathbf{m}$ is GFT-invariant;

2) if $V \subset \mathbb{S}\mathfrak{M}$ is a measurable subset, then $\mathbf{m}(\mathbf{G}_{\mathrm{reg}}(V)) = \mu_L(V)$;

3) let $j \in \{1, \dots, d\}$, and let $\Psi = \{\gamma : (-\infty, x_\gamma) \to \mathfrak{M}\}$ be a set of one-sided regular geodesics such that for each $\gamma \in \Psi$ we have $\gamma(x_\gamma) \in \Omega$, the vector $\gamma'_-(x_\gamma)$ is almost orthogonal to X, and $\gamma'_-(x_\gamma) \in T_{\gamma(x_\gamma)}\Upsilon_j$. Furthermore, we assume that none of the geodesics in Ψ is a continuation of another geodesic. For $i \in \{1, \dots, d\} \setminus \{j\}$, let Ψ_i denote the set of continuations of all geodesics in Ψ to the simplex Υ_i and further, in all possible ways, up to complete regular geodesics in $\mathbf{G}_{\mathrm{reg}}$, i.e.,

$$\Psi_i := \{\gamma \in \mathbf{G}_{\mathrm{reg}} : \gamma|_{(-\infty, x_\gamma)} \in \Psi \ \&\ \gamma'_+(x_\gamma) \in T_{\gamma(x_\gamma)}\Upsilon_i\}.$$

Thus, $\overline{\Psi} := \bigcup_i \Psi_i$ is the set of all possible continuations of geodesics in Ψ to complete geodesics in $\mathbf{G}_{\mathrm{reg}}$. Then

$$\mathbf{m}(\Psi_i) = \frac{\mathbf{m}\left(\overline{\Psi}\right)}{d-1}, \quad i \in \{1, \dots, d\} \setminus \{j\}.$$

4.3. Estimating the measure of a set of geodesics. We denote by $\widetilde{\mathbf{G}}$ the space of complete generic geodesics $\gamma : \mathbb{R} \to \widetilde{\mathfrak{M}}$, we denote by $\widetilde{\mathbf{G}_{\mathrm{reg}}}$ the preimage of the set $\mathbf{G}_{\mathrm{reg}}$ in $\widetilde{\mathbf{G}}$, and we let $\pi : \widetilde{\mathbf{G}_{\mathrm{reg}}} \to \mathbf{G}_{\mathrm{reg}}$ be the projection map, which is a covering.

We say that a geodesic γ in $\widetilde{\mathfrak{M}}$ is *regular* if γ is a lifting of a regular geodesic.

Let $\widetilde{\mathbf{m}}$ denote the lifting of the measure $\mathbf{m}$ to $\widetilde{\mathbf{G}_{\mathrm{reg}}}$ under the covering π.

For a set $\mathcal{A}$ of regular generic geodesics in $\widetilde{\mathfrak{M}}$ that are defined on segments, we denote by $\widetilde{\mathbf{m}}(\mathcal{A})$ the $\widetilde{\mathbf{m}}$-measure of the set of all possible continuations of the geodesics in $\mathcal{A}$ to complete geodesics in $\widetilde{\mathbf{G}_{\mathrm{reg}}}$. If for $\gamma \in \mathcal{A}$ we have $\gamma(a) \in \mathfrak{M}_0$, then the projection of $\mathcal{A}$ to the space of geodesics in $\mathfrak{M}$ is injective, and the measure $\widetilde{\mu}_L(\mathcal{A}^\dagger(a))$ of the multiset $\mathcal{A}^\dagger(a)$ is well defined as the measure of the projection. The following lemma relates the measure $\widetilde{\mathbf{m}}(\mathcal{A})$ of some set $\mathcal{A}$ of geodesics defined on a segment to $\widetilde{\mu}_L(\mathcal{A}^\dagger)$.

Lemma 2. *Let $l \in \mathbb{N}$, and let $\mathcal{A} = \{\gamma : [a,b] \to \widetilde{\mathfrak{M}}\}$ be a set of regular generic geodesics with initial points in a fundamental domain $\mathfrak{M}_0$. Suppose that for every $\gamma \in \mathcal{A}$ there exist parameters $t_1(\gamma) < \cdots < t_l(\gamma) \in (a,b)$ such that $\gamma(t_i(\gamma)) \in \widetilde{\Omega}$ for each i, and $\gamma'_-(t_i(\gamma))$ is almost orthogonal to $\widetilde{X}$ and goes in the direction of $\widetilde{\Upsilon}_1$. Then*

$$\widetilde{\mathbf{m}}(\mathcal{A}) \le \frac{\widetilde{\mu}_L(\mathcal{A}^\dagger(a))}{(d-1)^l} = \frac{\widetilde{\mu}_L(\mathcal{A}^\dagger(b))}{(d-1)^l}. \tag{$*$}$$

(*The identity* $\widetilde{\mu}_L(\mathcal{A}^\dagger(a)) = \widetilde{\mu}_L(\mathcal{A}^\dagger(b))$ *was proved earlier.*)

Proof. At a point $t_s(\gamma)$, the geodesic $\gamma \in \mathcal{A}$ passes to one of the simplices $\widetilde{\Upsilon}_2, \dots, \widetilde{\Upsilon}_d$. With each geodesic $\gamma \in \mathcal{A}$ we associate a sequence $\{i_1, \dots, i_l\}$ so that γ passes to $\widetilde{\Upsilon}_{i_k}$ at the point $t_k(\gamma)$. Since the number of such sequences is finite, $\mathcal{A}$ splits into finitely many subsets $\mathcal{A}_i$ such that for each i the geodesics in $\mathcal{A}_i$ determine one and the same sequence. $\square$

Claim. *The subsets $\mathcal{A}_i$ satisfy the required inequality, namely,*

$$\widetilde{\mathbf{m}}(\mathcal{A}_i) \le \frac{\widetilde{\mu}_L(\mathcal{A}'_i(a))}{(d-1)^l} \tag{$*_i$}$$

for each i.

We denote by $\mathcal{A}_{i,s}$ the set consisting of all complete geodesics in $\widetilde{\mathbf{G}_{\mathrm{reg}}}$ that are continuations of the restrictions of the geodesics $\gamma \in \mathcal{A}_i$ to the interval $(a, t_s(\gamma))$ (this interval is specific for each geodesic γ):

$$\mathcal{A}_{i,s} := \overline{\{\gamma|_{(a,t_s(\gamma))} : \gamma \in \widetilde{\mathbf{G}_{\mathrm{reg}}}\}}.$$

Since the initial points of the geodesics in $\mathcal{A}$ lie in $\mathfrak{M}_0$, it follows that the projection $\pi|_{\mathcal{A}} : \mathcal{A} \to \mathbf{G}_{\mathrm{reg}}$ is an injective mapping, whence $\mathbf{m}(\pi(\mathcal{A})) = \widetilde{\mathbf{m}}(\mathcal{A})$. Therefore, by property 3) of the measure $\mathbf{m}$, where we put $\Psi_i := \mathcal{A}_{i,s+1}$ and $\overline{\Psi} := \mathcal{A}_{i,s}$, we have

$$\widetilde{\mathbf{m}}(\mathcal{A}_{i,s+1}) \le \frac{\widetilde{\mathbf{m}}(\mathcal{A}_{i,s})}{d-1}, \quad s = 1, \dots, l.$$

Property 2) of the measure $\mathbf{m}$ implies that

$$\widetilde{\mathbf{m}}(\mathcal{A}_{i,1}) \le \widetilde{\mu}_L(\mathcal{A}'_i(a)).$$

Combining the above $l+1$ inequalities, we obtain

$$\widetilde{\mathbf{m}}(\mathcal{A}_{i,l}) \le \frac{\widetilde{\mu}_L(\mathcal{A}'_i(a))}{(d-1)^l}.$$

Since $\widetilde{\mathbf{m}}(\mathcal{A}_i) \le \widetilde{\mathbf{m}}(\mathcal{A}_{i,l})$, we arrive at $(*_i)$.

Summing the inequalities $(*_i)$, $i \in \mathbb{N}$, we obtain

$$\widetilde{\mathbf{m}}(\mathcal{A}) \le \frac{1}{(d-1)^l} \sum_i \widetilde{\mu}_L(\mathcal{A}'_i(a)).$$

Since $\mathcal{A} = \bigcup_i \mathcal{A}_i$, we have

$$\sum_i \mathbf{1}_{\mathcal{A}'_i(a)}(\mathbf{e}) \le \mathbf{1}_{\mathcal{A}^\dagger(a)}(\mathbf{e}), \quad \mathbf{e} \in \mathbb{S}\mathfrak{M}.$$

Integration over $\mathbb{S}\mathfrak{M}$ yields

$$\sum_i \widetilde{\mu}_L(\mathcal{A}'_i(a)) \le \widetilde{\mu}_L(\mathcal{A}^\dagger(a)),$$

which proves $(*)$.

4.4. The set A_ε of often-branching geodesics. The next step of the proof is construction of the set of sufficiently-often-branching geodesics, to which we apply Lemma 2.

We describe an auxiliary subset of $\mathbf{G}_{\mathrm{reg}}$. Let $\Theta \subset \mathbb{S}\Omega$ be the set of all unit tangent vectors at points in Ω that go in the direction of Υ_1 and are almost orthogonal to X. We assume that the closure of Ω lies strictly inside X.

There exists $\delta_0 > 0$ such that for every $\mathbf{e} \in \Theta$ there exists a unique geodesic $\gamma_{\mathbf{e}} : [0, \delta_0] \to \mathfrak{M}$ with $\gamma(0) = \mathbf{e}$, i.e., $\gamma_{\mathbf{e}}$ does not intersect the $(n-1)$-simplexes on which branching is possible. We define

$$g_0 := \{-\gamma'_{\mathbf{e}}(t) \mid \mathbf{e} \in \Theta, 0 < t < \delta_0\} \subset \mathbb{S}\mathfrak{M}$$

and set

$$\mathbb{G}_0 := \mathbf{G}_{\mathrm{reg}}(g_0).$$

We have $\mu_L(g_0) \neq 0$. Then property 2) of the measure $\mathbf{m}$ and Remark 2 imply that $\mathbf{m}(\mathbb{G}_0) \neq 0$. For $\gamma \in \mathbf{G}_{\mathrm{reg}}$ and $k > 0$, we let $N_\gamma(k)$ be the number of connected components of the set $[0, k] \cap (\gamma')^{-1}(g_0)$, i.e., $N_\gamma(k)$ is the number of comings of γ into $\mathbb{G}_0$ under the action of the geodesic flow transformation within the time k.

The set g_0 is chosen so that the duration of the stay of a geodesic in the set $\mathbb{G}_0$ under the action of the geodesic flow be at least δ_0. Then, by the definition of $N_\gamma(k)$, we have

$$N_\gamma(k)\delta_0 \geq \int_0^k \mathbf{1}_{\mathbb{G}_0}(\varphi_s\gamma)ds. \tag{4.1}$$

Lemma 3. *For any $\varepsilon > 0$, there is a set $A_\varepsilon \subset \mathbf{G}_{\mathrm{reg}}$ and positive numbers N and δ with the following properties:*

1) $\mathbf{m}(A_\varepsilon) \neq 0$*;*

2) $\mathrm{diam}(A_\varepsilon(0)) < \varepsilon$*. (Here and below, we use the natural notation $A_\varepsilon(0) := \{\gamma(0) \mid \gamma \in A_\varepsilon\} \subset \mathfrak{M}$, etc.);*

3) $N_\gamma(k) > \delta k$ *for all $k > N$ and all $\gamma \in A_\varepsilon$.*

Proof. We use a general result for measure spaces, the proof of which involves the ergodic theorem. □

Proposition 2. *Suppose D is a space with a measure m and $\{T_t\}$ is a one-parametric semigroup of measure-preserving transformations of D, where t takes nonnegative real values and $T_{s+t} = T_s \cdot T_t$. Furthermore, suppose that*

$$D \times \mathbb{R}_{\geq 0} \to D, \quad (x, t) \mapsto T_t(x)$$

is a measurable mapping.

Then for every set $\Delta \subset D$ of nonzero finite measure there is a set $D_0 \subset D$ of nonzero measure such that for the points in D_0 the average duration of the stay in Δ under the action of the transformation T_t during the time t is uniformly bounded away from zero as $t \to \infty$. This means that there exist $s_0 > 0$ and $\varepsilon_0 > 0$ such that for any $x \in D_0$ and any $s > s_0$ we have

$$\frac{1}{s}\int_0^s \mathbf{1}_\Delta(T_t x)dt > \varepsilon_0.$$

Proof. Let $\mathrm{Ave}_\Delta(x)$ denote the average value of $\mathbf{1}_\Delta$ on the trajectory of the geodesic flow with initial value x:

$$\mathrm{Ave}_\Delta(x) := \lim_{t\to\infty} \frac{1}{t}\int_0^t \mathbf{1}_\Delta(T_s(x))ds.$$

Applying the ergodic theorem, we obtain

$$\int_D \mathrm{Ave}_\Delta(x)dm(x) = \int_D \mathbf{1}_\Delta(x)dm(x) = m(\Delta) > 0.$$

Consequently, there is $\varepsilon_2 > 0$ and a set $D_1 \subset D$ of nonzero measure such that $\mathrm{Ave}_\Delta(x) > \varepsilon_2$ for each $x \in D_1$. Then there exist numbers $s_0 > 0$ and $\varepsilon_0 > 0$ and a set $D_0 \subset D_1$ such that for all $x \in D_0$ and $s > s_0$ we have

$$\frac{1}{s}\int_0^s \mathbf{1}_\Delta(T_t x)dt > \varepsilon_0.$$

□

Since the measure $\mathbf{m}$ is invariant under the geodesic flow on $\mathbf{G}_{\mathrm{reg}}$, and $\mathbf{m}(\mathbb{G}_0) \neq 0$, we can apply Proposition 2 to the case where

$$(D, m, \Delta, T_t) := (\mathbf{G}_{\mathrm{reg}}, \mathbf{m}, \mathbb{G}_0, \varphi_t).$$

Thus, there is a set $A_0 \subset \mathbf{G}_{\mathrm{reg}}$ of nonzero measure and positive numbers s_0 and ε_0 such that

$$\frac{1}{T}\int_0^T \mathbf{1}_{\mathbb{G}_0}(\varphi_s\gamma)ds > \varepsilon_0$$

for any $\gamma \in A_0$ and any $T > s_0$. Applying inequality (4.1) and letting $\delta = \varepsilon_0/\delta_0$ and $N = s_0$, we see that $N_\gamma(k) > \delta k$ for each geodesic $\gamma \in A_0$ and each $k > N$. Moreover, passing if necessary to a subset of A_0 of nonzero measure, we may assume that $\mathrm{diam}(A_0(0)) < \varepsilon$. Lemma 3 is proved.

Proposition 3. *If the volume entropy of $\widetilde{\mathfrak{M}}$ is zero, then for every $\varepsilon > 0$ there are two complete generic geodesics γ_1 and γ_2 in $\widetilde{\mathfrak{M}}$ and a number $t_0 > 1$ with the following properties:*

1) $\widetilde{\rho}(\gamma_1(0), \gamma_2(0)) < \varepsilon$;
2) $\gamma_1(t_0) = \gamma_2(t_0) \in \widetilde{\Omega}$;
3) ${\gamma_1}'_+(t_0) = {\gamma_2}'_+(t_0)$, *and the vector* ${\gamma_1}'_+(t_0)$ *is almost orthogonal to* $\widetilde{X}$ *and goes in the direction of* $\widetilde{\Upsilon}_j$ *for some* $j \in \{1, \dots, d\}$;
4) *at the point* t_0, *the geodesics* γ_1 *and* γ_2 *pass to* $\widetilde{\Upsilon}_j$ *from distinct* n*-simplexes adjacent to* $\widetilde{X}$, *i.e.,* $\gamma_1^-(t_0) \neq \gamma_2^-(t_0)$.

Proof. Applying Lemma 3, we obtain a set $A_\varepsilon \subset \mathbf{G}_{\mathrm{reg}}$ and numbers N and δ.

We fix a fundamental domain $\mathfrak{M}_0 \subset \widetilde{\mathfrak{M}}$ and a point $x_0 \in \mathfrak{M}_0$. Let $\mathbb{S}B_r(x_0)$ denote the set of unit tangent vectors at the points of the ball $B_r(x_0)$, and let $\mathcal{A}$ be the set of the geodesics in $\widetilde{\mathbf{G}_{\mathrm{reg}}}$ that are the liftings of the geodesics in A_ε with initial points in $\mathfrak{M}_0$.

For $k > N$, we define

$$\mathcal{A}_k := \{\gamma|_{[0,k]} : \gamma \in \mathcal{A}\}.$$

Assertion 3) of Lemma 3 implies that $N_\gamma(k) \geq \delta k$ for $\gamma \in \mathcal{A}$. We apply Lemma 2 to the set $\mathcal{A}_k$, letting $l := [\delta k] + 1$. Lemma 3 and the inequality $\widetilde{\mathbf{m}}(\mathcal{A}) \leq \widetilde{\mathbf{m}}(\mathcal{A}_k)$ show that

$$\widetilde{\mathbf{m}}(\mathcal{A}) \leq \frac{\widetilde{\mu}_L(\mathcal{A}_k^\dagger(k))}{(d-1)^{\delta k}}. \tag{4.2}$$

We assume that $\mathrm{diam}\,\mathfrak{M}_0 < 1$. Then the function $\mathbf{1}_{\mathcal{A}_k^\dagger(k)}(\mathbf{v})$ vanishes outside $\mathbb{S}B_{k+1}(x_0)$, and (4.2) takes the form

$$\frac{1}{(d-1)^{\delta k}}\int_{\mathbb{S}B_{k+1}(x_0)} \mathbf{1}_{\mathcal{A}_k^\dagger(k)}(\mathbf{v})d\widetilde{\mu}_L \geq \widetilde{\mathbf{m}}(\mathcal{A}).$$

We define

$$f(k) := \max\{\mathbf{1}_{\mathcal{A}_k^\dagger(k)}(\mathbf{v}) \mid \mathbf{v} \in \mathbb{S}B_{k+1}(x_0)\}.$$

Since the volume entropy of $\widetilde{\mathfrak{M}}$ is zero, we have $\widetilde{\mu}_L(\mathbb{S}B_{k+1}(x_0)) = o((d-1)^{\delta k})$. Then

$$\frac{o((d-1)^{\delta k})f(k)}{(d-1)^{\delta k}} \geq \widetilde{\mathbf{m}}(\mathcal{A}) =: c > 0,$$

whence

$$f(k) \geq c\frac{(d-1)^{\delta k}}{o((d-1)^{\delta k})}. \tag{4.3}$$

Estimate (4.3) implies the existence of $k_1 > N$ such that $f(k_1) \geq f(N) + 1$. This means that $\mathcal{A}$ contains $f(k_1)$ geodesics that are distinct on the interval $(0, k_1)$ and have equal velocity vectors at k_1. At least two of them are distinct on the interval (N, k_1). (Indeed, otherwise there were $f(k_1)$ geodesics that are distinct on the interval $(0, N)$ and have equal velocity vectors at N. This would mean that $f(N) \geq f(k_1)$, which contradicts our choice of k_1.) Consequently, these geodesics meet at a point $t_0 \in (N, k_1]$. These two geodesics and the parameter t_0 satisfy all the requirements of the proposition. □

4.5. End of the proof of Theorem 1. Suppose that the volume entropy of the universal cover $\widetilde{\mathfrak{M}}$ is zero. The remaining part of the proof proceeds in $\widetilde{\mathfrak{M}}$. For short, the distance function in $\widetilde{\mathfrak{M}}$ is denoted by $|\cdot\cdot|$. For $a, b \in \widetilde{\mathfrak{M}}$, we denote by $[a, b]$ a unique geodesic segment joining a and b. Since $\widetilde{\mathfrak{M}}$ contains no conjugate points, the initial velocity vector $\mathbf{e}_{ab}$ of $[a, b]$ depends continuously on a and b. Furthermore, since $[a, b]$ is a shortest curve, the length of $[a, b]$ is equal to $|ab|$.

Applying Proposition 3 to a sufficiently small ε, we obtain geodesics γ_1 and γ_2 and a number t_0. We define

$$\begin{gathered} c := \gamma_1(t_0) = \gamma_2(t_0), \\ a := \gamma_1(0), \quad b := \gamma_2(0), \\ d := \gamma_1(t_0 + 1/2) = \gamma_2(t_0 + 1/2). \end{gathered}$$

In this notation, the geodesic segments $[a, d]$ and $[b, d]$ are the restrictions of the geodesics γ_1 and γ_2 to the interval $[0, t_0 + 1/2]$. We have $|ab| < \varepsilon$.

Our purpose is to join a and d by a polygonal line of length less than $|ad|$, which is a contradiction. More precisely, we find a point $f^* \in [a, b]$ such that $|df^*|$ is less than $|da|$ by a constant depending only on the geometry of $\widetilde{\mathfrak{M}}$. Now, we can make $|af^*|$ arbitrarily less than this constant by taking ε sufficiently small.

Consider geodesic segments $[c, f]$, where $f \in [a, b]$. The space $\mathbb{S}_c$ is the union of $(n-1)$-hemispheres glued together along their common $(n-2)$-sphere. By assertion 4) of Proposition 3, the vectors $\mathbf{e}_{ca}$ and $\mathbf{e}_{cb}$ lie in distinct $(n-1)$-hemispheres. By continuity, there is $f^* \in [a, b]$ such that $\mathbf{e}_{cf^*} \in T_{f^*}\widetilde{\Omega}$, i.e., $\mathbf{e}_{cf^*}$ is tangent to an $(n-1)$-simplex. There is $e \in [c, f^*]$ such that $|ce| = 1/2$. By compactness, there is a number $q > 0$ depending only on $\widetilde{\mathfrak{M}}$ and $\widetilde{\Omega}$ and such that

$$|dc| + |ce| - |de| > q.$$

In more detail, we consider the set $\mathfrak{A}$ of triplets of points (x, y, z) belonging to $\mathfrak{M}_0$ and such that the following conditions 1)–3) are fulfilled:

1) $x \in \widetilde{\Omega}$ and $|xy| = |xz| = 1/2$;
2) $\mathbf{e}_{xy}$ is almost orthogonal to $\widetilde{X}$;
3) $\mathbf{e}_{xz}$ is tangent to $\widetilde{X}$.

The set $\mathfrak{A}$ is compact. Therefore, the function $|xy| + |xz| - |yz|$ attains a minimum, which is positive because of the absence of conjugate points (since $x \notin [yz]$).

So, in Proposition 3 we put $\varepsilon := q/4$. The inequality $|dc| + |ce| - |de| > q$ and the triangle inequality for $\triangle def^*$ imply that

$$|df^*| < |dc| + |cf^*| - q.$$

By our choice of ε in Proposition 3, we have $|ab| < q/4$, and $|af^*| < q/4$ because $f^* \in ab$. By the triangle inequality,

$$|cf^*| < |ac| + \frac{q}{4}.$$

Adding the last two inequalities, we obtain

$$|ac| + |dc| > |df^*| + \frac{3}{4}q.$$

Again by the triangle inequality,

$$|df^*| > |ad| - \frac{q}{4}.$$

Adding the two inequalities obtained, we get

$$|ac| + |dc| > |ad| + \frac{q}{2},$$

i.e., a, c, and d do not lie on one shortest curve. Therefore, γ_1 is not a geodesic, a contradiction. Theorem 1 is proved.

§5. Appendix

5.1. Notation. Two tangent vectors at one point are *opposite* if they make an angle of π.

We set $\overset{\circ}{\mathfrak{M}}{}^{n-1} := \mathfrak{M}^{n-1} \setminus \mathfrak{M}^{n-2}$.

Let $\Lambda(\mathfrak{M}) \subset \mathbb{S}\mathfrak{M}$ denote the set of all unit vectors tangent to $\mathfrak{M}$ strictly inside the $(n-1)$-faces and transversal to these faces, and let $D \subset \Lambda(\mathfrak{M}) \times \Lambda(\mathfrak{M})$ be the set of all pairs of opposite vectors.

5.2. Definitions. 1) A measurable function

$$p : D \to [0, 1]$$

is called a *transition probability function* if it possesses the following property: if $v_1 \in \Lambda$ is a tangent vector at a point of an $(n-1)$-face to which m faces of dimension n are adjacent, and $v_2, \dots, v_m$ are all the vectors opposite to the vector v_1, then

$$p(v_1, v_2) + \dots + p(v_1, v_m) = 1 = p(v_2, v_1) + \dots + p(v_m, v_1).$$

2) We say that p is *single-valued* on a subset $D_0 \subset D \subset \Lambda \times \Lambda$ if $p(D_0) \subset \{0, 1\}$.

3) We say that a geodesic γ obeys p if for each point $\gamma(c)$ of transversal intersection with an $(n-1)$-face we have $p(\gamma'_-(c), \gamma'_+(c)) \neq 0$.

Proposition 4. *Let $D_0 \subset D$ be the subset consisting of all pairs of vectors such that the angle that they make with the corresponding $(n-1)$-faces is less than some positive constant θ, and of all pairs of tangent vectors at the points lying in the τ-neighborhood of the $(n-2)$-skeleton for some $\tau > 0$.*

Suppose that $p : D \to [0, 1]$ is a transition function single-valued on D_0.

Then the set of unit vectors v for which there exists a nongeneric geodesic with the initial velocity vector v and obeying the transition function p has zero Liouville measure in $\mathbb{S}\mathfrak{M}$.

This is similar to the corresponding fact of the theory of billiard dynamical systems, and we do not present a complete proof here. To prove this fact it suffices to estimate the measure of the ε-neighborhood of the tangent vectors of $(n-1)$-faces starting at the points of $(n-2)$-faces (this measure is $O(\varepsilon^2)$) and the duration of nongeneric geodesics in this neighborhood (it is at least $\mathrm{const}\cdot\varepsilon$).

Theorem 2. *Let p be as in Proposition 4. Then there exists a measure $\mathbf{m}_p$ on $\mathbf{G}$ with the following properties:*

1). *The measure $\mathbf{m}_p$ is GFT-invariant.*

2). *If $V \subset \mathbb{S}\mathfrak{M}$ is a measurable subset, then $\mathbf{m}_p(\mathbf{G}(V)) = \mu_L(V)$.*

3). *Let $\Psi = \{\gamma : (-\infty, x_\gamma) \to \mathfrak{M}\}$ be a set of one-sided geodesics none of which is a continuation of another geodesic, where $\gamma(x_\gamma) \in \mathring{\mathfrak{M}}^{n-1}$. Suppose that for each vector $\gamma'_-(x_\gamma)$ one opposite vector $\gamma'_+(x_\gamma)$ is chosen. Furthermore, suppose that for all $\gamma \in \Psi$ we have*

$$p(\gamma'_-(x_\gamma), \gamma'_+(x_\gamma)) = \mathrm{const} = p'.$$

Then

$$\mathbf{m}_p(\Psi_+) = p'\mathbf{m}_p\left(\overline{\Psi}\right),$$

where Ψ_+ is the set of all possible continuations of the geodesics $\gamma \in \Psi$ to complete generic geodesics that have the chosen velocity vectors $\gamma'_+(x_\gamma)$ at the points x_γ, and $\overline{\Psi}$ is the set of all possible continuations of the geodesics in Ψ to complete generic geodesics.

Proof. Suppose that $A \subset \mathbf{G}$ is an open subset. We define

$$\mathbf{m}_p(A) = \int_{\mathbb{S}\mathfrak{M}} h^A(\mathbf{v})d\mu_L(\mathbf{v}),$$

where h^A is the function with values in $[0,1]$ and defined μ_L-almost everywhere on $\mathbb{S}\mathfrak{M}$ as follows. Let $\mathbf{v} \in \mathbb{S}(\mathfrak{M} \setminus \mathring{\mathfrak{M}}^{n-1})$ be a vector such that every complete geodesic γ in $\mathfrak{M}$ with $\gamma'(0) = \mathbf{v}$ and obeying the transition function p is generic. (See Proposition 4.) We fix some lifting of $\mathbf{v}$ to $\widetilde{\mathfrak{M}}$, which we also denote by $\mathbf{v}$, and consider the set $\mathbf{G}_\mathbf{v} \subset \widetilde{\mathbf{G}}$ of all generic geodesics with initial velocity vector $\mathbf{v}$. The support (the union of the images of geodesics) $T_\mathbf{v}$ of $\mathbf{G}_\mathbf{v}$ is a tree with an oriented distinguished edge e_0.

The function p (more precisely, a lifting of p to $\widetilde{\mathfrak{M}}$) is defined at the points of intersection of the geodesics in $\mathbf{G}_\mathbf{v}$ with $\mathfrak{M}^{n-1}$. Therefore, for all paths in $T_\mathbf{v}$ passing through e_0 in the given direction there are probabilities $p(w, w')$ of the passage from an edge $w \subset T_\mathbf{v}$ to the subsequent edge w'. For a path $W = w_1 \cdots w_k \subset T_\mathbf{v}$ containing e_0, let $\mathfrak{P}_W \subset \mathbf{G}_\mathbf{v}$ be the set of all paths containing W. We define

$$\mathbf{m}_{p,\mathbf{v}}(\mathfrak{P}_W) := p(w_1, w_2)\cdots p(w_{k-1}, w_k).$$

The properties of p imply that $\mathbf{m}_{p,\mathbf{v}}$ is canonically extended to a probability measure on the space $\mathbf{G}_\mathbf{v}$ of paths in $T_\mathbf{v}$; this extension is also denoted by $\mathbf{m}_{p,\mathbf{v}}$.

Every geodesic $\gamma \in A$ with $\gamma'(0) = \mathbf{v}$ is lifted to $T_\mathbf{v}$ canonically as one of the paths described above. Let $A_\mathbf{v} \subset \mathbf{G}_\mathbf{v}$ be the set of such liftings for all corresponding geodesics in A. We set $h^A(\mathbf{v}) = \mathbf{m}_{p,\mathbf{v}}(A_\mathbf{v})$.

We check the properties of the measure $\mathbf{m}_p$.

1. Let $t > 0$, let Δ_1 and Δ_1 be two neighboring n-simplices, and let A be the set of the geodesics γ with $\gamma(0) \in \Delta_1$ and $\gamma(t) \in \Delta_2$. We assume that the geodesics in A intersect $\mathfrak{M}^{n-1}$ once. If $\gamma \in A$ and $\mathbf{v}_t := \gamma'(t)$, then $h^A(\mathbf{v}_0) = h^{\varphi_t A}(\mathbf{v}_t)$. Now, since the

Liouville measure is GFT-invariant, we obtain

$$\begin{aligned}\int_{\mathbb{S}\mathfrak{M}} h^A(\mathbf{v})d\mu_L(\mathbf{v}) &= \int_{\mathbb{S}\Delta_1} h^A(\mathbf{v})d\mu_L(\mathbf{v}) \\ &= \int_{\mathbb{S}\Delta_2} h^{\varphi_t A}(\mathbf{v}_t)d\mu_L(\mathbf{v}_t) \\ &= \int_{\mathbb{S}\mathfrak{M}} h^{\varphi_t A}(\mathbf{v})d\mu_L(\mathbf{v}),\end{aligned}$$

whence

$$\mathbf{m}_p(A) = \mathbf{m}_p(\varphi_t A).$$

In the general case, we prove the invariance of the measure $\mathbf{m}_p$ by splitting A and the parameter along the flow.

2. If $V \subset \mathbb{S}\mathfrak{M}$, then $h^{\mathbf{G}(V)}$ is the indicator function of V. This implies property 2) of the measure $\mathbf{m}_p$.

3. We may assume that $x_\gamma > 0$ for all $\gamma \in \Psi$. Fixing a vector $\mathbf{v} \in \mathbb{S}\mathfrak{M}$, we put $\Psi_{\mathbf{v}} := \{\Psi \cap \mathbf{G}(\{\mathbf{v}\})$. The set $\Psi_{\mathbf{v}}$ splits into a countable number of subsets $\Psi^i_{\mathbf{v}}$ such that the point $\gamma(x_\gamma)$ and the vector $\gamma'_+(x_\gamma)$ are the same for the geodesics $\gamma \in \Psi^i_{\mathbf{v}}$. For each i, by the definition of the measure $\mathbf{m}_{p,\mathbf{v}}$, we have

$$h^{(\Psi^i_{\mathbf{v}})_+} = p' h^{(\Psi^i_{\mathbf{v}})'}.$$

Since none of the geodesics in Ψ is a continuation of another geodesic, it follows that the sets $(\Psi^i_{\mathbf{v}})'$ are disjoint, whence

$$h^{\Psi_+} = p' h^{\Psi'}.$$

This implies property 3) of $\mathbf{m}_p$. □

In order to apply this theorem in our case (see Subsection 4.2), it suffices to let the function p be equal to 0, 1, and $1/(d-1)$ at the corresponding points.

References

[1] W. Ballmann and M. Brin, *Orbihedra of nonpositive curvature*, Inst. Hautes Études Sci. Publ. Math. No. 82 (1995), 169–209 (1996). MR **97i:**53049

[2] C. Croke, *Volumes of balls in manifolds without conjugate points*, Internat. J. Math. **3** (1992), 455–467. MR **93e:**53048

[3] M. Gromov, *Asymptotic invariants of infinite groups*, Geometric Group Theory, Vol. 2 (Sussex, 1991), London Math. Soc. Lecture Note Ser., vol. 182, Cambridge Univ. Press, Cambridge, 1993, pp. 1–295. MR **95m:**20041

[4] I. P. Kornfel′d, Ya. G. Sinaĭ, and S. V. Fomin, *Ergodic theory*, "Nauka", Moscow, 1980; English transl., Grundlehren Math. Wiss., vol. 245, Springer-Verlag, New York, 1982. MR **83a:**28017

St. Petersburg State University, Department of Mathematics and Mechanics, Universitetskiĭ pr. 28, St. Petersburg 198504, Russia

Received 2/SEP/2002

Translated by N. YU. NETSVETAEV