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HEATING OF THE AHLFORS–BEURLING OPERATOR, AND ESTIMATES OF ITS NORM

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Abstract. A new estimate is established for the norm of the Ahlfors–Beurling transform $T\varphi(z) := \frac{1}{\pi}\iint \frac{\varphi(\zeta)\,dA(\zeta)}{(\zeta-z)^2}$ in $L^p(dA)$. Namely, it is proved that $\|T\|_{L^p\to L^p} \le 2(p-1)$ for all $p \ge 2$. The method of Bellman function is used; however, the exact Bellman function of the problem has not been found. Instead, a certain approximation to the Bellman function is employed, which leads to the factor 2 on the right (in place of the conjectural 1).

§0. Introduction

Notation.

$:=$ means "is equal by definition";
$x := (x_1, x_2)$;
$D(x,R)$ is the disk centered at x and of radius R;
$k(x,t) := \frac{1}{4\pi t}e^{-\frac{\|x\|^2}{4t}}$ is the heat kernel on the plane;
$\mathcal{D}$ is a collection of dyadic intervals.

Main objects and results. The main object in this paper is the Ahlfors–Beurling operator given by

$$T\varphi(z) := \frac{1}{\pi}\iint \frac{\varphi(\zeta)\,dA(\zeta)}{(\zeta-z)^2}.$$

Here dA denotes area Lebesgue measure on $\mathbb{C}$. Our goal is to present a new estimate of the norm of T. This estimate falls short of the proof of the well-known conjecture saying that

$$\|T\|_{L^p\to L^p} = p-1, \quad p \ge 2. \tag{0.1}$$

Here we show that $\|T\|_{L^p\to L^p} \le 2(p-1)$ for all $p \ge 2$, which is two times worse than (0.1). The estimate $\|T\|_{L^p\to L^p} \le 4(p-1)$ was established in [3]. After the first preprint version of the present paper appeared, Rodrigo Bañuelos and Pedro Méndez-Hernández [9] informed us that they also managed to improve 4 to 2 by modifying the methods used in [3].

Actually, this problem has a long history, and it has been reappearing in many papers on the regularity of quasiconformal homeomorphisms and quasiregular maps. Essentially, the L^p-theory of quasiregular mappings was started with the work of B. Bojarski [5, 6]. Later, this subject came under intensive investigation. In particular, the best integrability of K-quasiconformal mappings and the problem (dual in a sense; see [28]) on the minimal regularity of quasiregular mappings were discussed in many papers. Here we mention [15]–[17], [18, 19], [22]–[24], [27, 28]. The best integrability result was established finally

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in [2]. The best minimal regularity result was obtained recently in [30]. The method of [30] will be applied in the present paper to establish the inequality

$$\|T\|_{L^p\to L^p} \le 2(p-1), \quad p\ge 2. \tag{0.2}$$

By the same method, it is possible to prove that

$$\Big\|\Big(\sum_{j,k=1}^{2}\Big|\frac{\partial^2 f}{\partial x_j\partial x_k}\Big|^2\Big)^{1/2}\Big\|_p \le \sqrt{2}p\|\Delta f\|_p, \quad f\in W_2^p,\ p\ge 2, \tag{0.3}$$

which is better than in [24].

§1. Consequences of the "$(p-1)$-estimate"

Let us formulate analytic and geometric consequences of the elimination of the factor 2 in (0.2). We are dealing with (local) solutions of the Beltrami equation

$$f_{\bar z} - \mu f_z = 0. \tag{1.1}$$

We ask two questions.

1) Suppose $\|\mu\|_\infty = k < 1$. If a solution is *a priori* in W_1^2 locally, what is the ensured smoothness of this local solution? It is classical that f must belong to $W_1^{2+\varepsilon(k)}$ locally, where $\varepsilon(k) > 0$. Finding the best $\varepsilon(k)$ was the key point of the problem by F. Gehring solved by K. Astala [2]. The best $\varepsilon(k)$ turned out to be equal to $\frac{1-k}{k}$. It is not attainable in general.

2) Suppose $\|\mu\|_\infty = k < 1$. If a solution is *a priori* in W_1^q locally (now $q < 2$), what is the smallest q that ensures that $f\in W_1^2$ locally (and then, by [2], ensures that $f\in W_1^{1+1/k-\tau}$ for any positive τ)? The smallest q turns out to be $1+k$. It is attainable (see [30]).

These two questions are intimately related to estimate (0.1). We explain the reason for that. Consider (1.1) in a neighborhood W of the origin, and put $V=\frac12 W$. Let φ be a C^∞-function supported on W and equal to 1 on V. We set $g:=\varphi f_{\bar z}$ and consider $f-\frac{1}{\pi}\int_{\mathbb{C}}\frac{g(\zeta)}{z-\zeta}\,dA(\zeta)$. Here dA stands for planar Lebesgue measure. Application of the distributional operator $\bar\partial$ to the latter expression yields zero in V. So, we have a function h analytic in V and such that $f = h+\frac{1}{\pi}\int_{\mathbb{C}}\frac{g(\zeta)}{z-\zeta}\,dA(\zeta)$. Consequently,

$$f_z = h' + Tg$$

in V. If we multiply (1.1) by φ and use the notation $g:=\varphi f_{\bar z}$ and the previous expression for f_z, we get

$$g-\mu\varphi Tg = r := \mu\varphi h'$$

in V. On $U:=\frac12 V$ the function r is bounded, and therefore it belongs to any $L^p(U,dA)$. Let M denote the operator of multiplication by $\mu\varphi$ in $L^p(U,dA)$, $\|M\|\le k$. We denote $t(p):=\|T\|_{L^p(U,dA)\to L^p(U,dA)}$ and consider the identity

$$(I-MT)g = r$$

to conclude that the inequality

$$kt(p) < 1 \tag{1.2}$$

implies that $g\in L^p$ in U, which is the same as to say that $f\in W_1^p$ locally. Now we see that (0.1) would imply that for every p with $p<1+\frac1k$ the solution in question of (1.1) is in $W_1^p(dA)$ locally.

The above considerations yield also *a lower estimate for the norm of T on L^p*. This argument is borrowed from [19]. Suppose that $t(p)$ is strictly smaller than $p-1$. Using (1.2) in the same way as above, we see that if $\|\mu\|_\infty = k < 1$, then any solution of (1.1) that is *a priori* in W_1^2 locally is in fact in $W_1^{1+1/k+\varepsilon}$ for some $\varepsilon > 0$. But it is easy to compute that the function $f(z) := z|z|^{-\frac{2k}{1+k}}$ satisfies (1.1) with $\mu(z) = -k\frac{z}{\bar z}$. Thus, the L^∞-norm of μ is k. However, f is not in $W_1^{1+1/k+\varepsilon}$ near the origin. It is not even in $W_1^{1+1/k}$. In fact, it is readily computable that $|f_z| = C(k)|z|^{-\frac{2k}{1+k}}$, which does not belong to any $L^p(dA)$ for $p \geq 1 + 1/k$. Thus, $\|T\|_p \geq p - 1$.

We think that we have presented enough motivation for our interest in the estimation of the L^p-norm of such a particular Fourier multiplier as T, and of related multipliers to be considered here.

§2. Littlewood–Paley identity for heat extensions

For the Ahlfors–Beurling operator T we can write the identity $T = R_1^2 - R_2^2 + 2iR_1R_2$, where the R_i are the planar Riesz transforms. We fix, say, R_1^2 and two complex-valued test functions $\varphi, \psi \in C_0^\infty$. We use heat extensions. For a function f on the plane, its heat extension is given by the formula

$$f(y,t) := \frac{1}{4\pi t}\iint_{\mathbb{R}^2} f(x)\exp\Big(-\frac{|x-y|^2}{4t}\Big)\,dx_1\,dx_2, \quad (y,t) \in \mathbb{R}^3_+.$$

Usually, we employ the same letter to denote a function and its heat extension.

Lemma 2.1. *Let $\varphi, \psi \in C_0^\infty$. Then the integral $\iiint \frac{\partial\varphi}{\partial x_1}\cdot\frac{\partial\psi}{\partial x_1}\,dx_1\,dx_2\,dt$ converges absolutely and*

$$\iint R_1^2\varphi\cdot\psi\,dx_1\,dx_2 = -2\iiint \frac{\partial\varphi}{\partial x_1}\cdot\frac{\partial\psi}{\partial x_1}\,dx_1\,dx_2\,dt. \tag{2.1}$$

Proof. Actually, the proof of this lemma is trivial. It is based on the fact that a function is an integral of its derivative, and also involves Parseval's formula. Consider complex-valued functions $\varphi, \psi \in C_0^\infty$ and write

$$\begin{aligned}\iint \psi R_1^2\varphi\,dx_1\,dx_2 &= \iint \frac{\xi_1^2}{\xi_1^2+\xi_2^2}\hat\varphi(\xi_1,\xi_2)\hat\psi(-\xi_1,-\xi_2)\,d\xi_1\,d\xi_2\\ &= 2\iint\int_0^\infty e^{-2t(\xi_1^2+\xi_2^2)}\xi_1^2\hat\varphi(\xi_1,\xi_2)\hat\psi(-\xi_1,-\xi_2)\,dt\,d\xi_1\,d\xi_2\\ &= -2\int_0^\infty\iint i\xi_1\hat\varphi(\xi_1,\xi_2)e^{-t(\xi_1^2+\xi_2^2)}\cdot i\xi_1\hat\psi(-\xi_1,-\xi_2)e^{-t(\xi_1^2+\xi_2^2)}\,d\xi_1\,d\xi_2\,dt\\ &= 2\int_0^\infty\iint \frac{\partial\varphi}{\partial x_1}(x_1,x_2,t)\frac{\partial\psi}{\partial x_1}(x_1,x_2,t)\,dx_1\,dx_2\,dt\\ &= 2\iiint_{\mathbb{R}^3_+}\frac{\partial\varphi}{\partial x_1}(x_1,x_2,t)\frac{\partial\psi}{\partial x_1}(x_1,x_2,t)\,dx_1\,dx_2\,dt.\end{aligned}$$

We have used Parseval's formula twice, and also the absolute convergence of the integrals

$$\iiint_{\mathbb{R}^3_+} e^{-2t(\xi_1^2+\xi_2^2)}\xi_1^2\hat\varphi(\xi_1,\xi_2)\hat\psi(\xi_1,\xi_2)\,d\xi_1\,d\xi_2\,dt,$$
$$\iiint_{\mathbb{R}^3_+}\frac{\partial\varphi}{\partial x_1}(x_1,x_2,t)\frac{\partial\psi}{\partial x_1}(x_1,x_2,t)\,dx_1\,dx_2\,dt.$$

This is obvious for the first integral and easy for the second. We leave this as an exercise for the reader. □

§3. The Bellman function proof of (0.2)

We warn the reader that sometimes it will be convenient to think that $\mathbb{C}$ is $\mathbb{R}^2$, and that the absolute value $|\cdot|$ is the norm $\|\cdot\|$ of a vector in $\mathbb{R}^2$.

Let φ, ψ be complex-valued test functions in $C_0^\infty(\mathbb{R}^2)$. We denote their heat extensions to $\mathbb{R}^3_+$ by the same letters and use Lemma 2.1. It is easily seen that estimating combinations of $\langle R_i^2\varphi, \psi\rangle$ is reduced to estimating integrals that occur in the next theorem. Notice also that if U_ρ denotes the operator $U_\rho\varphi(z) := f(e^{i\rho}z)$, then $2R_1R_2 = U^*_{\pi/4}(R_1^2 - R_2^2)U_{\pi/4}$. Therefore, the proof of (0.2) follows immediately from Theorem 3.1 and Lemma 2.1.

Theorem 3.1. *For any complex-valued $\varphi, \psi \in C_0^\infty$ we have*

$$2\iiint_{\mathbb{R}^3_+}\left|\frac{\partial\varphi}{\partial x_1}\right|\left|\frac{\partial\psi}{\partial x_1}\right| dx_1\,dx_2\,dt + 2\iiint_{\mathbb{R}^3_+}\left|\frac{\partial\varphi}{\partial x_2}\right|\left|\frac{\partial\psi}{\partial x_2}\right| dx_1\,dx_2\,dt \le (p-1)\|\varphi\|_p\|\psi\|_q.$$

In particular,

$$\|R_1^2 - R_2^2\|_p \le p-1, \;\; \|2R_1R_2\|_p \le p-1 \quad \text{for all } p,\ 2 \le p < \infty.$$

In the proof of Theorem 3.1 we use the following key result. (In what follows, $d^2 f$ denotes the Hessian form that is the second differential form of f.)

Theorem 3.2. *For any $p \ge 2$ we define the domain $D_p := \{0 < (X, Y, \xi, \eta) \subset \mathbb{R}\times\mathbb{R}\times\mathbb{R}^2\times\mathbb{R}^2 : \|\xi\|^p < X, \|\eta\|^q < Y\}$. Let K be any compact subset of D_p, and let ε be an arbitrary positive number. Then there exists a function $B = B_{\varepsilon,p,K}(X, Y, x, y)$ infinitely differentiable in a small neighborhood of K and such that*

$$\begin{aligned} &1)\quad 0 \le B \le (1+\varepsilon)(p-1)X^{1/p}Y^{1/q},\\ &2)\quad -d^2B \ge 2\|d\xi\|\|d\eta\|. \end{aligned}$$

We prove Theorem 3.2 later. Now we use it to obtain the proof of Theorem 3.1.

Proof. We consider two functions $\varphi, \psi \in C_0^\infty$ and take $B = B_{\varepsilon,p,K}$, where a compact set K remains to be chosen.

We are interested in

$$b(x,t) := B(|\varphi|^p(x,t), |\psi|^q(x,t), \varphi(x,t), \psi(x,t)).$$

This is a well-defined function, because the Cauchy inequality ensures that the 6-vector

$$v := (|\varphi|^p(x,t), |\psi|^q(x,t), \varphi(x,t), \psi(x,t))$$

lies in D_p for any $(x,t) \in \mathbb{R}^3_+$. Also we can fix any compact subset M of the open set $\mathbb{R}^3_+$ and guarantee that for $(x,t) \in M$, the vector v lies in some compact set K. Indeed, observe that for compactly supported φ, ψ the mapping $(x,t) \to v(x,t)$ takes compact sets in $\mathbb{R}^3_+$ to compact sets in D_p. Now we simply take K sufficiently large in accordance with M; in our future considerations M will run over larger and larger compact sets in $\mathbb{R}^3_+$.

We want to apply Green's formula to $b(x,t)$. To do this, we introduce the Green function $G(x,t)$ as in [13]. Taking a sufficiently large cylinder $\Omega := \Omega_l := D(0,l)\times(0,l)$, we put $\partial'\Omega = \partial D(0,l)\times(0,l)$ and consider the following Green function:

$$\begin{cases} \left(\dfrac{\partial}{\partial t} + \Delta\right) G_\Omega = -\delta_{0,1} & \text{in } \Omega,\\ G_\Omega = 0 & \text{on } \partial'\Omega,\\ G_\Omega = 0 & \text{if } t = l. \end{cases}$$

Here $\delta_{0,1}$ is the δ-function at the point $(0,1)$.

Let $k(x,t) := \frac{1}{4\pi t}e^{-\frac{|x|^2}{4t}}$, $x := (x_1, x_2)$, be the heat kernel in $\mathbb{R}^3_+$. The quantity $k(0,t)$ can be understood as the temperature of the point $(0,0)$ on the plane at the time moment $t > 0$ if initially (at $t = 0$) the distribution of temperature coincided with the delta distribution concentrated at $(0,0)$. It is important to keep in mind that

$$G_\Omega(0,0) \to k(0,1) \quad \text{as } l \to \infty. \tag{3.1}$$

Indeed, it suffices to compare the interpretation of $k(0,1)$ with the fact that $G_\Omega(0,0)$ is the temperature at the moment 1 provided the same initial distribution is given but the temperature on $\partial'\Omega_l$ is kept to be 0. However, if l is large, it is clear that these two quantities are very close.

We also need the Green function in the cylinder $\Omega(R,R^2) = D(0,lR) \times (0,lR^2)$:

$$\begin{cases} \left(\dfrac{\partial}{\partial t} + \Delta\right) G^R_\Omega = -\delta_{0,R^2} & \text{in } \Omega(R,R^2) = D(0,lR) \times (0,lR^2), \\ G^R_\Omega = 0 & \text{on } \partial'\Omega(R,R^2) = \partial D(0,lR) \times (0,lR^2), \\ G^R_\Omega = 0 & \text{if } t = lR^2. \end{cases}$$

The following fact is easy.

Lemma 3.3. $G^R_\Omega(x,t) = \frac{1}{R^2}G_\Omega(x/R, t/R^2)$.

We are ready to apply Green's formula to $b(x,t)$. First we estimate $b(0,R^2) = B(|\varphi|^p(0,R^2),\dots,\psi(0,R^2))$. Using property 1) of B (see Theorem 3.2), we get ($x = (x_1,x_2)$ as always, and $1/p + 1/q = 1$):

$$b(0,R^2) \le (1+\varepsilon)(p-1)(|\varphi|^p(0,R^2))^{1/p}(|\psi|^q(0,R^2))^{1/q}.$$

Thus,

$$b(0,R^2) \le (1+\varepsilon)(p-1)\left(\frac{1}{4\pi R^2}\iint |\varphi|^p(x)e^{-\frac{|x|^2}{4R^2}}\right)^{1/p}\left(\frac{1}{4\pi R^2}\iint |\psi|^q(x)e^{-\frac{|x|^2}{4R^2}}\right)^{1/q}.$$

Now, by Green's formula in $C(R,R^2)$, we have

$$\begin{aligned} b(0,R^2) &= -\iiint_{\Omega(R,R^2)\cap\{t>\delta\}} b(x,t)\left(\frac{\partial}{\partial t}+\Delta\right)G^R_\Omega(x,t)\,dx_1\,dx_2\,dt \\ &= \iiint_{\Omega(R,R^2)\cap\{t>\delta\}} G^R_\Omega(x,t)\left(\frac{\partial}{\partial t}-\Delta\right)b(x,t)\,dx_1\,dx_2\,dt \\ &\quad + \iint_{D(0,R)} b(x,\delta)G^R_\Omega(x,\delta)\,dx_1\,dx_2 \\ &\quad + \iint_{\partial'\Omega(R,R^2)\cap\{t>\delta\}}\left(\frac{\partial b}{\partial n_{\text{outer}}}G^R_\Omega - \frac{\partial G^R_\Omega}{\partial n_{\text{outer}}}b\right)ds\,dt \\ &= \iiint_{\Omega(R,R^2)\cap\{t>\delta\}} G^R_\Omega(x,t)\left(\frac{\partial}{\partial t}-\Delta\right)b(x,t)\,dx_1\,dx_2\,dt \\ &\quad + \iint_{D(0,lR)} b(x,\delta)G^R_\Omega(x,\delta)\,dx_1\,dx_2 + \iint_{\partial'\Omega(R,R^2)\cap\{t>\delta\}}\frac{\partial G^R_\Omega}{\partial n_{\text{inner}}}b\,ds\,dt \\ &\ge \iiint_{\Omega(R,R^2)\cap\{t>\delta\}} G^R_\Omega(x,t)\left(\frac{\partial}{\partial t}-\Delta\right)b(x,t)\,dx_1\,dx_2\,dt. \end{aligned}$$

The last inequality is clear: the double integrals are both nonnegative, because b is nonnegative and because G^R_Ω is nonnegative and vanishes on the side boundary. We combine the estimates of $b(0,R^2)$ into the following (here $\Omega_{R,\delta} := \Omega(R,R^2)\cap\{t>\delta\}$):

$$\begin{aligned}&\iiint_{\Omega_{R,\varepsilon}} G^R_\Omega(x,t)\left(\frac{\partial}{\partial t}-\Delta\right)b(x,t)\\ &\quad\le \frac{(1+\varepsilon)(p-1)}{4\pi R^2}\left(\iint|\varphi|^p(x)e^{-\frac{|x|^2}{4R^2}}\right)^{1/p}\left(\iint|\psi|^q(x)e^{-\frac{|x|^2}{4R^2}}\right)^{1/q}.\end{aligned}\tag{3.2}$$

Fixing R and $\delta>0$, we take the compact set $M=\{(x,t) : x\in \mathrm{const}(D(0,lR)), \delta\le t\le lR^2\}$. The vector-valued function v maps M to a compact subset of the domain D_p. We denote this compact subset by K and choose $B=B_{\varepsilon,p,K}$ as in Theorem 3.2.

The next calculation is simple but it is key to the proof. In it

$$v=(|\varphi|^p(x,t),|\psi|^q(x,t),\varphi(x,t),\psi(x,t)).$$

Lemma 3.4. *If $(x,t)\in M$, then*

$$\left(\frac{\partial}{\partial t}-\Delta\right)b(x,t)=\left((-d^2B)\,\frac{\partial v}{\partial x_1},\frac{\partial v}{\partial x_1}\right)_{\mathbb{R}^6}+\left((-d^2B)\,\frac{\partial v}{\partial x_2},\frac{\partial v}{\partial x_2}\right)_{\mathbb{R}^6}.$$

Proof.

$$\begin{aligned}\frac{\partial}{\partial t}b&=\left(\nabla B,\frac{\partial v}{\partial t}\right)_{\mathbb{R}^6},\\ \Delta b&=\left((d^2B)\frac{\partial v}{\partial x_1},\frac{\partial v}{\partial x_1}\right)_{\mathbb{R}^6}+\left((d^2B)\frac{\partial v}{\partial x_2},\frac{\partial v}{\partial x_2}\right)_{\mathbb{R}^6}+(\nabla B,\Delta v)_{\mathbb{R}^6}.\end{aligned}$$

(Merely, we have used the chain rule.) Now,

$$\left(\frac{\partial}{\partial t}-\Delta\right)b=\left(\nabla B,\frac{\partial v}{\partial t}-\Delta v\right)_{\mathbb{R}^6}-\left((d^2B)\frac{\partial v}{\partial x_1},\frac{\partial v}{\partial x_1}\right)_{\mathbb{R}^6}-\left((d^2B)\frac{\partial v}{\partial x_2},\frac{\partial v}{\partial x_2}\right)_{\mathbb{R}^6}.$$

However, the first term is zero because all entries of the vector v are solutions of the heat equation. □

By Theorem 3.2, in M we have

$$-d^2B(X,Y,\xi,\eta)\ge 2\|d\xi\|\|d\eta\|.\tag{3.3}$$

For $(x,t)\in M$, Lemma 3.4 yields

$$\left(\frac{\partial}{\partial t}-\Delta\right)b(x,t)\ge 2\left|\frac{\partial\varphi}{\partial x_1}\right|\left|\frac{\partial\psi}{\partial x_1}\right|+\left|\frac{\partial\varphi}{\partial x_2}\right|\left|\frac{\partial\psi}{\partial x_2}\right|.\tag{3.4}$$

Combining (3.2) and (3.4), we get

$$\begin{aligned}&2\iiint_M G^R_\Omega(x,t)\left(\left|\frac{\partial\varphi}{\partial x_1}\right|\left|\frac{\partial\psi}{\partial x_1}\right|+\left|\frac{\partial\varphi}{\partial x_2}\right|\left|\frac{\partial\psi}{\partial x_2}\right|\right)\\ &\quad\le\frac{(1+\varepsilon)(p-1)}{4\pi R^2}\left(\iint|\varphi|^p(x)\right)^{1/p}\left(\iint|\psi|^q(x)\right)^{1/q}.\end{aligned}\tag{3.5}$$

Now it is time to use Lemma 3.3. So, (3.5) implies the inequality

$$\begin{aligned}&2\iiint_M G_\Omega\!\left(\frac{x}{R},\frac{t}{R^2}\right)\left(\left|\frac{\partial\varphi}{\partial x_1}\right|\left|\frac{\partial\psi}{\partial x_1}\right|+\left|\frac{\partial\varphi}{\partial x_2}\right|\left|\frac{\partial\psi}{\partial x_2}\right|\right)\\ &\quad\le\frac{(1+\varepsilon)(p-1)}{4\pi}\left(\iint|\varphi|^p(x)\right)^{1/p}\left(\iint|\psi|^q(x)\right)^{1/q}.\end{aligned}\tag{3.6}$$

But $M = \{(x,t) : x \in \mathrm{const}(D(0,R)), \delta \le t \le R^2\}$. We fix any compact M_0 in $\mathbb{R}^3_+$ and choose R and $\delta > 0$ in such a way that $M_0 \subset M$. Next, we restrict the integration in (3.6) to M_0 and let $R \to \infty$. Since

$$G_\Omega(x/R, t/R^2) \to G_\Omega(0,0),$$

we obtain

$$\begin{aligned} &2G_{\Omega_l}(0,0)\iiint_{M_0}\left(\left|\frac{\partial\varphi}{\partial x_1}\right|\left|\frac{\partial\psi}{\partial x_1}\right| + \left|\frac{\partial\varphi}{\partial x_2}\right|\left|\frac{\partial\psi}{\partial x_2}\right|\right) \\ &\qquad \le \frac{(1+\varepsilon)(p-1)}{4\pi}\left(\iint |\varphi|^p(x)\right)^{1/p}\left(\iint |\psi|^q(x)\right)^{1/q}. \end{aligned} \tag{3.7}$$

Now it is time to let $\Omega = \Omega_l$ tend to $\mathbb{R}^3_+$ by making l go to infinity. By (3.1), we conclude that $G_{\Omega_l}(0,0) \to \frac{1}{4\pi}$. Then (3.7) becomes

$$\begin{aligned} &2\iiint_{M_0}\left(\left|\frac{\partial\varphi}{\partial x_1}\right|\left|\frac{\partial\psi}{\partial x_1}\right| + \left|\frac{\partial\varphi}{\partial x_2}\right|\left|\frac{\partial\psi}{\partial x_2}\right|\right) \\ &\qquad \le (1+\varepsilon)(p-1)\left(\iint |\varphi|^p(x)\right)^{1/p}\left(\iint |\psi|^q(x)\right)^{1/q}. \end{aligned} \tag{3.8}$$

But M_0 is an arbitrary compact set in the upper half-space, and ε is an arbitrary positive number. Therefore,

$$\begin{aligned} &2\iiint_{\mathbb{R}^3_+}\left(\left|\frac{\partial\varphi}{\partial x_1}\right|\left|\frac{\partial\psi}{\partial x_1}\right| + \left|\frac{\partial\varphi}{\partial x_2}\right|\left|\frac{\partial\psi}{\partial x_2}\right|\right) \\ &\qquad \le (p-1)\left(\iint |\varphi|^p(x)\right)^{1/p}\left(\iint |\psi|^q(x)\right)^{1/q}. \end{aligned} \tag{3.9}$$

This proves Theorem 3.1. □

§4. The existence of a Bellman function. Proof of Theorem 3.2

We start with a simple "model" operator T_σ. To define it, we let $\mathcal{D}$ denote a family of dyadic intervals on the line. To each $I \in \mathcal{D}$ we assign its Haar function: $h_I = 1/\sqrt{|I|}$ on I_+ and $h_I = -1/\sqrt{|I|}$ on I_-, where I_+ and I_- are the right half and the left half of I, respectively. Every nice complex-valued function (continuous with compact support on one of I's, say on $[0,1]$) can be written as its Haar series: $f = \sum_I (f, h_I)h_I$. Consider the operator $T_\sigma f = \sum_I \sigma_I (f, h_I)h_I$, where σ_I is an arbitrary sequence of complex numbers with absolute value 1. Notation: we use $\langle f\rangle_I$ to denote $\frac{1}{|I|}\int_I f\,dx$.

The logic will be as follows. We want to get a sharp estimate of $\|T_\sigma\|_{L^p\to L^p}$ in terms of p. This problem has been solved by Burkholder. In [10] he found that ($p \ge 2$)

$$\sup_\sigma \|T_\sigma\|_p \le p-1. \tag{4.1}$$

He proved (4.1) by constructing a certain function of two real variables (actually, another Bellman function) that has certain convexity and size properties. The reader is referred to the papers by Burkholder [10, 11], or to the book by D. Stroock [32] for the description of his approach. In particular, the following is written about (4.1) in [32, p. 344]: "Quite recently Burkholder has discovered *the right argument*: ... it is completely elementary. Unfortunately, it is also completely opaque. Indeed, his new argument is nothing but an elementary verification that he has got the right answer; it gives no hint about how he came to that answer". Further on "for those who want to know the secret behind his proof, Burkholder has written an explanation in his article" [10]. Here is Burkholder's function ($p \geq 2$):

$$b(x,y) = (|x| - (p-1)|y|)(|x| + |y|)^{p-1}.$$

Actually, a stochastic Bellman PDE readily explains the way to write this function, and this was done, for example, in [33].

We wish we could use this Bellman function of Burkholder in our problem. But we are unable to do that. The reason is simple. The variables in Burkholder's function stand for certain martingales, which in his case are related to each other: one is subordinate to the other. In our case we replace this variables not by martingales but by functions: the first is $R_1^2\varphi$, and the second is φ, where φ is a test function. There is no subordination here. The only differential relation between these two functions (actually, between their heat extensions) is the following:

$$\frac{\partial}{\partial t} R_1^2 \varphi = \frac{\partial^2}{\partial x_1^2} \varphi.$$

This is a second order differential identity and, as such, it is in no relationship with the subordination property, which interplayed so essentially with a very special convexity property of the Burkholder function (see [10]). It would have been related to subordination (and then to convexity), should it be a differential identity of the first order. What we mean can be illustrated by the following oversimplified example. Obviously, the composition of a convex function a with a linear function l is convex, but the composition of a convex a and a convex l may fail to be convex ($a(x) = e^{-x}$, $l(x) = x^2$). That is exactly the obstruction to using Burkholder's function and composing it with our second order Riesz transforms.

We do not see the way to win over this difficulty. We prefer another approach, which follows the approach in [30].

The idea: we formulate Burkholder's inequality in an equivalent form (simply in its dual form). The resulting inequality generates another Bellman function. This will be our B in Theorem 3.2.

As has already been said, we shall use the following lemma due to Burkholder.

Lemma 4.1. *Let H be a separable Hilbert space. Suppose (X_n, F_n, P) and (Y_n, F_n, P) are H-valued martingales. If*

$$\|X_0(\omega)\|_H \leq \|Y_0(\omega)\|_H, \quad \|X_n(\omega) - X_{n-1}(\omega)\|_H \leq \|Y_n(\omega) - Y_{n-1}(\omega)\|_H$$

for almost every ω and all n, then

$$\|X_n\|_{L^p(P,H)} \leq \max(p-1, 1/(p-1))\|Y_n\|_{L^p(P,H)}$$

for each $p \in (1,\infty)$.

From the lemma we can easily deduce the following theorem.

Theorem 4.2. *Suppose $J \in \mathcal{D}$, $f \in L^p(J)$, and $g \in L^{p'}(J)$. Let $p \geq 2$. Then*

$$\frac{1}{4}\frac{1}{|J|}\sum_{I\in\mathcal{D}, I\subset J} |\langle f\rangle_{I_+} - \langle f\rangle_{I_-}||\langle g\rangle_{I_+} - \langle g\rangle_{I_-}||I| \leq (p-1)\langle |f|^p\rangle_J^{1/p}\langle |g|^{p'}\rangle_J^{1/p'}.$$

Proof. Without loss of generality, let $J = [0,1]$. Let F_n be the σ-algebra generated by all dyadic subintervals in J of length at least 2^{-n}. Consider $\omega \in [0,1]$ and put

$$Y_n(\omega) := \sum_{I\subset J, |I|\geq 2^{-n}} (f, h_I)h_I(\omega).$$

We fix any sequence of complex numbers $\sigma_I = e^{i\alpha_I}$, $\alpha_I \in \mathbb{R}$, and consider

$$X_n(\omega) := \sum_{I\subset J, |I|\geq 2^{-n}} \sigma_I (f, h_I) h_I(\omega).$$

Both (X_n, F_n, dx) and (Y_n, F_n, dx) are martingales. Clearly,

$$Y_n - Y_{n-1} = \sum_{I\subset J, |I|=2^{-n}} (f, h_I) h_I,$$

and

$$X_n - X_{n-1} := \sum_{I\subset J, |I|=2^{-n}} \sigma_I (f, h_I) h_I.$$

These martingales satisfy the assumptions of Burkholder's lemma. The Hilbert space $H = \mathbb{R}^2$ is identified naturally with $\mathbb{C}$. Now $\|f\|_{L^p} = \lim_{n\to\infty} \|Y_n\|_{L^p(H)}$, $\|T_\sigma f\|_{L^p} = \lim_{n\to\infty} \|X_n\|_{L^p(H)}$; Burkholder's lemma implies that

$$\|T_\sigma f\|_{L^p} \leq (p-1)\|f\|_{L^p} \tag{4.2}$$

for $p \geq 2$ and for any sequence σ as above.

We reformulate (4.2) as $|(T_\sigma f, g)| \leq (p-1)\|f\|_{L^p}\|g\|_{L^{p'}}$. Now the definition of T_σ implies

$$\frac{1}{|J|}\left|\sum_I \sigma_I (f, h_I)\overline{(g, h_I)}\right| \leq (p-1)\langle |f|^p\rangle_J^{1/p} \langle |g|^{p'}\rangle_J^{1/p'}, \quad \sigma_I = e^{i\alpha_I}. \tag{4.3}$$

Notice that $(f, h_I) = \frac{1}{2}\sqrt{|I|}(\langle f\rangle_{I_+} - \langle f\rangle_{I_-})$. Since we also may vary σ_I, the theorem follows. $\square$

Theorem 4.3. *In the domain* $G = \{(\Phi, \Psi, \phi, \psi) \in \mathbb{R}\times\mathbb{R}\times\mathbb{C}\times\mathbb{C} : |\phi|^p < \Phi, |\psi|^{p'} < \Psi\}$, *there exists a function* $B(\Phi, \Psi, \phi, \psi)$ *such that for any quadruples* $a = (\Phi, \Psi, \phi, \psi)$, $a_- = (\Phi_-, \Psi_-, \phi_-, \psi_-)$, *and* $a_+ = (\Phi_+, \Psi_+, \phi_+, \psi_+)$ *with* $a = \frac{a_- + a_+}{2}$ *we have*

$$B(a) - \frac{1}{2}(B(a_-) + B(a_+)) \geq \frac{1}{4}|\phi_- - \phi_+||\psi_- - \psi_+|.$$

Also,

$$0 \leq B(a) \leq (p-1)\Phi^{1/p}\Psi^{1/p'}$$

everywhere in G.

For every compact subset K *in* G *we can find an infinitely smooth function* B_K *on* K *such that the first estimate is fulfilled. Consider* $\phi = \xi_1 + i\eta_1, \psi = \xi_2 + i\eta_2$, *where the* ξ*'s and* η*'s are real. If we view* B_K *as a function of* 6 *real variables, then we can consider its* 6×6 *Jacobi matrix and the corresponding second differential form, i.e., the Hessian. Then the Hessian of* B_K *must satisfy the following inequality:*

$$-d^2 B_K \geq 2|d\phi||d\psi| = 2((d\xi_1)^2 + (d\eta_1)^2)^{1/2}((d\xi_2)^2 + (d\eta_2)^2)^{1/2}.$$

At the same time, for any positive ε, B_K *can be chosen to satisfy*

$$0 \leq B_K(a) \leq (1+\varepsilon)(p-1)\Phi^{1/p}\Psi^{1/p'}.$$

Proof. Fix $(\Phi, \Psi, \phi, \psi) \in G$ and consider all complex-valued functions f, g on J such that $\Phi = \langle |f|^p\rangle_J$, $\Psi = \langle |g|^{p'}\rangle_J$, $\phi = \langle f\rangle_J$, $\psi = \langle g\rangle_J$. Let

$$B(\Phi, \Psi, \phi, \psi) := \sup\Big\{\frac{1}{4}\frac{1}{|J|}\sum_{I\in\mathcal{D}, I\subset J} |\langle f\rangle_{I_+} - \langle f\rangle_{I_-}||\langle g\rangle_{I_+} - \langle g\rangle_{I_-}||I|\Big\},$$

where the supremum is taken over all f and g as above. This supremum does not depend on the interval J. This observation helps to prove the first inequality $B(a) - \frac{1}{2}(B(a_-) + B(a_+)) \geq \frac{1}{4}|\phi_- - \phi_+||\psi_- - \psi_+|$ exactly as this is done in any paper on Bellman functions.

On the other hand, the second inequality $0 \le B(a) \le (p-1)\Phi^{1/p}\Psi^{1/p'}$ has already been proved—this is the claim of Theorem 4.2.

If we fix a compact set K, we can also fix a very small ε (much smaller than the distance from K to the boundary of G), and we can consider $\frac{1}{\varepsilon^6}S(\frac{a}{\varepsilon})$, where S is a C_0^∞-function supported on the unit ball of $\mathbb{R}\times\mathbb{R}\times\mathbb{C}\times\mathbb{C}=\mathbb{R}^6$ centered at zero. It remains to mollify B by convolving it with $\frac{1}{\varepsilon^6}S(\frac{a}{\varepsilon})$. The concavity inequality will be satisfied with no change for the new function. The size inequality can become $(1+C_K\varepsilon)$ times worse. $\square$

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