# DISCRETE SPECTRUM OF A TWO-DIMENSIONAL PERIODIC ELLIPTIC SECOND ORDER OPERATOR PERTURBED BY A DECAYING POTENTIAL. II. INTERNAL GAPS 

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#### Abstract

The discrete spectrum in the spectral gaps is studied in the case of a twodimensional periodic elliptic second order operator perturbed by a decaying potential. The main goal is to find asymptotics (for the large coupling constant) of the number of eigenvalues that have been "born" (or have "died") at the edges of the gap. The high-energy (Weyl) asymptotics and the threshold asymptotics are distinguished. At the right edge of the gap, a competition between the Weyl contribution and the threshold contribution may occur. The case of a semiinfinite gap was studied in part I of the paper.


## Introduction

1. Let $A$ be an elliptic periodic second order operator in $L_{2}\left(\mathbb{R}^{d}\right), d \geq 2$, given by the expression $A=-\operatorname{div} g(\mathbf{x}) \nabla+p(\mathbf{x})$, and let $V$ be the operator of multiplication by a function $V(\mathbf{x}) \geq 0$ that tends to zero at infinity. Suppose an interval ( $\lambda_{-}, \lambda_{+}$) is a gap in the spectrum of $A$. We put

$$
A_{ \pm}(\alpha)=A \mp \alpha V(\mathbf{x}), \quad \alpha>0 .
$$

Let $\mathfrak{N}_{+}\left(\alpha, \lambda_{+}\right)$denote the number of eigenvalues of the operator $A_{+}(t)$ that have been "born" at the point $\lambda_{+}$as the coupling constant $t$ has been growing from 0 to $\alpha$. The function $\mathfrak{N}_{-}\left(\alpha, \lambda_{-}\right)$is defined similarly for the operator $A_{-}$. We are interested in the asymptotics of these functions as $\alpha \rightarrow \infty$ (in the large coupling constant limit). The corresponding asymptotics (which may be fairly diverse, depending both on $A$ and $V$ ) were studied in a number of papers. We mention [B3], B5], (BL) and especially the survey [B4] and the references therein. Another approach to the problems under discussion was proposed in IV].

Usually, for the study of the functions $\mathfrak{N}_{ \pm}\left(\alpha, \lambda_{ \pm}\right)$in an internal gap of $A$, certain conditions on the structure of the edges of the gap are imposed (see Conditions 1.3(土) below). The asymptotic behavior of the functions $\mathfrak{N}_{ \pm}\left(\alpha, \lambda_{ \pm}\right)$depends on the dimension $d$, the character of decay of $V$, and also the signs " $\pm$ ". The case where $d \geq 3$ is rather well studied $([\boxed{B 3}]-[\mathbf{B 5}])$. If $d \geq 3$ and $V \in L_{d / 2}\left(\mathbb{R}^{d}\right)$, then the function $\mathfrak{N}_{+}\left(\alpha, \lambda_{+}\right)$has

[^0]the Weyl asymptotics
\[

$$
\begin{equation*}
\mathfrak{N}_{+}\left(\alpha, \lambda_{+}\right) \sim(2 \pi)^{-d} \omega_{d} \alpha^{d / 2} \int V^{d / 2}(\operatorname{det} g)^{-1 / 2} d \mathbf{x}, \quad \alpha \rightarrow \infty \tag{0.1}
\end{equation*}
$$

\]

here $\omega_{d}$ is the volume of the unit ball in $\mathbb{R}^{d}$. If $V \notin L_{d / 2}\left(\mathbb{R}^{d}\right)$, then the estimate $\mathfrak{N}_{+}\left(\alpha, \lambda_{+}\right)=O\left(\alpha^{d / 2}\right)$ fails, and $\mathfrak{N}_{+}\left(\alpha, \lambda_{+}\right)$may have an arbitrary order of growth greater than $d / 2$. Essentially, the asymptotics (0.1) has a "high-energy" origin, while the behavior of $\mathfrak{N}_{+}\left(\alpha, \lambda_{+}\right)$with $V \notin L_{d / 2}\left(\mathbb{R}^{d}\right)$ is determined by the "threshold effect" near the edge of the gap of the unperturbed operator (see the discussion in [B4, §2]).
2. If $d=2$, the situation is much more complicated. Already for $A=-\Delta$ in the case of the semiinfinite gap $(-\infty, 0)$, the condition $V \in L_{1}\left(\mathbb{R}^{2}\right)$ does not ensure an asymptotics of the form (0.1). Due to the threshold effects, $\mathfrak{N}_{+}(\alpha, 0)$ may have an arbitrary order of growth greater than $d / 2$. Moreover, it may happen that $\mathfrak{N}_{+}(\alpha, 0)=O(\alpha)$, but the asymptotics is not of Weyl type. In the latter case the asymptotic coefficient is the sum of the Weyl term and the "threshold" term. Thus, for $d=2$ a "competition" between the Weyl contribution and the threshold contribution is possible; this competition cannot occur for $d \geq 3$. A "special channel" is responsible for the threshold effect, by which we mean the problem on the semiaxis that is obtained by restricting $-\Delta-\alpha V$ to the subspace of functions depending only on $|\mathbf{x}|$. At the same time, the potential $V$ is averaged over the polar angle. These effects were investigated in [BL] in detail. At the level of estimates, the special channel was discovered in [ $\underline{]}$.

In BLSu (i.e., in part I of the present paper), the same phenomena were analyzed in the case where $A$ is a periodic elliptic operator of the form $A=-\operatorname{div} g(\mathbf{x}) \nabla+p(\mathbf{x})$. Adding an appropriate constant to $p$ allows us to assume that the lower edge of the spectrum is the point $\lambda=0$. In BLSu , the negative discrete spectrum of the operator $A-$ $\alpha V$, i.e., the case of the semiinfinite gap $(-\infty, 0)$ in the spectrum of $A$, was studied. The description of the special channel was given in terms of the Floquet-Bloch decomposition for the unperturbed operator $A$. The answer involves the so-called tensor of effective masses at the edge of the spectrum and a positive periodic solution $\varphi$ of the equation $A \varphi=0$. The function $\varphi$ can be eliminated from the answer under a certain additional "regularity" condition imposed on $V$.

The present paper is a continuation of [BLSu], but now we study the case of an internal gap in the spectrum of $A$. For this, we need to change the technique of investigation considerably. The presentation is independent of BLSu. At the same time, we use some technical results from [BLSu].
3. As has already been mentioned, for the study of the asymptotics of $\mathfrak{N}_{ \pm}\left(\alpha, \lambda_{ \pm}\right)$as $\alpha \rightarrow \infty$ in an internal gap of $A$, we are forced to impose certain restrictions on the structure of the edges of the gap (see Conditions $1.3( \pm)$ below). (For the lower edge of the spectrum $\lambda=0$, these conditions are fulfilled automatically.) The answers are given in terms of the model operators, which are simpler than $A_{ \pm}(\alpha)$. The model operators involve the tensors of effective masses at the edges of the gap, and the corresponding eigenfunctions. As in the case of the semiinfinite gap, it is possible to eliminate the eigenfunctions from the answer under an additional "regularity" condition imposed on $V$. The main results are formulated in Theorems $2.2( \pm)$ and $2.5( \pm)$. On the right (but not on the left) edge of the gap the Weyl contribution and the threshold contribution to the asymptotics may compete.

The asymptotics of $\mathfrak{N}_{+}\left(\alpha, \lambda_{+}\right)$at the right edge $\lambda_{+}$of the gap can be obtained by the same method as in BLSu]. However, this method fails in the case of $\mathfrak{N}_{-}\left(\alpha, \lambda_{-}\right)$, i.e., for the left edge of the gap $\lambda_{-}$. Therefore, we modify the approach so as to make it applicable for both edges of the gap.

A short version of the present paper was published in Su .
4. Acknowledgements. The author is grateful to M. Sh. Birman for useful discussions and permanent attention. The author also thanks R. G. Shterenberg who read the manuscript and made useful comments.
5. Notation. In what follows, $\mathbb{Q}^{2}$ is an open unit square in $\mathbb{R}^{2}$. The symbol $\langle\cdot, \cdot\rangle_{m}$ stands for the standard inner product in $\mathbb{C}^{m}$; sometimes we omit the index $m$. Further, $\mathbf{1}_{m}=\mathbf{1}$ is the unit $(m \times m)$-matrix. The symbol $\asymp$ means a two-sided estimate. Any integral without indication of the integration domain is over $\mathbb{R}^{2}$. Further, $\nabla=\operatorname{grad}$, $\nabla^{*}=-\operatorname{div}$. We denote by $H^{s}, s \geq 0$, the Sobolev classes. The operator of multiplication by a function $f$ is denoted either by the same symbol $f$, or by the symbol $[f]$, depending on the context. Various constants in estimates are denoted by $C$ or $c$, possibly, with indices. For a sequence of operators in a Hilbert space, (s)-lim, (u)-lim, and (S)-lim denote (respectively) the strong limit, the limit in the operator norm, and the limit relative to a norm in some symmetrically (quasi)normed ideal $\mathfrak{S}$ (see GoKr ) of compact operators.

Many statements and formulas contain the double indices $\pm$. Unless otherwise is stated explicitly, the upper and the lower index versions should be read independently.

## §1. Setting of the problem. Preliminaries

1. Differential operators. By an unperturbed operator we mean a periodic elliptic second order operator in $\mathbb{R}^{2}$. There is no loss of generality in assuming that the lattice of periods is $\mathbb{Z}^{2}$. Let $g$ be a $(2 \times 2)$-matrix-valued function, and let $p$ be a real-valued function; we assume that

$$
\begin{align*}
& \bar{g}=g>0, \quad \bar{p}=p, \quad g+g^{-1} \in L_{\infty}\left(\mathbb{R}^{2}\right), \quad p \in L_{\infty}\left(\mathbb{R}^{2}\right) \\
& g(\mathbf{x}+\mathbf{n})=g(\mathbf{x}), \quad p(\mathbf{x}+\mathbf{n})=p(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^{2}, \quad \mathbf{n} \in \mathbb{Z}^{2} \tag{1.1}
\end{align*}
$$

Formally, an unperturbed operator $A$ is given by the differential expression

$$
\begin{equation*}
\mathcal{A} u=\nabla^{*} g \nabla u+p u \tag{1.2}
\end{equation*}
$$

The precise definition of $A$ as a selfadjoint operator in the Hilbert space $L_{2}\left(\mathbb{R}^{2}\right)$ is given in terms of the closed semibounded quadratic form

$$
\begin{equation*}
a[u, u]=\int\left(\langle g \nabla u, \nabla u\rangle+p|u|^{2}\right) d \mathbf{x}, \quad u \in H^{1}\left(\mathbb{R}^{2}\right) \tag{1.3}
\end{equation*}
$$

Adding an appropriate constant to $p$ allows us to assume that

$$
\begin{equation*}
\inf \operatorname{spec} A=0 \tag{1.4}
\end{equation*}
$$

Under condition (1.4), in $H^{1}\left(\mathbb{R}^{2}\right)$ the form $a[u, u]+\gamma \int|u|^{2} d \mathbf{x}, \gamma>0$, determines a metric equivalent to the standard one.

A perturbation is introduced as the operator of multiplication by a function $V$ such that

$$
\begin{equation*}
V(\mathbf{x}) \geq 0, \quad \mathbf{x} \in \mathbb{R}^{2} \tag{1.5}
\end{equation*}
$$

We impose the following condition on $V$ (cf., e.g., [BL]).
Condition 1.1. For some $\sigma>1$,

$$
\begin{equation*}
\left(\int_{|\mathbf{x}| \leq 1}|V|^{\sigma} d \mathbf{x}\right)^{1 / \sigma}+\sum_{k \geq 1}\left(\int_{e^{k-1} \leq|\mathbf{x}| \leq e^{k}}|V|^{\sigma}|\mathbf{x}|^{2(\sigma-1)} d \mathbf{x}\right)^{1 / \sigma}<\infty, \quad \sigma>1 \tag{1.6}
\end{equation*}
$$

It should be mentioned at once that (1.6) implies that

$$
\begin{equation*}
((V))_{\sigma}:=\sum_{\mathbf{n} \in \mathbb{Z}^{2}}\left(\int_{\mathbb{Q}^{2}+\mathbf{n}} V^{\sigma} d \mathbf{x}\right)^{1 / \sigma}<\infty, \quad \sigma>1 \tag{1.7}
\end{equation*}
$$

and, moreover,

$$
V \in L_{1}\left(\mathbb{R}^{2}\right)
$$

Consider the quadratic form

$$
v[u, u]=\int V|u|^{2} d \mathbf{x}
$$

Under condition (1.7) (and, moreover, under condition (1.6)), this form is compact in $H^{1}\left(\mathbb{R}^{2}\right)$. Consequently, the form

$$
a_{ \pm}(\alpha)[u, u]:=a[u, u] \mp \alpha v[u, u], \quad u \in H^{1}\left(\mathbb{R}^{2}\right), \quad \alpha>0
$$

is lower semibounded and closed in $L_{2}\left(\mathbb{R}^{2}\right)$. The form $a_{ \pm}(\alpha)$ gives rise to a selfadjoint operator $A_{ \pm}(\alpha)$ in $L_{2}\left(\mathbb{R}^{2}\right)$. Thus, in the sense of form-sums,

$$
\begin{equation*}
A_{ \pm}(\alpha)=A \mp \alpha V, \quad \alpha>0 \tag{1.8}
\end{equation*}
$$

Formally, the operator $A_{ \pm}(\alpha)$ corresponds to the differential expression

$$
\mathcal{A}_{ \pm}(\alpha) u=\nabla^{*} g \nabla u+p u \mp \alpha V u
$$

The spectrum of $A_{ \pm}(\alpha)$ in the spectral gaps of $A$ is discrete.
First, we recall a result for a semiinfinite gap. Let

$$
\mathfrak{N}_{+}(\alpha, \lambda ; A, V), \quad \alpha>0, \quad \lambda \leq 0
$$

denote the number of eigenvalues of the operator $A_{+}(\alpha)$ that lie to the left of the point $\lambda$. For the Weyl asymptotic coefficient we introduce the notation

$$
\begin{equation*}
J(V, g):=(4 \pi)^{-1} \int V(\operatorname{det} g)^{-1 / 2} d \mathbf{x} \tag{1.9}
\end{equation*}
$$

Proposition 1.2. Under condition (1.7), we have

$$
\begin{array}{rlrl}
\mathfrak{N}_{+}(\alpha, \lambda ; A, V) \leq C \alpha((V))_{\sigma}, & C=C(g, p, \sigma, \lambda), & \sigma>1, & \\
\lim _{\alpha \rightarrow \infty} \alpha^{-1} \mathfrak{N}_{+}(\alpha, \lambda ; A, V) & =J(V, g), & &  \tag{1.11}\\
\lambda_{\alpha},
\end{array}
$$

Comments on Proposition 1.2 and the necessary references can be found in [BLSu].
For $\lambda=0$, the Weyl asymptotics (1.11) may fail even under condition (1.6) because of spectral "threshold" effects. These phenomena were studied in BL for the operator $-\Delta-\alpha V$ and in $\overline{\mathrm{BLSu}}$ in the general case of a periodic operator (1.2). In the sequel we shall impose yet another condition on $V$ (see Condition $2.1(q)$ ), which ensures that $\mathfrak{N}_{+}(\alpha, 0 ; A, V)=O\left(\alpha^{q}\right), \alpha \rightarrow \infty, q \geq 1$. In the present paper we treat the discrete spectrum of the operators $A_{ \pm}(\alpha)$ in the internal gaps of $A$.
2. The Floquet decomposition. As usual, for the spectral analysis of periodic operators we employ partial diagonalization (the Floquet-Bloch theory). Here we recall the necessary facts. Let $\widetilde{H}^{1}\left(\mathbb{Q}^{2}\right)$ be the subspace formed by the functions in $H^{1}\left(\mathbb{Q}^{2}\right)$ such that their $\mathbb{Z}^{2}$-periodic extensions belong to the class $H_{\text {loc }}^{1}\left(\mathbb{R}^{2}\right)$. Next, we denote by $\widetilde{H}_{\xi}^{1}\left(\mathbb{Q}^{2}\right)$ the subspace of functions of the form

$$
\begin{equation*}
u(\mathbf{x})=e^{i\langle\mathbf{x}, \boldsymbol{\xi}\rangle} \widetilde{u}(\mathbf{x}), \quad \widetilde{u} \in \widetilde{H}^{1}\left(\mathbb{Q}^{2}\right), \quad \boldsymbol{\xi} \in \mathbb{R}^{2} \tag{1.12}
\end{equation*}
$$

In $L_{2}\left(\mathbb{Q}^{2}\right)$, we consider the family of quadratic forms

$$
\begin{equation*}
a_{\boldsymbol{\xi}}[u, u]=\int_{\mathbb{Q}^{2}}\left(\langle g \nabla u, \nabla u\rangle+p|u|^{2}\right) d \mathbf{x}, \quad u \in \widetilde{H}_{\boldsymbol{\xi}}^{1}\left(\mathbb{Q}^{2}\right), \quad \boldsymbol{\xi} \in \mathbb{R}^{2} \tag{1.13}
\end{equation*}
$$

The selfadjoint operator in $L_{2}\left(\mathbb{Q}^{2}\right)$ generated by the form (1.13) is denoted by $A(\boldsymbol{\xi})$. The operator $A(\boldsymbol{\xi})$ corresponds to the expression (1.2) with $(\boldsymbol{\xi})$-quasiperiodic boundary conditions. Translating $\boldsymbol{\xi}$ by a vector of the lattice $(2 \pi \mathbb{Z})^{2}$ turns the operator $A(\boldsymbol{\xi})$ into a unitarily equivalent one. Therefore, usually it suffices to consider $\boldsymbol{\xi} \in \mathbb{T}^{2}=$ $\mathbb{R}^{2} /(2 \pi \mathbb{Z})^{2}$. The parameter $\boldsymbol{\xi}$ is called the quasimomentum. All operators $A(\boldsymbol{\xi})$ have discrete spectrum. Let $E_{k}(\boldsymbol{\xi}), k \in \mathbb{N}$, be the consecutive eigenvalues (counted with multiplicities) of the operator $A(\boldsymbol{\xi})$, and let $\psi_{k}(\mathbf{x}, \boldsymbol{\xi})$ be the corresponding eigenfunctions normalized in $L_{2}\left(\mathbb{Q}^{2}\right)$. Then

$$
0 \leq E_{1}(\boldsymbol{\xi}) \leq E_{2}(\boldsymbol{\xi}) \leq \cdots \leq E_{k}(\boldsymbol{\xi}) \leq \cdots
$$

The functions $E_{k}$ are continuous and $(2 \pi \mathbb{Z})^{2}$-periodic. The spectrum of $A$ coincides with the union of the intervals (bands) that are the ranges of the functions $E_{k}$. By (1.12), the eigenfunctions $\psi_{k}$ admit representation in the form

$$
\psi_{k}(\mathbf{x}, \boldsymbol{\xi})=e^{i\langle\mathbf{x}, \boldsymbol{\xi}\rangle} \varphi_{k}(\mathbf{x}, \boldsymbol{\xi}), \quad \varphi_{k}(\cdot, \boldsymbol{\xi}) \in \widetilde{H}^{1}\left(\mathbb{Q}^{2}\right)
$$

The functions $\psi_{k}, \varphi_{k}$ are Hölder continuous with respect to $\mathbf{x}$.
Now we consider the integral operators

$$
\begin{equation*}
\left(\Psi_{k} u\right)(\boldsymbol{\xi})=(2 \pi)^{-1} \int \overline{\psi_{k}(\mathbf{x}, \boldsymbol{\xi})} u(\mathbf{x}) d \mathbf{x}, \quad k \in \mathbb{N} . \tag{1.14}
\end{equation*}
$$

The mappings

$$
\Psi_{k}: L_{2}\left(\mathbb{R}^{2}\right) \rightarrow L_{2}\left(\mathbb{T}^{2}\right)
$$

are partially isometric and surjective. The operators $\Psi_{k}^{*} \Psi_{k}, k \in \mathbb{N}$, are orthoprojections in $L_{2}\left(\mathbb{R}^{2}\right)$. They are pairwise orthogonal and

$$
\sum_{k \in \mathbb{N}} \Psi_{k}^{*} \Psi_{k}=I
$$

The operators (1.14) provide a partial diagonalization of the operator $A$. Namely, denoting by $\left[E_{k}\right]$ the operator of multiplication by the function $E_{k}(\boldsymbol{\xi})$ in $L_{2}\left(\mathbb{T}^{2}\right)$, we obtain

$$
A=\sum_{k \in \mathbb{N}} \Psi_{k}^{*}\left[E_{k}\right] \Psi_{k}
$$

3. A gap. The spectrum of $A$ may have gaps other than the semiinfinite gap $(-\infty, 0)$. Let $\Lambda=\left(\lambda_{-}, \lambda_{+}\right)$be a gap; we assume that $\lambda_{ \pm} \in \operatorname{spec} A$. Clearly,

$$
\begin{align*}
& \lambda_{+}=\min _{\boldsymbol{\xi} \in \mathbb{T}^{2}} E_{l}(\boldsymbol{\xi}),  \tag{1.15+}\\
& \lambda_{-}=\max _{\boldsymbol{\xi} \in \mathbb{T}^{2}} E_{l-1}(\boldsymbol{\xi}) \tag{1.15-}
\end{align*}
$$

for some number $l \in \mathbb{N}$. As usual (cf., e.g., [B3]-[B5]), we impose some "regularity" conditions on the edges $\lambda_{+}, \lambda_{-}$of the gap. The conditions for $\lambda_{+}$look like this.

Condition 1.3(+). a) $\min _{\boldsymbol{\xi} \in \mathbb{T}^{2}} E_{l+1}(\boldsymbol{\xi})>\lambda_{+}$; b) the minimum in 1.15+ is only attained at finitely many points $\boldsymbol{\xi}_{j}^{(+)} \in \mathbb{T}^{2}, j=1, \ldots, m_{+}$, each being a nondegenerate minimum point for $E_{l}(\cdot)$.

Remark 1.4. For the semiinfinite gap $(-\infty, 0)$, Condition $1.3(+)$ at $\lambda_{+}=0$ is fulfilled automatically with $l=1, m_{+}=1$, and $\boldsymbol{\xi}_{1}^{(+)}=0$. This fact was used in [BLSu].

By Condition $1.3(+), \lambda_{+}$is a simple eigenvalue of the operator $A(\boldsymbol{\xi})$ with $\boldsymbol{\xi}=\boldsymbol{\xi}_{j}^{(+)}$, $j=1, \ldots, m_{+}$. It follows that, for some (sufficiently small) $\delta>0$, the eigenvalue $E_{l}(\boldsymbol{\xi})$,
$\left|\boldsymbol{\xi}-\boldsymbol{\xi}_{j}^{(+)}\right| \leq \delta$, is simple. This implies the real analyticity of $E_{l}(\boldsymbol{\xi})$ in these neighborhoods of the points $\boldsymbol{\xi}_{j}^{(+)}, j=1, \ldots, m_{+}$. Then Condition $\left.1.3(+), \mathrm{b}\right)$ means that

$$
\begin{align*}
E_{l}(\boldsymbol{\xi})-\lambda_{+}= & \mathfrak{b}_{j}^{(+)}\left(\boldsymbol{\xi}-\boldsymbol{\xi}_{j}^{(+)}\right)+O\left(\left|\boldsymbol{\xi}-\boldsymbol{\xi}_{j}^{(+)}\right|^{3}\right) \\
& \left|\boldsymbol{\xi}-\boldsymbol{\xi}_{j}^{(+)}\right| \leq \delta, \quad j=1, \ldots, m_{+} \tag{1.16+}
\end{align*}
$$

where $\mathfrak{b}_{j}^{(+)}$is a positive definite quadratic form.
The conditions on $\lambda_{-}$are similar.
Condition 1.3(-). a) $\max _{\boldsymbol{\xi} \in \mathbb{T}^{2}} E_{l-2}(\boldsymbol{\xi})<\lambda_{-}$; b) the maximum in (1.15-1 is only attained at finitely many points $\boldsymbol{\xi}_{j}^{(-)} \in \mathbb{T}^{2}, j=1, \ldots, m_{-}$, each being a nondegenerate maximum point for $E_{l-1}(\cdot)$.

We note that condition a) makes sense only for $l>2$. Like (1.16+, Condition 1.3(-), b) means that

$$
\begin{align*}
\lambda_{-}-E_{l-1}(\boldsymbol{\xi})= & \mathfrak{b}_{j}^{(-)}\left(\boldsymbol{\xi}-\boldsymbol{\xi}_{j}^{(-)}\right)+O\left(\left|\boldsymbol{\xi}-\boldsymbol{\xi}_{j}^{(-)}\right|^{3}\right),  \tag{1.16-}\\
& \left|\boldsymbol{\xi}-\boldsymbol{\xi}_{j}^{(-)}\right| \leq \delta, \quad j=1, \ldots, m_{-},
\end{align*}
$$

where $\mathfrak{b}_{j}^{(-)}$is a positive definite quadratic form.
We agree that the points $\boldsymbol{\xi}_{j}^{( \pm)} \in \mathbb{T}^{2}$ are represented as points of the half-open cube

$$
\boldsymbol{\xi}_{j}^{( \pm)} \in[-\pi, \pi)^{2}, \quad j=1, \ldots, m_{ \pm}
$$

Accordingly, the points $\boldsymbol{\xi} \in \mathbb{T}^{2}$ close to $\boldsymbol{\xi}^{ \pm}$are realized as points of $\mathbb{R}^{2}$-neighborhoods of the points (1.17土).

In terms of the inner product $\langle\cdot, \cdot\rangle$, the form $\mathfrak{b}_{j}^{( \pm)}\left(\boldsymbol{\xi}-\boldsymbol{\xi}_{j}^{( \pm)}\right)$can be written as

$$
\begin{array}{r}
\mathfrak{b}_{j}^{( \pm)}\left(\boldsymbol{\xi}-\boldsymbol{\xi}_{j}^{( \pm)}\right)=\left\langle b_{j}^{( \pm)}\left(\boldsymbol{\xi}-\boldsymbol{\xi}_{j}^{( \pm)}\right), \boldsymbol{\xi}-\boldsymbol{\xi}_{j}^{( \pm)}\right\rangle=\left|\beta_{j}^{( \pm)}\left(\boldsymbol{\xi}-\boldsymbol{\xi}_{j}^{( \pm)}\right)\right|^{2} \\
\beta_{j}^{( \pm)}=\left(b_{j}^{( \pm)}\right)^{1 / 2}, \quad j=1, \ldots, m_{ \pm}
\end{array}
$$

where $b_{j}^{( \pm)}$is a constant positive definite $(2 \times 2)$-matrix. (The matrix $\left(b_{j}^{( \pm)}\right)^{-1}$ determines the so-called tensor of effective masses for the point $\boldsymbol{\xi}_{j}^{( \pm)}$.)

We put $E_{+}:=E_{l}, E_{-}:=E_{l-1}, \psi^{(+)}:=\psi_{l}, \varphi^{(+)}:=\varphi_{l}, \psi^{(-)}:=\psi_{l-1}, \varphi^{(-)}:=\varphi_{l-1}$. The functions $\psi^{( \pm)}, \varphi^{( \pm)}$can be chosen as real-analytic $H^{1}\left(\mathbb{Q}^{2}\right)$-valued functions of $\boldsymbol{\xi}$ for $\left|\boldsymbol{\xi}-\boldsymbol{\xi}_{j}^{( \pm)}\right| \leq \delta, j=1, \ldots, m_{ \pm}$. The functions $\psi^{( \pm)}, \varphi^{( \pm)}$are Hölder continuous in $\mathbf{x}$. We shall use the notation

$$
\psi_{j}^{( \pm)}(\mathbf{x}):=\psi^{( \pm)}\left(\mathbf{x}, \boldsymbol{\xi}_{j}^{( \pm)}\right), \quad \varphi_{j}^{( \pm)}(\mathbf{x}):=\varphi^{( \pm)}\left(\mathbf{x}, \boldsymbol{\xi}_{j}^{( \pm)}\right), \quad j=1, \ldots, m_{ \pm}
$$

Remark 1.5. In the recent paper [He], the following was shown for the operator $-h^{2} \Delta+$ $V(\mathbf{x})$ with $d=2$ and orthogonal lattice of periods (under certain restrictions on $V$ ). For any $j \in \mathbb{N}$ and any sufficiently small $h \in\left(0, h_{0}(j)\right]$, there are at least $j$ gaps in the spectrum; moreover, the edges of these gaps are regular and $m_{ \pm}=1$.
4. Suppose that an "observation point" $\lambda$ lies in the gap $\Lambda: \lambda_{-}<\lambda<\lambda_{+}$. We denote by

$$
\begin{equation*}
\mathfrak{N}_{ \pm}(\alpha, \lambda ; A, V), \quad \alpha>0, \quad \lambda \in \Lambda, \tag{1.20}
\end{equation*}
$$

the number of eigenvalues of $A_{ \pm}(t)$ that have crossed the point $\lambda$ as $t$ has been growing from 0 to $\alpha$. Note that, as $t$ grows, the eigenvalues of $A_{+}(t)$ move from the right to the left, while the eigenvalues of $A_{-}(t)$ move from the left to the right. Therefore, the function
$\lambda \mapsto \mathfrak{N}_{+}(\alpha, \lambda ; A, V)$ is monotone nondecreasing, and the function $\lambda \mapsto \mathfrak{N}_{-}(\alpha, \lambda ; A, V)$ is monotone nonincreasing (for $\alpha>0$ fixed).

The following statement was proved in [B2] (see also Theorem 3.2 in the survey [B4]).
Proposition 1.6. Under condition (1.7), we have

$$
\begin{align*}
\lim _{\alpha \rightarrow \infty} \alpha^{-1} \mathfrak{N}_{+}(\alpha, \lambda ; A, V) & =J(V, g), \quad \lambda \in \Lambda  \tag{1.21}\\
\lim _{\alpha \rightarrow \infty} \alpha^{-1} \mathfrak{N}_{-}(\alpha, \lambda ; A, V) & =0, \quad \lambda \in \Lambda
\end{align*}
$$

Thus, if the observation point $\lambda$ lies inside the gap, then $\mathfrak{N}_{+}$has the Weyl asymptotics (1.21). For $\lambda=\lambda_{+}$the asymptotics (1.21) may fail even under condition (1.6). Later we impose an additional condition (Condition $2.1(q)$ ) on $V$ that ensures that the following limits are finite:

$$
\begin{align*}
& \mathfrak{N}_{ \pm}\left(\alpha, \lambda_{+} ; A, V\right)=\lim _{\lambda \rightarrow \lambda_{+}-0} \mathfrak{N}_{ \pm}(\alpha, \lambda ; A, V)  \tag{1.22+}\\
& \mathfrak{N}_{ \pm}\left(\alpha, \lambda_{-} ; A, V\right)=\lim _{\lambda \rightarrow \lambda_{-}+0} \mathfrak{N}_{ \pm}(\alpha, \lambda ; A, V) \tag{1.22-}
\end{align*}
$$

We are interested in the behavior of the functions (1.22土) as $\alpha \rightarrow \infty$.
5. On compact operators. Here we collect the necessary facts about compact operators. Let $\mathfrak{H}, \mathfrak{G}$ be separable Hilbert spaces. The space of continuous linear operators is denoted by $\mathfrak{R}$, and that of compact operators by $\mathfrak{S}_{\infty}$. If necessary, we write in more detail: $\mathfrak{R}(\mathfrak{H}), \mathfrak{S}_{\infty}(\mathfrak{H}, \mathfrak{G})$, etc. Let $T \in \mathfrak{S}_{\infty}$, and let $s_{k}(T)$ be the singular numbers of $T$, i.e., the consecutive eigenvalues (counted with multiplicities) of the operator $\left(T^{*} T\right)^{1 / 2}$. We denote

$$
n(s, T):=\operatorname{card}\left\{k: s_{k}(T)>s\right\}, \quad s>0
$$

If $T=T^{*}$, we put $2 T_{ \pm}=|T| \pm T$ and

$$
n_{ \pm}(s, T):=n\left(s, T_{ \pm}\right), \quad s>0
$$

Clearly, $n_{+}(\cdot, T)$ is the counting function for the sequence $\left\{\lambda_{k}^{(+)}(T)\right\}$ of positive eigenvalues of $T$. For the sequence $\left\{\lambda_{k}^{(-)}(T)\right\}$ a similar role is played by $n_{-}(\cdot, T)$, where $\lambda_{k}^{(-)}(T)=\lambda_{k}^{(+)}(-T)$. We have $n(s, T)=n_{+}(s, T)+n_{-}(s, T), s>0$. If $T_{1}=T_{1}^{*} \in \mathfrak{S}_{\infty}$, $T_{2}=T_{2}^{*} \in \mathfrak{S}_{\infty}$, then

$$
n_{ \pm}\left(\lambda+\mu, T_{1}+T_{2}\right) \leq n_{ \pm}\left(\lambda, T_{1}\right)+n_{ \pm}\left(\mu, T_{2}\right), \quad \lambda>0, \quad \mu>0
$$

We denote by $\Sigma_{q}, 0<q<\infty$, the space (ideal) of compact operators distinguished by the condition

$$
\mid T \mathbf{|}_{q}^{q}:=\sup _{s>0} s^{q} n(s, T)<\infty, \quad q>0
$$

The space $\Sigma_{q}$ is complete in the quasinorm $|\cdot|_{q}$; for $q>1$ it is normable. The space $\Sigma_{q}$ is nonseparable. We introduce the separable subspace (ideal)

$$
\Sigma_{q}^{0}=\left\{T \in \Sigma_{q}: n(s, T)=o\left(s^{-q}\right), s \rightarrow 0\right\}
$$

On the space $\Sigma_{q}$, we consider the functionals

$$
\begin{align*}
\Delta_{q}(T) & :=\limsup _{s \rightarrow 0} s^{q} n(s, T)  \tag{1.23}\\
\delta_{q}(T) & :=\liminf _{s \rightarrow 0} s^{q} n(s, T) \tag{1.24}
\end{align*}
$$

For $T=T^{*} \in \Sigma_{q}$, we put

$$
\begin{equation*}
\Delta_{q}^{( \pm)}(T):=\Delta_{q}\left(T_{ \pm}\right), \quad \delta_{q}^{( \pm)}(T):=\delta_{q}\left(T_{ \pm}\right) \tag{1.25}
\end{equation*}
$$

Let $D_{q}$ be any of the functionals (1.23)-(1.25). The following inequality can be found in BS1:

$$
\begin{equation*}
\left|\left(D_{q}\left(T_{2}\right)\right)^{\tau}-\left(D_{q}\left(T_{1}\right)\right)^{\tau}\right| \leq\left(\Delta_{q}\left(T_{2}-T_{1}\right)\right)^{\tau}, \quad \tau=(q+1)^{-1} \tag{1.26}
\end{equation*}
$$

In particular, (1.26) implies the following statement.
Proposition 1.7. All six functionals (1.23)-(1.25) are continuous on $\Sigma_{q}$. They do not change under adding a summand of class $\Sigma_{q}^{0}$ to $T$. In other words, these functionals are well defined on the factor-space $\Sigma_{q} / \Sigma_{q}^{0}$. They are continuous with respect to the factor-quasinorm $\left(\Delta_{q}(\cdot)\right)^{1 / q}$ on $\Sigma_{q} / \Sigma_{q}^{0}$.

We shall also use the following simple statement.
Proposition 1.8. Let $T_{j}, \widetilde{T}_{j} \in \Sigma_{2 q}, j=1, \ldots, N$, and let $T:=\sum_{j=1}^{N} \widetilde{T}_{j} T_{j}$. Then $T \in \Sigma_{q}$ and

$$
\left(\Delta_{q}(T)\right)^{2} \leq C(q, N) \sum_{j=1}^{N} \Delta_{2 q}\left(\widetilde{T}_{j}\right) \Delta_{2 q}\left(T_{j}\right)
$$

As usual,

$$
\mathfrak{S}_{q}:=\left\{T \in \mathfrak{S}_{\infty}: \sum_{k} s_{k}^{q}(T)<\infty\right\}, \quad 0<q<\infty
$$

Note that $\mathfrak{S}_{q} \subset \Sigma_{q}^{0}$. The class $\mathfrak{S}_{2}$ is formed by the Hilbert-Schmidt operators, and the class $\mathfrak{S}_{1}$ by the nuclear operators.

If $T^{*}=T \in \mathfrak{S}_{\infty}(\mathfrak{H})$ and $\mathfrak{t}[u, u]=(T u, u)_{\mathfrak{H}}$, then the numbers $\lambda_{k}^{(+)}(T)$ (the numbers $\left.\left(-\lambda_{k}^{(-)}(T)\right)\right)$ coincide with the consecutive positive maxima (respectively, the consecutive negative minima) of the ratio of quadratic forms

$$
\begin{equation*}
\mathfrak{t}[u, u] /\|u\|_{\mathfrak{H}}^{2}, \quad u \in \mathfrak{H} . \tag{1.27}
\end{equation*}
$$

Passage from $T$ to the ratio (1.27) facilitates application of variational arguments. Therefore, we shall use the simpler notation $n_{ \pm}(s,(1.27))$ in place of $n_{ \pm}(s, T),|(1.27)|_{q}$ in place of $\backslash T \mathbf{\}_{q}, D_{q}(1.27)$ in place of $D_{q}(T)$, etc.

We shall also need the following easy technical fact about convergence in symmetrically normed ideals.

Lemma 1.9. Suppose that a sequence of operators $X_{n}$ converges strongly:

$$
\underset{n \rightarrow \infty}{(\mathrm{~s})-\lim _{n \rightarrow \infty}} X_{n}=X_{0} \in \mathfrak{R} .
$$

Let $T \in \mathfrak{S}$, where $\mathfrak{S}$ is some separable symmetrically normed ideal. Then

$$
\underset{n \rightarrow \infty}{(\mathfrak{S})-\lim } X_{n} T=X_{0} T, \quad \underset{n \rightarrow \infty}{(\mathfrak{S})-\lim } T X_{n}^{*}=T X_{0}^{*}
$$

6. An auxiliary problem on the semiaxis. The following auxiliary problem on the semiaxis will be used below. Let $\bar{f}=f \in L_{1, \operatorname{loc}\left(\mathbb{R}_{+}\right) \text {. For some } R \geq 1 \text {, we consider the }}^{\text {s }}$ ratio of quadratic forms

$$
\begin{equation*}
\int_{R}^{\infty} f(r)|z(r)|^{2} r d r / \int_{R}^{\infty}\left|z^{\prime}(r)\right|^{2} r d r, \quad z(R)=0, \quad R \geq 1 \tag{1.28}
\end{equation*}
$$

The ratio (1.28) is considered for all functions $z$ that are absolutely continuous on $\mathbb{R}_{+}$ and such that the integral in the denominator is finite. On $f$ we impose the following "implicit" condition: for some $q>1 / 2$

$$
\begin{equation*}
\mathbf{\|}(1.28) \mathbf{|}_{q}<\infty, \quad q>1 / 2 . \tag{q}
\end{equation*}
$$

This condition is fulfilled (or not fulfilled) simultaneously for all $R \geq 1$. Moreover, under condition $1.29_{q}$ all six functionals $D_{q}(1.28)$ are independent of $R \geq 1$.

We can give (see [BL], and also [BS2], [BLSu]) an elementary sufficient condition for $1.29_{q}$. This condition becomes necessary for the nonnegative $f$. Namely, we put

$$
\begin{aligned}
\zeta_{0}(f) & :=\int_{0}^{1}\left|f\left(e^{t}\right)\right| e^{2 t} d t, \\
\zeta_{n}(f) & :=\int_{e^{n-1}}^{e^{n}} t\left|f\left(e^{t}\right)\right| e^{2 t} d t, \quad n \in \mathbb{N}, \\
\zeta(f) & :=\left\{\zeta_{n}(f)\right\}, \quad n \in \mathbb{Z}_{+},
\end{aligned}
$$

and introduce the notation

$$
\begin{aligned}
\|\boldsymbol{\zeta}(f)\|_{q, \infty}^{q} & :=\sup _{s>0} s^{q} \operatorname{card}\left\{n: \zeta_{n}(f)>s\right\}, \quad 2 q>1 \\
\Delta_{q}(\boldsymbol{\zeta}(f)) & :=\limsup _{s \rightarrow 0}^{q} s^{q} \operatorname{card}\left\{n: \zeta_{n}(f)>s\right\}, \quad 2 q>1, \\
\delta_{q}(\boldsymbol{\zeta}(f)) & :=\liminf _{s \rightarrow 0} s^{q} \operatorname{card}\left\{n: \zeta_{n}(f)>s\right\}, \quad 2 q>1
\end{aligned}
$$

Proposition 1.10. a) Assume that

$$
\begin{equation*}
\|\boldsymbol{\zeta}(f)\|_{q, \infty}<\infty, \quad 2 q>1 \tag{q}
\end{equation*}
$$

Then $1.29_{q}$ is true and

$$
\Delta_{q}(1.28) \leq C(q) \Delta_{q}(\boldsymbol{\zeta}(f)), \quad 2 q>1
$$

b) Assume that $f(r) \geq 0, r \geq R_{0}$, for some $R_{0} \geq 1$. Then $1.29_{q}$ implies $1.30_{q}$ and also the inequalities

$$
\partial_{q}(\overline{1.28}) \geq c(q) \partial_{q}(\boldsymbol{\zeta}(f)), \quad \partial=\Delta, \delta, \quad 2 q>1
$$

Remark 1.11. An elementary criterion for the spectrum of the ratio (1.28) to be discrete can be found, e.g., in [BS3]. We shall not use it.

## §2. Formulation of the main results

1. Our goal is to study the asymptotics of $\mathfrak{N}_{ \pm}\left(\alpha, \lambda_{+} ; A, V\right)$ and $\mathfrak{N}_{ \pm}\left(\alpha, \lambda_{-} ; A, V\right)$ (see $(1.22 \pm))$ as $\alpha \rightarrow \infty$. We introduce the following quantities:

$$
\begin{align*}
\Delta_{q}^{( \pm)}\left(\lambda_{+} ; A, V\right):=\limsup _{\alpha \rightarrow \infty} \alpha^{-q} \mathfrak{N}_{ \pm}\left(\alpha, \lambda_{+} ; A, V\right), & q \geq 1  \tag{2.1+}\\
\delta_{q}^{( \pm)}\left(\lambda_{+} ; A, V\right):=\liminf _{\alpha \rightarrow \infty} \alpha^{-q} \mathfrak{N}_{ \pm}\left(\alpha, \lambda_{+} ; A, V\right), & q \geq 1  \tag{2.2+}\\
\Delta_{q}^{( \pm)}\left(\lambda_{-} ; A, V\right):=\limsup _{\alpha \rightarrow \infty} \alpha^{-q} \mathfrak{N}_{ \pm}\left(\alpha, \lambda_{-} ; A, V\right), & q \geq 1  \tag{2.1-}\\
\delta_{q}^{( \pm)}\left(\lambda_{-} ; A, V\right):=\liminf _{\alpha \rightarrow \infty} \alpha^{-q} \mathfrak{N}_{ \pm}\left(\alpha, \lambda_{-} ; A, V\right), & q \geq 1 \tag{2.2-}
\end{align*}
$$

Relation (1.21) shows that there is no point in considering $q<1$ (at least, for $\mathfrak{N}_{+}$).
For a function $F(\mathbf{x}), \mathbf{x} \in \mathbb{R}^{2}$, we put $F_{\beta}(\mathbf{x})=F(\beta \mathbf{x})$, where $\beta$ is a positive constant matrix. Let $(r, \theta)$ be the polar coordinates of a point $\mathbf{x} \in \mathbb{R}^{2}$; we write $F(\mathbf{x})=F(r, \theta)$. By $\langle F\rangle$ we denote the "mean value of $F$ over the angle":

$$
\langle F\rangle(r)=(2 \pi)^{-1} \int_{-\pi}^{\pi} F(r, \theta) d \theta
$$

Also, we shall use the composition $\left\langle F_{\beta}\right\rangle$ of the above transformations.
Along with Conditions 1.1 and (1.5), we impose the following condition on $V$ :

Condition 2.1(q). For some $q \geq 1$,

$$
\begin{equation*}
|(1.28)|_{q}<\infty \quad \text { for } f=\langle V\rangle, \quad q \geq 1 \tag{q}
\end{equation*}
$$

By Proposition 1.10 and condition (1.5), relation $2.3 q$ is equivalent to the condition

$$
\|\boldsymbol{\zeta}(\langle V\rangle)\|_{q, \infty}<\infty
$$

Examples (for any $q \geq 1$ ) demonstrating that Conditions 1.1 and $2.1(q)$ are compatible can be found in [BL] and [BLSu §8].

Let $\beta$ be a constant positive matrix, and let $\varphi(\mathbf{x}), \mathbf{x} \in \mathbb{R}^{2}$, be a bounded function. We introduce the notation

$$
f_{\beta, \varphi}:=\left\langle\left(|\varphi|^{2} V\right)_{\beta}\right\rangle .
$$

Let

$$
\begin{equation*}
\Delta_{q}^{(+)}(V, \beta, \varphi), \quad \delta_{q}^{(+)}(V, \beta, \varphi), \quad q \geq 1 \tag{2.4}
\end{equation*}
$$

denote the functionals $\Delta_{q}^{(+)}(1.28)$ and $\delta_{q}^{(+)}(1.28)$ for $f=f_{\beta, \varphi}$. We mention that condition (2.3q) with $f=\langle V\rangle$ is equivalent to the same condition with $f=f_{\beta, \varphi}$. Moreover, we have

$$
\begin{equation*}
\Delta_{q}^{(+)}(V, \beta, \varphi) \leq C \Delta_{q}^{(+)}(V, \mathbf{1}, 1), \quad q \geq 1 \tag{2.5}
\end{equation*}
$$

This can be checked in an elementary way with the help of Proposition 1.10. In what follows, we shall use the quantities (2.4) with $\beta=\beta_{j}^{( \pm)}$(see (1.18土) and $\varphi=\varphi_{j}^{( \pm)}$(see (1.19土) or $\varphi=1$.

We also note that the functionals (2.4) coincide for potentials $V$ that are asymptotically close as $|\mathbf{x}| \rightarrow \infty$ (see Proposition 2.2 in $[\overline{\mathrm{BLSu}}]$ ).
2. In $[\mathrm{BLSu}]$ it was shown that if $V$ satisfies Conditions 1.1 and $2.1(q)$, then

$$
\begin{equation*}
\mathfrak{N}_{+}(\alpha, 0 ; A, V)=O\left(\alpha^{q}\right), \quad \alpha \rightarrow \infty \tag{2.6}
\end{equation*}
$$

and the corresponding asymptotic formulas were established. Earlier the same results were obtained in BL in the case where $A=-\Delta$.

Below we formulate two theorems (Theorems $2.2(+)$ and $2.5(+)$ ) on the asymptotics of $\mathfrak{N}_{ \pm}\left(\alpha, \lambda_{+} ; A, V\right)$ and two theorems (Theorems $2.2(-)$ and $\left.2.5(-)\right)$ on the asymptotics of $\mathfrak{N}_{ \pm}\left(\alpha, \lambda_{-} ; A, V\right)$. In Theorems $2.2( \pm)$ the answers are formulated in terms of the model Schrödinger operators with, generally speaking, matrix-valued potentials. In Theorems $2.5( \pm)$ the answers are formulated in terms of the auxiliary problem on the semiaxis, but $V$ is subject to an additional restriction.

The description of the model operators (cf. [B5]) involves the quadratic forms $\mathfrak{b}_{j}^{( \pm)}$(see $(1.16 \pm)$ and the corresponding eigenfunctions $\psi_{j}^{( \pm)}$(see (1.19) $\pm$). In the Hilbert space $\mathfrak{H}_{ \pm}=L_{2}\left(\mathbb{R}^{2} ; \mathbb{C}^{m_{ \pm}}\right)$, we consider the following diagonal second order elliptic operator with constant coefficients:

$$
\mathcal{B}_{ \pm}(\mathbf{D})=\operatorname{diag}\left(\mathfrak{b}_{1}^{( \pm)}(\mathbf{D}), \ldots, \mathfrak{b}_{m_{ \pm}}^{( \pm)}(\mathbf{D})\right), \quad \mathbf{D}=-i \nabla
$$

The expression $(2.7 \pm)$ gives rise to a positive selfadjoint operator $B_{ \pm}$in $\mathfrak{H}_{ \pm}$. Now, we introduce the following row matrix and column matrix:

$$
\Pi_{ \pm}(\mathbf{x}):=\left\{\psi_{j}^{( \pm)}(\mathbf{x})\right\}_{j=1}^{m_{ \pm}}, \quad \Pi_{ \pm}^{*}(\mathbf{x}):=\operatorname{col}\left\{\overline{\psi_{j}^{( \pm)}(\mathbf{x})}\right\}_{j=1}^{m_{ \pm}}
$$

The square Hermitian matrix

$$
\mathcal{P}_{ \pm}(\mathbf{x})=\Pi_{ \pm}^{*}(\mathbf{x}) \Pi_{ \pm}(\mathbf{x})=\left\{\overline{\psi_{k}^{( \pm)}(\mathbf{x})} \psi_{j}^{( \pm)}(\mathbf{x})\right\}_{k, j=1}^{m_{ \pm}}
$$

is of rank 1. The trace of it coincides with its only nonzero eigenvalue:

$$
\operatorname{tr} \mathcal{P}_{ \pm}(\mathbf{x})=\sum_{j=1}^{m_{ \pm}}\left|\psi_{j}^{( \pm)}(\mathbf{x})\right|^{2}
$$

We denote

$$
W(\mathbf{x})=(V(\mathbf{x}))^{1 / 2}
$$

and define the nonnegative matrix potential

$$
\mathcal{U}_{ \pm}(\mathbf{x}):=V(\mathbf{x}) \mathcal{P}_{ \pm}(\mathbf{x})=\left(W(\mathbf{x}) \Pi_{ \pm}(\mathbf{x})\right)^{*} W(\mathbf{x}) \Pi_{ \pm}(\mathbf{x})
$$

The function $(2.8 \pm)$ is bounded; therefore, the potential $\mathcal{U}_{ \pm}(\mathbf{x})$ admits a pointwise estimate in terms of $V(\mathbf{x})$.

In the space $\mathfrak{H}_{ \pm}$, we consider the quadratic form

$$
\begin{array}{r}
\sum_{j=1}^{m_{ \pm}} \int\left\langle b_{j}^{( \pm)} \nabla v_{j}, \nabla v_{j}\right\rangle_{2} d \mathbf{x}-\alpha \int\left\langle\mathcal{U}_{ \pm}(\mathbf{x}) \mathbf{v}, \mathbf{v}\right\rangle_{m_{ \pm}} d \mathbf{x} \\
\mathbf{v}=\left(v_{1}, \ldots, v_{m_{ \pm}}\right) \in H^{1}\left(\mathbb{R}^{2} ; \mathbb{C}^{m_{ \pm}}\right)
\end{array}
$$

Under condition (1.7) (and, moreover, under condition (1.6)) on $V$, this form is lower semibounded and closed in $\mathfrak{H}_{ \pm}$. The corresponding selfadjoint operator in $\mathfrak{H}_{ \pm}$(the model operator) is denoted by $B_{ \pm}(\alpha)$. In the sense of form-sums,

$$
B_{ \pm}(\alpha):=B_{ \pm}-\alpha \mathcal{U}_{ \pm}, \quad \alpha>0
$$

By $\mathfrak{N}_{+}\left(\alpha, \lambda ; B_{ \pm}, \mathcal{U}_{ \pm}\right), \alpha>0, \lambda \leq 0$, we denote the number of eigenvalues of the operator $B_{ \pm}(\alpha)$ that lie to the left of the point $\lambda$. Clearly, estimate (2.6) for $A=-\Delta$ can be carried over to the operator (2.9土):

$$
\mathfrak{N}_{+}\left(\alpha, 0 ; B_{ \pm}, \mathcal{U}_{ \pm}\right)=O\left(\alpha^{q}\right), \quad \alpha \rightarrow \infty
$$

We consider the ratio of (finite-dimensional) forms

$$
\frac{\left\langle\mathcal{U}_{ \pm}(\mathbf{x}) \mathbf{c}, \mathbf{c}\right\rangle_{m_{ \pm}}}{\left\langle\mathcal{B}_{ \pm}(\boldsymbol{\eta}) \mathbf{c}, \mathbf{c}\right\rangle_{m_{ \pm}}}, \quad \mathbf{c} \in \mathbb{C}^{m_{ \pm}} ; \mathbf{x} \in \mathbb{R}^{2}, \quad \boldsymbol{\eta} \in \mathbb{R}^{2}
$$

Let $n^{( \pm)}(\mu ; \mathbf{x}, \boldsymbol{\eta})$ denote the number of eigenvalues of the ratio (2.10 $)$ that are greater than $\mu$, where $\mu>0$. We introduce the following notation for the Weyl coefficient corresponding to the operator (2.9土):

$$
\widetilde{J}\left(\mathcal{B}_{ \pm}, \mathcal{U}_{ \pm}\right):=(2 \pi)^{-2} \iint n^{( \pm)}(1 ; \mathbf{x}, \boldsymbol{\eta}) d \mathbf{x} d \boldsymbol{\eta}
$$

For $\lambda<0$, the function $\mathfrak{N}_{+}\left(\alpha, \lambda ; B_{ \pm}, \mathcal{U}_{ \pm}\right)$has Weyl asymptotics:

$$
\lim _{\alpha \rightarrow \infty} \alpha^{-1} \mathfrak{N}_{+}\left(\alpha, \lambda ; B_{ \pm}, \mathcal{U}_{ \pm}\right)=\widetilde{J}\left(\mathcal{B}_{ \pm}, \mathcal{U}_{ \pm}\right), \quad \lambda<0
$$

Moreover,

$$
\mathfrak{N}_{+}\left(\alpha, \lambda ; B_{ \pm}, \mathcal{U}_{ \pm}\right) \leq C \alpha((V))_{\sigma}, \quad \sigma>1, \quad \lambda<0
$$

We put

$$
\begin{gather*}
\Delta_{q}\left(B_{ \pm}, \mathcal{U}_{ \pm}\right):=\limsup _{\alpha \rightarrow \infty} \alpha^{-q} \mathfrak{N}_{+}\left(\alpha, 0 ; B_{ \pm}, \mathcal{U}_{ \pm}\right), \quad q \geq 1 \\
\delta_{q}\left(B_{ \pm}, \mathcal{U}_{ \pm}\right):=\liminf _{\alpha \rightarrow \infty} \alpha^{-q} \mathfrak{N}_{+}\left(\alpha, 0 ; B_{ \pm}, \mathcal{U}_{ \pm}\right), \quad q \geq 1 \\
\widetilde{\Delta}_{1}\left(B_{ \pm}, \mathcal{U}_{ \pm}\right):=\Delta_{1}\left(B_{ \pm}, \mathcal{U}_{ \pm}\right)-\widetilde{J}\left(\mathcal{B}_{ \pm}, \mathcal{U}_{ \pm}\right) \\
\widetilde{\delta}_{1}\left(B_{ \pm}, \mathcal{U}_{ \pm}\right):=\delta_{1}\left(B_{ \pm}, \mathcal{U}_{ \pm}\right)-\widetilde{J}\left(\mathcal{B}_{ \pm}, \mathcal{U}_{ \pm}\right)
\end{gather*}
$$

Theorem 2.2(+). Let the operator $A$ be generated by the form (1.3) under conditions (1.1). Let $\left(\lambda_{-}, \lambda_{+}\right)$be a gap in the spectrum of the operator $A$, and let Condition $1.3(+)$ be satisfied. Suppose that the potential $V$ for the operators (1.8) satisfies condition (1.5) and also Conditions 1.1, and 2.1(q). Then the following is true for the quantities (2.1+) and (2.2+).
(a) If $q=1$, then

$$
\begin{align*}
\partial_{1}^{(+)}\left(\lambda_{+} ; A, V\right) & =J(V, g)+\widetilde{\partial}_{1}\left(B_{+}, \mathcal{U}_{+}\right), \quad \partial=\Delta, \delta,  \tag{2.17}\\
\Delta_{1}^{(-)}\left(\lambda_{+} ; A, V\right) & =0 \tag{2.18}
\end{align*}
$$

Here $J(V, g)$ is as in (1.9), and $\widetilde{\partial}_{1}\left(B_{+}, \mathcal{U}_{+}\right)$is as in (2.15+), (2.16+).
(b) If $q>1$, then

$$
\begin{equation*}
\partial_{q}^{(+)}\left(\lambda_{+} ; A, V\right)=\partial_{q}\left(B_{+}, \mathcal{U}_{+}\right), \quad \partial=\Delta, \delta \tag{2.19}
\end{equation*}
$$

and (2.18) is valid. Here $\partial_{q}\left(B_{+}, \mathcal{U}_{+}\right)$is the quantity defined in $(2.13+),(2.14+)$.
(c) For the validity of the Weyl asymptotics

$$
\begin{equation*}
\Delta_{1}^{(+)}\left(\lambda_{+} ; A, V\right)=\delta_{1}^{(+)}\left(\lambda_{+} ; A, V\right)=J(V, g) \tag{2.20}
\end{equation*}
$$

it suffices that the following condition be fulfilled in addition to $2.3_{q}$ with $q=1$ :

$$
\Delta_{1}^{(+)}(V, \mathbf{1}, 1)=0
$$

Here $\Delta_{1}^{(+)}(V, \mathbf{1}, 1)$ is the quantity defined in (2.4).
Theorem 2.2(-). Let the operator $A$ be generated by the form (1.3) under conditions (1.1). Let $\left(\lambda_{-}, \lambda_{+}\right)$be a gap in the spectrum of $A$, and let Condition 1.3(-) be satisfied. Suppose that the potential $V$ for the operators (1.8) satisfies condition (1.5) and also Conditions 1.1, and 2.1 $q$ ). Then the following is true for the quantities (2.1- and (2.2-).
(a) If $q=1$, then

$$
\begin{align*}
\Delta_{1}^{(+)}\left(\lambda_{-} ; A, V\right) & =\delta_{1}^{(+)}\left(\lambda_{-} ; A, V\right)=J(V, g),  \tag{2.21}\\
\partial_{1}^{(-)}\left(\lambda_{-} ; A, V\right) & =\widetilde{\partial}_{1}\left(B_{-}, \mathcal{U}_{-}\right), \quad \partial=\Delta, \delta \tag{2.22}
\end{align*}
$$

Here $J(V, g)$ is as in (1.9), and $\widetilde{\partial}_{1}\left(B_{-}, \mathcal{U}_{-}\right)$is as in $(2.15-),(2.16-)$.
(b) If $q>1$, then

$$
\begin{align*}
\Delta_{1}^{(+)}\left(\lambda_{-} ; A, V\right) & =\delta_{1}^{(+)}\left(\lambda_{-} ; A, V\right)=J(V, g)  \tag{2.23}\\
\partial_{q}^{(-)}\left(\lambda_{-} ; A, V\right) & =\partial_{q}\left(B_{-}, \mathcal{U}_{-}\right), \quad \partial=\Delta, \delta \tag{2.24}
\end{align*}
$$

Here $\partial_{q}\left(B_{-}, \mathcal{U}_{-}\right)$is as in (2.13-), (2.14-).
3. The model operator ( $2.9 \pm$ ) involves the forms $\mathfrak{b}_{j}^{( \pm)}$or, equivalently, the matrices $\beta_{j}^{( \pm)}$(and, therefore, the tensors of effective masses at the points $\boldsymbol{\xi}_{j}^{( \pm)}$), and also the eigenfunctions $\psi_{j}^{( \pm)}$. It is impossible to avoid the dependence on $\beta_{j}^{( \pm)}$in the asymptotic formulas (2.17), (2.19), (2.22) and (2.24). As to the more unpleasant dependence on the functions $\psi_{j}^{( \pm)}$, the situation is different. These functions can be eliminated from the asymptotic formulas under some supplementary conditions of "regular" behavior of the perturbation $V$. The problem is solved by Theorems $2.5( \pm)$ below; to state them, we need some preparations.

In addition to condition (1.5) and Conditions 1.1, 2.1 $(q)$, we impose the following condition on $V$ (cf. Condition 2.4 in [BLSu] $)$.

Condition 2.3( $\pm$ ). There exists a function $\mathcal{S}=\overline{\mathcal{S}}$ satisfying conditions (1.5), (1.6) (with $V$ replaced by $S$ ) and such that

$$
\begin{equation*}
V(\mathbf{x})=\mathcal{S}(\mathbf{x})(1+o(1)) \quad \text { as }|\mathbf{x}| \rightarrow \infty \tag{2.25}
\end{equation*}
$$

Suppose that every point of the countable set

$$
\left\{\boldsymbol{\xi}=2 \pi \mathbf{n}, \mathbf{n} \in \mathbb{Z}^{2} \backslash\{0\}\right\} \cup\left\{\boldsymbol{\xi}=\boldsymbol{\xi}_{j}^{( \pm)}-\boldsymbol{\xi}_{k}^{( \pm)}+2 \pi \mathbf{n}, k, j=1, \ldots, m_{ \pm} ; j \neq k, \mathbf{n} \in \mathbb{Z}^{2}\right\}
$$

possesses a neighborhood $\mathcal{O}$ such that the Fourier image $\Phi \mathcal{S}$ of $\mathcal{S}$ has the following property: for some $\varkappa>1$,

$$
\Phi \mathcal{S} \in H^{\varkappa}(\mathcal{O}), \quad \varkappa>1
$$

Condition $2.3( \pm)$ is implied by the following one, which is easier to verify.
Condition 2.4. There exists a function $\mathcal{S}=\overline{\mathcal{S}}$ satisfying (1.5), (1.6), and (2.25) and such that the Fourier image $\Phi \mathcal{S}$ of $\mathcal{S}$ has the following property: for some $\varkappa>1$ and $0<\varepsilon<1$,

$$
\Phi \mathcal{S} \in H^{\varkappa}\left(\mathbb{R}^{2} \backslash B_{\varepsilon}\right), \quad \varkappa>1
$$

where $B_{\varepsilon}=\left\{\boldsymbol{\xi} \in \mathbb{R}^{2}:|\boldsymbol{\xi}| \leq \varepsilon\right\}$.
Theorem 2.5(+). Under the assumptions of Theorem 2.2(+), suppose also that Condition $2.3(+)$ (or the more restrictive Condition 2.4) is satisfied. Then the following is true for the quantities $2.1+$ and (2.2+).
(a) If $q=1$, then (2.18) is fulfilled, and

$$
\begin{equation*}
\partial_{1}^{(+)}\left(\lambda_{+} ; A, V\right)=J(V, g)+\sum_{j=1}^{m_{+}} \partial_{1}^{(+)}\left(V, \beta_{j}^{(+)}, 1\right), \quad \partial=\Delta, \delta . \tag{2.26}
\end{equation*}
$$

Here $J(V, g)$ is as in (1.9), and the quantities $\partial_{1}^{(+)}\left(V, \beta_{j}^{(+)}, 1\right)$ are defined in accordance with (2.4).
(b) If $q>1$, then (2.18) is fulfilled, and

$$
\begin{equation*}
\partial_{q}^{(+)}\left(\lambda_{+} ; A, V\right)=\sum_{j=1}^{m_{+}} \partial_{q}^{(+)}\left(V, \beta_{j}^{(+)}, 1\right), \quad \partial=\Delta, \delta \tag{2.27}
\end{equation*}
$$

Theorem 2.5(-). Under the conditions of Theorem 2.2(-), suppose also that Condition $2.3(-)$ (or the more restrictive Condition 2.4) is satisfied. Then the following is true for the quantities 2.1- and (2.2-).
(a) If $q=1$, then (2.21) is true, and

$$
\begin{equation*}
\partial_{1}^{(-)}\left(\lambda_{-} ; A, V\right)=\sum_{j=1}^{m_{-}} \partial_{1}^{(+)}\left(V, \beta_{j}^{(-)}, 1\right), \quad \partial=\Delta, \delta \tag{2.28}
\end{equation*}
$$

(b) If $q>1$, then (2.23) is true, and

$$
\begin{equation*}
\partial_{q}^{(-)}\left(\lambda_{-} ; A, V\right)=\sum_{j=1}^{m_{-}} \partial_{q}^{(+)}\left(V, \beta_{j}^{(-)}, 1\right), \quad \partial=\Delta, \delta \tag{2.29}
\end{equation*}
$$

Remark 2.6. From (2.26) and (2.27) it is clear that the contributions of different points $\boldsymbol{\xi}_{j}^{(+)}$are independent of one another and enter the asymptotic formula additively. This is not so in the "parallel" formulas (2.17) and (2.19). The same refers to the expressions (2.28) and (2.29).

## §3. Model integral operators

The model integral operators responsible for the non-Weyl contribution to the asymptotic formulas (2.17), (2.19), (2.22) and (2.24) were studied in detail in [BLSu]. Here we state the corresponding results.

Let $\mathcal{W}$ be a complex-valued function on $\mathbb{R}^{2}$ such that the function $\mathcal{V}:=|\mathcal{W}|^{2}$ satisfies conditions (1.6) and (2.3q) (with $V$ replaced by $\mathcal{V})$. Next, let $\chi_{0}(\boldsymbol{\eta})=\chi_{0}(|\boldsymbol{\eta}|)$ denote the characteristic function of the disk $|\boldsymbol{\eta}| \leq \delta$ with some $\delta>0$, and let $\mathcal{G}(\gamma ; \mathcal{V})=\mathcal{G}(\gamma)$ be the following integral operator in $L_{2}\left(\mathbb{R}^{2}\right)$ :

$$
\begin{equation*}
(\mathcal{G}(\gamma) v)(\mathbf{y})=(2 \pi)^{-1} \mathcal{W}(\mathbf{y}) \int e^{i\langle\mathbf{y}, \boldsymbol{\eta}\rangle} \chi_{0}(\boldsymbol{\eta})\left(|\boldsymbol{\eta}|^{2}+\gamma^{2}\right)^{-1 / 2} v(\boldsymbol{\eta}) d \boldsymbol{\eta}, \quad \gamma>0 \tag{3.1}
\end{equation*}
$$

The operator (3.1) has the same form as the operator (3.2) in BLSu; the only difference is that in BLSu it was assumed that $\mathcal{W}(\mathbf{y}) \geq 0$, while in the present setting the function $\mathcal{W}(\mathbf{y})$ may be complex-valued. Obviously, the results of BLSu can be carried over to this case. The following statement is a consequence of [BLSu, Proposition 3.6].

Proposition 3.1. Under the above conditions on $\mathcal{V}$, we have

$$
\mathcal{G}(\gamma)=\widehat{\mathcal{G}}(\gamma)+Z(\gamma), \quad \operatorname{rank} Z(\gamma)=1
$$

and the following limit exists:

$$
\begin{equation*}
\left(\underset{\gamma \rightarrow 0}{(\mathrm{u})-\lim } \widehat{\mathcal{G}}(\gamma)=: \widehat{\mathcal{G}}(0) \in \Sigma_{2 q}\right. \tag{3.2}
\end{equation*}
$$

We have

$$
\widehat{\mathcal{G}}(\gamma)=\widehat{\mathcal{G}}(\gamma)\left[\chi_{0}\right] .
$$

Remark 3.2. The operator $\widehat{\mathcal{G}}(0)$ depends on the coefficient $\mathcal{W}$ linearly (this follows from the construction described in [BLSu §3]).

Consider the operator

$$
\mathcal{H}=\widehat{\mathcal{G}}(0)^{*} \widehat{\mathcal{G}}(0)
$$

As was shown in [BLSu, Proposition 3.7], the asymptotic functionals $\partial_{q}^{(+)}(\mathcal{H}), \partial=\Delta, \delta$, coincide with the functionals $\partial_{q}^{(+)}$for the ratio

$$
\begin{equation*}
\int_{1}^{\infty}\langle\mathcal{V}\rangle(r)|z(r)|^{2} r d r / \int_{1}^{\infty}\left|z^{\prime}(r)\right|^{2} r d r, \quad z(1)=0 \tag{3.3}
\end{equation*}
$$

The ratio (3.3) coincides with (1.28) for $f=\langle\mathcal{V}\rangle$ and $R=1$. Thus, the following statement is true.

Proposition 3.3. Under the above conditions on $\mathcal{V}$, we have

$$
\partial_{q}^{(+)}(\mathcal{H})=\partial_{q}^{(+)}(3.3), \quad \partial=\Delta, \delta .
$$

Remark 3.4. If $\mathcal{V}_{1}(\mathbf{y})=\mathcal{V}_{2}(\mathbf{y})$ for $|\mathbf{y}| \geq N$, then the difference between the corresponding operators $\widehat{\mathcal{G}}\left(0 ; \mathcal{V}_{1}\right)$ and $\widehat{\mathcal{G}}\left(0 ; \mathcal{V}_{2}\right)$ is of class $\Sigma_{2 q}^{0}$. This follows from [BLSu, Proposition 2.2 and §3].

## §4. Reduction to compact operators

1. We denote

$$
\begin{align*}
G(\lambda) & =W|A-\lambda I|^{-1 / 2}, \quad \lambda \in \Lambda  \tag{4.1}\\
X(\lambda) & :=G(\lambda)(\operatorname{sgn}(A-\lambda I)) G(\lambda)^{*}=W(A-\lambda I)^{-1} W, \quad \lambda \in \Lambda \tag{4.2}
\end{align*}
$$

Recall that $W=V^{1 / 2}$ ．The following observation（see［B1］，BS2］），which relates the functions（1．20）and the counting functions for the spectrum of the operator $X(\lambda)$ ，is well known：

$$
\begin{equation*}
\mathfrak{N}_{ \pm}(\alpha, \lambda ; A, V)=n_{ \pm}(t, X(\lambda)), \quad t \alpha=1, \quad \lambda \in \Lambda \tag{4.3}
\end{equation*}
$$

In（4．3）we cannot pass to the limit as $\lambda \rightarrow \lambda_{ \pm}$，because the operators（4．1）and（4．2）do not have limits．Therefore，we need an appropriate regularization．

Proposition 4．1（土）．Suppose that，for $\lambda$ close to $\lambda_{ \pm}$，the operator $X(\lambda)$ is represented in the form

$$
X(\lambda)=\Gamma_{ \pm}(\lambda)+Y_{ \pm}(\lambda)
$$

where $\left(\Gamma_{ \pm}(\lambda)\right)^{*}=\Gamma_{ \pm}(\lambda)$ ，the limit

$$
\underset{\substack{\mathrm{u}) \\ \lambda \rightarrow \lambda_{ \pm}}}{(\mathrm{lim}} \Gamma_{ \pm}(\lambda)=: \Gamma_{ \pm}
$$

exists，and（uniformly in $\lambda$ ）

$$
\operatorname{rank} Y_{ \pm}(\lambda) \leq r_{ \pm}<\infty
$$

Suppose also that $\Gamma_{ \pm} \in \Sigma_{q}$ for some $q \geq 1$ ．Then

$$
\begin{array}{r}
\partial_{q}^{(+)}\left(\lambda_{ \pm} ; A, V\right)=\partial_{q}^{(+)}\left(\Gamma_{ \pm}\right) \\
\partial_{q}^{(-)}\left(\lambda_{ \pm} ; A, V\right)=\partial_{q}^{(-)}\left(\Gamma_{ \pm}\right) \\
\partial=\Delta, \delta, \quad q \geq 1
\end{array}
$$

Proof．We prove the statement for $\lambda_{+}$．Relations（4．3），（4．4＋）and（4．5＋）imply that

$$
\begin{equation*}
\left|\mathfrak{N}_{ \pm}(\alpha, \lambda ; A, V)-n_{ \pm}\left(t, \Gamma_{+}(\lambda)\right)\right| \leq r_{+}, \quad t \alpha=1 \tag{4.7+}
\end{equation*}
$$

for $\lambda$ close to $\lambda_{+}$．In（4．7＋），we can pass to the limit as $\lambda \rightarrow \lambda_{+}$，at least at the points of continuity of the functions $n_{ \pm}\left(\cdot, \Gamma_{+}\right)$．We obtain

$$
\begin{equation*}
\left|\mathfrak{N}_{ \pm}\left(\alpha, \lambda_{+} ; A, V\right)-n_{ \pm}\left(t, \Gamma_{+}\right)\right| \leq r_{+}, \quad t \alpha=1 \tag{4.8+}
\end{equation*}
$$

Multiplying 4．8＋）by $\alpha^{-q}=t^{q}$ and passing to the limit as $\alpha \rightarrow \infty$ ，we arrive at （4．6＋）．

Note that relations $(4.6 \pm)$ are preserved under adding an operator of class $\Sigma_{q}^{0}$ to $\Gamma_{ \pm}$．
2．Let $\zeta_{N}(\mathbf{x})$ denote the characteristic function of the disk $|\mathbf{x}| \leq N, N>0$ ，and let $\widetilde{\zeta}_{N}(\mathbf{x}):=1-\zeta_{N}(\mathbf{x})$ ．We put

$$
\begin{aligned}
W_{N}(\mathbf{x}) & :=\zeta_{N}(\mathbf{x}) W(\mathbf{x}), & \widetilde{W}_{N}(\mathbf{x}) & :=\widetilde{\zeta}_{N}(\mathbf{x}) W(\mathbf{x}) \\
V_{N}(\mathbf{x}) & :=\zeta_{N}(\mathbf{x}) V(\mathbf{x}), & \widetilde{V}_{N}(\mathbf{x}) & :=\widetilde{\zeta}_{N}(\mathbf{x}) V(\mathbf{x})
\end{aligned}
$$

The operator $X(\lambda)$ can be represented as

$$
\begin{equation*}
X(\lambda)=L_{N}(\lambda)+K_{N}(\lambda)+2 \operatorname{Re} M_{N}(\lambda) \tag{4.9}
\end{equation*}
$$

where

$$
\begin{align*}
& L_{N}(\lambda):=W_{N}(A-\lambda I)^{-1} W_{N}  \tag{4.10}\\
& K_{N}(\lambda):=\widetilde{W}_{N}(A-\lambda I)^{-1} \widetilde{W}_{N}  \tag{4.11}\\
& M_{N}(\lambda):=\widetilde{W}_{N}(A-\lambda I)^{-1} W_{N} \tag{4.12}
\end{align*}
$$

We are going to regularize the operators（4．10）（4．12）separately and to examine the contribution of each of them to the limit quantities（2．1土）and（2．2土）．

3．Now，we fulfill this program for the model operator（2．9土）．We introduce the operators

$$
\mathfrak{G}_{ \pm}(\gamma):=W \Pi_{ \pm}\left(B_{ \pm}+\gamma^{2} I\right)^{-1 / 2}, \quad \gamma>0
$$

acting from $\mathfrak{H}_{ \pm}=L_{2}\left(\mathbb{R}^{2} ; \mathbb{C}^{m_{ \pm}}\right)$to $L_{2}\left(\mathbb{R}^{2}\right)$ ，and the operators

$$
\mathcal{X}_{ \pm}(\gamma):=\mathfrak{G}_{ \pm}(\gamma)\left(\mathfrak{G}_{ \pm}(\gamma)\right)^{*}=W \Pi_{ \pm}\left(B_{ \pm}+\gamma^{2} I\right)^{-1} \Pi_{ \pm}^{*} W, \quad \gamma>0
$$

acting in $L_{2}\left(\mathbb{R}^{2}\right)$ ．The role of $\lambda$ is played by $-\gamma^{2}$ ．We have

$$
\begin{array}{ll}
\mathfrak{N}_{ \pm}\left(\alpha,-\gamma^{2} ; B_{+}, \mathcal{U}_{+}\right)=n_{ \pm}\left(t, \mathcal{X}_{+}(\gamma)\right), & t \alpha=1,  \tag{4.14+}\\
\mathfrak{N}_{ \pm}\left(\alpha,-\gamma^{2} ; B_{-}, \mathcal{U}_{-}\right)=n_{ \pm}\left(t, \mathcal{X}_{-}(\gamma)\right), & t \alpha=1, \\
\end{array}
$$

Suppose that，for sufficiently small $\gamma$ ，the operator $\mathcal{X}_{ \pm}(\gamma)$ is represented as

$$
\mathcal{X}_{ \pm}(\gamma)=\widehat{\Gamma}_{ \pm}(\gamma)+\mathcal{Y}_{ \pm}(\gamma)
$$

where $\left(\widehat{\Gamma}_{ \pm}(\gamma)\right)^{*}=\widehat{\Gamma}_{ \pm}(\gamma)$ ，the limit

$$
\underset{\gamma \rightarrow 0}{(\mathrm{u})-\lim } \widehat{\Gamma}_{ \pm}(\gamma)=: \widehat{\Gamma}_{ \pm}
$$

exists，and（uniformly in $\gamma$ ） $\operatorname{rank} \mathcal{Y}_{ \pm}(\gamma) \leq \widehat{r}_{ \pm}<\infty$ ．Suppose also that $\widehat{\Gamma}_{ \pm} \in \Sigma_{q}$ for some $q \geq 1$ ．Then

$$
\partial_{q}\left(B_{ \pm}, \mathcal{U}_{ \pm}\right)=\partial_{q}^{(+)}\left(\widehat{\Gamma}_{ \pm}\right), \quad \partial=\Delta, \delta, \quad q \geq 1
$$

The operator $\mathcal{X}_{ \pm}(\gamma)$ can be written as

$$
\mathcal{X}_{ \pm}(\gamma)=\mathcal{L}_{N}^{( \pm)}(\gamma)+\mathcal{K}_{N}^{( \pm)}(\gamma)+2 \operatorname{Re} \mathcal{M}_{N}^{( \pm)}(\gamma)
$$

where

$$
\begin{align*}
\mathcal{L}_{N}^{( \pm)}(\gamma) & :=W_{N} \Pi_{ \pm}\left(B_{ \pm}+\gamma^{2} I\right)^{-1} \Pi_{ \pm}^{*} W_{N} \\
\mathcal{K}_{N}^{( \pm)}(\gamma) & :=\widetilde{W}_{N} \Pi_{ \pm}\left(B_{ \pm}+\gamma^{2} I\right)^{-1} \Pi_{ \pm}^{*} \widetilde{W}_{N} \\
\mathcal{M}_{N}^{( \pm)}(\gamma) & :=\widetilde{W}_{N} \Pi_{ \pm}\left(B_{ \pm}+\gamma^{2} I\right)^{-1} \Pi_{ \pm}^{*} W_{N}
\end{align*}
$$

In $\S 5$ we consider the operators（4．10）and（4．18土），in $\S 6$ the operators 4．11）and （4．19士），and in $\S 7$ the operators 4．12 and（4．20土）．
§5．The operators $L_{N}(\lambda)$ and $\mathcal{L}_{N}^{( \pm)}(\gamma)$
1．We put $L_{N}(-1):=W_{N}(A+I)^{-1} W_{N}$ ．Obviously，$L_{N}(-1) \geq 0$ ．By（4．3）and（1．11） （with $W$ replaced by $W_{N}$ ），condition（1．7）implies the Weyl asymptotics

$$
\begin{equation*}
\Delta_{1}\left(L_{N}(-1)\right)=\delta_{1}\left(L_{N}(-1)\right)=J\left(V_{N}, g\right) \tag{5.1}
\end{equation*}
$$

Consider the difference

$$
\begin{equation*}
L_{N}(\lambda)-L_{N}(-1)=(\lambda+1) W_{N}(A-\lambda I)^{-1}(A+I)^{-1} W_{N} . \tag{5.2}
\end{equation*}
$$

Let $\delta>0$ be small enough that $E_{ \pm}(\boldsymbol{\xi})$ is a simple eigenvalue of the operator $A(\boldsymbol{\xi})$ for all $\boldsymbol{\xi}$ lying in the ellipses

$$
\mathcal{E}_{j}^{( \pm)}:=\left\{\boldsymbol{\xi}:\left|\beta_{j}^{( \pm)}\left(\boldsymbol{\xi}-\boldsymbol{\xi}_{j}^{( \pm)}\right)\right| \leq \delta\right\}, \quad j=1, \ldots, m_{ \pm}
$$

and $\mathcal{E}_{j}^{( \pm)} \cap \mathcal{E}_{k}^{( \pm)}=\varnothing$ for $j \neq k$ ．Let $\chi_{j}^{( \pm)}$denote the characteristic function of the ellipse $\mathcal{E}_{j}^{( \pm)}$．We put

$$
\chi^{( \pm)}(\boldsymbol{\xi}):=\sum_{j=1}^{m_{ \pm}} \chi_{j}^{( \pm)}(\boldsymbol{\xi}), \quad \widetilde{\chi}^{( \pm)}(\boldsymbol{\xi}):=1-\chi^{( \pm)}(\boldsymbol{\xi})
$$

and introduce the projections

$$
\Xi^{( \pm)}:=\Psi_{ \pm}^{*}\left[\chi^{( \pm)}\right] \Psi_{ \pm}, \quad \widetilde{\Xi}^{( \pm)}:=I-\Xi^{( \pm)}
$$

which commute with $A$. Here $\Psi_{+}:=\Psi_{l}, \Psi_{-}:=\Psi_{l-1}$. We write the operator (15.2) as

$$
L_{N}(\lambda)-L_{N}(-1)=(\lambda+1)\left(Z_{N}^{( \pm)}(\lambda)+\widetilde{Z}_{N}^{( \pm)}(\lambda)\right)
$$

where

$$
\begin{align*}
& Z_{N}^{( \pm)}(\lambda):=W_{N}(A-\lambda I)^{-1} \Xi^{( \pm)}(A+I)^{-1} W_{N} \\
& \widetilde{Z}_{N}^{( \pm)}(\lambda):=W_{N}(A-\lambda I)^{-1} \widetilde{\Xi}^{( \pm)}(A+I)^{-1} W_{N}
\end{align*}
$$

Proposition 5.1( $\pm$ ). The following limit exists:

$$
\underset{\lambda \rightarrow \lambda_{ \pm}}{\left(\Sigma_{1}\right)-\lim } \widetilde{Z}_{N}^{( \pm)}(\lambda)=: \widetilde{Z}_{N}^{( \pm)}\left(\lambda_{ \pm}\right) \in \Sigma_{1}^{0}
$$

Proof. We write the operator (5.6土) as

$$
\widetilde{Z}_{N}^{( \pm)}(\lambda)=\left(W_{N}(A+I)^{-1}\right) F^{( \pm)}(\lambda)\left(W_{N}(A+I)^{-1}\right)^{*}
$$

where $F^{( \pm)}(\lambda):=(A+I)(A-\lambda I)^{-1} \widetilde{\Xi}^{( \pm)}$. It is easy to see that the limit

$$
\underset{\lambda \rightarrow \lambda_{ \pm}}{(\mathrm{u})-\lim ^{( \pm)}}(\lambda)=(A+I)\left(A-\lambda_{ \pm} I\right)^{-1} \widetilde{\Xi}^{( \pm)}=: F^{( \pm)}\left(\lambda_{ \pm}\right) \in \mathfrak{R}
$$

exists. As was shown in [BLSu, §4], we have

$$
W_{N}(A+I)^{-1} \in \Sigma_{2}^{0}
$$

Hence, the limit

$$
\underset{\lambda \rightarrow \lambda_{ \pm}}{\left(\Sigma_{1}\right)-\lim } \widetilde{Z}_{N}^{( \pm)}(\lambda)=\left(W_{N}(A+I)^{-1}\right) F^{( \pm)}\left(\lambda_{ \pm}\right)\left(W_{N}(A+I)^{-1}\right)^{*} \in \Sigma_{1}^{0}
$$

also exists.
2. We introduce the operators

$$
G_{N}^{( \pm)}(\lambda):=W_{N}|A-\lambda I|^{-1 / 2} \Xi^{( \pm)}=W_{N} \Psi_{ \pm}^{*}\left[\chi^{( \pm)}\left|E_{ \pm}-\lambda\right|^{-1 / 2}\right] \Psi_{ \pm}
$$

and write the operator (5.5土) as

$$
Z_{N}^{( \pm)}(\lambda)= \pm G_{N}^{( \pm)}(\lambda)(A+I)^{-1}\left(G_{N}^{( \pm)}(\lambda)\right)^{*}
$$

Proposition 5.2( $\pm$ ). For any $N>0$, we have

$$
G_{N}^{( \pm)}(\lambda)=\widehat{G}_{N}^{( \pm)}(\lambda)+\check{G}_{N}^{( \pm)}(\lambda), \quad \operatorname{rank} \check{G}_{N}^{( \pm)}(\lambda)=m_{ \pm}
$$

and the limit

$$
\underset{\lambda \rightarrow \lambda_{ \pm}}{\left(\mathfrak{S}_{2}\right)-\lim } \widehat{G}_{N}^{( \pm)}(\lambda)=: \widehat{G}_{N}^{( \pm)}\left(\lambda_{ \pm}\right) \in \mathcal{S} S_{2}
$$

exists.

Proof．Since the operator $G_{N}^{( \pm)}(\lambda)$ has the form（5．8土），temporarily we can ignore the operator $\Psi_{ \pm}$on the right．Thus，we study the integral operator with the kernel

$$
\begin{aligned}
& \sum_{j=1}^{m_{ \pm}} W_{N}(\mathbf{x}) \psi^{( \pm)}(\mathbf{x}, \boldsymbol{\xi}) \chi_{j}^{( \pm)}(\boldsymbol{\xi})\left|E_{ \pm}(\boldsymbol{\xi})-\lambda\right|^{-1 / 2} \\
&= \sum_{j=1}^{m_{ \pm}} W_{N}(\mathbf{x})\left(\psi^{( \pm)}(\mathbf{x}, \boldsymbol{\xi})-\psi^{( \pm)}\left(\mathbf{x}, \boldsymbol{\xi}_{j}^{( \pm)}\right)\right) \chi_{j}^{( \pm)}(\boldsymbol{\xi})\left|E_{ \pm}(\boldsymbol{\xi})-\lambda\right|^{-1 / 2} \\
&+\sum_{j=1}^{m_{ \pm}} W_{N}(\mathbf{x}) \psi^{( \pm)}\left(\mathbf{x}, \boldsymbol{\xi}_{j}^{( \pm)}\right) \chi_{j}^{( \pm)}(\boldsymbol{\xi})\left|E_{ \pm}(\boldsymbol{\xi})-\lambda\right|^{-1 / 2}
\end{aligned}
$$

The second sum on the right corresponds to an operator of rank $m_{ \pm}$．Each term of the first sum on the right can be written as

$$
\begin{align*}
& \left(W_{N}(\mathbf{x})\left(\psi^{( \pm)}(\mathbf{x}, \boldsymbol{\xi})-\psi^{( \pm)}\left(\mathbf{x}, \boldsymbol{\xi}_{j}^{( \pm)}\right)\right) \chi_{j}^{( \pm)}(\boldsymbol{\xi})\left|E_{ \pm}(\boldsymbol{\xi})-\lambda_{ \pm}\right|^{-1 / 2}\right) \\
& \quad \times\left(\chi_{j}^{( \pm)}(\boldsymbol{\xi})\left|E_{ \pm}(\boldsymbol{\xi})-\lambda_{ \pm}\right|^{1 / 2}\left|E_{ \pm}(\boldsymbol{\xi})-\lambda\right|^{-1 / 2}\right) .
\end{align*}
$$

The second bracketed expression in（5．10土）represents an operator family strongly con－ verging to the operator $\left[\chi_{j}^{( \pm)}\right]$as $\lambda \rightarrow \lambda_{ \pm}$．The first bracketed expression corresponds to a Hilbert－Schmidt kernel．This follows from the relations $W_{N} \in L_{2}\left(\mathbb{R}^{2}\right)$ and（1．16土）， combined with the estimate

$$
\begin{array}{r}
\left|\psi^{( \pm)}(\mathbf{x}, \boldsymbol{\xi})-\psi^{( \pm)}\left(\mathbf{x}, \boldsymbol{\xi}_{j}^{( \pm)}\right)\right| \leq C(N)\left|\boldsymbol{\xi}-\boldsymbol{\xi}_{j}^{( \pm)}\right|, \\
|\mathbf{x}| \leq N, \quad\left|\beta_{j}^{( \pm)}\left(\boldsymbol{\xi}-\boldsymbol{\xi}_{j}^{( \pm)}\right)\right| \leq \delta .
\end{array}
$$

In order to prove（5．11土），we use the representation

$$
\begin{align*}
\varphi^{( \pm)}(\mathbf{x}, \boldsymbol{\xi})-\varphi^{( \pm)}\left(\mathbf{x}, \boldsymbol{\xi}_{j}^{( \pm)}\right) & =\left\langle\boldsymbol{\xi}-\boldsymbol{\xi}_{j}^{( \pm)}, \mathbf{e}_{1}\right\rangle \vartheta_{1}(\mathbf{x}, \boldsymbol{\xi})+\left\langle\boldsymbol{\xi}-\boldsymbol{\xi}_{j}^{( \pm)}, \mathbf{e}_{2}\right\rangle \vartheta_{2}(\mathbf{x}, \boldsymbol{\xi}) \\
& \left|\beta_{j}^{( \pm)}\left(\boldsymbol{\xi}-\boldsymbol{\xi}_{j}^{( \pm)}\right)\right| \leq \delta
\end{align*}
$$

where $\left\{\mathbf{e}_{1}, \mathbf{e}_{2}\right\}$ is the standard basis in $\mathbb{C}^{2}$ ，and the functions $\vartheta_{1}, \vartheta_{2}$ are uniformly bounded （see［B5］for the details）．Then

$$
\begin{aligned}
& \left|\psi^{( \pm)}(\mathbf{x}, \boldsymbol{\xi})-\psi^{( \pm)}\left(\mathbf{x}, \boldsymbol{\xi}_{j}^{( \pm)}\right)\right| \\
& \quad \leq\left|\varphi^{( \pm)}(\mathbf{x}, \boldsymbol{\xi})-\varphi^{( \pm)}\left(\mathbf{x}, \boldsymbol{\xi}_{j}^{( \pm)}\right)\right|+\left|\varphi^{( \pm)}\left(\mathbf{x}, \boldsymbol{\xi}_{j}^{( \pm)}\right)\right|\left|e^{i\langle\mathbf{x}, \boldsymbol{\xi}\rangle}-e^{i\left\langle\mathbf{x}, \boldsymbol{\xi}_{j}^{( \pm)}\right\rangle}\right|
\end{aligned}
$$

Combining this with（ $5.12 \pm$ ），we obtain estimate（5．11土）．It remains to use Lemma 1.9 ．

Relation（5．9土）and Proposition $5.2( \pm)$ directly imply the following statement．
Proposition 5．3（土）．For any $N>0$ ，we have

$$
Z_{N}^{( \pm)}(\lambda)=\widehat{Z}_{N}^{( \pm)}(\lambda)+\check{Z}_{N}^{( \pm)}(\lambda), \quad \operatorname{rank} \check{Z}_{N}^{( \pm)}(\lambda) \leq 2 m_{ \pm}
$$

and the following limit exists：

$$
\underset{\lambda \rightarrow \lambda_{ \pm}}{\left(\mathfrak{S}_{1}\right)-\lim } \widehat{Z}_{N}^{( \pm)}(\lambda)=: \widehat{Z}_{N}^{( \pm)}\left(\lambda_{ \pm}\right) \in \mathfrak{S}_{1}
$$

3．The following statement is a consequence of relations（5．1）and（5．4土）and Proposi－ tions 5．1 $\pm$ ）and 5．3（ $\pm$ ）．
Proposition 5．4（土）．For any $N>0$ ，the following representation is valid：

$$
L_{N}(\lambda)=\widehat{L}_{N}^{( \pm)}(\lambda)+\check{L}_{N}^{( \pm)}(\lambda), \quad \operatorname{rank} \check{L}_{N}^{( \pm)}(\lambda) \leq 2 m_{ \pm}
$$

where the limit

$$
\left(\underset{\lambda \rightarrow \lambda_{ \pm}}{\left(\Sigma_{1}\right)-\lim } \widehat{L}_{N}^{( \pm)}(\lambda)=: \widehat{L}_{N}^{( \pm)}\left(\lambda_{ \pm}\right) \in \Sigma_{1}\right.
$$

exists．We have

$$
\begin{align*}
& \Delta_{1}^{(+)}\left(\widehat{L}_{N}^{( \pm)}\left(\lambda_{ \pm}\right)\right)=\delta_{1}^{(+)}\left(\widehat{L}_{N}^{( \pm)}\left(\lambda_{ \pm}\right)\right)=J\left(V_{N}, g\right) \\
& \Delta_{1}^{(-)}\left(\widehat{L}_{N}^{( \pm)}\left(\lambda_{ \pm}\right)\right)=0
\end{align*}
$$

$$
\widehat{L}_{N}^{( \pm)}\left(\lambda_{ \pm}\right)=\zeta_{N} \widehat{L}_{N}^{( \pm)}\left(\lambda_{ \pm}\right) \zeta_{N}
$$

For the proof it suffices to put

$$
\begin{aligned}
& \widehat{L}_{N}^{( \pm)}(\lambda)=L_{N}(-1)+(\lambda+1)\left(\widehat{Z}_{N}^{( \pm)}(\lambda)+\widetilde{Z}_{N}^{( \pm)}(\lambda)\right) \\
& \check{L}_{N}^{( \pm)}(\lambda)=(\lambda+1) \check{Z}_{N}^{( \pm)}(\lambda)
\end{aligned}
$$

4．The operator $\mathcal{L}_{N}^{( \pm)}(\gamma)$（see（4．18 $\left.\pm\right)$ ）is analyzed by analogy with $L_{N}(\lambda)$ ．By（4．14 $\pm$ ） and（2．11 $\pm$ ）with $\gamma=1$ ，we have the Weyl asymptotics

$$
\Delta_{1}\left(\mathcal{L}_{N}^{( \pm)}(1)\right)=\delta_{1}\left(\mathcal{L}_{N}^{( \pm)}(1)\right)=\widetilde{J}\left(\mathcal{B}_{ \pm}, \mathcal{U}_{ \pm}^{(N)}\right)
$$

where $\mathcal{U}_{ \pm}^{(N)}(\mathbf{x}):=V_{N}(\mathbf{x}) \mathcal{P}_{ \pm}(\mathbf{x})$ ．Consider the difference

$$
\mathcal{L}_{N}^{( \pm)}(\gamma)-\mathcal{L}_{N}^{( \pm)}(1)=\left(1-\gamma^{2}\right) W_{N} \Pi_{ \pm}\left(B_{ \pm}+\gamma^{2} I\right)^{-1}\left(B_{ \pm}+I\right)^{-1} \Pi_{ \pm}^{*} W_{N}
$$

Since $B_{ \pm}$is a differential operator with constant coefficients，it follows that，in the Fourier representation，$B_{ \pm}$turns into multiplication by the matrix－valued symbol $\mathcal{B}_{ \pm}(\boldsymbol{\eta})$ ：

$$
B_{ \pm}=\Phi^{*}\left[\mathcal{B}_{ \pm}\right] \Phi
$$

For $\left(B_{ \pm}+\gamma^{2} I\right)^{-1}$ ，we have

$$
\left(B_{ \pm}+\gamma^{2} I\right)^{-1}=\Phi^{*}\left[\mathcal{R}_{ \pm}(\gamma)\right] \Phi
$$

where

$$
\mathcal{R}_{ \pm}(\gamma ; \boldsymbol{\eta}):=\operatorname{diag}\left\{\left(\mathfrak{b}_{j}^{( \pm)}(\boldsymbol{\eta})+\gamma^{2}\right)^{-1}\right\}_{j=1}^{m_{ \pm}} .
$$

Let $\rho_{j}^{( \pm)}(\boldsymbol{\eta})$ denote the characteristic function of the ellipse $\left\{\boldsymbol{\eta}:\left|\beta_{j}^{( \pm)} \boldsymbol{\eta}\right| \leq \delta\right\}$ ．We put

$$
\rho^{( \pm)}(\boldsymbol{\eta}):=\operatorname{diag}\left\{\rho_{1}^{( \pm)}(\boldsymbol{\eta}), \ldots, \rho_{m_{ \pm}}^{( \pm)}(\boldsymbol{\eta})\right\}, \quad \widetilde{\rho}^{( \pm)}(\boldsymbol{\eta}):=\mathbf{1}_{m_{ \pm}}-\rho^{( \pm)}(\boldsymbol{\eta})
$$

and introduce the projections $\Xi_{0}^{( \pm)}=\Phi^{*}\left[\rho^{( \pm)}\right] \Phi, \widetilde{\Xi}_{0}^{( \pm)}=I-\Xi_{0}^{( \pm)}$in the space $\mathfrak{H}_{ \pm}=$ $L_{2}\left(\mathbb{R}^{2} ; \mathbb{C}^{m_{ \pm}}\right)$．The operator（5．18土）is represented as

$$
\mathcal{L}_{N}^{( \pm)}(\gamma)-\mathcal{L}_{N}^{( \pm)}(1)=\left(1-\gamma^{2}\right)\left(\mathcal{Z}_{N}^{( \pm)}(\gamma)+\widetilde{\mathcal{Z}}_{N}^{( \pm)}(\gamma)\right)
$$

where

$$
\begin{align*}
\mathcal{Z}_{N}^{( \pm)}(\gamma) & :=W_{N} \Pi_{ \pm}\left(B_{ \pm}+\gamma^{2} I\right)^{-1} \Xi_{0}^{( \pm)}\left(B_{ \pm}+I\right)^{-1} \Pi_{ \pm}^{*} W_{N} \\
\widetilde{\mathcal{Z}}_{N}^{( \pm)}(\gamma) & :=W_{N} \Pi_{ \pm}\left(B_{ \pm}+\gamma^{2} I\right)^{-1} \widetilde{\Xi}_{0}^{( \pm)}\left(B_{ \pm}+I\right)^{-1} \Pi_{ \pm}^{*} W_{N}
\end{align*}
$$

The operators $(5.20 \pm)$ and（5．21 $\pm$ ）are studied by the same method as the operators $(5.5 \pm)$ and $(5.6 \pm)$ ．In order to avoid repetition of similar arguments，we omit the proofs
of the following statements，which are analogs of Propositions 5．1 $( \pm)$ and $5.3( \pm)$ ．Note that the proofs become simpler somewhat，because，instead of the operators $\Psi_{ \pm}$，we now deal with the simpler Fourier operator $\Phi$ ．At the same time，the matrix character of the operators（ $5.20 \pm$ ）and（ $5.21 \pm$ ）does not add serious difficulties．

Proposition 5．5（土）．The following limit exists：

$$
\underset{\gamma \rightarrow 0}{\left(\Sigma_{1}\right)-\lim } \underset{N}{ } \widetilde{\mathcal{Z}}_{N}^{( \pm)}(\gamma)=: \widetilde{\mathcal{Z}}_{N}^{( \pm)}(0) \in \Sigma_{1}^{0}
$$

Proposition 5．6（土）．For any $N>0$ ，we have

$$
\mathcal{Z}_{N}^{( \pm)}(\gamma)=\widehat{\mathcal{Z}}_{N}^{( \pm)}(\gamma)+\check{\mathcal{Z}}_{N}^{( \pm)}(\gamma), \quad \operatorname{rank} \check{\mathcal{Z}}_{N}^{( \pm)}(\gamma) \leq 2 m_{ \pm}
$$

and the limit

$$
\underset{\gamma \rightarrow 0}{\left(\mathfrak{S}_{1}\right) \lim } \widehat{\mathcal{Z}}_{N}^{( \pm)}(\gamma)=: \widehat{\mathcal{Z}}_{N}^{( \pm)}(0) \in \mathfrak{S}_{1}
$$

exists．
5．The following statement is a consequence of relations（5．17 $\pm$ ）and（5．19 $\pm$ ）and Propo－ sitions 5．5（土）and 5．6（土）．

Proposition $5.7( \pm)$ ．For any $N>0$ ，the following representation is valid：

$$
\mathcal{L}_{N}^{( \pm)}(\gamma)=\widehat{\mathcal{L}}_{N}^{( \pm)}(\gamma)+\check{\mathcal{L}}_{N}^{( \pm)}(\gamma), \quad \operatorname{rank} \check{\mathcal{L}}_{N}^{( \pm)}(\gamma) \leq 2 m_{ \pm}
$$

The limit

$$
\underset{\gamma \rightarrow 0}{\left(\Sigma_{1}\right)-\lim } \widehat{\mathcal{L}}_{N}^{( \pm)}(\gamma)=: \widehat{\mathcal{L}}_{N}^{( \pm)}(0) \in \Sigma_{1}
$$

exists，and

$$
\begin{align*}
\Delta_{1}^{(+)}\left(\widehat{\mathcal{L}}_{N}^{( \pm)}(0)\right) & =\delta_{1}^{(+)}\left(\widehat{\mathcal{L}}_{N}^{( \pm)}(0)\right)=\widetilde{J}\left(\mathcal{B}_{ \pm}, \mathcal{U}_{ \pm}^{(N)}\right) \\
\Delta_{1}^{(-)}\left(\widehat{\mathcal{L}}_{N}^{( \pm)}(0)\right) & =0
\end{align*}
$$

We have

$$
\widehat{\mathcal{L}}_{N}^{( \pm)}(0)=\zeta_{N} \widehat{\mathcal{L}}_{N}^{( \pm)}(0) \zeta_{N}
$$

For the proof，it suffices to put

$$
\begin{aligned}
& \widehat{\mathcal{L}}_{N}^{( \pm)}(\gamma)=\mathcal{L}_{N}^{( \pm)}(1)+\left(1-\gamma^{2}\right)\left(\widehat{\mathcal{Z}}_{N}^{( \pm)}(\gamma)+\widetilde{\mathcal{Z}}_{N}^{( \pm)}(\gamma)\right) \\
& \check{\mathcal{L}}_{N}^{( \pm)}(\gamma)=\left(1-\gamma^{2}\right) \check{\mathcal{Z}}_{N}^{( \pm)}(\gamma)
\end{aligned}
$$

§6．The operators $K_{N}(\lambda)$ and $\mathcal{K}_{N}^{( \pm)}(\gamma)$
1．The operator（4．11）can be written as

$$
K_{N}(\lambda)=Q_{N}^{( \pm)}(\lambda)+\widetilde{K}_{N}^{( \pm)}(\lambda)
$$

where

$$
\begin{align*}
& Q_{N}^{( \pm)}(\lambda):=\widetilde{W}_{N}(A-\lambda I)^{-1} \Xi^{( \pm)} \widetilde{W}_{N} \\
& \widetilde{K}_{N}^{( \pm)}(\lambda):=\widetilde{W}_{N}(A-\lambda I)^{-1} \widetilde{\Xi}^{( \pm)} \widetilde{W}_{N}
\end{align*}
$$

The projections $\Xi^{( \pm)}$and $\widetilde{\Xi}^{( \pm)}$were introduced in（5．3土）．

Proposition 6．1（ $\pm$ ）．The limit

$$
\underset{\lambda \rightarrow \lambda_{ \pm}}{\left(\Sigma_{1}\right)-\lim } \widetilde{K}_{N}^{( \pm)}(\lambda)=: \widetilde{K}_{N}^{( \pm)}\left(\lambda_{ \pm}\right) \in \Sigma_{1}
$$

exists，and

$$
\begin{align*}
& \Delta_{1}^{(+)}\left(\widetilde{K}_{N}^{( \pm)}\left(\lambda_{ \pm}\right)\right)=\delta_{1}^{(+)}\left(\widetilde{K}_{N}^{( \pm)}\left(\lambda_{ \pm}\right)\right)=J\left(\widetilde{V}_{N}, g\right) \\
& \Delta_{1}^{(-)}\left(\widetilde{K}_{N}^{( \pm)}\left(\lambda_{ \pm}\right)\right)=0
\end{align*}
$$

We have

$$
\widetilde{K}_{N}^{( \pm)}\left(\lambda_{ \pm}\right)=\widetilde{\zeta}_{N} \widetilde{K}_{N}^{( \pm)}\left(\lambda_{ \pm}\right) \widetilde{\zeta}_{N}
$$

Proof．By Hilbert＇s identity，

$$
\widetilde{K}_{N}^{( \pm)}(\lambda)=\widetilde{K}_{N}^{( \pm)}(-1)+(\lambda+1) \widetilde{W}_{N}(A-\lambda I)^{-1}(A+I)^{-1} \widetilde{\Xi}^{( \pm)} \widetilde{W}_{N}
$$

Arguing as in the case of $(5.7 \pm)$ ，we easily check that the limit

$$
\left(\underset{\lambda \rightarrow \lambda_{ \pm}}{\left.\Sigma_{1}\right)-\lim }\left(\widetilde{K}_{N}^{( \pm)}(\lambda)-\widetilde{K}_{N}^{( \pm)}(-1)\right) \in \Sigma_{1}^{0}\right.
$$

exists．We show that

$$
\widetilde{K}_{N}^{( \pm)}(-1)-\widetilde{W}_{N}(A+I)^{-1} \widetilde{W}_{N} \in \Sigma_{1}^{0}
$$

Indeed，

$$
\begin{gathered}
\widetilde{K}_{N}^{( \pm)}(-1)-\widetilde{W}_{N}(A+I)^{-1} \widetilde{W}_{N}=-\widetilde{W}_{N}(A+I)^{-1} \Xi^{( \pm)} \widetilde{W}_{N} \\
=-\left(\widetilde{W}_{N}(A+I)^{-1}\right)\left((A+I) \Xi^{( \pm)}\right)\left(\widetilde{W}_{N}(A+I)^{-1}\right)^{*}
\end{gathered}
$$

The relations $(A+I) \Xi^{( \pm)} \in \Re$ and $\widetilde{W}_{N}(A+I)^{-1} \in \Sigma_{2}^{0}$（see BLSu，§4］）imply（6．9土）．
By（4．3）with $\lambda=-1$ and the asymptotic formula（1．11），we have

$$
\begin{equation*}
\Delta_{1}^{(+)}\left(\widetilde{W}_{N}(A+I)^{-1} \widetilde{W}_{N}\right)=\delta_{1}^{(+)}\left(\widetilde{W}_{N}(A+I)^{-1} \widetilde{W}_{N}\right)=J\left(\widetilde{V}_{N}, g\right) \tag{6.10}
\end{equation*}
$$

Obviously，$\Delta_{1}^{(-)}\left(\widetilde{W}_{N}(A+I)^{-1} \widetilde{W}_{N}\right)=0$ ．Now，（6．8土），（6．9土）and（6．10）imply（6．4土）－ （6．6 $\pm$ ）．Relation（ $6.7 \pm$ ）follows directly from（ $6.3 \pm$ ）and（ $6.4 \pm$ ）．

2．We write the operator（6．2土）as

$$
Q_{N}^{( \pm)}(\lambda)= \pm \widetilde{G}_{N}^{( \pm)}(\lambda)\left(\widetilde{G}_{N}^{( \pm)}(\lambda)\right)^{*}
$$

where

$$
\widetilde{G}_{N}^{( \pm)}(\lambda):=\widetilde{W}_{N}|A-\lambda I|^{-1 / 2} \Xi^{( \pm)}
$$

Consider the operator

$$
\widetilde{G}_{N}^{( \pm)}(\lambda)=\widetilde{W}_{N} \Psi_{ \pm}^{*}\left[\chi^{( \pm)}\left|E_{ \pm}-\lambda\right|^{-1 / 2}\right] \Psi_{ \pm}=\sum_{j=1}^{m_{ \pm}} T_{j N}^{( \pm)}(\lambda) \Psi_{ \pm}
$$

where

$$
T_{j N}^{( \pm)}(\lambda):=\widetilde{W}_{N} \Psi_{ \pm}^{*}\left[\chi_{j}^{( \pm)}\left|E_{ \pm}-\lambda\right|^{-1 / 2}\right], \quad j=1, \ldots, m_{ \pm}
$$

We start with the study of the operators（6．14土）．The kernel of the integral operator $T_{j N}^{( \pm)}(\lambda)$ has the form

$$
(2 \pi)^{-1} \widetilde{W}_{N}(\mathbf{x}) \chi_{j}^{( \pm)}(\boldsymbol{\xi})\left|E_{ \pm}(\boldsymbol{\xi})-\lambda\right|^{-1 / 2} e^{i\langle\mathbf{x}, \boldsymbol{\xi}\rangle} \varphi^{( \pm)}(\mathbf{x}, \boldsymbol{\xi})
$$

Along with $T_{j N}^{( \pm)}(\lambda)$ ，we consider the operator $\widehat{T}_{j N}^{( \pm)}(\lambda)$ with a simpler kernel，namely，

$$
(2 \pi)^{-1} \widetilde{W}_{N}(\mathbf{x}) \chi_{j}^{( \pm)}(\boldsymbol{\xi})\left| \pm \mathfrak{b}_{j}^{( \pm)}\left(\boldsymbol{\xi}-\boldsymbol{\xi}_{j}^{( \pm)}\right)+\lambda_{ \pm}-\lambda\right|^{-1 / 2} e^{i\langle\mathbf{x}, \boldsymbol{\xi}\rangle} \varphi_{j}^{( \pm)}(\mathbf{x})
$$

Recall that $\mathfrak{b}_{j}^{( \pm)}$is the quadratic form introduced in（1．16土），and $\varphi_{j}^{( \pm)}$is the periodic function defined in $(1.19 \pm)$ ．The following statement shows that，under regularization， the operator $T_{j N}^{( \pm)}(\lambda)$ can be replaced by $\widehat{T}_{j N}^{( \pm)}(\lambda)$ ．

Proposition 6．2（ $\pm$ ）．The following limit exists：

$$
\underset{\lambda \rightarrow \lambda_{ \pm}}{\left(\mathfrak{S}_{2}\right)-\lim }\left(T_{j N}^{( \pm)}(\lambda)-\widehat{T}_{j N}^{( \pm)}(\lambda)\right), \quad j=1, \ldots, m_{ \pm}
$$

Proof．We proceed in two steps．First，we replace the functions $\varphi^{( \pm)}(\mathbf{x}, \boldsymbol{\xi})$ by $\varphi_{j}^{( \pm)}(\mathbf{x})=$ $\varphi^{( \pm)}\left(\mathbf{x}, \boldsymbol{\xi}_{j}^{( \pm)}\right)$in $(6.15 \pm)$ ．Here we use the representation（5．12土）．The difference of the corresponding kernels can be written as

$$
\begin{aligned}
& \left((2 \pi)^{-1} \widetilde{W}_{N}(\mathbf{x}) \chi_{j}^{( \pm)}(\boldsymbol{\xi}) e^{i\langle\mathbf{x}, \boldsymbol{\xi}\rangle}\left(\varphi^{( \pm)}(\mathbf{x}, \boldsymbol{\xi})-\varphi_{j}^{( \pm)}(\mathbf{x})\right)\left|E_{ \pm}(\boldsymbol{\xi})-\lambda_{ \pm}\right|^{-1 / 2}\right) \\
& \quad \times\left(\chi_{j}^{( \pm)}(\boldsymbol{\xi})\left|E_{ \pm}(\boldsymbol{\xi})-\lambda_{ \pm}\right|^{1 / 2}\left|E_{ \pm}(\boldsymbol{\xi})-\lambda\right|^{-1 / 2}\right)
\end{aligned}
$$

Here the first expression in parentheses corresponds to a kernel of the Hilbert－Schmidt class．Indeed，$\widetilde{W}_{N} \in L_{2}\left(\mathbb{R}^{2}\right)$ ，and，by（5．12土），the function $\left(\varphi^{( \pm)}(\mathbf{x}, \boldsymbol{\xi})-\varphi_{j}^{( \pm)}(\mathbf{x})\right)$ elimi－ nates the singularity $\left|E_{ \pm}(\boldsymbol{\xi})-\lambda_{ \pm}\right|^{-1 / 2} \asymp\left|\boldsymbol{\xi}-\boldsymbol{\xi}_{j}^{( \pm)}\right|^{-1}$ ．The second expression in paren－ theses represents the kernel of an operator family strongly converging to the operator $\left[\chi_{j}^{( \pm)}\right]$as $\lambda \rightarrow \lambda_{ \pm}$．By Lemma 1．9，this yields $\left(\mathfrak{S}_{2}\right)$－convergence．It remains to replace $E_{ \pm}$by $\left(\lambda_{ \pm} \pm \mathfrak{b}_{j}^{( \pm)}\right)$．Now，the difference of the corresponding kernels can be represented as

$$
\begin{aligned}
& \left((2 \pi)^{-1} \widetilde{W}_{N}(\mathbf{x}) \chi_{j}^{( \pm)}(\boldsymbol{\xi}) \varphi_{j}^{( \pm)}(\mathbf{x}) e^{i\langle\mathbf{x}, \boldsymbol{\xi}\rangle}\right) \\
& \quad \times\left(\chi_{j}^{( \pm)}(\boldsymbol{\xi})\left(\left|E_{ \pm}(\boldsymbol{\xi})-\lambda\right|^{-1 / 2}-\left| \pm \mathfrak{b}_{j}^{( \pm)}\left(\boldsymbol{\xi}-\boldsymbol{\xi}_{j}^{( \pm)}\right)+\lambda_{ \pm}-\lambda\right|^{-1 / 2}\right)\right)
\end{aligned}
$$

Here，obviously，the first expression in parentheses represents an operator of class $\mathfrak{S}_{2}$ ， and the second generates a strongly convergent operator family（by（1．16土））．Referring to Lemma 1.9 once again，we obtain（ $6.17 \pm$ ）．

3．It is elementary to reduce the operator $\widehat{T}_{j N}^{( \pm)}(\lambda)$ to the operator $\mathcal{G}(\gamma)$ treated in $\S 3$ ， with $\gamma^{2}= \pm\left(\lambda_{ \pm}-\lambda\right)$ ．Indeed，the change of variables $\boldsymbol{\eta}=\beta_{j}^{( \pm)}\left(\boldsymbol{\xi}-\boldsymbol{\xi}_{j}^{( \pm)}\right), \mathbf{y}=\left(\beta_{j}^{( \pm)}\right)^{-1} \mathbf{x}$ in the kernel（6．16土）results in the kernel of the operator（3．1）with

$$
\begin{align*}
\mathcal{W}(\mathbf{y}) & =\mathcal{W}_{j}^{( \pm)}(\mathbf{y})=\widetilde{W}_{N}\left(\beta_{j}^{( \pm)} \mathbf{y}\right) \varphi_{j}^{( \pm)}\left(\beta_{j}^{( \pm)} \mathbf{y}\right) \exp \left(i\left\langle\beta_{j}^{( \pm)} \mathbf{y}, \boldsymbol{\xi}_{j}^{( \pm)}\right\rangle\right) \\
\mathcal{V}(\mathbf{y}) & =\widetilde{V}_{N}\left(\beta_{j}^{( \pm)} \mathbf{y}\right)\left|\varphi_{j}^{( \pm)}\left(\beta_{j}^{( \pm)} \mathbf{y}\right)\right|^{2}=\left(\left|\varphi_{j}^{( \pm)}\right|^{2} \widetilde{V}_{N}\right)_{\beta_{j}^{( \pm)}}(\mathbf{y})
\end{align*}
$$

We recall that conditions（1．6）and（2．3q）for $V$ imply similar conditions for $\mathcal{V}$ ．Obvi－ ously，the corresponding operator $\mathcal{G}(\gamma)=\mathcal{G}_{j N}^{( \pm)}(\gamma), \gamma^{2}= \pm\left(\lambda_{ \pm}-\lambda\right)$ ，is unitarily equivalent to the operator $\widehat{T}_{j N}^{( \pm)}(\lambda)$ ．Propositions 3.1 and $6.2( \pm)$ imply the following statement．

Proposition 6．3（土）． $1^{\circ}$ ．The following representations are valid：

$$
\begin{aligned}
\widehat{T}_{j N}^{( \pm)}(\lambda) & =\widehat{\mathcal{T}}_{j N}^{( \pm)}(\lambda)+Y_{j N}^{( \pm)}(\lambda), & & j=1, \ldots, m_{ \pm} \\
T_{j N}^{( \pm)}(\lambda) & =\mathcal{T}_{j N}^{( \pm)}(\lambda)+Y_{j N}^{( \pm)}(\lambda), & & j=1, \ldots, m_{ \pm} \\
\operatorname{rank} Y_{j N}^{( \pm)}(\lambda) & =1, & & j=1, \ldots, m_{ \pm}
\end{aligned}
$$

where the limits

$$
\begin{aligned}
& \underset{\substack{\mathrm{u})-\lim }}{\left(\widehat{\mathcal{T}}_{j N}^{( \pm)}\right.}(\lambda)=: \widehat{\mathcal{T}}_{j N}^{( \pm)}\left(\lambda_{ \pm}\right) \in \Sigma_{2 q}, \quad j=1, \ldots, m_{ \pm} \\
& (\mathrm{u})-\lim \\
& \lambda \rightarrow \lambda_{ \pm} \\
& \mathcal{T}_{j N}^{( \pm)}(\lambda)=: \mathcal{T}_{j N}^{( \pm)}\left(\lambda_{ \pm}\right) \in \Sigma_{2 q}, \quad j=1, \ldots, m_{ \pm}
\end{aligned}
$$

exist，and

$$
\mathcal{T}_{j N}^{( \pm)}\left(\lambda_{ \pm}\right)=\widehat{\mathcal{T}}_{j N}^{( \pm)}\left(\lambda_{ \pm}\right) \quad\left(\bmod \Sigma_{2 q}^{0}\right), \quad j=1, \ldots, m_{ \pm}
$$

$2^{\circ}$ ．The operator $\widehat{\mathcal{T}}_{j N}^{( \pm)}\left(\lambda_{ \pm}\right)$is unitarily equivalent to the operator $\widehat{\mathcal{G}}(0)=\widehat{\mathcal{G}}_{j N}^{( \pm)}(0)$ defined as in（3．2）with $\mathcal{G}(\gamma)=\mathcal{G}_{j N}^{( \pm)}(\gamma)$ ，i．e．，in the case where（6．18土）is fulfilled． $3^{\circ}$ ．We have

$$
\begin{array}{ll}
\mathcal{T}_{j N}^{( \pm)}\left(\lambda_{ \pm}\right)=\widetilde{\zeta}_{N} \mathcal{T}_{j N}^{( \pm)}\left(\lambda_{ \pm}\right), & \widehat{\mathcal{T}}_{j N}^{( \pm)}\left(\lambda_{ \pm}\right)=\widetilde{\zeta}_{N} \widehat{\mathcal{T}}_{j N}^{( \pm)}\left(\lambda_{ \pm}\right), \\
\mathcal{T}_{j N}^{( \pm)}\left(\lambda_{ \pm}\right)=\mathcal{T}_{j N}^{( \pm)}\left(\lambda_{ \pm}\right)\left[\chi_{j}^{( \pm)}\right], & \widehat{\mathcal{T}}_{j N}^{( \pm)}\left(\lambda_{ \pm}\right)=\widehat{\mathcal{T}}_{j N}^{( \pm)}\left(\lambda_{ \pm}\right)\left[\chi_{j}^{( \pm)}\right]
\end{array}
$$

4．We turn to the operator $(6.11 \pm)$ ，which we write in the following form，in accordance with（6．13士）：

$$
Q_{N}^{( \pm)}(\lambda)= \pm \sum_{j, k=1}^{m_{ \pm}} T_{j N}^{( \pm)}(\lambda) \Psi_{ \pm}\left(T_{k N}^{( \pm)}(\lambda) \Psi_{ \pm}\right)^{*}
$$

Simultaneously，we consider the operator

$$
\widehat{Q}_{N}^{( \pm)}(\lambda)= \pm \sum_{j, k=1}^{m_{ \pm}} \widehat{T}_{j N}^{( \pm)}(\lambda) \Psi_{ \pm}\left(\widehat{T}_{k N}^{( \pm)}(\lambda) \Psi_{ \pm}\right)^{*}
$$

Proposition 6．3（土）implies the following statement．
Proposition 6．4（ $\pm$ ）．We have

$$
\begin{array}{ll}
Q_{N}^{( \pm)}(\lambda)=\mathfrak{Q}_{N}^{( \pm)}(\lambda)+Y_{N}^{( \pm)}(\lambda), & \operatorname{rank} Y_{N}^{( \pm)}(\lambda) \leq 2 m_{ \pm} \\
\widehat{Q}_{N}^{( \pm)}(\lambda)=\widehat{\mathfrak{Q}}_{N}^{( \pm)}(\lambda)+\widehat{Y}_{N}^{( \pm)}(\lambda), & \operatorname{rank} \widehat{Y}_{N}^{( \pm)}(\lambda) \leq 2 m_{ \pm}
\end{array}
$$

where the limits

$$
\begin{aligned}
& \underset{\lambda \rightarrow \lambda_{ \pm}}{(\mathrm{u})-\lim } \mathfrak{Q}_{N}^{( \pm)}(\lambda)=: \mathfrak{Q}_{N}^{( \pm)}\left(\lambda_{ \pm}\right) \in \Sigma_{q} \\
& \underset{\lambda \rightarrow \lambda_{ \pm}}{(\mathrm{u})-\lim } \widehat{\mathfrak{Q}}_{N}^{( \pm)}(\lambda)=: \widehat{\mathfrak{Q}}_{N}^{( \pm)}\left(\lambda_{ \pm}\right)= \pm \sum_{j, k=1}^{m_{ \pm}} \widehat{\mathcal{T}}_{j N}^{( \pm)}\left(\lambda_{ \pm}\right) \Psi_{ \pm}\left(\widehat{\mathcal{T}}_{k N}^{( \pm)}\left(\lambda_{ \pm}\right) \Psi_{ \pm}\right)^{*} \in \Sigma_{q}
\end{aligned}
$$

exist，and

$$
\mathfrak{Q}_{N}^{( \pm)}\left(\lambda_{ \pm}\right)=\widehat{\mathfrak{Q}}_{N}^{( \pm)}\left(\lambda_{ \pm}\right) \quad\left(\bmod \Sigma_{q}^{0}\right)
$$

We have

$$
\mathfrak{Q}_{N}^{( \pm)}\left(\lambda_{ \pm}\right)=\widetilde{\zeta}_{N} \mathfrak{Q}_{N}^{( \pm)}\left(\lambda_{ \pm}\right) \widetilde{\zeta}_{N}, \quad \widehat{\mathfrak{Q}}_{N}^{( \pm)}\left(\lambda_{ \pm}\right)=\widetilde{\zeta}_{N} \widehat{\mathfrak{Q}}_{N}^{( \pm)}\left(\lambda_{ \pm}\right) \widetilde{\zeta}_{N}
$$

Observe that the operator $\mathfrak{Q}_{N}^{(+)}\left(\lambda_{+}\right)$is nonnegative，and the operator $\mathfrak{Q}_{N}^{(-)}\left(\lambda_{-}\right)$is nonpositive，because

$$
\mathfrak{Q}_{N}^{( \pm)}\left(\lambda_{ \pm}\right)= \pm\left(\sum_{j=1}^{m_{ \pm}} \mathcal{T}_{j N}^{( \pm)}\left(\lambda_{ \pm}\right) \Psi_{ \pm}\right)\left(\sum_{k=1}^{m_{ \pm}} \mathcal{T}_{k N}^{( \pm)}\left(\lambda_{ \pm}\right) \Psi_{ \pm}\right)^{*}
$$

Therefore，

$$
\begin{align*}
& n_{-}\left(t, \mathfrak{Q}_{N}^{(+)}\left(\lambda_{+}\right)\right)=0, \quad t>0  \tag{6.23}\\
& n_{+}\left(t, \mathfrak{Q}_{N}^{(-)}\left(\lambda_{-}\right)\right)=0, \quad t>0 \tag{6.24}
\end{align*}
$$

5．Instead of $\widehat{\mathfrak{Q}}_{N}^{( \pm)}\left(\lambda_{ \pm}\right)$，it is more convenient to study the operator $\pm \Psi_{ \pm}^{*} P_{N}^{( \pm)}\left(\lambda_{ \pm}\right) \Psi_{ \pm}$ with the same nonzero spectrum．Here

$$
P_{N}^{( \pm)}\left(\lambda_{ \pm}\right):=\sum_{j, k=1}^{m_{ \pm}}\left(\widehat{\mathcal{T}}_{k N}^{( \pm)}\left(\lambda_{ \pm}\right)\right)^{*} \widehat{\mathcal{T}}_{j N}^{( \pm)}\left(\lambda_{ \pm}\right)
$$

Since the operators $\Psi_{ \pm}$are partially isometric and surjective，the nonzero spectra of the operators $\widehat{\mathfrak{Q}}_{N}^{( \pm)}\left(\lambda_{ \pm}\right)$and $\pm P_{N}^{( \pm)}\left(\lambda_{ \pm}\right)$coincide．Hence，

$$
\begin{array}{ll}
\partial_{q}^{(+)}\left(\widehat{\mathfrak{Q}}_{N}^{(+)}\left(\lambda_{+}\right)\right)=\partial_{q}^{(+)}\left(P_{N}^{(+)}\left(\lambda_{+}\right)\right), & \partial=\Delta, \delta,  \tag{6.26+}\\
\partial_{q}^{(-)}\left(\widehat{\mathfrak{Q}}_{N}^{(-)}\left(\lambda_{-}\right)\right)=\partial_{q}^{(+)}\left(P_{N}^{(-)}\left(\lambda_{-}\right)\right), & \partial=\Delta, \delta
\end{array}
$$

By Proposition $6.3( \pm), 2^{\circ}$ ，the operator $\widehat{\mathcal{T}}_{j N}^{( \pm)}\left(\lambda_{ \pm}\right)$is unitarily equivalent to the oper－ ator $\widehat{\mathcal{G}}_{j N}^{( \pm)}(0)$ ．Therefore，Remark 3.4 implies that

$$
\widehat{\mathcal{T}}_{j N_{1}}^{( \pm)}\left(\lambda_{ \pm}\right)=\widehat{\mathcal{T}}_{j N_{2}}^{( \pm)}\left(\lambda_{ \pm}\right) \quad\left(\bmod \Sigma_{2 q}^{0}\right), \quad N_{1}, N_{2}>0
$$

Consequently，the quantities $\partial_{q}^{(+)}\left(P_{N}^{( \pm)}\left(\lambda_{ \pm}\right)\right), \partial=\Delta, \delta$ ，do not depend on $N$ ．Combining this with $(6.21 \pm)$ and $(6.26 \pm)$ ，we obtain the following statement．

Proposition 6．5（土）．The quantities

$$
\begin{equation*}
\partial_{q}^{(+)}\left(\mathfrak{Q}_{N}^{(+)}\left(\lambda_{+}\right)\right)=: \partial_{q}^{(+)}(*), \quad \partial_{q}^{(-)}\left(\mathfrak{Q}_{N}^{(-)}\left(\lambda_{-}\right)\right)=: \partial_{q}^{(-)}(*) \tag{6.27}
\end{equation*}
$$

are independent of $N$ ．
We introduce the notation

$$
H_{j N}^{( \pm)}:=\left(\widehat{\mathcal{T}}_{j N}^{( \pm)}\left(\lambda_{ \pm}\right)\right)^{*} \widehat{\mathcal{T}}_{j N}^{( \pm)}\left(\lambda_{ \pm}\right)
$$

By（6．19土），Propositions $6.3( \pm)$（item $2^{\circ}$ ）and 3．3，and the definition of the quantities （2．4），we obtain

$$
\begin{array}{r}
\partial_{q}^{(+)}\left(H_{j N}^{( \pm)}\right)=\partial_{q}^{(+)}\left(\widetilde{V}_{N}, \beta_{j}^{( \pm)}, \varphi_{j}^{( \pm)}\right)=\partial_{q}^{(+)}\left(V, \beta_{j}^{( \pm)}, \varphi_{j}^{( \pm)}\right) \\
\partial=\Delta, \delta, \quad j=1, \ldots, m_{ \pm}
\end{array}
$$

We represent $P_{N}^{( \pm)}\left(\lambda_{ \pm}\right)$as

$$
\begin{align*}
P_{N}^{( \pm)}\left(\lambda_{ \pm}\right) & =\widehat{P}_{N}^{( \pm)}\left(\lambda_{ \pm}\right)+\check{P}_{N}^{( \pm)}\left(\lambda_{ \pm}\right) \\
\widehat{P}_{N}^{( \pm)}\left(\lambda_{ \pm}\right) & =\sum_{j=1}^{m_{ \pm}} H_{j N}^{( \pm)} \\
\check{P}_{N}^{( \pm)}\left(\lambda_{ \pm}\right) & =\sum_{j \neq k}\left(\widehat{\mathcal{T}}_{k N}^{( \pm)}\left(\lambda_{ \pm}\right)\right)^{*} \widehat{\mathcal{T}}_{j N}^{( \pm)}\left(\lambda_{ \pm}\right)
\end{align*}
$$

By（6．21土），（6．26土）and（6．27）we have

$$
\begin{equation*}
\partial_{q}^{(+)}(*)=\partial_{q}^{(+)}\left(P_{N}^{(+)}\left(\lambda_{+}\right)\right), \quad \partial_{q}^{(-)}(*)=\partial_{q}^{(+)}\left(P_{N}^{(-)}\left(\lambda_{-}\right)\right) \tag{6.33}
\end{equation*}
$$

We calculate $\partial_{q}^{(+)}\left(\widehat{P}_{N}^{( \pm)}\left(\lambda_{ \pm}\right)\right)$．By（6．20 $\pm$）and（6．28土），the summands in（6．31士）are pairwise orthogonal．Therefore，by（6．29土），

$$
\partial_{q}^{(+)}\left(\widehat{P}_{N}^{( \pm)}\left(\lambda_{ \pm}\right)\right)=\sum_{j=1}^{m_{ \pm}} \partial_{q}^{(+)}\left(H_{j N}^{( \pm)}\right)=\sum_{j=1}^{m_{ \pm}} \partial_{q}^{(+)}\left(V, \beta_{j}^{( \pm)}, \varphi_{j}^{( \pm)}\right), \quad \partial=\Delta, \delta
$$

6．We summarize the results for the operator $K_{N}(\lambda)$ ．We put

$$
\widehat{K}_{N}^{( \pm)}(\lambda)=\widetilde{K}_{N}^{( \pm)}(\lambda)+\mathfrak{Q}_{N}^{( \pm)}(\lambda)
$$

The following statement is a consequence of（6．16土）and Propositions $6.1( \pm), 6.4( \pm)$ ， and $6.5( \pm)$ ．
Proposition 6．6（土）．We have

$$
K_{N}(\lambda)=\widehat{K}_{N}^{( \pm)}(\lambda)+Y_{N}^{( \pm)}(\lambda), \quad \operatorname{rank} Y_{N}^{( \pm)}(\lambda) \leq 2 m_{ \pm}
$$

where the limit

$$
\underset{\lambda \rightarrow \lambda_{ \pm}}{(\mathrm{u})-\lim } \widehat{K}_{N}^{( \pm)}(\lambda)=: \widehat{K}_{N}^{( \pm)}\left(\lambda_{ \pm}\right)=\widetilde{K}_{N}^{( \pm)}\left(\lambda_{ \pm}\right)+\mathfrak{Q}_{N}^{( \pm)}\left(\lambda_{ \pm}\right) \in \Sigma_{q}
$$

exists．If $q>1$ ，then

$$
\begin{align*}
& \partial_{q}^{(+)}\left(\widehat{K}_{N}^{(+)}\left(\lambda_{+}\right)\right)=\partial_{q}^{(+)}\left(\mathfrak{Q}_{N}^{(+)}\left(\lambda_{+}\right)\right)=: \partial_{q}^{(+)}(*),  \tag{6.36+}\\
& \partial_{q}^{(-)}\left(\widehat{K}_{N}^{(-)}\left(\lambda_{-}\right)\right)=\partial_{q}^{(-)}\left(\mathfrak{Q}_{N}^{(-)}\left(\lambda_{-}\right)\right)=: \partial_{q}^{(-)}(*), \tag{6.36-}
\end{align*}
$$

$$
\begin{aligned}
& \lim _{N \rightarrow \infty} \partial_{1}^{(+)}\left(\widehat{K}_{N}^{(+)}\left(\lambda_{+}\right)\right)=\partial_{1}^{(+)}(*) \\
& \lim _{N \rightarrow \infty} \partial_{1}^{(-)}\left(\widehat{K}_{N}^{(-)}\left(\lambda_{-}\right)\right)=\partial_{1}^{(-)}(*)
\end{aligned}
$$

For $q \geq 1$ ，

$$
\begin{align*}
\Delta_{1}^{(-)}\left(\widehat{K}_{N}^{(+)}\left(\lambda_{+}\right)\right) & =0  \tag{6.38+}\\
\lim _{N \rightarrow \infty} \Delta_{1}^{(+)}\left(\widehat{K}_{N}^{(-)}\left(\lambda_{-}\right)\right) & =0
\end{align*}
$$

We have

$$
\widehat{K}_{N}^{( \pm)}\left(\lambda_{ \pm}\right)=\widetilde{\zeta}_{N} \widehat{K}_{N}^{( \pm)}\left(\lambda_{ \pm}\right) \widetilde{\zeta}_{N}
$$

Proof．It remains to prove（6．38土）and（6．39土）．Relation（6．38＋）follows from（6．35＋）， $(6.6+)$ and（6．23）．Next，relations $(6.35-),(6.5-)$ and（6．24）imply the inequality

$$
\Delta_{1}^{(+)}\left(\widehat{K}_{N}^{(-)}\left(\lambda_{-}\right)\right) \leq \Delta_{1}^{(+)}\left(\widetilde{K}_{N}^{(-)}\left(\lambda_{-}\right)\right)=J\left(\widetilde{V}_{N}, g\right)
$$

It remains to observe that the right－hand side tends to zero as $N \rightarrow \infty$ ．This proves （6．38－）．

From $(6.7 \pm),(6.22 \pm)$ and（ $6.35 \pm)$ we deduce（ $6.39 \pm$ ）．
7．We pass to the operator（4．19土），which is represented in the form

$$
\mathcal{K}_{N}^{( \pm)}(\gamma)=\mathcal{Q}_{N}^{( \pm)}(\gamma)+\widetilde{\mathcal{K}}_{N}^{( \pm)}(\gamma)
$$

where

$$
\begin{align*}
& \mathcal{Q}_{N}^{( \pm)}(\gamma):=\widetilde{W}_{N} \Pi_{ \pm}\left(B_{ \pm}+\gamma^{2} I\right)^{-1} \Xi_{0}^{( \pm)} \Pi_{ \pm}^{*} \widetilde{W}_{N} \\
& \widetilde{\mathcal{K}}_{N}^{( \pm)}(\gamma):=\widetilde{W}_{N} \Pi_{ \pm}\left(B_{ \pm}+\gamma^{2} I\right)^{-1} \widetilde{\Xi}_{0}^{( \pm)} \Pi_{ \pm}^{*} \widetilde{W}_{N}
\end{align*}
$$

The projections $\Xi_{0}^{( \pm)}$and $\widetilde{\Xi}_{0}^{( \pm)}$were introduced in Subsection 5．4．

Proposition 6．7（ $\pm$ ）．The limit

$$
\underset{\gamma \rightarrow 0}{\left(\Sigma_{1}\right)-\lim } \widetilde{\mathcal{K}}_{N}^{( \pm)}(\gamma)=: \widetilde{\mathcal{K}}_{N}^{( \pm)}(0) \in \Sigma_{1}
$$

exists．The following asymptotics is valid：

$$
\Delta_{1}^{(+)}\left(\widetilde{\mathcal{K}}_{N}^{( \pm)}(0)\right)=\delta_{1}^{(+)}\left(\widetilde{\mathcal{K}}_{N}^{( \pm)}(0)\right)=\widetilde{J}\left(\mathcal{B}_{ \pm}, \widetilde{\mathcal{U}}_{ \pm}^{(N)}\right)
$$

where $\widetilde{\mathcal{U}}_{ \pm}^{(N)}(\mathbf{x})=\widetilde{V}_{N}(\mathbf{x}) \mathcal{P}_{ \pm}(\mathbf{x})$ ．We have $\widetilde{\mathcal{K}}_{N}^{( \pm)}(0)=\widetilde{\zeta}_{N} \widetilde{\mathcal{K}}_{N}^{( \pm)}(0) \widetilde{\zeta}_{N}$ ．
Proof．Since the proof is similar to that of Proposition $6.1( \pm)$ ，we omit it in order to avoid repetition．

Obviously，the operator（6．42土）is nonnegative，so that

$$
n_{-}\left(t, \widetilde{\mathcal{K}}_{N}^{( \pm)}(0)\right)=0
$$

8．Now，we find a relationship between the operators $Q_{N}^{( \pm)}(\lambda)$ and $\mathcal{Q}_{N}^{( \pm)}(\gamma)$ ．Denote

$$
\widehat{T}_{N}^{( \pm)}(\lambda):=\sum_{j=1}^{m_{ \pm}} \widehat{T}_{j N}^{( \pm)}(\lambda)
$$

where $\widehat{T}_{j N}^{( \pm)}(\lambda)$ is the integral operator with the kernel（6．16土）．Then

$$
\begin{aligned}
& \left(\widehat{T}_{N}^{( \pm)}(\lambda) f\right)(\mathbf{x}) \\
& (6.45 \pm)=(2 \pi)^{-1} \sum_{j=1}^{m_{ \pm}} \widetilde{W}_{N}(\mathbf{x}) \\
& \quad \times \int \chi_{j}^{( \pm)}(\boldsymbol{\xi})\left(\mathfrak{b}_{j}^{( \pm)}\left(\boldsymbol{\xi}-\boldsymbol{\xi}_{j}^{( \pm)}\right)+\gamma^{2}\right)^{-1 / 2} e^{i\langle\mathbf{x}, \boldsymbol{\xi}\rangle} \varphi_{j}^{( \pm)}(\mathbf{x}) f(\boldsymbol{\xi}) d \boldsymbol{\xi}
\end{aligned}
$$

where $\gamma^{2}= \pm\left(\lambda_{ \pm}-\lambda\right)$ ．We transform $(6.45 \pm)$ ，making the change of variables $\boldsymbol{\eta}=\boldsymbol{\xi}-\boldsymbol{\xi}_{j}^{( \pm)}$ in the $j$ th summand and introducing the notation $f\left(\boldsymbol{\eta}+\boldsymbol{\xi}_{j}^{( \pm)}\right)=: h_{j}^{( \pm)}(\boldsymbol{\eta})$ ．Then

$$
\begin{aligned}
& \left(\widehat{T}_{N}^{( \pm)}(\lambda) f\right)(\mathbf{x}) \\
& =(2 \pi)^{-1} \sum_{j=1}^{m_{ \pm}} \widetilde{W}_{N}(\mathbf{x}) \\
& \quad \times \int \rho_{j}^{( \pm)}(\boldsymbol{\eta})\left(\mathfrak{b}_{j}^{( \pm)}(\boldsymbol{\eta})+\gamma^{2}\right)^{-1 / 2} e^{i\langle\mathbf{x}, \boldsymbol{\eta}\rangle} e^{i\left\langle\mathbf{x}, \boldsymbol{\xi}_{j}^{( \pm)}\right\rangle} \varphi_{j}^{( \pm)}(\mathbf{x}) h_{j}^{( \pm)}(\boldsymbol{\eta}) d \boldsymbol{\eta}
\end{aligned}
$$

Let $h^{( \pm)}=\operatorname{col}\left\{h_{j}^{( \pm)}\right\}_{j=1}^{m_{ \pm}}=: \Upsilon^{( \pm)} f$ ．Clearly，

$$
\widehat{T}_{N}^{( \pm)}(\lambda) f=\widetilde{W}_{N} \Pi_{ \pm} \Phi^{*}\left[\rho^{( \pm)}\left(\mathcal{R}_{ \pm}(\gamma)\right)^{1 / 2}\right] h^{( \pm)}=\widetilde{\mathfrak{G}}_{N}^{( \pm)}(\gamma) \Phi^{*}\left[\rho^{( \pm)}\right] \Upsilon^{( \pm)} f
$$

where

$$
\widetilde{\mathfrak{G}}_{N}^{( \pm)}(\gamma):=\widetilde{W}_{N} \Pi_{ \pm}\left(B_{ \pm}+\gamma^{2} I\right)^{-1 / 2} \Xi_{0}^{( \pm)}
$$

It is easy to check that

$$
\Phi^{*}\left[\rho^{( \pm)}\right] \Upsilon^{( \pm)}\left(\Upsilon^{( \pm)}\right)^{*}\left[\rho^{( \pm)}\right] \Phi=\Xi_{0}^{( \pm)}
$$

Then（6．41土）and（6．46土）－（6．48土）imply that

$$
\mathcal{Q}_{N}^{( \pm)}(\gamma)=\widetilde{\mathfrak{G}}_{N}^{( \pm)}(\gamma)\left(\widetilde{\mathfrak{G}}_{N}^{( \pm)}(\gamma)\right)^{*}=\widehat{T}_{N}^{( \pm)}(\lambda)\left(\widehat{T}_{N}^{( \pm)}(\lambda)\right)^{*}, \quad \gamma^{2}= \pm\left(\lambda_{ \pm}-\lambda\right)
$$

Proposition $6.3( \pm)$ and relation（6．44土）yield the following statement．

Proposition 6.8( $\pm$ ). The following representation is true:

$$
\mathcal{Q}_{N}^{( \pm)}(\gamma)=\widehat{\mathcal{Q}}_{N}^{( \pm)}(\gamma)+\mathcal{Y}_{N}^{( \pm)}(\gamma), \quad \operatorname{rank} \mathcal{Y}_{N}^{( \pm)}(\gamma) \leq 2 m_{ \pm}
$$

where the limit

$$
\left(\underset{\gamma \rightarrow 0}{(\mathrm{u})-\lim } \widehat{\mathcal{Q}}_{N}^{( \pm)}(\gamma)=: \widehat{\mathcal{Q}}_{N}^{( \pm)}(0) \in \Sigma_{q}\right.
$$

exists, and

$$
\widehat{\mathcal{Q}}_{N}^{( \pm)}(0)=\widehat{\mathcal{T}}_{N}^{( \pm)}\left(\lambda_{ \pm}\right)\left(\widehat{\mathcal{T}}_{N}^{( \pm)}\left(\lambda_{ \pm}\right)\right)^{*}
$$

where

$$
\widehat{\mathcal{T}}_{N}^{( \pm)}\left(\lambda_{ \pm}\right):=\sum_{j=1}^{m_{ \pm}} \widehat{\mathcal{T}}_{j N}^{( \pm)}\left(\lambda_{ \pm}\right)
$$

We have $\widehat{\mathcal{Q}}_{N}^{( \pm)}(0)=\widetilde{\zeta}_{N} \widehat{\mathcal{Q}}_{N}^{( \pm)}(0) \widetilde{\zeta}_{N}$.
From ( $6.49 \pm$ ) and ( $6.25 \pm$ ) we deduce that

$$
\partial_{q}^{(+)}\left(\widehat{\mathcal{Q}}_{N}^{( \pm)}(0)\right)=\partial_{q}^{(+)}\left(\widehat{\mathcal{T}}_{N}^{( \pm)}\left(\lambda_{ \pm}\right)\left(\widehat{\mathcal{T}}_{N}^{( \pm)}\left(\lambda_{ \pm}\right)\right)^{*}\right)=\partial_{q}^{(+)}\left(P_{N}^{( \pm)}\left(\lambda_{ \pm}\right)\right)
$$

Taking (6.21 $\pm$ ) and (6.26土) into account, we arrive at the relations

$$
\begin{align*}
& \partial_{q}^{(+)}\left(\mathfrak{Q}_{N}^{(+)}\left(\lambda_{+}\right)\right)=\partial_{q}^{(+)}\left(\widehat{\mathfrak{Q}}_{N}^{(+)}\left(\lambda_{+}\right)\right)=\partial_{q}^{(+)}\left(\widehat{\mathcal{Q}}_{N}^{(+)}(0)\right)  \tag{6.50+}\\
& \partial_{q}^{(-)}\left(\mathfrak{Q}_{N}^{(-)}\left(\lambda_{-}\right)\right)=\partial_{q}^{(-)}\left(\widehat{\mathfrak{Q}}_{N}^{(-)}\left(\lambda_{-}\right)\right)=\partial_{q}^{(+)}\left(\widehat{\mathcal{Q}}_{N}^{(-)}(0)\right) .
\end{align*}
$$

Obviously, the operator ( $6.49 \pm$ ) is nonnegative, whence

$$
n_{-}\left(t, \widehat{\mathcal{Q}}_{N}^{( \pm)}(0)\right)=0
$$

9. We summarize the results for the operator $\mathcal{K}_{N}^{( \pm)}(\gamma)$. We put

$$
\widehat{\mathcal{K}}_{N}^{( \pm)}(\gamma)=\widetilde{\mathcal{K}}_{N}^{( \pm)}(\gamma)+\widehat{\mathcal{Q}}_{N}^{( \pm)}(\gamma)
$$

Relation $(6.40 \pm)$, Propositions $6.5( \pm), 6.7( \pm)$ and $6.8( \pm)$, and also relations ( $6.43 \pm$ ), ( $6.50 \pm$ ) and ( $6.51 \pm$ ) yield the following statement.

Proposition 6.9( $\pm$ ). The following representation is valid:

$$
\mathcal{K}_{N}^{( \pm)}(\gamma)=\widehat{\mathcal{K}}_{N}^{( \pm)}(\gamma)+\mathcal{Y}_{N}^{( \pm)}(\gamma), \quad \operatorname{rank} \mathcal{Y}_{N}^{( \pm)}(\gamma) \leq 2 m_{ \pm}
$$

where the limit

$$
\underset{\gamma \rightarrow 0}{(\mathrm{u})-\lim } \widehat{\mathcal{K}}_{N}^{( \pm)}(\gamma)=: \widehat{\mathcal{K}}_{N}^{( \pm)}(0) \in \Sigma_{q}
$$

exists. If $q>1$, then

$$
\begin{align*}
& \partial_{q}^{(+)}\left(\widehat{\mathcal{K}}_{N}^{(+)}(0)\right)=\partial_{q}^{(+)}(*)  \tag{6.52+}\\
& \partial_{q}^{(+)}\left(\widehat{\mathcal{K}}_{N}^{(-)}(0)\right)=\partial_{q}^{(-)}(*) \tag{6.52-}
\end{align*}
$$

If $q=1$, then

$$
\begin{align*}
& \lim _{N \rightarrow \infty} \partial_{1}^{(+)}\left(\widehat{\mathcal{K}}_{N}^{(+)}(0)\right)=\partial_{1}^{(+)}(*)  \tag{6.53+}\\
& \lim _{N \rightarrow \infty} \partial_{1}^{(+)}\left(\widehat{\mathcal{K}}_{N}^{(-)}(0)\right)=\partial_{1}^{(-)}(*) \tag{6.53-}
\end{align*}
$$

We have

$$
\begin{align*}
n_{-}\left(t, \widehat{\mathcal{K}}_{N}^{( \pm)}(0)\right) & =0 \\
\widehat{\mathcal{K}}_{N}^{( \pm)}(0) & =\widetilde{\zeta}_{N} \widehat{\mathcal{K}}_{N}^{( \pm)}(0) \widetilde{\zeta}_{N}
\end{align*}
$$

$$
\text { §7. The operators } M_{N}(\lambda) \text { and } \mathcal{M}_{N}^{( \pm)}(\gamma)
$$

1．We represent the operator（4．12）as follows：

$$
M_{N}(\lambda)=M_{N}^{( \pm)}(\lambda)+\widetilde{M}_{N}^{( \pm)}(\lambda)
$$

where

$$
\begin{align*}
M_{N}^{( \pm)}(\lambda) & :=\widetilde{W}_{N}(A-\lambda I)^{-1} \Xi^{( \pm)} W_{N} \\
\widetilde{M}_{N}^{( \pm)}(\lambda) & :=\widetilde{W}_{N}(A-\lambda I)^{-1} \widetilde{\Xi}^{( \pm)} W_{N}
\end{align*}
$$

Then

$$
\widetilde{M}_{N}^{( \pm)}(\lambda):=\widetilde{\Theta}_{N}^{( \pm)}(\lambda)\left(\Theta_{N}^{( \pm)}(\lambda)\right)^{*}
$$

where

$$
\begin{align*}
& \widetilde{\Theta}_{N}^{( \pm)}(\lambda):=\widetilde{W}_{N}|A-\lambda I|^{-1 / 2} \widetilde{\Xi}^{( \pm)}, \\
& \Theta_{N}^{( \pm)}(\lambda):=W_{N}|A-\lambda I|^{1 / 2}(A-\lambda I)^{-1} \widetilde{\Xi}^{( \pm)} .
\end{align*}
$$

Proposition 7．1（ $\pm$ ）．The limits

$$
\begin{align*}
& \underset{\lambda \rightarrow \lambda_{ \pm}}{\left(\Sigma_{2}\right)-\lim } \widetilde{\Theta}_{N}^{( \pm)}(\lambda)=: \widetilde{\Theta}_{N}^{( \pm)}\left(\lambda_{ \pm}\right) \in \Sigma_{2} \\
& \underset{\substack{\left(\Sigma_{2}\right)-\lim \\
\lambda \rightarrow \lambda_{ \pm}}}{ } \Theta_{N}^{( \pm)}(\lambda)=: \Theta_{N}^{( \pm)}\left(\lambda_{ \pm}\right) \in \Sigma_{2}
\end{align*}
$$

exist，and

$$
\begin{align*}
\lim _{N \rightarrow \infty} \Delta_{2}\left(\widetilde{\Theta}_{N}^{( \pm)}\left(\lambda_{ \pm}\right)\right) & =0 \\
\Delta_{2}\left(\Theta_{N}^{( \pm)}\left(\lambda_{ \pm}\right)\right) & \leq C((V))_{\sigma}
\end{align*}
$$

Proof．The operator（7．4土）can be written as

$$
\widetilde{\Theta}_{N}^{( \pm)}(\lambda)=\left(\widetilde{W}_{N}(A+I)^{-1 / 2}\right)\left((A+I)^{1 / 2}|A-\lambda I|^{-1 / 2} \widetilde{\Xi}^{( \pm)}\right)
$$

The second expression in parentheses converges in the operator norm to the bounded operator

$$
(A+I)^{1 / 2}\left|A-\lambda_{ \pm} I\right|^{-1 / 2} \widetilde{\Xi}^{( \pm)}
$$

as $\lambda \rightarrow \lambda_{ \pm}$．By（4．3）and（with $\lambda=-1$ ），we have $\widetilde{W}_{N}(A+I)^{-1 / 2} \in \Sigma_{2}$ ．This implies（7．5土）and the estimate

$$
\Delta_{2}\left(\widetilde{\Theta}_{N}^{( \pm)}\left(\lambda_{ \pm}\right)\right) \leq C_{1} \Delta_{2}\left(\widetilde{W}_{N}(A+I)^{-1 / 2}\right) \leq C_{2}\left(\left(\widetilde{V}_{N}\right)\right)_{\sigma}
$$

Obviously，$\left(\left(\widetilde{V}_{N}\right)\right)_{\sigma} \rightarrow 0$ as $N \rightarrow \infty$ ．This proves（7．7土）．Relation（7．6 $\pm$ ）and the estimate

$$
\Delta_{2}\left(\Theta_{N}^{( \pm)}\left(\lambda_{ \pm}\right)\right) \leq C\left(\left(V_{N}\right)\right)_{\sigma} \leq C((V))_{\sigma}
$$

are proved in a similar way．
The following statement is a consequence of（7．3土）and Proposition 7．1（ $\pm$ ）．
Proposition $7.2( \pm)$ ．The limit

$$
\underset{\lambda \rightarrow \lambda_{ \pm}}{\left(\Sigma_{1}\right)-\lim } \widetilde{M}_{N}^{( \pm)}(\lambda)=: \widetilde{M}_{N}^{( \pm)}\left(\lambda_{ \pm}\right) \in \Sigma_{1}
$$

exists，and

$$
\lim _{N \rightarrow \infty} \Delta_{1}\left(\widetilde{M}_{N}^{( \pm)}\left(\lambda_{ \pm}\right)\right)=0
$$

2．In accordance with（5．8土）and（6．12土），we write the operator（7．2土）in the form

$$
M_{N}^{( \pm)}(\lambda)= \pm \widetilde{G}_{N}^{( \pm)}(\lambda)\left(G_{N}^{( \pm)}(\lambda)\right)^{*}
$$

Using（ $7.8 \pm$ ），Propositions $5.2( \pm)$ and $6.3( \pm)$ ，and relation（ $6.13 \pm$ ），we obtain the fol－ lowing statement．

Proposition 7．3（ $\pm$ ）．We have

$$
M_{N}^{( \pm)}(\lambda)=\widehat{M}_{N}^{( \pm)}(\lambda)+\check{M}_{N}^{( \pm)}(\lambda), \quad \operatorname{rank} \check{M}_{N}^{( \pm)}(\lambda) \leq 2 m_{ \pm}
$$

and the following limit exists：

$$
\underset{\lambda \rightarrow \lambda_{ \pm}}{(\mathrm{u})-\lim } \widehat{M}_{N}^{( \pm)}(\lambda)=: \widehat{M}_{N}^{( \pm)}\left(\lambda_{ \pm}\right) \in \Sigma_{q}^{0}
$$

We put

$$
\mathfrak{M}_{N}^{( \pm)}(\lambda):=\widehat{M}_{N}^{( \pm)}(\lambda)+\widetilde{M}_{N}^{( \pm)}(\lambda)
$$

By（7．1 $\pm$ ）and Propositions $7.2( \pm)$ and $7.3( \pm)$ ，we obtain the following statement．
Proposition 7．4（土）．We have

$$
M_{N}(\lambda)=\mathfrak{M}_{N}^{( \pm)}(\lambda)+\check{M}_{N}^{( \pm)}(\lambda), \quad \operatorname{rank} \check{M}_{N}^{( \pm)}(\lambda) \leq 2 m_{ \pm}
$$

and the limit

$$
\underset{\lambda \rightarrow \lambda_{ \pm}}{(\mathrm{u})-\lim } \mathfrak{M}_{N}^{( \pm)}(\lambda)=: \mathfrak{M}_{N}^{( \pm)}\left(\lambda_{ \pm}\right) \in \Sigma_{q}
$$

exists．If $q>1$ ，then

$$
\mathfrak{M}_{N}^{( \pm)}\left(\lambda_{ \pm}\right) \in \Sigma_{q}^{0}, \quad q>1
$$

If $q=1$ ，then

$$
\lim _{N \rightarrow \infty} \Delta_{1}\left(\mathfrak{M}_{N}^{( \pm)}\left(\lambda_{ \pm}\right)\right)=0
$$

3．The operator $(4.20 \pm)$ can be studied by analogy with the operator（4．12）．In order to avoid repetition，we omit the details and formulate the result．The following statement is an analog of Proposition $7.4( \pm)$ ．

Proposition 7．5（ $\pm$ ）．We have

$$
\mathcal{M}_{N}^{( \pm)}(\gamma)=\widehat{\mathfrak{M}}_{N}^{( \pm)}(\gamma)+\check{\mathcal{M}}_{N}^{( \pm)}(\gamma), \quad \operatorname{rank} \check{\mathcal{M}}_{N}^{( \pm)}(\gamma) \leq 2 m_{ \pm}
$$

and the limit

$$
\left(\underset{\gamma \rightarrow 0}{(\mathrm{u})-\lim } \widehat{\mathfrak{M}}_{N}^{( \pm)}(\gamma)=: \widehat{\mathfrak{M}}_{N}^{( \pm)}(0) \in \Sigma_{q}\right.
$$

exists．If $q>1$ ，then

$$
\widehat{\mathfrak{M}}_{N}^{( \pm)}(0) \in \Sigma_{q}^{0}, \quad q>1
$$

If $q=1$ ，then

$$
\lim _{N \rightarrow \infty} \Delta_{1}\left(\widehat{\mathfrak{M}}_{N}^{( \pm)}(0)\right)=0
$$

## §8．Proof of Theorems 2．2（土）

1．Now everything is prepared for applying the general method of $\S 4$ ．In accordance with（4．9），regularization of the operator $X(\lambda)$（see（4．2））reduces to Propositions 5．4（土）， $6.6( \pm)$ and $7.4( \pm)$ ．The operator $X(\lambda)$ is representable as in $(4.4 \pm)$ ，with the operators

$$
\begin{array}{ll}
\Gamma_{N}^{( \pm)}(\lambda):=\widehat{L}_{N}^{( \pm)}(\lambda)+\widehat{K}_{N}^{( \pm)}(\lambda)+2 \operatorname{Re} \mathfrak{M}_{N}^{( \pm)}(\lambda), & N>0 \\
\mathfrak{Y}_{N}^{( \pm)}(\lambda):=\check{L}_{N}^{( \pm)}(\lambda)+Y_{N}^{( \pm)}(\lambda)+2 \operatorname{Re} \check{M}_{N}^{( \pm)}(\lambda), & N>0,
\end{array}
$$

in the role of $\Gamma_{ \pm}(\lambda)$ and $Y_{ \pm}(\lambda)$ ．
We have

$$
\operatorname{rank} \mathfrak{Y}_{N}^{( \pm)}(\lambda) \leq 6 m_{ \pm}
$$

and the limit

$$
\underset{\lambda \rightarrow \lambda_{ \pm}}{(\mathrm{u})-\lim } \Gamma_{N}^{( \pm)}(\lambda)=: \Gamma_{N}^{( \pm)}\left(\lambda_{ \pm}\right) \in \Sigma_{q}
$$

exists，where

$$
\Gamma_{N}^{( \pm)}\left(\lambda_{ \pm}\right)=\widehat{L}_{N}^{( \pm)}\left(\lambda_{ \pm}\right)+\widehat{K}_{N}^{( \pm)}\left(\lambda_{ \pm}\right)+2 \operatorname{Re} \mathfrak{M}_{N}^{( \pm)}\left(\lambda_{ \pm}\right)
$$

The parameter $N$ will play an important role in what follows．We write relations （4．6 $\pm$ ）for the operator $\Gamma_{N}^{( \pm)}\left(\lambda_{ \pm}\right)$：

$$
\begin{array}{ll}
\partial_{q}^{(+)}\left(\lambda_{ \pm} ; A, V\right)=\partial_{q}^{(+)}\left(\Gamma_{N}^{( \pm)}\left(\lambda_{ \pm}\right)\right), & \partial=\Delta, \delta \\
\partial_{q}^{(-)}\left(\lambda_{ \pm} ; A, V\right)=\partial_{q}^{(-)}\left(\Gamma_{N}^{( \pm)}\left(\lambda_{ \pm}\right)\right), & \partial=\Delta, \delta
\end{array}
$$

Since the left－hand side is independent of $N$ ，so is the right－hand side．Moreover，the right－hand side does not change under adding an operator of class $\Sigma_{q}^{0}$ to $\Gamma_{N}^{( \pm)}\left(\lambda_{ \pm}\right)$．

2．For the model operator，by（4．17 $\pm$ ），regularization of $\mathcal{X}_{ \pm}(\gamma)$ in（4．13 $\pm$ ）reduces to Propositions $5.7( \pm), 6.9( \pm)$ and $7.5( \pm)$ ．The operator $\mathcal{X}_{ \pm}(\gamma)$ is representable as in （4．15土），with

$$
\begin{array}{ll}
\widehat{\Gamma}_{N}^{( \pm)}(\gamma):=\widehat{\mathcal{L}}_{N}^{( \pm)}(\gamma)+\widehat{\mathcal{K}}_{N}^{( \pm)}(\gamma)+2 \operatorname{Re} \widehat{\mathfrak{M}}_{N}^{( \pm)}(\gamma), \quad N>0 \\
\widehat{\mathfrak{Y}}_{N}^{( \pm)}(\gamma):=\check{\mathcal{L}}_{N}^{( \pm)}(\gamma)+\mathcal{Y}_{N}^{( \pm)}(\gamma)+2 \operatorname{Re} \check{\mathcal{M}}_{N}^{( \pm)}(\gamma), \quad N>0
\end{array}
$$

in the role of $\widehat{\Gamma}_{ \pm}(\gamma)$ and $\mathcal{Y}_{ \pm}(\gamma)$ ．We have

$$
\operatorname{rank} \widehat{\mathfrak{Y}}_{N}^{( \pm)}(\gamma) \leq 6 m_{ \pm}
$$

and the limit

$$
\underset{\gamma \rightarrow 0}{(\mathrm{u})-\lim } \widehat{\Gamma}_{N}^{( \pm)}(\gamma)=: \widehat{\Gamma}_{N}^{( \pm)}(0) \in \Sigma_{q}
$$

exists，where

$$
\widehat{\Gamma}_{N}^{( \pm)}(0)=\widehat{\mathcal{L}}_{N}^{( \pm)}(0)+\widehat{\mathcal{K}}_{N}^{( \pm)}(0)+2 \operatorname{Re} \widehat{\mathfrak{M}}_{N}^{( \pm)}(0)
$$

We write relations $(4.16 \pm)$ for $\widehat{\Gamma}_{N}^{( \pm)}(0)$ ：

$$
\partial_{q}\left(B_{ \pm}, \mathcal{U}_{ \pm}\right)=\partial_{q}^{(+)}\left(\widehat{\Gamma}_{N}^{( \pm)}(0)\right), \quad \partial=\Delta, \delta
$$

3．It is convenient to start the proof of Theorems $2.2( \pm)$ with checking statement（b） （the case of $q>1$ ）．In this case，from（8．1土），（5．13土）and（7．9土）it follows that

$$
\Gamma_{N}^{( \pm)}\left(\lambda_{ \pm}\right)-\widehat{K}_{N}^{( \pm)}\left(\lambda_{ \pm}\right) \in \Sigma_{q}^{0}, \quad q>1
$$

Hence，by（8．2＋）and（8．3－），

$$
\begin{array}{ll}
\partial_{q}^{(+)}\left(\lambda_{+} ; A, V\right)=\partial_{q}^{(+)}\left(\widehat{K}_{N}^{(+)}\left(\lambda_{+}\right)\right), & \partial=\Delta, \delta  \tag{8.6+}\\
\partial_{q}^{(-)}\left(\lambda_{-} ; A, V\right)=\partial_{q}^{(-)}\left(\widehat{K}_{N}^{(-)}\left(\lambda_{-}\right)\right), & \partial=\Delta, \delta
\end{array}
$$

For the model operator，relations（8．4土），（5．22土）and（7．11土）imply that

$$
\widehat{\Gamma}_{N}^{( \pm)}(0)-\widehat{\mathcal{K}}_{N}^{( \pm)}(0) \in \Sigma_{q}^{0}, \quad q>1
$$

Then，by（8．5土），

$$
\begin{array}{ll}
\partial_{q}\left(B_{+}, \mathcal{U}_{+}\right)=\partial_{q}^{(+)}\left(\widehat{\mathcal{K}}_{N}^{(+)}(0)\right), & \partial=\Delta, \delta \\
\partial_{q}\left(B_{-}, \mathcal{U}_{-}\right)=\partial_{q}^{(+)}\left(\widehat{\mathcal{K}}_{N}^{(-)}(0)\right), & \partial=\Delta, \delta \tag{8.7-}
\end{array}
$$

By（ $8.7 \pm$ ）and（6．52土），we have

$$
\partial_{q}\left(B_{ \pm}, \mathcal{U}_{ \pm}\right)=\partial_{q}^{( \pm)}(*)
$$

Comparing（8．6 $),(8.7 \pm)$ and（8．8土）with（6．36土），we obtain（2．19）and（2．24）．
Relations（2．18）and（2．23）will be proved below in Subsection 8．5．
4．Proof of statement（a）（the case of $q=1$ ）．For $q=1$ ，in（8．18土）the terms $\widehat{L}_{N}^{( \pm)}\left(\lambda_{ \pm}\right), \widehat{K}_{N}^{( \pm)}\left(\lambda_{ \pm}\right), \mathfrak{M}_{N}^{( \pm)}\left(\lambda_{ \pm}\right)$are of class $\Sigma_{1}$ ．Thus，
$(8.9+) \quad \partial_{1}^{( \pm)}\left(\Gamma_{N}^{(+)}\left(\lambda_{+}\right)\right)=\partial_{1}^{( \pm)}\left(\widehat{L}_{N}^{(+)}\left(\lambda_{+}\right)+\widehat{K}_{N}^{(+)}\left(\lambda_{+}\right)+2 \operatorname{Re} \mathfrak{M}_{N}^{(+)}\left(\lambda_{+}\right)\right), \quad \partial=\Delta, \delta$,
$(8.9-) \quad \partial_{1}^{( \pm)}\left(\Gamma_{N}^{(-)}\left(\lambda_{-}\right)\right)=\partial_{1}^{( \pm)}\left(\widehat{L}_{N}^{(-)}\left(\lambda_{-}\right)+\widehat{K}_{N}^{(-)}\left(\lambda_{-}\right)+2 \operatorname{Re} \mathfrak{M}_{N}^{(-)}\left(\lambda_{-}\right)\right), \quad \partial=\Delta, \delta$.
In（ $8.9 \pm$ ）we pass to the limit as $N \rightarrow \infty$ ．As has already been mentioned，the terms on the left－hand side of（ $8.9 \pm$ ）do not depend on $N$ ．Relation（7．10土）allows us to apply Proposition 1．7．Next，we take into account that，by（5．6土）and（6．39土），the contributions of $\widehat{L}_{N}^{( \pm)}\left(\lambda_{ \pm}\right)$and $\widehat{K}_{N}^{( \pm)}\left(\lambda_{ \pm}\right)$to $\partial_{1}^{( \pm)}$can be added．Then，using（8．2 $\pm$）and （8．3土），we obtain

$$
\begin{aligned}
& \partial_{1}^{( \pm)}\left(\lambda_{+} ; A, V\right)=\lim _{N \rightarrow \infty} \partial_{1}^{( \pm)}\left(\widehat{L}_{N}^{(+)}\left(\lambda_{+}\right)\right)+\lim _{N \rightarrow \infty} \partial_{1}^{( \pm)}\left(\widehat{K}_{N}^{(+)}\left(\lambda_{+}\right)\right) \\
& \partial_{1}^{( \pm)}\left(\lambda_{-} ; A, V\right)=\lim _{N \rightarrow \infty} \partial_{1}^{( \pm)}\left(\widehat{L}_{N}^{(-)}\left(\lambda_{-}\right)\right)+\lim _{N \rightarrow \infty} \partial_{1}^{( \pm)}\left(\widehat{K}_{N}^{(-)}\left(\lambda_{-}\right)\right)
\end{aligned}
$$

Now $(5.14 \pm),(5.15 \pm),(6.37 \pm)$ and（6．38土）imply that

$$
\begin{align*}
\partial_{1}^{(+)}\left(\lambda_{+} ; A, V\right) & =J(V, g)+\partial_{1}^{(+)}(*), \quad \partial=\Delta, \delta  \tag{8.10}\\
\Delta_{1}^{(-)}\left(\lambda_{+} ; A, V\right) & =0 \\
\Delta_{1}^{(+)}\left(\lambda_{-} ; A, V\right) & =\delta_{1}^{(+)}\left(\lambda_{-} ; A, V\right)=J(V, g), \quad \partial=\Delta, \delta  \tag{8.11}\\
\partial_{1}^{(-)}\left(\lambda_{-} ; A, V\right) & =\partial_{1}^{(-)}(*), \quad \partial=\Delta, \delta \tag{8.12}
\end{align*}
$$

For the model operator，if $q=1$ ，then the terms in（8．4 $\pm$ ）are of class $\Sigma_{1}$ ．Then，by $(8.4 \pm),(8.5 \pm),(7.12 \pm),(5.24 \pm)$ and $(6.54 \pm)$ ，we have

$$
\partial_{1}\left(B_{ \pm}, \mathcal{U}_{ \pm}\right)=\lim _{N \rightarrow \infty} \partial_{1}^{(+)}\left(\widehat{\mathcal{L}}_{N}^{( \pm)}(0)\right)+\lim _{N \rightarrow \infty} \partial_{1}^{(+)}\left(\widehat{\mathcal{K}}_{N}^{( \pm)}(0)\right)
$$

Combining this with（ $5.23 \pm$ ）and（6．53土），we arrive at the relations

$$
\begin{array}{ll}
\partial_{1}\left(B_{+}, \mathcal{U}_{+}\right)=\widetilde{J}\left(\mathcal{B}_{+}, \mathcal{U}_{+}\right)+\partial_{1}^{(+)}(*), & \partial=\Delta, \delta \\
\partial_{1}\left(B_{-}, \mathcal{U}_{-}\right)=\widetilde{J}\left(\mathcal{B}_{-}, \mathcal{U}_{-}\right)+\partial_{1}^{(-)}(*), & \partial=\Delta, \delta \tag{8.13-}
\end{array}
$$

Comparing（8．10）and 8．13＋），and also 8．12）and 8．13－，and recalling（2．15土）and （2．16土），we obtain（2．17）and（2．22）．Relation（8．11）implies（2．21）．Relation（2．18）will be proved in Subsection 8．5．

5．Now we prove（2．18）and（2．23）．Since $\mathfrak{N}_{-}(\alpha, \lambda ; A, V)$ is a monotone nonincreasing function of $\lambda \in \Lambda$（with $\alpha$ fixed），from（1．22＋it follows that

$$
\mathfrak{N}_{-}\left(\alpha, \lambda_{+} ; A, V\right) \leq \mathfrak{N}_{-}(\alpha, \lambda ; A, V), \quad \lambda \in \Lambda
$$

Combining this with Proposition 1．6，we obtain（2．18）（with $q \geq 1$ ）．
Similarly， $\mathfrak{N}_{+}(\alpha, \lambda ; A, V)$ is a monotone nonincreasing function of $\lambda \in \Lambda$（with $\alpha$ fixed）． Then（1．22－）yields the inequality

$$
\mathfrak{N}_{+}\left(\alpha, \lambda_{-} ; A, V\right) \leq \mathfrak{N}_{-}(\alpha, \lambda ; A, V), \quad \lambda \in \Lambda
$$

Combining this with（1．21），we obtain the estimate

$$
\Delta_{1}^{(+)}\left(\lambda_{-} ; A, V\right) \leq J(V, g)
$$

Next，observe that the potential $V_{N}$ satisfies the conditions of Theorem 2．2（－）with $q=1$ ．Therefore，by（2．21）with $V=V_{N}$,

$$
\Delta_{1}^{(+)}\left(\lambda_{-} ; A, V_{N}\right)=\delta_{1}^{(+)}\left(\lambda_{-} ; A, V_{N}\right)=J\left(V_{N}, g\right)
$$

Since $V_{N}(\mathbf{x}) \leq V(\mathbf{x})$ ，we have

$$
\mathfrak{N}_{+}\left(\alpha, \lambda_{-} ; A, V_{N}\right) \leq \mathfrak{N}_{+}\left(\alpha, \lambda_{-} ; A, V\right)
$$

Thus，

$$
J\left(V_{N}, g\right)=\delta_{1}^{(+)}\left(\lambda_{-} ; A, V_{N}\right) \leq \delta_{1}^{(+)}\left(\lambda_{-} ; A, V\right) \leq \Delta_{1}^{(+)}\left(\lambda_{-} ; A, V\right) \leq J(V, g)
$$

Letting $N \rightarrow \infty$ ，we arrive at（2．23）．
6．Proof of statement（c）of Theorem $\mathbf{2 . 2 ( + )}$ ．Let $q=1$ ．Relation（2．17）shows that the Weyl asymptotics（2．20）occurs if and only if

$$
\widetilde{\Delta}_{1}\left(B_{+}, \mathcal{U}_{+}\right)=\widetilde{\delta}_{1}\left(B_{+}, \mathcal{U}_{+}\right)=0
$$

By $(2.15+)$ and $(2.16+)$ ，this is equivalent to the Weyl asymptotics for the model oper－ ator：

$$
\begin{equation*}
\Delta_{1}\left(B_{+}, \mathcal{U}_{+}\right)=\delta_{1}\left(B_{+}, \mathcal{U}_{+}\right)=\widetilde{J}\left(\mathcal{B}_{+}, \mathcal{U}_{+}\right) \tag{8.14}
\end{equation*}
$$

By（8．13＋），relation（8．14）is equivalent to

$$
\begin{equation*}
\Delta_{1}^{(+)}(*)=0 \tag{8.15}
\end{equation*}
$$

Next，（6．33）shows that（8．15）is equivalent to the relation

$$
\begin{equation*}
P_{N}^{(+)}\left(\lambda_{+}\right) \in \Sigma_{1}^{0} \tag{8.16}
\end{equation*}
$$

where $P_{N}^{(+)}\left(\lambda_{+}\right)$is the operator defined by $(6.25+)$ ．For the proof of（8．16），it suffices to check that

$$
\widehat{\mathcal{T}}_{j N}^{(+)}\left(\lambda_{+}\right) \in \Sigma_{2}^{0}, \quad j=1, \ldots, m_{+}
$$

or，equivalently，

$$
\begin{equation*}
H_{j N}^{(+)} \in \Sigma_{1}^{0}, \quad j=1, \ldots, m_{+} \tag{8.17}
\end{equation*}
$$

The operators $H_{j N}^{(+)}$are defined by（6．28＋）．By（6．29＋），condition 8．17）means that

$$
\begin{equation*}
\Delta_{1}^{(+)}\left(V, \beta_{j}^{(+)}, \varphi_{j}^{(+)}\right)=0, \quad j=1, \ldots, m_{+} \tag{8.18}
\end{equation*}
$$

Finally，from（2．5）it is clear that the condition $\Delta_{1}^{(+)}(V, \mathbf{1}, 1)=0$ ensures（8．18），and， consequently，also（2．20）．The proof of Theorems $2.2( \pm)$ is complete．

## §9．Proof of Theorems 2．5（土）

1．By（ $8.8 \pm$ ）and（ $8.13 \pm$ ），Theorems $2.5( \pm)$ will follow directly from Theorems $2.2( \pm)$ if we prove the relations

$$
\begin{align*}
\partial_{q}^{(+)}(*) & =\sum_{j=1}^{m_{+}} \partial_{q}^{(+)}\left(V, \beta_{j}^{(+)}, 1\right), \quad \partial=\Delta, \delta, \quad q \geq 1  \tag{9.1+}\\
\partial_{q}^{(-)}(*) & =\sum_{j=1}^{m_{-}} \partial_{q}^{(+)}\left(V, \beta_{j}^{(-)}, 1\right), \quad \partial=\Delta, \delta, \quad q \geq 1 \tag{9.1-}
\end{align*}
$$

We recall that the quantities $\partial_{q}^{( \pm)}(*)$ were introduced in 6．27）．By（6．33），（6．30土）， $(6.31 \pm)$ and $(6.34 \pm)$ ，for the proof of $(9.1 \pm)$ it suffices to establish the following two propositions．

Proposition 9．1（ $\pm$ ）．Let

$$
P_{k j N}^{( \pm)}\left(\lambda_{ \pm}\right):=\left(\widehat{\mathcal{T}}_{k N}^{( \pm)}\left(\lambda_{ \pm}\right)\right)^{*} \widehat{\mathcal{T}}_{j N}^{( \pm)}\left(\lambda_{ \pm}\right)
$$

Under the conditions of Theorem $2.5( \pm)$ ，we have

$$
\Delta_{q}\left(P_{k j N}^{( \pm)}\left(\lambda_{ \pm}\right)\right)=0, \quad j \neq k
$$

Proposition 9．2（土）．Under the conditions of Theorem 2．5（土），we have

$$
\partial_{q}^{(+)}\left(V, \beta_{j}^{( \pm)}, \varphi_{j}^{( \pm)}\right)=\partial_{q}^{(+)}\left(V, \beta_{j}^{( \pm)}, 1\right), \quad \partial=\Delta, \delta, \quad j=1, \ldots, m_{ \pm}
$$

Proposition 9．2（土）follows from［BLSu，Lemma 7．1］．
2．All of what follows is devoted to the proof of Proposition $9.1( \pm)$ ．We recall that the operator $\widehat{T}_{j N}^{( \pm)}(\lambda)$ was introduced in Subsection 6.2 as the integral operator with the kernel $(6.16 \pm)$ ，and the operator $\widehat{\mathcal{T}}_{j N}^{( \pm)}\left(\lambda_{ \pm}\right)$was introduced in Proposition $6.3( \pm)$ as the result of the regularization of $\widehat{T}_{j N}^{( \pm)}(\lambda)$ ．

We introduce some new notation in order to reflect the dependence of operators on the coefficients explicitly．Let $\mathfrak{W}(\mathbf{x})$ be a function on $\mathbb{R}^{2}$ such that $|\mathfrak{W}|^{2}$ satisfies conditions （1．6）and $\left[2.3_{q}\right)$（with $V$ replaced by $|\mathfrak{W}|^{2}$ ），and let $\varphi(\mathbf{x})$ be a continuous periodic function on $\mathbb{R}^{2}$ ．By $\mathfrak{T}_{j}^{( \pm)}(\gamma ; \mathfrak{W}, \varphi)$ we denote the integral operator with the kernel（cf．$\left.(6.16 \pm)\right)$

$$
\begin{array}{r}
(2 \pi)^{-1} \mathfrak{W}(\mathbf{x}) \varphi(\mathbf{x}) \chi_{j}^{( \pm)}(\boldsymbol{\xi})\left(\mathfrak{b}_{j}^{( \pm)}\left(\boldsymbol{\xi}-\boldsymbol{\xi}_{j}^{( \pm)}\right)+\gamma^{2}\right)^{-1 / 2} e^{i\langle\mathbf{x}, \boldsymbol{\xi}\rangle}, \\
j=1, \ldots, m_{ \pm}, \quad \gamma>0 .
\end{array}
$$

From Proposition 3.1 it follows（cf．Proposition 6．3（ $\pm$ ））that

$$
\begin{align*}
\mathfrak{T}_{j}^{( \pm)}(\gamma ; \mathfrak{W}, \varphi) & =\widehat{\mathfrak{T}}_{j}^{( \pm)}(\gamma ; \mathfrak{W}, \varphi)+\check{\mathfrak{T}}_{j}^{( \pm)}(\gamma ; \mathfrak{W}, \varphi), \quad j=1, \ldots, m_{ \pm} \\
\operatorname{rank} \check{\mathfrak{T}}_{j}^{( \pm)}(\gamma ; \mathfrak{W}, \varphi) & =1
\end{align*}
$$

and the limit

$$
\underset{\gamma \rightarrow 0}{(\mathrm{u})-\lim } \widehat{\mathfrak{T}}_{j}^{( \pm)}(\gamma ; \mathfrak{W}, \varphi)=\widehat{\mathfrak{T}}_{j}^{( \pm)}(0 ; \mathfrak{W}, \varphi) \in \Sigma_{2 q}
$$

exists．Remark 3.2 shows that the operator $\widehat{\mathfrak{T}}_{j}^{( \pm)}(0 ; \mathfrak{W}, \varphi)$ depends linearly both on $\mathfrak{W}$ and on $\varphi$ ．

Let $\psi$ be a function satisfying the same conditions as $\varphi$ ．We put
$(9.7 \pm) \quad \mathfrak{P}_{k j}^{( \pm)}(\gamma ; \mathfrak{W}, \varphi, \psi):=\left(\mathfrak{T}_{k}^{( \pm)}(\gamma ; \mathfrak{W}, \varphi)\right)^{*} \mathfrak{T}_{j}^{( \pm)}(\gamma ; \mathfrak{W}, \psi), \quad k, j=1, \ldots, m_{ \pm}$.
Relations（9．4土）－（9．6土）imply that

$$
\mathfrak{P}_{k j}^{( \pm)}(\gamma ; \mathfrak{W}, \varphi, \psi)=\widehat{\mathfrak{P}}_{k j}^{( \pm)}(\gamma ; \mathfrak{W}, \varphi, \psi)+\mathfrak{Z}_{k j}^{( \pm)}(\gamma ; \mathcal{W}, \varphi, \psi)
$$

where

$$
\widehat{\mathfrak{P}}_{k j}^{( \pm)}(\gamma ; \mathfrak{W}, \varphi, \psi)=\left(\widehat{\mathfrak{T}}_{k}^{( \pm)}(\gamma ; \mathfrak{W}, \varphi)\right)^{*} \widehat{\mathfrak{T}}_{j}^{( \pm)}(\gamma ; \mathfrak{W}, \psi), \quad k, j=1, \ldots, m_{ \pm}
$$

We have

$$
\operatorname{rank} \mathfrak{Z}_{k j}^{( \pm)}(\gamma ; \mathfrak{W}, \varphi, \psi) \leq 2
$$

and the limit

$$
\underset{\gamma \rightarrow 0}{(\mathrm{u})-\lim } \widehat{\mathfrak{P}}_{k j}^{( \pm)}(\gamma ; \mathfrak{W}, \varphi, \psi)=\widehat{\mathfrak{P}}_{k j}^{( \pm)}(0 ; \mathfrak{W}, \varphi, \psi) \in \Sigma_{q}
$$

exists．Next，

$$
\widehat{\mathfrak{P}}_{k j}^{( \pm)}(0 ; \mathfrak{W}, \varphi, \psi)=\left(\widehat{\mathfrak{T}}_{k}^{( \pm)}(0 ; \mathfrak{W}, \varphi)\right)^{*} \widehat{\mathfrak{T}}_{j}^{( \pm)}(0 ; \mathfrak{W}, \psi), \quad k, j=1, \ldots, m_{ \pm}
$$

By Proposition 3.3 with $k=j$ and $\varphi=\psi$ ，we obtain（cf．（6．28土）and（6．29土））
$(9.13 \pm) \quad \partial_{q}^{(+)}\left(\widehat{\mathfrak{P}}_{j j}^{( \pm)}(0 ; \mathfrak{W}, \varphi, \varphi)\right)=\partial_{q}^{(+)}\left(|\mathfrak{W}|^{2}, \beta_{j}^{( \pm)}, \varphi\right), \quad \partial=\Delta, \delta, \quad j=1, \ldots, m_{ \pm}$.
3．In accordance with the new notation，

$$
\begin{align*}
\widehat{\mathcal{T}}_{j N}^{( \pm)}\left(\lambda_{ \pm}\right) & =\widehat{\mathfrak{T}}_{j}^{( \pm)}\left(0 ; \widetilde{W}_{N}, \varphi_{j}^{( \pm)}\right), \quad j=1, \ldots, m_{ \pm} \\
P_{k j N}^{( \pm)}\left(\lambda_{ \pm}\right) & =\widehat{\mathfrak{P}}_{k j}^{( \pm)}\left(0 ; \widetilde{W}_{N}, \varphi_{k}^{( \pm)}, \varphi_{j}^{( \pm)}\right), \quad k, j=1, \ldots, m_{ \pm}
\end{align*}
$$

Proposition 9．3（ $\pm$ ）．Let $\mathcal{S}$ be the function introduced in（2．25）．Then

$$
\Delta_{q}\left(\widehat{\mathfrak{P}}_{k j}^{( \pm)}\left(0 ; \widetilde{W}_{N}, \varphi, \psi\right)\right)=\Delta_{q}\left(\widehat{\mathfrak{P}}_{k j}^{( \pm)}(0 ; \sqrt{\mathcal{S}}, \varphi, \psi)\right)
$$

Proof．By Proposition 1．8，we obtain

$$
\begin{aligned}
\left(\Delta_{q}\right. & \left.\left(\widehat{\mathfrak{P}}_{k j}^{( \pm)}\left(0 ; \widetilde{W}_{N}, \varphi, \psi\right)-\widehat{\mathfrak{P}}_{k j}^{( \pm)}(0 ; \sqrt{\mathcal{S}}, \varphi, \psi)\right)\right)^{2} \\
\leq \leq & C_{q} \Delta_{2 q}\left(\widehat{\mathfrak{T}}_{k}^{( \pm)}\left(0 ; \widetilde{W}_{N}, \varphi\right)\right) \Delta_{2 q}\left(\widehat{\mathfrak{T}}_{j}^{( \pm)}\left(0 ; \widetilde{W}_{N}, \psi\right)-\widehat{\mathfrak{T}}_{j}^{( \pm)}(0 ; \sqrt{\mathcal{S}}, \psi)\right) \\
& +C_{q} \Delta_{2 q}\left(\widehat{\mathfrak{T}}_{j}^{( \pm)}(0 ; \sqrt{\mathcal{S}}, \psi)\right) \Delta_{2 q}\left(\widehat{\mathfrak{T}}_{k}^{( \pm)}\left(0 ; \widetilde{W}_{N}, \varphi\right)-\widehat{\mathfrak{T}}_{k}^{( \pm)}(0 ; \sqrt{\mathcal{S}}, \varphi)\right)
\end{aligned}
$$

We have

$$
\widehat{\mathfrak{T}}_{j}^{( \pm)}\left(0 ; \widetilde{W}_{N}, \psi\right)-\widehat{\mathfrak{T}}_{j}^{( \pm)}(0 ; \sqrt{\mathcal{S}}, \psi)=\widehat{\mathfrak{T}}_{j}^{( \pm)}\left(0 ; \widetilde{W}_{N}-\sqrt{\mathcal{S}}, \psi\right)
$$

By（9．12土）and（9．13土），

$$
\begin{aligned}
\Delta_{2 q} & \left(\widehat{\mathfrak{T}}_{j}^{( \pm)}\left(0 ; \widetilde{W}_{N}-\sqrt{\mathcal{S}}, \psi\right)\right)=\Delta_{q}\left(\widehat{\mathfrak{P}}_{j j}^{( \pm)}\left(0 ; \widetilde{W}_{N}-\sqrt{\mathcal{S}}, \psi, \psi\right)\right) \\
& =\Delta_{q}^{(+)}\left(\left|\widetilde{W}_{N}-\sqrt{\mathcal{S}}\right|^{2}, \beta_{j}^{( \pm)}, \psi\right)
\end{aligned}
$$

Here the right－hand side is equal to zero（see the proof of Proposition 2.2 in［BLSu］）．
Similarly，

$$
\Delta_{2 q}\left(\widehat{\mathfrak{T}}_{k}^{( \pm)}\left(0 ; \widetilde{W}_{N}-\sqrt{\mathcal{S}}, \varphi\right)\right)=0
$$

As a result，we obtain

$$
\Delta_{q}\left(\widehat{\mathfrak{P}}_{k j}^{( \pm)}\left(0 ; \widetilde{W}_{N}, \varphi, \psi\right)-\widehat{\mathfrak{P}}_{k j}^{( \pm)}(0 ; \sqrt{\mathcal{S}}, \varphi, \psi)\right)=0
$$

Combining this with（1．26），we arrive at（9．15土）．
Proposition 9．4（ $\pm$ ）．Let $t_{s}^{(r)}(\mathbf{x}), s \in \mathbb{N}, r=1,2$ ，be a sequence of periodic functions of class $L_{\infty}\left(\mathbb{R}^{2}\right)$ such that

$$
\lim _{s \rightarrow \infty}\left\|t_{s}^{(1)}-\varphi\right\|_{L_{\infty}}=0, \quad \lim _{s \rightarrow \infty}\left\|t_{s}^{(2)}-\psi\right\|_{L_{\infty}}=0
$$

Then

$$
\lim _{s \rightarrow \infty} \Delta_{q}\left(\widehat{\mathfrak{P}}_{k j}^{( \pm)}\left(0 ; \mathfrak{W}, t_{s}^{(1)}, t_{s}^{(2)}\right)\right)=\Delta_{q}\left(\widehat{\mathfrak{P}}_{k j}^{( \pm)}(0 ; \mathfrak{W}, \varphi, \psi)\right) .
$$

Proof．By Proposition 1．8，

$$
\begin{aligned}
&\left(\Delta_{q}\left(\widehat{\mathfrak{P}}_{k j}^{( \pm)}(0 ; \mathfrak{W}, \varphi, \psi)-\widehat{\mathfrak{P}}_{k j}^{( \pm)}\left(0 ; \mathfrak{W}, t_{s}^{(1)}, t_{s}^{(2)}\right)\right)\right)^{2} \\
& \leq C_{q} \Delta_{2 q}\left(\widehat{\mathfrak{T}}_{k}^{( \pm)}(0 ; \mathfrak{W}, \varphi)\right) \Delta_{2 q}\left(\widehat{\mathfrak{T}}_{j}^{( \pm)}(0 ; \mathfrak{W}, \psi)-\widehat{\mathfrak{T}}_{j}^{( \pm)}\left(0 ; \mathfrak{W}, t_{s}^{(2)}\right)\right) \\
&+C_{q} \Delta_{2 q}\left(\widehat{\mathfrak{T}}_{j}^{( \pm)}\left(0 ; \mathfrak{W}, t_{s}^{(2)}\right)\right) \Delta_{2 q}\left(\widehat{\mathfrak{T}}_{k}^{( \pm)}(0 ; \mathfrak{W}, \varphi)-\widehat{\mathfrak{T}}_{k}^{( \pm)}\left(0 ; \mathfrak{W}, t_{s}^{(1)}\right)\right) .
\end{aligned}
$$

We have

$$
\widehat{\mathfrak{T}}_{j}^{( \pm)}(0 ; \mathfrak{W}, \psi)-\widehat{\mathfrak{T}}_{j}^{( \pm)}\left(0 ; \mathfrak{W}, t_{s}^{(2)}\right)=\widehat{\mathfrak{T}}_{j}^{( \pm)}\left(0 ; \mathfrak{W}, \psi-t_{s}^{(2)}\right)
$$

By（ $9.12 \pm)$ and $(9.13 \pm)$ ，

$$
\Delta_{2 q}\left(\widehat{\mathfrak{T}}_{j}^{( \pm)}\left(0 ; \mathfrak{W}, \psi-t_{s}^{(2)}\right)\right)=\Delta_{q}\left(|\mathfrak{W}|^{2}, \beta_{j}^{( \pm)}, \psi-t_{s}^{(2)}\right)
$$

The definition of the quantities（2．4）and the standard variational arguments show that

$$
\Delta_{q}\left(|\mathfrak{W}|^{2}, \beta_{j}^{( \pm)}, \psi-t_{s}^{(2)}\right) \leq\left\|\psi-t_{s}^{(2)}\right\|_{L_{\infty}}^{2 q} \Delta_{q}\left(|\mathfrak{W}|^{2}, \beta_{j}^{( \pm)}, 1\right)
$$

By $(9.16 \pm)$ ，the right－hand side tends to zero as $s \rightarrow \infty$ ．Similarly，

$$
\Delta_{2 q}\left(\widehat{\mathfrak{T}}_{j}^{( \pm)}\left(0 ; \mathfrak{W}, \varphi-t_{s}^{(1)}\right)\right) \rightarrow 0 \quad \text { as } s \rightarrow \infty
$$

The quantities $\Delta_{2 q}\left(\widehat{\mathfrak{T}}_{j}^{( \pm)}\left(0 ; \mathfrak{W}, t_{s}^{(2)}\right)\right)$ are dominated by $\left\|t_{s}^{(2)}\right\|_{L_{\infty}}^{2 q} \Delta_{q}\left(|\mathfrak{W}|^{2}, \beta_{j}^{( \pm)}, 1\right)$ ；there－ fore，they are uniformly bounded for $s \in \mathbb{N}$ ．As a result，we see that

$$
\Delta_{q}\left(\widehat{\mathfrak{P}}_{k j}^{( \pm)}(0 ; \mathfrak{W}, \varphi, \psi)-\widehat{\mathfrak{P}}_{k j}^{( \pm)}\left(0 ; \mathfrak{W}, t_{s}^{(1)}, t_{s}^{(2)}\right)\right) \rightarrow 0 \quad \text { as } s \rightarrow \infty .
$$

It remains to refer to（1．26）．
The following proposition is a direct consequence of Propositions 9．3（ $\pm$ ）and $9.4( \pm)$ and relations（9．14土）．

Proposition 9．5（土）．Let $\mathcal{S}$ be a function satisfying（2．25）．Suppose that for every trigonometric polynomial of the form

$$
\begin{equation*}
t^{(r)}(\mathbf{x})=\sum_{\mathbf{m} \in \mathbb{Z}^{2}:|\mathbf{m}| \leq M_{r}} c_{\mathbf{m}}^{(r)} e^{2 \pi i\langle\mathbf{m}, \mathbf{x}\rangle}, \quad r=1,2, \tag{9.17}
\end{equation*}
$$

we have

$$
\Delta_{q}\left(\widehat{\mathfrak{P}}_{k j}^{( \pm)}\left(0 ; \sqrt{\mathcal{S}}, t^{(1)}, t^{(2)}\right)\right)=0, \quad j \neq k .
$$

Then relations $(9.2 \pm)$ are valid．

4．Now we attack the problem＂from the other end＂．For $k \neq j$ ，consider the operator $(9.7 \pm)$ with $\mathfrak{W}=\sqrt{\mathcal{S}}, \varphi=t^{(1)}, \psi=t^{(2)}$ ：

$$
\mathfrak{P}_{k j}^{( \pm)}(\gamma)=\mathfrak{P}_{k j}^{( \pm)}\left(\gamma ; \sqrt{\mathcal{S}}, t^{(1)}, t^{(2)}\right)=\left(\mathfrak{T}_{k}^{( \pm)}\left(\gamma ; \sqrt{\mathcal{S}}, t^{(1)}\right)\right)^{*} \mathfrak{T}_{j}^{( \pm)}\left(\gamma ; \sqrt{\mathcal{S}}, t^{(2)}\right)
$$

where $t^{(1)}$ and $t^{(2)}$ are trigonometric polynomials of the form（9．17）．In this subsection， to the operator $(9.19 \pm)$ we apply another method of regularization，different from that used in Subsection 9．2．（Cf．［BLSu，§7］．）By（9．3土），the kernel of the operator（9．19土） has the form

$$
\begin{align*}
& (2 \pi)^{-1}\left(\Phi\left(\overline{\mathcal{S} t^{(1)}} t^{(2)}\right)\right)(\boldsymbol{\xi}-\boldsymbol{\eta}) \chi_{k}^{( \pm)}(\boldsymbol{\xi}) \chi_{j}^{( \pm)}(\boldsymbol{\eta}) \\
& \quad \times\left(\mathfrak{b}_{k}^{( \pm)}\left(\boldsymbol{\xi}-\boldsymbol{\xi}_{k}^{( \pm)}\right)+\gamma^{2}\right)^{-1 / 2}\left(\mathfrak{b}_{j}^{( \pm)}\left(\boldsymbol{\eta}-\boldsymbol{\xi}_{j}^{( \pm)}\right)+\gamma^{2}\right)^{-1 / 2}
\end{align*}
$$

where $\Phi\left(\overline{\mathcal{S} t^{(1)}} t^{(2)}\right)$ is the Fourier image of the function $\overline{\mathcal{S} t^{(1)}} t^{(2)}$ ．
In what follows it is convenient to change some notation in（9．20土）．Let $\widehat{\mathfrak{A}}$ be a function defined in some neighborhood $\mathcal{O}_{k j}^{( \pm)}$of the point $\boldsymbol{\xi}_{k}^{( \pm)}-\boldsymbol{\xi}_{j}^{( \pm)}$and satisfying the condition

$$
\begin{equation*}
\widehat{\mathfrak{A}} \in H^{\varkappa}\left(\mathcal{O}_{k j}^{( \pm)}\right), \quad \varkappa>1 \tag{9.21}
\end{equation*}
$$

We fix a sufficiently small number $\varepsilon>0$ and put $B_{k j}^{( \pm)}(\varepsilon):=\left\{\boldsymbol{\xi}:\left|\boldsymbol{\xi}-\boldsymbol{\xi}_{k}^{( \pm)}+\boldsymbol{\xi}_{j}^{( \pm)}\right| \leq \varepsilon\right\}$ ． Suppose that $B_{k j}^{( \pm)}(3 \varepsilon) \subset \mathcal{O}_{k j}^{( \pm)}$and that the number $\delta$ is so small that $\boldsymbol{\xi}-\boldsymbol{\eta} \in B_{k j}^{( \pm)}(\varepsilon)$ for $\boldsymbol{\xi} \in \operatorname{supp} \chi_{k}^{( \pm)}, \boldsymbol{\eta} \in \operatorname{supp} \chi_{j}^{( \pm)}$．Next，let $\vartheta \in C_{0}^{\infty}\left(\mathbb{R}^{2}\right)$ be such that $\vartheta(\boldsymbol{\xi})=1$ for $\boldsymbol{\xi} \in B_{k j}^{( \pm)}(2 \varepsilon)$ and $\operatorname{supp} \vartheta \subset B_{k j}^{( \pm)}(3 \varepsilon)$ ．In $L_{2}\left(\mathbb{R}^{2}\right)$ ，we consider the integral operator $\mathcal{I}_{k j}^{( \pm)}(\gamma)$ with the kernel

$$
\begin{align*}
& \mathcal{I}_{k j}^{( \pm)}(\boldsymbol{\xi}, \boldsymbol{\eta} ; \gamma) \\
&=(2 \pi)^{-1} \widehat{\mathfrak{A}}(\boldsymbol{\xi}-\boldsymbol{\eta}) \chi_{k}^{( \pm)}(\boldsymbol{\xi}) \chi_{j}^{( \pm)}(\boldsymbol{\eta}) \\
& \times\left(\mathfrak{b}_{k}^{( \pm)}\left(\boldsymbol{\xi}-\boldsymbol{\xi}_{k}^{( \pm)}\right)+\gamma^{2}\right)^{-1 / 2}\left(\mathfrak{b}_{j}^{( \pm)}\left(\boldsymbol{\eta}-\boldsymbol{\xi}_{j}^{( \pm)}\right)+\gamma^{2}\right)^{-1 / 2}
\end{align*}
$$

$\gamma>0, k \neq j$ ．We regularize the operator $\mathcal{I}_{k j}^{( \pm)}(\gamma)$ in order to pass to the limit as $\gamma \rightarrow 0$ ． We put

$$
\vartheta \widehat{\mathfrak{A}}=\widehat{\mathfrak{A}}_{0}, \quad \mathfrak{A}_{0}=\Phi^{*} \widehat{\mathfrak{A}}_{0}
$$

Then

$$
\begin{equation*}
\widehat{\mathfrak{A}}_{0} \in H^{\varkappa}\left(\mathbb{R}^{2}\right), \quad \varkappa>1 \tag{9.23}
\end{equation*}
$$

and in the kernel $(9.22 \pm)$ it is possible to replace $\widehat{\mathfrak{A}}$ by $\widehat{\mathfrak{A}}_{0}$ ：

$$
\begin{align*}
\mathcal{I}_{k j}^{( \pm)}(\boldsymbol{\xi}, \boldsymbol{\eta} ; \gamma)= & (2 \pi)^{-1} \widehat{\mathfrak{A}}_{0}(\boldsymbol{\xi}-\boldsymbol{\eta}) \chi_{k}^{( \pm)}(\boldsymbol{\xi}) \chi_{j}^{( \pm)}(\boldsymbol{\eta}) \\
& \times\left(\mathfrak{b}_{k}^{( \pm)}\left(\boldsymbol{\xi}-\boldsymbol{\xi}_{k}^{( \pm)}\right)+\gamma^{2}\right)^{-1 / 2}\left(\mathfrak{b}_{j}^{( \pm)}\left(\boldsymbol{\eta}-\boldsymbol{\xi}_{j}^{( \pm)}\right)+\gamma^{2}\right)^{-1 / 2} \\
& \gamma>0, \quad k \neq j
\end{align*}
$$

We regularize the operator $\mathcal{I}_{k j}^{( \pm)}(\gamma)$ ，replacing the kernel $(9.24 \pm)$ by the kernel

$$
\begin{align*}
& \widehat{\mathcal{I}}_{k j}^{( \pm)}(\boldsymbol{\xi}, \boldsymbol{\eta} ; \gamma) \\
& \quad=(2 \pi)^{-1}\left(\widehat{\mathfrak{A}}_{0}(\boldsymbol{\xi}-\boldsymbol{\eta})-\widehat{\mathfrak{A}}_{0}\left(\boldsymbol{\xi}-\boldsymbol{\xi}_{j}^{( \pm)}\right)-\widehat{\mathfrak{A}}_{0}\left(\boldsymbol{\xi}_{k}^{( \pm)}-\boldsymbol{\eta}\right)+\widehat{\mathfrak{A}}_{0}\left(\boldsymbol{\xi}_{k}^{( \pm)}-\boldsymbol{\xi}_{j}^{( \pm)}\right)\right) \\
& \quad \times \chi_{k}^{( \pm)}(\boldsymbol{\xi}) \chi_{j}^{( \pm)}(\boldsymbol{\eta}) \\
& \quad \times\left(\mathfrak{b}_{k}^{( \pm)}\left(\boldsymbol{\xi}-\boldsymbol{\xi}_{k}^{( \pm)}\right)+\gamma^{2}\right)^{-1 / 2}\left(\mathfrak{b}_{j}^{( \pm)}\left(\boldsymbol{\eta}-\boldsymbol{\xi}_{j}^{( \pm)}\right)+\gamma^{2}\right)^{-1 / 2}, \quad \gamma>0, \quad k \neq j .
\end{align*}
$$

The corresponding operator is denoted by $\widehat{\mathcal{I}}_{k j}^{( \pm)}(\gamma)$. Clearly,

$$
\operatorname{rank}\left(\mathcal{I}_{k j}^{( \pm)}(\gamma)-\widehat{\mathcal{I}}_{k j}^{( \pm)}(\gamma)\right)=3
$$

The operator $\widehat{\mathcal{I}}_{k j}^{( \pm)}(\gamma)$ makes sense also for $\gamma=0$. Moreover, the following statement is true (cf. Proposition 7.2 in [BLSu]).
Proposition 9.6( $\pm$ ). If condition (9.21) is fulfilled, then the operator $\widehat{\mathcal{I}}_{k j}^{( \pm)}(0)$ is well defined and

$$
\begin{gather*}
\widehat{\mathcal{I}}_{k j}^{( \pm)}(0) \in \mathfrak{S}_{1} \\
\left(\underset{\gamma \rightarrow 0}{\left(\mathfrak{S}_{1}\right)-\lim } \widehat{\mathcal{I}}_{k j}^{( \pm)}(\gamma)=\widehat{\mathcal{I}}_{k j}^{( \pm)}(0)\right.
\end{gather*}
$$

Proof. For $\gamma=0$, the kernel (9.25 $\pm$ ) can be rewritten in the form

$$
\begin{aligned}
\widehat{\mathcal{I}}_{k j}^{( \pm)}(\boldsymbol{\xi}, \boldsymbol{\eta} ; 0)= & (2 \pi)^{-2} \chi_{k}^{( \pm)}(\boldsymbol{\xi}) \chi_{j}^{( \pm)}(\boldsymbol{\eta})\left(\mathfrak{b}_{k}^{( \pm)}\left(\boldsymbol{\xi}-\boldsymbol{\xi}_{k}^{( \pm)}\right)\right)^{-1 / 2}\left(\mathfrak{b}_{j}^{( \pm)}\left(\boldsymbol{\eta}-\boldsymbol{\xi}_{j}^{( \pm)}\right)\right)^{-1 / 2} \\
& \times \int \mathfrak{A}_{0}(\mathbf{x})\left(e^{-i\langle\mathbf{x}, \boldsymbol{\xi}\rangle}-e^{-i\left\langle\mathbf{x}, \boldsymbol{\xi}_{k}^{( \pm)}\right\rangle}\right)\left(e^{i\langle\mathbf{x}, \boldsymbol{\eta}\rangle}-e^{i\left\langle\mathbf{x}, \boldsymbol{\xi}_{j}^{( \pm)}\right\rangle}\right) d \mathbf{x} .
\end{aligned}
$$

Thus, the operator $\widehat{\mathcal{I}}_{k j}^{( \pm)}(0)$ is the composition of three operators:

$$
\widehat{\mathcal{I}}_{k j}^{( \pm)}(0)=\left(\mathcal{J}_{k}^{( \pm)}\right)^{*}\left[\check{\mathfrak{A}}_{0}\right] \mathcal{J}_{j}^{( \pm)}
$$

Here $\mathcal{J}_{k}^{( \pm)}$and $\mathcal{J}_{j}^{( \pm)}$are the integral operators with the kernels

$$
\begin{aligned}
& \mathcal{J}_{k}^{( \pm)}(\mathbf{x}, \boldsymbol{\xi})=(2 \pi)^{-1} \chi_{k}^{( \pm)}(\boldsymbol{\xi})\left(\mathfrak{b}_{k}^{( \pm)}\left(\boldsymbol{\xi}-\boldsymbol{\xi}_{k}^{( \pm)}\right)\right)^{-1 / 2}\left|\mathfrak{A}_{0}(\mathbf{x})\right|^{1 / 2}\left(e^{i\langle\mathbf{x}, \boldsymbol{\xi}\rangle}-e^{i\left\langle\mathbf{x}, \boldsymbol{\xi}_{k}^{( \pm)}\right\rangle}\right) \\
& \mathcal{J}_{j}^{( \pm)}(\mathbf{x}, \boldsymbol{\eta})=(2 \pi)^{-1} \chi_{j}^{( \pm)}(\boldsymbol{\eta})\left(\mathfrak{b}_{j}^{( \pm)}\left(\boldsymbol{\eta}-\boldsymbol{\xi}_{j}^{( \pm)}\right)\right)^{-1 / 2}\left|\mathfrak{A}_{0}(\mathbf{x})\right|^{1 / 2}\left(e^{i\langle\mathbf{x}, \boldsymbol{\eta}\rangle}-e^{i\left\langle\mathbf{x}, \boldsymbol{\xi}_{j}^{( \pm)}\right\rangle}\right)
\end{aligned}
$$

and $\mathscr{\mathfrak { A }}_{0}(\mathbf{x})=\mathfrak{A}_{0}(\mathbf{x})\left|\mathfrak{A}_{0}(\mathbf{x})\right|^{-1}$ if $\mathfrak{A}_{0}(\mathbf{x}) \neq 0$, and $\mathfrak{A}_{0}(\mathbf{x})=1$ if $\mathfrak{A}_{0}(\mathbf{x})=0$. For the proof of $(9.27 \pm)$, it suffices to check that $\mathcal{J}_{k}^{( \pm)}, \mathcal{J}_{j}^{( \pm)} \in \mathfrak{S}_{2}$. We have

$$
\begin{aligned}
& \iint\left|\mathcal{J}_{k}^{( \pm)}(\mathbf{x}, \boldsymbol{\xi})\right|^{2} d \mathbf{x} d \boldsymbol{\xi} \\
& \quad \leq \pi^{-2} \int\left|\mathfrak{A}_{0}(\mathbf{x})\right|\left(\int_{\mathcal{E}_{k}^{( \pm)}}\left(\mathfrak{b}_{k}^{( \pm)}\left(\boldsymbol{\xi}-\boldsymbol{\xi}_{k}^{( \pm)}\right)\right)^{-1} \sin ^{2}\left(2^{-1} \mathbf{x}\left(\boldsymbol{\xi}-\boldsymbol{\xi}_{k}^{( \pm)}\right)\right) d \boldsymbol{\xi}\right) d \mathbf{x} \\
& \quad \leq C_{1} \int(1+\log (1+|\mathbf{x}|))\left|\mathfrak{A}_{0}(\mathbf{x})\right| d \mathbf{x}
\end{aligned}
$$

Here $\mathcal{E}_{k}^{( \pm)}=\operatorname{supp} \chi_{k}^{( \pm)}$is the ellipse introduced in Subsection 5.1. Thus, we arrive at the condition

$$
\begin{equation*}
\int(1+\log (1+|\mathbf{x}|))\left|\mathfrak{A}_{0}(\mathbf{x})\right| d \mathbf{x}<\infty \tag{9.29}
\end{equation*}
$$

On the other hand, condition (9.23) means that

$$
\left\|\widehat{\mathfrak{A}}_{0}\right\|_{H^{\varkappa\left(\mathbb{R}^{2}\right)}}^{2}=\int\left|\mathfrak{A}_{0}(\mathbf{x})\right|^{2}\left(1+|\mathbf{x}|^{2}\right)^{\varkappa} d \mathbf{x}<\infty, \quad \varkappa>1
$$

Therefore, (9.29) is fulfilled and, moreover,

$$
\begin{aligned}
& \left\|\mathcal{J}_{k}^{( \pm)}\right\|_{\mathfrak{S}_{2}} \leq C_{2}\left\|\widehat{\mathfrak{A}}_{0}\right\|_{H^{\varkappa( }\left(\mathbb{R}^{2}\right)}^{1 / 2} \leq C_{2}\|\widehat{\mathfrak{A}}\|_{H^{\varkappa}\left(\mathcal{O}_{k j}^{( \pm)}\right)}^{1 / 2} \\
& \left\|\mathcal{J}_{j}^{( \pm)}\right\|_{\mathfrak{S}_{2}} \leq C_{2}\left\|\widehat{\mathfrak{A}}_{0}\right\|_{H^{\varkappa( }\left(\mathbb{R}^{2}\right)}^{1 / 2} \leq C_{2}\|\widehat{\mathfrak{A}}\|_{H^{\varkappa}\left(\mathcal{O}_{k j}^{( \pm)}\right)}^{1 / 2}
\end{aligned}
$$

This yields $(9.27 \pm)$.

We rewrite the kernel（9．25土）as follows：

$$
\begin{aligned}
\widehat{\mathcal{I}}_{k j}^{( \pm)}(\boldsymbol{\xi}, \boldsymbol{\eta} ; \gamma)= & \left(\left(\mathfrak{b}_{k}^{( \pm)}\left(\boldsymbol{\xi}-\boldsymbol{\xi}_{k}^{( \pm)}\right)\right)^{1 / 2}\left(\mathfrak{b}_{k}^{( \pm)}\left(\boldsymbol{\xi}-\boldsymbol{\xi}_{k}^{( \pm)}\right)+\gamma^{2}\right)^{-1 / 2} \chi_{k}^{( \pm)}(\boldsymbol{\xi})\right) \widehat{\mathcal{I}}_{k j}^{( \pm)}(\boldsymbol{\xi}, \boldsymbol{\eta} ; 0) \\
& \times\left(\left(\mathfrak{b}_{j}^{( \pm)}\left(\boldsymbol{\eta}-\boldsymbol{\xi}_{j}^{( \pm)}\right)\right)^{1 / 2}\left(\mathfrak{b}_{j}^{( \pm)}\left(\boldsymbol{\eta}-\boldsymbol{\xi}_{j}^{( \pm)}\right)+\gamma^{2}\right)^{-1 / 2} \chi_{j}^{( \pm)}(\boldsymbol{\eta})\right) .
\end{aligned}
$$

The first expression in parentheses represents the kernel of an operator family strongly convergent to $\left[\chi_{k}^{( \pm)}\right]$as $\gamma \rightarrow 0$ ，and the second expression in parentheses is the kernel of an operator family strongly convergent to $\left[\chi_{j}^{( \pm)}\right]$．Combining this with（9．27 $\pm$）and applying Lemma 1．9，we obtain（9．28土）．

5．Now we compare the results of two different ways of regularization for the operator （ $9.19 \pm$ ）．One way was described in Subsection 9.2 （see（ $9.8 \pm)-(9.12 \pm)$ ），and the other in Subsection 9．4．The operator $\mathfrak{P}_{k j}^{( \pm)}(\gamma)$ defined by（ $9.19 \pm$ ）has the kernel $(9.20 \pm)$ ， which coincides with $(9.22 \pm)$ if $\mathfrak{A}=\mathcal{S} \overline{t^{(1)}} t^{(2)}$ ．The function $\overline{t^{(1)}} t^{(2)}$ is a trigonometric polynomial of the form

$$
\overline{t^{(1)}}(\mathbf{x}) t^{(2)}(\mathbf{x})=\sum_{\mathbf{m} \in \mathbb{Z}^{2}:|\mathbf{m}| \leq M} c_{\mathbf{m}} e^{2 \pi i\langle\mathbf{m}, \mathbf{x}\rangle} .
$$

Then the Fourier image of the function $\mathfrak{A}$ is

$$
\widehat{\mathfrak{A}}(\boldsymbol{\xi})=\sum_{\mathbf{m} \in \mathbb{Z}^{2}:|\mathbf{m}| \leq M} c_{\mathbf{m}}(\Phi \mathcal{S})(\boldsymbol{\xi}-2 \pi \mathbf{m}) .
$$

By Condition $2.3( \pm)$ ，

$$
\widehat{\mathfrak{A}} \in H^{\varkappa}\left(\mathcal{O}_{k j}^{( \pm)}\right), \quad \varkappa>1, \quad k \neq j
$$

where $\mathcal{O}_{k j}^{( \pm)}$is a neighborhood of the point $\boldsymbol{\xi}_{k}^{( \pm)}-\boldsymbol{\xi}_{j}^{( \pm)}, k \neq j$ ．Proposition $9.6( \pm)$ is applicable．

By（9．8土），（9．10土）and（9．26土），we have

$$
\operatorname{rank}\left(\widehat{\mathfrak{P}}_{k j}^{( \pm)}\left(\gamma ; \sqrt{\mathcal{S}}, t^{(1)}, t^{(2)}\right)-\widehat{\mathcal{I}}_{k j}^{( \pm)}(\gamma)\right) \leq 5, \quad k \neq j
$$

Using（9．11士）and（9．28土），in（9．30土）we can pass to the limit as $\gamma \rightarrow 0$ ：

$$
\operatorname{rank}\left(\widehat{\mathfrak{P}}_{k j}^{( \pm)}\left(0 ; \sqrt{\mathcal{S}}, t^{(1)}, t^{(2)}\right)-\widehat{\mathcal{I}}_{k j}^{( \pm)}(0)\right) \leq 5, \quad k \neq j
$$

Now（9．31土）and（9．27土）imply（9．18土）．Applying Proposition 9．5（土），we obtain（9．2土）． The proof of Proposition $9.1( \pm)$ and，with it，of Theorem $2.5( \pm)$ is complete．

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Received 14/JAN/2003
Translated by THE AUTHOR


[^0]:    2000 Mathematics Subject Classification. Primary 35P20.
    Key words and phrases. Periodic operator, perturbation, discrete spectrum, spectral gap, threshold effect.

    Supported by RFBR (grant no. 02-01-00798).

