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SEPARATION OF SINGULARITIES OF ANALYTIC FUNCTIONS WITH PRESERVATION OF BOUNDEDNESS

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Dedicated to Mikhail Shlemovich Birman on the occasion of his 75th birthday

Abstract. For which pairs (O_1, O_2) of open sets on the complex plane is it true that the operator

$$J : (f_1, f_2) \mapsto (f_1 + f_2)|(O_1 \cap O_2)$$

from $H^\infty(O_1) \times H^\infty(O_2)$ to $H^\infty(O_1 \cap O_2)$ is a surjection? In the first part of the paper, a method is indicated for constructing pairs without this property. In the second part, for some classes of pairs (O_1, O_2) a right inverse for J is constructed explicitly. The paper continues the previous studies of the author jointly with A. H. Nersessian and J. Ortega Cedrá.

Introduction

Let O be an open subset of the extended complex plane $\widehat{\mathbb{C}}$. The space of all functions holomorphic in O will be denoted by $\mathrm{Hol}(O)$, and the symbol $H^\infty(O)$ will denote the set of all *bounded* functions $f \in \mathrm{Hol}(O)$.

Let S_1 and S_2 be relatively closed subsets of O, and let $S := S_1 \cup S_2$. Suppose that an arbitrary function $f \in H^\infty(O \setminus S)$ coincides in $O \setminus S$ with $f_1 + f_2$, where $f_j \in H^\infty(O \setminus S_j)$, $j = 1, 2$. Then the pair (S_1, S_2) is said to *admit separation* in O.

Sometimes, when saying that a pair (S_1, S_2) of closed subsets of $\mathbb{C}$ (which are not necessarily subsets of O) *admits separation* in O, we mean that so does the pair $(S_1 \cap O, S_2 \cap O)$ in O.

Our aim in this paper is to look for *geometric* criteria of separation.

Alice Roth's well-known fusion lemma [4, 5, 6]) answers a question of this kind. The separation problem arises in connection with interpolation by bounded analytic functions in multidimensional domains [15, 14]. In the author's opinion, this problem is also interesting in itself.

Passing to the complements $G_j = O \setminus S_j$, $j = 1, 2$, $G = O \setminus S$, we can formulate the separation problem as follows: *describe the pairs (G_1, G_2) of open sets ($G_j \subset \widehat{\mathbb{C}}$) such that an arbitrary function $f \in H^\infty(G)$, $G := G_1 \cap G_2$, decomposes in G in the sum $f_1 + f_2$ with $f_j \in H^\infty(G_j)$.* In 1983, P. L. Polyakov [15] considered this version of the problem in connection with interpolation questions mentioned above. For $G_1 = \{|z| < 1, \mathrm{Im}\, z > 0\}$, $G_2 = \{|z| < 1, \mathrm{Re}\, z > 0\}$ he proved that every $f \in H^\infty(G)$ coincides in G with $f_1 + f_2$, where f_j is analytic in G_j and bounded in $G_j \cap \{|z| < 1/2\}$, $j = 1, 2$.

Systematically, the separation problem was treated in the papers [10, 11]; the present paper is a continuation of them.

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We preface the survey of the results of [10, 11] by the remark that if we discard *boundedness* and only retain analyticity in the definition of pairs that admit separation, then we arrive at a much simpler question, which was answered long ago. Poincaré [13] was the first to study it in the course of a discussion of analytic continuation with Borel (see [16, Chapter 3, §21] about that). With the help of quite an elegant construction, Poincaré showed that every $f \in \mathrm{Hol}(\mathbb{C} \setminus \mathbb{R})$ is the sum of the restrictions to $\mathbb{C} \setminus \mathbb{R}$ of two functions analytic in $\mathbb{C} \setminus [-1, 1]$ and $\mathbb{C} \setminus ((-\infty, -1] \cup [1, +\infty))$, respectively (we note that the pair (S_1, S_2), where $S_1 = [-1, 1]$, $S_2 = \mathbb{R} \setminus S_1$, does not admit separation in $\mathbb{C}$; see [10]). Later, separation of singularities for (arbitrary) analytic functions was studied by Fréchet [3]. The problem was solved completely by Aronszajn [1]. His result reads as follows: *for every pair of open sets G_1, $G_2 \subset \widehat{\mathbb{C}}$*, every function $f \in \mathrm{Hol}(G_1 \cap G_2)$ *is representable as the sum of two functions analytic in G_1 and G_2, respectively.* The papers [13, 3, 1] have sunk into oblivion by now (which is unjust to a certain extent), and the Aronszajn theorem occurs without reference in the monographs [2, pp. 225] and [18] as an illustration (of little interest) to sheaf $\overline{\partial}$-methods. The papers [8, 9, 12, 7] were devoted to various aspects of separation of singularities in the spirit of [13, 3, 1].

Returning to our topic, we can say that in the present article a quantitative version of the Aronszajn theorem is discussed (as was also in [10, 11]): assuming that a function $f \in \mathrm{Hol}(O \setminus S)$ is bounded, we want to find conditions of geometric nature ensuring that always (i.e., for an *arbitrary* bounded f) the summands f_1 and f_2 in the Aronszajn decomposition can be chosen bounded in $O \setminus S_1$ and $O \setminus S_2$, respectively.

In a sense, the separation problem is *local.* Let $O \subset \mathbb{C}$ be a bounded open set, and let K_1 and K_2 be closed in $\widehat{\mathbb{C}}$. Putting $k := K_1 \cap K_2$ and taking an arbitrarily small neighborhood v of k, we define $k_j = \mathrm{Clos}(v \cap K_j)$, $j = 1, 2$. Then the following localization theorem can be proved (see [10, p. 156]) with the help of the Vitushkin operator [19, 4], which is well known in approximation theory: *if the pair (k_1, k_2) admits separation in O, then the pair (K_1, K_2) also admits separation in O.* Thus, only the "germs" of K_1 and K_2 near k are responsible for the property of the pair (K_1, K_2) to admit separation. As in [10, 11], in the present paper principal attention is paid to the case of a *finite* set k, which reduces easily to $k = \{0\}$.

We observe that if K_1 and K_2 are compact in $\mathbb{C}$ and $K_1 \cap K_2 = \varnothing$, then the Cauchy integral formula shows immediately that (K_1, K_2) admits separation in $\mathbb{C}$. The more general case where $K_1 \setminus k$ and $K_2 \setminus k$ are at a positive distance from each other is less obvious. However, (K_1, K_2) admits separation in $\mathbb{C}$ even under this condition (this is a version of the A. Roth lemma mentioned above; see [10, p. 157]).

The property of (K_1, K_2) to admit separation in O is conformally invariant in the sense that the pair $(\varphi(K_1), \varphi(K_2))$ still admits separation in $\varphi(O)$ for every conformal homeomorphism φ of O. This observation justifies the interest to the separation property for certain "model" triples (K_1, K_2, O). Below, special attention will be paid to separation for pairs (K_1, K_2) in the upper half-plane $\mathbb{C}_+ := \{\mathrm{Im}\, z > 0\}$.

As has already been mentioned, mainly we shall be busy with sets K_1 and K_2 that "meet" at a unique point ($k = \{0\}$). In [10] it was shown that if these sets meet *transversally*, then the pair (K_1, K_2) *nearly* admits separation in $\mathbb{C}$. We give the precise statement.

A compact set $\varkappa \subset \mathbb{C}$ is said to be *regular* if

(a) $\varkappa \subset \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_N$, where the Γ_j are simple compact rectifiable arcs such that the sets $\Gamma_j \setminus \{0\}$ are mutually disjoint; and

(b) $s(\varkappa \cap r\mathbb{D}) = O(r)$ as $r \to 0$, where $\mathbb{D}$ is the open unit disk and s stands for length.

In order to state the result about near separation for sets meeting transversally (see [10, Theorem 4.2]), we consider the ray $\mathcal{L} := \{p + te^{i\psi} : t \geq 0\}$ originating at a point

$p \in \mathbb{C}$. We denote by $A(\mathcal{L}, \delta)$ the angle with vertex at p, with bisector $\mathcal{L}$, and of opening 2δ. The ray $\mathcal{L}$ is said to be *tangent to E at p* if for every $\varepsilon > 0$ there exists $\sigma > 0$ such that $E \cap \mathbb{D}(p, \sigma) \subset A(\mathcal{L}, \varepsilon)$ (here $\mathbb{D}(p, \sigma) := p + \sigma\mathbb{D}$).

Now, consider a family $\Gamma = \{\gamma_1, \dots, \gamma_N\}$ of compact sets such that $0 \in \gamma_j$, $j = 1, \dots, N$, and the differences $\gamma_j \setminus \{0\}$ are mutually disjoint. Suppose γ_j has a tangent ray $\mathcal{L}_j$ at the origin and $\mathcal{L}_i \neq \mathcal{L}_j$ for $i \neq j$ (this means that γ_i and γ_j meet *transversally* at the origin), and the germs of the sets γ_j at the origin are *regular*, i.e., all the sets $\gamma_j \cap \{|z| \leq \Delta\}$ are regular for some $\Delta > 0$.

For instance, these conditions are satisfied by any family Γ of simple smooth arcs emanating from the origin and forming nonzero angles with one another. By abuse of language (an arc to be confused with its equation), we may say that the γ_j are one-to-one complex C^1-functions such that $\gamma_j(0) = 0$, γ_j' does not vanish, and the numbers $\gamma_j'(0)/\gamma_i'(0)$ are not positive reals for $i \neq j$ (transversality).

Returning from *arcs* to the general setting, consider a compact circular sector Σ with vertex at the origin and such that

$$\Sigma \cap (\gamma_j \setminus \{0\}) = \varnothing, \quad j = 1, \dots, N.$$

Theorem 0.1. *For every $f \in H^\infty(\mathbb{C} \setminus (\gamma_1 \cup \cdots \cup \gamma_N))$ there exist functions $f_j \in H^\infty(\mathbb{C} \setminus (\gamma_j \cup \Sigma))$, $j = 1, \dots, N$, such that*

$$(*) \qquad f = f_1 + f_2 + \cdots + f_N \quad \text{in } \mathbb{C} \setminus (\gamma_1 \cup \cdots \cup \gamma_N \cup \Sigma).$$

Formula $(*)$ "nearly separates" the singularities of f with preservation of boundedness. This is done at the expense of a slight expansion (arbitrarily small) of the singular set for f_j compared to the desired outcome: instead of γ_j, the summand f_j has singularities in $\gamma_j \cup \Sigma$, where Σ is a small circular sector having no points in common with $\gamma_1 \cup \cdots \cup \gamma_N$ in $\mathbb{C} \setminus \{0\}$. Simple examples show that, in general, it is impossible to discard the sector Σ and make the f_j to be elements of $H^\infty(\mathbb{C} \setminus \gamma_j)$. However, in some *domains* formula $(*)$ does yield *complete* separation of singularities (with preservation of boundedness). For instance, with the help of $(*)$ it is easy to see that the operator $(f_1, f_2) \mapsto (f_1 + f_2)|G$ is surjective if, e.g., G_1 and G_2 are two disks. Surely, the same is true for a much wider class of pairs of domains: the only requirement is that the "crescent" G be formed with *transversal* intersection of the boundaries (see [10, p. 165]; see also Figure 4 in Subsection 3.4.3 at the end of §I.4 of the present paper).

It should be noted that some smoothness conditions on the sets γ_j in Theorem 0.1 make it possible to replace the sector Σ by a smaller set (e.g., by an arbitrarily short segment emanating from the origin and not intersecting $\gamma_j \setminus \{0\}$ for $j = 1, \dots, N$; see [10, pp. 166–169]).

The transversality condition is essential in Theorem 0.1. Suppose two compact and smooth simple arcs γ_1, γ_2 lie in a domain O except for their common end p that belongs to ∂O; next, suppose $\gamma_1 \cap \gamma_2 \cap O = \varnothing$ and the arcs γ_1 and γ_2 have a common tangent at p. Separation for such pairs is the principal topic of the present paper (in distinction to the paper [10] devoted entirely to "transversal" pairs); tangent pairs were also treated (with different tools) in the paper [11], about which I shall report later in more detail.

In particular, we shall see that, for arcs γ_1 and γ_2 as above, separation depends on the relationship between the velocity of their mutual approach near the tangency point p and the velocity of their approach to ∂O near p.

In the first part of the paper, a general method will be developed for constructing pairs that do not admit separation in a given domain. In particular, we shall describe some pairs of *arcs* that meet tangentially at a boundary point and do not admit separation.

For a pair (S_1, S_2) in G, separation may be "good" or "bad". In §I.1 we introduce the quantity $b(S_1, S_2, O)$, which indicates the "quality of separation". This quantity (called the "separation constant of (S_1, S_2) in O") becomes smaller when separation improves; by definition, the identity $b(S_1, S_2, O) = +\infty$ means that the pair (S_1, S_2) fails to admit separation in O.

The main result of the first part is Theorem 1 in §I.3. It provides a lower estimate for the separation constant $b(K_1, K_2, O)$ of disjoint continua in O in terms of metric characteristics of their size and closeness. In §§I.1 and I.2 we prepare the statement and the proof of Theorem 1.

On the basis of Theorem 1, some pairs of sets not admitting separation are described in Theorem 2. The latter theorem is illustrated by specific examples of nonseparability in the upper half-plane $\mathbb{C}_+$ realized by pairs of arcs with common tangent at the origin (Theorems 3 and 4). In the case of $C^{1+\varepsilon}$-arcs, the conditions obtained in Theorems 3 and 4 and sufficient for the absence of separation differ by a certain logarithmic factor from the necessary and sufficient conditions obtained in [11]. From [11] it follows that $\log y$ can be removed from formula (32) of the present paper and that the function $l(x)$ in (35) may in fact be an arbitrary infinitesimal as $x \to 0$ (without the condition $l(x) = o(1/|\log \varphi_1(x)|)$. These (extraneous) logarithmic factors are the payment for generality: Theorems 2 and 3 apply not only to pairs of smooth arcs, in distinction to [11].

In the second part of the paper it is shown that certain pairs of arcs with common tangent ray $[0, +\infty)$ at the origin admit separation. The main result of the second part is Theorem 5, which gives a fairly simple sufficient condition of separation for such pairs in $\mathbb{C}_+$. This condition coincides with the necessary one (found in [11] by different techniques—see the discussion in Subsection 2.4 of the first part) and has a simple geometric meaning: the hyperbolic width of the "corridor" whose "walls" are formed by the arcs in question must be bounded away from zero. The second part ends with some specific examples, of which we mention only Example 4 related to the Poincaré pair (S_+, S_-) $(S_+ = [0, +\infty], S_-(-[\infty, 0])$; see the beginning of the Introduction). This pair fails to admit separation in $\mathbb{C}$, but Theorems 5 and $5'$ imply that every function $f \in H^\infty(\mathbb{C}_+)$ coincides in $\mathbb{C}_+$ with $f_+ + f_-$, where $f_\pm \in H^\infty(\mathbb{C} \setminus S_\pm)$.

This paper is intimately related to [10, 11]. In some respects, its results are cruder then those in [11], but otherwise they are finer. The "negative" results of the first part (about the failure of separation), i.e., Theorems 2 and 3, concern highly more general sets than the corresponding "negative" results in [11], which are applicable only to arcs in $\mathbb{C}_+$ (moreover, unlike the present paper, the arcs must be of class $C^{1+\varepsilon}$). The stronger assumptions in [11] make it possible to get rid of the logarithms in (32) and (35); however, the estimates in Theorem 1 may turn out to be sharp in the class of proper continua considered here.

As for "positive" results (i.e., conditions ensuring separation in $\mathbb{C}_+$), the progress achieved in [11] is fundamental. Both for transversal and for tangent intersection, the separation criteria obtained in [11] are applicable to much more general classes of sets than in [10] or in the present paper. The success is due to the invocation of B. Berndtson's deep theorems about bounded solutions of the $\overline{\partial}$-problem in $\mathbb{C}_+$. However, the mere fact that a pair (S_1, S_2) admits separation does not finish the story in the problem in question: also of interest is a linear operator $f \mapsto (f_1, f_2)$ $(f \in H^\infty(O \setminus S), f_j \in H^\infty(O \setminus S_j))$ that realizes separation of singularities. A natural, explicit , and very simple construction of such an operator was presented in [10] (informally, it will be discussed in Subsection 1.3 of §II.1), and this construction does not involve the $\overline{\partial}$-problem. Invented in the "transversal" paper [10], this construction is applicable also in the "tangent" setting of Theorem 5.

The applicability of the simple splitting operator in the paper [10] to separation of *tangent pairs* is a new result, which cannot be found in [11]. Besides simplicity and elementary character, our operator has yet another merit: as was shown in [20], it is suitable for separation of singularities with preservation of continuity up to the boundary.

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I. NEGATIVE RESULTS

The main results of this part are Theorems 1 and 2 in §I.3. They are accompanied by several examples (in the same section). The first two sections are devoted to technical preparations to the proof of Theorem 1.

§I.1. Separation of a pair of sets and uniform approximation by analytic functions with prescribed singularities

In this section, by O we denote an open set in $\mathbb{C}$, and by S_1 and S_2 its relatively closed subsets; we put

$$S := S_1 \cup S_2, \quad s := S_1 \cap S_2.$$

1.1. The separation constant of the pair (S_1, S_2) (definition)**.**

Lemma 1. *If (S_1, S_2) admits separation in O, then there exists a nonnegative constant $c = c(S_1, S_2, O)$ such that every function $f \in H^\infty(O \setminus S)$ can be represented on $O \setminus S$ in the form*

$$f = f_1 + f_2, \;\; f_j \in H^\infty(O \setminus S_j), \quad \text{where } \|f_j\|_{\infty, O\setminus S_j} \le c\|f\|_{\infty, O\setminus S}, \; j = 1, 2. \tag{1}$$

Proof. By assumption, the linear operator $(f_1, f_2) \mapsto (f_1 + f_2)|(O \setminus S)$ is a surjection of $H^\infty(O \setminus S_1) \oplus H^\infty(O \setminus S_2)$ onto $H^\infty(O \setminus S)$. By the Banach theorem, the claim follows. □

The Montel theorem shows that the infimum of the constants c occurring in (1) is their minimum. This minimum will be called the *separation constant* of the pair (S_1, S_2) in O, and it will be denoted by $b(S_1, S_2, O)$. If (S_1, S_2) does not admit separation in O, we put $b(S_1, S_2, O) = +\infty$ by definition.

Our principal aim in §§1 and 2 is to estimate the separation constant from below in terms of metric characteristics responsible for the mutual closeness of the sets that form the pair in question.

1.2. Some lower bounds for separation constants. We associate with a function $\varphi \in \mathrm{Hol}(O \setminus S_1)$, its best uniform approximation by elements of $\mathrm{Hol}(O \setminus S_2)$, i.e., we consider the quantity

$$d(\varphi, O, S_2) := \inf \|\varphi - h\|_{\infty, O\setminus S},$$

where the infimum is taken over all $h \in \mathrm{Hol}(O \setminus S_2)$. Let $d(\varphi, O)$ denote $d(\varphi, O, \varnothing)$. The following lemma yields an estimate for the separation constant of the pair (S_1, S_2) in terms of the best approximations by functions with prescribed singularities.

Lemma 2. *For every $\Phi \in \mathrm{Hol}(O \setminus S_1)$, we have*

$$d(\Phi, O, s) \le b(S_1, S_2, O) d(\Phi, O, S_2).$$

Proof. If $\delta > d(\Phi, O, S_2)$, then there exists $h \in \mathrm{Hol}(O \setminus S_2)$ such that $\|\Phi - h\|_{\infty, O\setminus S} < \delta$. Then $\Phi - h = \varphi_1 + \varphi_2$ in $O \setminus S$, where $\varphi_j \in H^\infty(O \setminus S_j)$, $\|\varphi_j\|_{\infty, O\setminus S_j} \le b(S_1, S_2, O)\delta$, $j = 1, 2$. But

$$\varphi_1 - \Phi = -h - \varphi_2 \quad \text{in } O \setminus S,$$

and we can define a function $H \in \mathrm{Hol}(O \setminus s)$ by

$$\begin{aligned} H(\zeta) &= \varphi_1(\zeta) - \Phi(\zeta) &&\text{for } \zeta \in O \setminus S_1, \\ H(\zeta) &= -h(\zeta) - \varphi_2(\zeta) &&\text{for } \zeta \in O \setminus S_2, \end{aligned}$$

so that

$$d(\Phi, O, s) \le \|\Phi + H\|_{\infty, O\setminus S_1} = \|\varphi_1\|_{\infty, O\setminus S_1} \le b(S_1, S_2, O)\delta. \qquad \square$$

1.3. A lower estimate of $d(\varphi, g)$ for a cell g. By a *cell* we mean a bounded Jordan domain g with rectifiable boundary ∂g. Let $|\partial g|$ denote the length of the boundary. For $\zeta \in g$, the quantity

$$\rho_g(\zeta) := \frac{2\pi \operatorname{dist}(\zeta, \partial g)}{|\partial g|}$$

will be called the *rotundity* of g relative to ζ (often, we call ζ a *center* of g). It is easily seen that $\rho_g(\zeta) \le 1$ (this pictorially obvious inequality is a consequence of the identity $1 = (2\pi i)^{-1} \int_{\partial g} \frac{dz}{z-\zeta}$, or of the isoperimetric inequality). The rotundity of a cell is equal to 1 if and only if g is a disk and ζ is its usual center.

In what follows, we shall deal with infinite families of cells g with marked centers and with rotundities uniformly bounded away from zero. In the simplest case, these cells will be disks with usual centers (so that $\rho_g(\zeta) \equiv 1$), but sometimes it will also be convenient to consider rectangles g with ratios of the side lengths uniformly bounded and bounded away from zero (again, with usual centers).

For some functions $\varphi \in \mathrm{Hol}(g \setminus K)$, where $K \subset g$ is a compact set, the best approximation $d(\varphi, g)$ by functions analytic in g admits a lower estimate in terms of $|\varphi(\zeta)|$ and $\rho_g(\zeta)$.

Lemma 3. *Suppose $\varphi \in H^\infty(\widehat{\mathbb{C}} \setminus K)$, $\varphi(\infty) = 0$, and $\zeta \in g \setminus K$. Then*

$$\rho_g(\zeta)|\varphi(\zeta)| \le 2d(\varphi, g). \tag{2}$$

Proof. If $\delta > d(\varphi, g)$, then there exists a function $h \in \mathrm{Hol}(g)$ with $\|\varphi - h\|_{\infty, g\setminus K} < \delta$. In particular, $h \in H^\infty(g)$, whence it follows that h is representable by the Cauchy formula (see [17]; $h(z)$ denotes the nontangential boundary value of h at the point $z \in \partial g$):

$$h(\zeta) = \frac{1}{2\pi i} \int_{\partial g} \frac{h(z)\, dz}{z - \zeta} = \frac{1}{2\pi i} \int_{\partial g} \frac{h(z) - \varphi(z)}{z - \zeta}\, dz, \quad \zeta \in g$$

(the second identity follows from the fact that $\varphi(\infty) = 0$). Therefore, $|h(\zeta)| \le (\rho_g(\zeta))^{-1}\delta$, and if $\zeta \in g \setminus K$, then

$$|\varphi(\zeta)| \le |h(\zeta)| + |\varphi(\zeta) - h(\zeta)| \le (\rho_g(\zeta))^{-1}\delta + \delta,$$

The claim follows from the estimate $\rho_g(\zeta) \le 1$. $\square$

Corollary. *If K, g, and φ satisfy the assumptions of Lemma* 3, *and $A \in \partial K$, then*

$$\rho_g(A)\overline{\lim}_A|\varphi| \le 2d(\varphi, g). \tag{3}$$

1.4. Separation constant and interference between large functions. Now, we turn to lower estimates of the separation constant of a pair (K_1, K_2), where the sets K_1, $K_2 \subset O$ are *compact* and disjoint.

Lemma 4. *Let g be a cell included in O, and let $K_1 \subset g$, $K_2 \subset O$. Suppose K_1 and K_2 are compact, $K_1 \cap K_2 = \varnothing$, and $A \in \partial K_1$. Put $K = K_1 \cup K_2$. Then for every function $\psi_1 \in \mathrm{Hol}(\widehat{\mathbb{C}} \setminus K_1)$ vanishing at infinity and every function $\psi_2 \in \mathrm{Hol}(O \setminus K_2)$ with $\psi_1 - \psi_2 \in H^\infty(O \setminus K)$ we have*

$$2\|\psi_1 - \psi_2\|_{\infty, O\setminus K} b(K_1, K_2, O) \geq \rho_g(A) \overline{\lim}_A |\psi_1|. \tag{4}$$

Proof. Clearly, $\psi_1 \in H^\infty(\widehat{\mathbb{C}} \setminus K_1)$ (the boundedness of ψ_1 follows from that of $\psi_1 - \psi_2$ and ψ_2 near K_1 and from the maximum principle). Comparing (3) (with $\varphi = \psi_1$) and Lemma 2, we obtain

$$\rho_g(A) \overline{\lim}_A |\psi_1| \leq 2d(\psi_1, g) \leq 2d(\psi_1, O) \leq 2b(K_1, K_2, O) d(\psi_1, O, K_2).$$

But $d(\psi_1, O, K_2) \leq \|\psi_1 - \psi_2\|_{\infty, O\setminus K}$. □

By Lemma 4, in order to show that the quantity $b(K_1, K_2, O)$ is large, it suffices to construct a pair (ψ_1, ψ_2) of functions such that $\psi_1 \in \mathrm{Hol}(\widehat{\mathbb{C}} \setminus K_1)$, $\psi_1(\infty) = 0$, and ψ_1 is very large near a point $A \in \partial K_1$, whereas $\psi_2 \in \mathrm{Hol}(O \setminus K_2)$ and the difference $\psi_1 - \psi_2$ is uniformly small in $O \setminus K$ by the reason of *interference* between ψ_1 and ψ_2. If this is done, a satisfactory lower estimate for $b(K_1, K_2, O)$ follows from (4) provided that the rotundity (relative to A) of a cell $g \subset O$ that includes K_1 is not too small.

1.5. A lower estimate of the separation constant of a pair of sets in terms of the separation constant of a pair of their compact subsets. We return to the sets S_1, S_2, and O (see the beginning of this section).

Lemma 5. *Suppose $S_1 \cap S_2 = \varnothing$, and let $K_j \subset S_j$ be compact sets, $j = 1, 2$. Then*

$$1 + b(S_1, S_2, O) \geq b(K_1, K_2, O);$$

if the S_j have empty interior, then

$$b(S_1, S_2, O) \geq b(K_1, K_2, O).$$

Proof. We put $K = K_1 \cup K_2$, $S = S_1 \cup S_2$. If $f \in H^\infty(O \setminus K)$, then $f = f_1 + f_2$ in $O \setminus S$, $f_j \in H^\infty(O \setminus S_j)$, $\|f_j\|_{\infty, O\setminus S_j} \leq b(S_1, S_2, O)\|f\|_{\infty, O\setminus S}$ (we assume that the pair (S_1, S_2) admits separation in O), $j = 1, 2$. The function f_j on $O \setminus S_j$ coincides with a function analytic in $O \setminus K_j$. Indeed, the functions f_1 and $f - f_2$ for instance, which are analytic in $O \setminus S$, fuse to yield a function analytic in $O \setminus K_1$. If S_1 is nowhere dense in O, then $\|f_1\|_{\infty, O\setminus K_1} = \|f_1\|_{\infty, O\setminus S_1} \leq b(S_1, S_2, O)\|f\|_{\infty, O\setminus K}$. In the general case, $|f_1(\zeta)| \leq b(S_1, S_2, O)\|f\|_{\infty, O\setminus K}$ for $\zeta \in O \setminus S_1$, and $|f_1(\zeta)| \leq |f(\zeta)| + |f_2(\zeta)| \leq (1 + b(S_1, S_2, O))\|f\|_{\infty, O\setminus K}$ for $\zeta \in S_1 \setminus K_1$. □

Lemma 5 suggests a method that allows us to decide whether a given disjoint pair (S_1, S_2) admits separation in O. For this, it suffices to be able to construct pairs (K_1, K_2) of compact subsets in S_1 and S_2 (respectively) with arbitrarily large $b(K_1, K_2, O)$. In its turn, to do this we need pairs (ψ_1, ψ_2) of mutually interfering functions as in Lemma 4. In §I.2 we shall show that such functions exist if K_1 and K_2 are sufficiently close to each other and are "proper"; additionally, it is required that there exist a sufficiently rotund cell $g \subset O$ including one of these sets.

§I.2. Proper continua; the functions ψ_q

The results of this section will be applied to very simple sets S_1 and S_2 (smooth arcs). However, our approach works for much more general pairs (S_1, S_2). To better understand the essence, here we proceed under less restrictive assumptions than in the final §I.3.

2.1. Some logarithmic functions. In this section, K will always denote a compact *connected* subset of $\mathbb{C}$ (a bounded *continuum*) with *connected* complement $\widehat{\mathbb{C}} \setminus K$.

The symbol log will denote the principal branch of the logarithm: the function log is defined in $\mathbb{C} \setminus (-\infty, 0]$ and

$$e^{\operatorname{Re} \log \zeta} = |\zeta|, \quad \operatorname{Im} \log \zeta \in (-\pi, \pi] \quad (\zeta \in \mathbb{C} \setminus (-\infty, 0]).$$

The domain $\widehat{\mathbb{C}} \setminus K$ is simply connected. Therefore, for every A, $B \in K$ with $A \neq B$ there exists a function $\mathcal{L}_{K,A,B} \in \operatorname{Hol}(\widehat{\mathbb{C}} \setminus K)$ such that

$$\exp \mathcal{L}_{K,A,B}(\zeta) = \frac{B - \zeta}{A - \zeta}, \quad \zeta \in \mathbb{C} \setminus K, \ \ \mathcal{L}_{K,A,B}(\infty) = 0.$$

Clearly,

$$\mathcal{L}_{K,A,B}(\zeta) = \log \frac{B - \zeta}{A - \zeta}$$

for all sufficiently large $|\zeta|$.

2.2. Proper continua. We formulate two condition to be imposed on the continuum K.

Condition 1. *There is a number $T > 1$ such that for every A, $B \in K$ with $A \neq B$ we have*

$$\mathcal{L}_{K,A,B}(\zeta) = \log \frac{B - \zeta}{A - \zeta} \quad \text{if } |\zeta - A| > T|B - A| \text{ and } \zeta \notin K. \tag{5}$$

The inequality $|\zeta - A| > |B - A|$ implies the estimate $|(\zeta - A)/(\zeta - B) - 1| < 1$, so that the right-hand side of (5) makes sense.

Condition 2. *The function $\operatorname{Im} \mathcal{L}_{K,A,B}$ is bounded in $\mathbb{C} \setminus K$ uniformly in all pairs A, $B \in K$ with $A \neq B$.*

In other words, in Condition 2 we require the existence of a constant T such that

$$|\operatorname{Im} \mathcal{L}_{K,A,B}(\zeta)| \leq T$$

for every $\zeta \in \mathbb{C} \setminus K$ and every A, $B \in K$, $A \neq B$.

Definition. If a continuum K satisfies Conditions 1 and 2 (with one and the same constant T in both cases), then K is called *T-proper* (or *proper* if the value of T is immaterial).

2.3. Lipschitz graphs are proper.

Lemma 6. *Let K be a Lipschitz graph (relative to some orthogonal basis in $\mathbb{R}^2$). Then K is a proper continuum.*

Proof. There is no loss of generality in assuming that

$$K = \{t + if(t) : \alpha \leq t \leq \beta\},$$

where f is a real function satisfying $|f(t') - f(t'')| \leq L|t' - t''|$ for some $L > 0$ and arbitrary t', $t'' \in [\alpha, \beta]$. We shall assume that $L > 1$.

To verify Condition 1 in Subsection 2.2 for K, we put $A = a + if(a)$, $B = b + if(b)$, $\alpha \le a < b \le \beta$, $l = |B - A|$ and consider the rectangle

$$\Pi = [a - l, a + l] \times [f(a) - Ll, f(a) + Ll],$$

which contains the graph of $f|[a-l, a+l]$ so that $K \setminus \Pi$ is the union of at most two arcs of K (of the graphs of $f|[\alpha, a-l)$ and $f|(a+l, \beta]$). Clearly, the set $\widehat{\mathbb{C}} \setminus (\Pi \cap K)$ is connected. The function $z \mapsto \log \frac{B-z}{A-z}$ is analytic in $\widehat{\mathbb{C}} \setminus \Pi$ because $\{|z - A| \le l\} \subset \Pi$. Next, this function coincides with $\mathcal{L}_{K,A,B}$ near infinity and hence everywhere in $\widehat{C} \setminus (\Pi \cup K)$. We can put $T = \sqrt{1 + L^2}$, because $\Pi \subset \{|z - A| \le Tl\}$.

We turn to Condition 2. Note that

$$\mathcal{L}_{K,A,B}(\zeta) = \int_{K_{A,B}} \frac{dz}{z - \zeta}, \quad \zeta \in \widehat{\mathbb{C}} \setminus K,$$

where $K_{A,B}$ is the graph of $f|[a, b]$. Indeed, the right-hand side J of this identity vanishes at infinity and $\exp J = (z - B)/(z - A)$ everywhere in $\mathbb{C} \setminus K_{A,B}$, which can easily be verified by differentiation. Consequently,

$$\operatorname{Im} \mathcal{L}_{K,A,B}(\zeta) = \int_a^b \frac{f'(t)(t - x) + f(t) - y}{(t - x)^2 + (f(t) - y)^2}\, dt, \quad \zeta = x + iy \in \mathbb{C} \setminus K.$$

For $t \ne x$ the integrand coincides with $(\arctan((f(t) - y)/(t - x)))'$, whence

$$|\operatorname{Im} \mathcal{L}_{K,A,B}(\zeta)| \le 2\pi \quad \text{for every } A, B \in K \text{ and } \zeta \notin K. \qquad \square$$

2.4. The functions ψ_q (definition). The functions ψ mentioned at the end of §I.1 will be labeled by quintuplets

$$q = (K, A, B, C, D),$$

where K is the continuum in question, and A, B, C, D are pairwise different points in K. We shall need the following linear functions $\lambda_{A,B}$:

$$\lambda_{A,B} := \frac{z - A}{B - A}, \quad A, B \in \mathbb{C}, \ \ A \ne B.$$

Now, we introduce functions $\psi_q \in \operatorname{Hol}(\widehat{\mathbb{C}} \setminus K)$ by the formula

$$\psi_q := \lambda_{A,B}\mathcal{L}_{K,A,B} + \mathcal{L}_{K,B,C} + \lambda_{D,C}\mathcal{L}_{K,D,C}, \tag{6}$$

$\psi_q(\infty) = 0$. Next, we put

$$\begin{aligned} \psi_q^* &:= \lambda_{A,B} \operatorname{Re} \mathcal{L}_{K,A,B} + \operatorname{Re} \mathcal{L}_{K,B,C} + \lambda_{D,C} \operatorname{Re} \mathcal{L}_{K,D,C}, \\ \psi_q^{**} &:= \psi_q - \psi_q^* = i(\lambda_{A,B} \operatorname{Im} \mathcal{L}_{K,A,B} + \operatorname{Im} \mathcal{L}_{K,B,C} + \lambda_{D,C} \operatorname{Im} \mathcal{L}_{K,D,C}). \end{aligned} \tag{7}$$

The functions (7) are defined (but not analytic) in $\widehat{\mathbb{C}} \setminus K$ (ψ_q^* makes sense even in $\widehat{C} \setminus \{A, B, C, D\}$).

Our choice of ψ_q is dictated by the following considerations. In the simplest case where K is a segment in $\mathbb{R}$ and A, B, C, D are points of K ($A < B < C < D$), the function ψ_q is representable by a Cauchy-type integral:

$$\psi_q(\zeta) = \int_A^D \frac{f(x)\, dx}{x - \zeta}, \quad \zeta \in \mathbb{C} \setminus K,$$

where f is the piecewise-linear function depicted in Figure 1.

It is easily seen that $\psi_q \in H^\infty(\widehat{\mathbb{C}} \setminus K)$ because f is Lipschitzian and $f(A) = f(D) = 0$. Moreover, $\psi_q(\infty) = 0$ and the absolute value $|\psi_q(A)|$ grows unboundedly as $B \to A$ (and C, D remain fixed). Thus, if $B \approx A$, then ψ_q possesses the properties of the function ψ_1 (the latter was discussed at the end of §I.1) for $K_1 = K$. Putting $K_2 = K + i\varepsilon$ with small $\varepsilon > 0$, and then $q' = (K_2, A + i\varepsilon, B + i\varepsilon, C + i\varepsilon, D + i\varepsilon)$, we may expect that $\psi_{q'}$

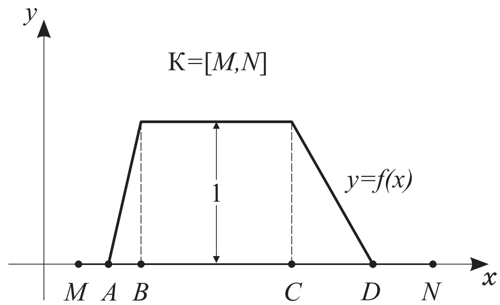

FIGURE 1.

will play the part of the "quenching" function ψ_2 that compensates for the growth of ψ_1 near A, so that $|\psi_1 - \psi_2|$ becomes uniformly bounded in $\widehat{\mathbb{C}} \setminus (K_1 \cup K_2)$.

The function ψ_q can be defined by a Cauchy-type integral with a trapezoid-like density f not only for a *segment* K but also for every rectifiable arc. This approach was chosen in [11], and it has some advantages. However, here we shall act in accordance with the definition (6), which does not involve integrals and, therefore, is applicable to arbitrary proper continua K rather than rectifiable arcs only. It should be noted that some additional assumptions about the smoothness of these arcs were required in [11].

The functions ψ_q^* are more convenient to work with than ψ_q: the logarithms involved in ψ_q^* obey the rule $\log XY = \log X + \log Y$, which allows us to express ψ_q^* in a very simple way in terms of the following function l:

$$l(\zeta) := \zeta \log|\zeta| \quad (\zeta \in \mathbb{C} \setminus \{0\}), \;\; l(0) := 0. \tag{8}$$

Namely,

$$\psi_q^*(\zeta) = \frac{l(\zeta - A)}{A - B} + \frac{l(\zeta - B)}{B - A} + \frac{l(\zeta - C)}{D - C} + \frac{l(\zeta - D)}{C - D}, \quad \zeta \in \mathbb{C} \setminus K. \tag{9}$$

2.5. Preliminary estimates of $|\psi_q|$, $|\psi_q^{**}|$. For a quintuplet $q = (K, A, B, C, D)$ and a number $T > 1$, put

$$\mathcal{D}_q := \{|z - A| > T|B - A|\} \cap \{|z - B| > T|C - B|\} \cap \{|z - C| > T|D - C|\}. \tag{10}$$

Lemma 7. *If the continuum K is T-proper, then*

(i) *the function $|\psi_q|$ is bounded in $\mathcal{D}_q \setminus K$ by a constant depending only on T;*

(ii) *the function $|\psi_q^{**}|$ is bounded in $\widehat{\mathbb{C}} \setminus K$ by a constant depending only on T.*

Proof. (i) By (5), we have

$$\mathcal{L}_{K,B,C}(\zeta) = \log\left(1 + \frac{C - B}{B - \zeta}\right)$$

if $\zeta \in \mathcal{D}_q \setminus K$ because $|\zeta - B| > T|B - C|$, so that the absolute value of the second summand in (6) at ζ does not exceed $\max\{|\log w| : |w - 1| \le 1/T\}$. If $\zeta \in \mathcal{D}_q \setminus K$, then $|\zeta - A| > T|B - A|$, and by (5) we have

$$\begin{aligned} \big|\lambda_{A,B}(\zeta)\mathcal{L}_{K,A,B}(\zeta)\big| &= \left|\frac{\zeta - A}{B - A}\log\left(1 + \frac{A - B}{\zeta - A}\right)\right| \\ &\le \max\{|w\log(1 + w^{-1})| : |w| \ge T\} =: c(T). \end{aligned} \tag{11}$$

The third summand in (6) is estimated in the same way.

(ii) Now we use Condition 2 in Subsection 2.2. It ensures the boundedness (by T in $\widehat{\mathbb{C}} \setminus K$) of the absolute value of the second summand in the expression for ψ_q^{**} in (7). We turn to the first summand: if $\zeta \notin K$ and $|\zeta - A| \le T|B-A|$, then $|\lambda_{A,B}(\zeta)| \le T$, and Condition 2 yields $|\lambda_{A,B}(\zeta)||\operatorname{Im}\mathcal{L}_{K,A,B}(\zeta)| \le T^2$. If $|\zeta - A| > T|B-A|$, then by (5) we have

$$\big|\lambda_{A,B}(\zeta)\big|\big|\operatorname{Im}\mathcal{L}_{K,A,B}(\zeta)\big| \le \left|\lambda_{A,B}(\zeta)\log\frac{\zeta - B}{\zeta - A}\right| \le c(T)$$

(see (11)). The third summand in (7) is estimated in the same way. □

2.6. The modules of continuity of l. In the next lemma we shall need the following remark: if $M > 0$ and $0 < x < 1/e$, then

$$|l(Mx)| \le M|l(x)| + M|\log M|x \le (M + M|\log M|)|l(x)|, \tag{12}$$

because $|\log x| > 1$.

Lemma 8. *If w_1, $w_2 \in \mathbb{C}$ and $|w_j| < 1/100$, $j = 1, 2$, then*

$$|l(w_1) - l(w_2)| \le c|l(|w_1 - w_2|)|, \tag{13}$$

where c is an absolute constant.

Proof. The function $x \mapsto |l(x)|$ increases on $[0, 1/e]$, and the function $x \mapsto |\log x|$ decreases on $(0, 1]$. Furthermore, $|\log|1+u|| \le 2|u|$ for $u \in \mathbb{C}$ with $|u| \le \frac{1}{2}$.

Let $|w_1| \le 2|w_1 - w_2|$. Then $|w_2| \le 3|w_1 - w_2|$ and

$$\Delta := |l(w_1) - l(w_2)| \le |l(|w_1|)| + |l(|w_2|)| \le 2|l(3|w_1 - w_2|)|.$$

Thus, (13) follows from (12). If $|w_1| > 2|w_1 - w_2|$, then

$$|w_2|\left|\log\left|\frac{w_2}{w_1}\right|\right| = |w_2|\left|\log\left|1 + \frac{w_1 - w_2}{w_2}\right|\right| \le 2|w_1 - w_2|$$

and

$$\begin{aligned}\Delta = |w_1 \log|w_1| - w_2 \log|w_2|| &\le |w_1 - w_2||\log|w_1|| + |w_2|\left|\log\frac{|w_2|}{|w_1|}\right| \\ &\le |w_1 - w_2||\log(2|w_1 - w_2|)| + 2|w_1 - w_2|.\end{aligned}$$

So, (13) follows from (12) and the inequality $|w_1 - w_2| < \frac{1}{e}$ (i.e., $|\log|w_1 - w_2|| > 1$). □

2.7. Closeness of ψ_q^* and $\psi_{q'}^*$ for $q \approx q'$ (a local estimate)**.** Consider two quintuplets

$$q = (K, A, B, C, D), \quad q' = (K', A', B', C', D'), \tag{14}$$

where K, K' are continua with connected complements, A, B, C, $D \in K$, and A', B', C', $D' \in K'$. The points A', B', C', D' are viewed as close to the points with similar notation without primes if

$$\max(|A - A'|, |B - B'|, |C - C'|, |D - D'|) < \beta, \tag{15}$$

where

$$0 < \beta < |A - B|/4. \tag{16}$$

Fixing $T > 1$, we assume that

$$|A - C| = |A - D|/2 =: a < (5000T)^{-1}, \quad |A - B| < a/2. \tag{17}$$

Put

$$\mathcal{U} = \{|z - A| < 4T|B - A|\} \cup \{|z - B| < 4T|C - B|\} \cup \{|z - C| < 4T|D - C|\}. \tag{18}$$

Lemma 9. *Under conditions* (15)–(17), *for $\zeta \in \mathcal{U}$ we have*

$$|\psi_q^*(\zeta) - \psi_{q'}^*(\zeta)| \le C(T)\left[\frac{|l(\beta)|}{|A - B|} + \frac{|l(a)|\beta}{|A - B|^2}\right]. \tag{19}$$

Proof. Let m denote the maximal among the eight quantities $|\zeta - A|$, $|\zeta - B|, \dots,$ $|\zeta - A'|, \dots, |\zeta - D'|$, where $\zeta \in \mathcal{U}$. From (15), (17) and the inequality $|D - C| \le 3a$ we deduce that

$$m \le 50Ta < 1/100. \tag{20}$$

Let $X(\zeta) := |\psi_q^*(\zeta) - \psi_{q'}^*(\zeta)|$. By (9) and (13), the identity

$$\frac{l(\zeta - P)}{Q - P} - \frac{l(\zeta - P')}{Q' - P'} = \frac{l(\zeta - P) - l(\zeta - P')}{Q - P} + l(\zeta - P')\frac{(P - P') + (Q - Q')}{(Q - P)(Q' - P')}$$

implies

$$\begin{aligned} X(\zeta) \le c|l(\beta)|\Big[\frac{1}{|A - B|} + \frac{1}{|D - C|}\Big] + \frac{|l(\zeta - A')|2\beta}{|A - B||A' - B'|} \\ + \frac{|l(\zeta - B')| \cdot 2\beta}{|A - B||A' - B'|} + \frac{|l(\zeta - C')|2\beta}{|D - C||D' - C'|} + \frac{|l(\zeta - D')|2\beta}{|D - C||D' - C'|}. \end{aligned}$$

But $|D - C| \ge 2|A - B|$, $|A' - B'| \ge |A - B| - 2\beta \ge |A - B|/2$, and $|D' - C'| \ge |D - C| - 2\beta \ge 2a - a/2 \ge 3|A - B|/2$.

These estimates and (20) show that

$$X(\zeta) \le c'\frac{|l(\beta)|}{|A - B|} + c''\frac{|l(c(T)a)|\beta}{|A - B|^2} \tag{21}$$

for $\zeta \in \mathcal{U}$, where c', c'' are absolute constants; now (19) is a consequence of (21) and (12). $\square$

2.8. A global estimate of $|\psi_q - \psi_{q'}|$ for $q \approx q'$.

Lemma 10. *Suppose that q and q' satisfy the assumptions of Lemma 9 and that the two continua K and K' are T-proper. Then for every $\zeta \in \mathbb{C} \setminus (K \cup K')$ we have*

$$|\psi_q(\zeta) - \psi_{q'}(\zeta)| \le c(T)\Big[\frac{|l(\beta)|}{|A - B|} + \frac{|l(a)|\beta}{|A - B|^2} + 1\Big]. \tag{22}$$

Proof. Let $\zeta \in \mathcal{U} \setminus (K \cup K')$ (see (18)). Then

$$|\psi_q(\zeta) - \psi_{q'}(\zeta)| \le |\psi_q^*(\zeta) - \psi_{q'}^*(\zeta)| + |\psi_q^{**}(\zeta)| + |\psi_{q'}^{**}(\zeta)|, \tag{23}$$

and (22) follows from Lemma 9 and Lemma 7 (ii). If $\zeta \in \mathbb{C} \setminus (\mathcal{U} \cup K \cup K')$, then ζ belongs to $\mathcal{D}_{q'}$ (i.e., to the complement of the union of three disks centered at A', B', C' and with radii $T|B' - A'|$, $T|B' - C'|$, and $T|D' - C'|$; see (10)). Indeed, if $|\zeta - A'| < T|B' - A'|$, then $|\zeta - A| \le |\zeta - A'| + |A' - A| < T|A' - B'| + \beta \le T(|A - B| + 2\beta) + \beta < T|A - B| + 3T\beta < 4T|A - B|$, whence $\zeta \in \mathcal{U}$; if $|\zeta - B'| < T|\mathcal{D}' - C'|$, then $|\zeta - B| < T|B - C| + 3T\beta \le T|B - C| + 3T|A - B|$, but $|B - C| \ge |C - A| - |B - A| \ge a - a/2 > |B - A|$, whence it follows that $|\zeta - T| < 4T|C - B|$ and $\zeta \in \mathcal{U}$; if $|\zeta - C'| < T|\mathcal{D}' - C'|$, then $|\zeta - C| < T|D - C| + 3T\beta \le T|C - D| + 3Ta \le 4T|C - D|$, so that $|C - D| \ge |A - D| - |A - C| = a$, and again $\zeta \in \mathcal{U}$. Thus, if $\zeta \notin \mathcal{U} \cup K \cup K'$, then $\zeta \notin \mathcal{D}_q \cup \mathcal{D}_{q'}$, and Lemma 7 applied to q and q' yields in combination with (23):

$$|\psi_q(\zeta) - \psi_{q'}(\zeta)| \le |\psi_q^*(\zeta)| + |\psi_{q'}^*(\zeta)| + |\psi_q^{**}(\zeta)| + |\psi_{q'}^{**}(\zeta)| \le \tilde{c}(T). \qquad \square$$

§I.3. Main theorems. Examples

Now we are in a position to prove the main technical result of the first part, namely, a lower estimate for $b(K, K', O)$ in purely geometric terms. We assume that $K \subset g \subset O$, where g is a cell with center $A \in \partial K$. The lower estimate for $b(K, K', O)$ will involve the rotundity $\rho_g(A)$ (see Subsection 1.3), the "amplitude" a of K:

$$a := \frac{1}{2} \max\{|\zeta - A| : \zeta \in K\}, \tag{24}$$

and a positive number $\beta(a)$ describing the closeness of K' and K:

$$\max\{\operatorname{dist}(\zeta, K') : \zeta \in K\} < \beta(a). \tag{25}$$

Anticipating the precise statements, we reveal at once that $\beta(a)$ will be taken negligibly small compared to a:

$$\beta(a) \ll a. \tag{26}$$

We specify this in the next subsection.

3.1. The smallness of $\beta(a)$. Now β denotes a *function* defined on $(0, b)$.

It would be natural to interpret (26) as $\beta(a) = o(a)$ $(a \to 0)$, but I do not know if this matches our purposes (specifically, it is not clear whether Theorem 2 is true under this assumption). Although the results of [11] give hope for a positive answer, here we must assume that

$$\beta(a) = o(a/|\log a|), \quad a \to 0, \tag{27}$$

i.e., that

$$\beta(a) = a\varepsilon(a)/|\log a|, \quad 0 < a < b, \;\; \varepsilon(a) = o(1), \;\; a \to 0. \tag{28}$$

We observe that

$$\sqrt{\varepsilon(a)}a|l(\beta(a))| = o(|l(a)|\beta(a)), \quad a \to 0, \tag{29}$$

which is a consequence of the relation

$$\sqrt{\varepsilon(a)}(|\log a| + \log|\log a|) = o(|\log a|), \quad a \to 0.$$

In what follows, O denotes a domain in $\mathbb{C}$; K, K', g, A, and β were defined at the beginning of the section and in Subsection 3.1.

Theorem 1. *There exist positive constants $c(T)$ and $a(\beta, T)$ such that for every two T-proper continua K, $K' \subset O$ and every number a in the interval $(0, a(\beta, T))$ satisfying* (25), *we have*

$$b(K, K', O) \geq c(T)\rho_g(A) \log \frac{1}{\varepsilon(a)}. \tag{30}$$

For example, by Lemma 6 we may take Lipschitz graphs (without common points) for K and K'. In this case $c(T)$ will depend on the Lipschitz constants of these graphs in the long run.

Proof (reduction to Lemma 4). We deduce (30) from (4) with a special choice of $\psi_1 = \psi_q$, $\psi_2 = \psi_{q'}$.

There is a point $D \in K$ with $|A - D| = 2a$. Since K is connected, there is a point $C \in K$ with $|C - A| = a$. If $a < a(\beta)$, then $\varepsilon(a) < 1$, and for some $B \in K$ we have

$$|A - B| = a\sqrt{\varepsilon(a)}.$$

Then we find four pairwise distinct points A', B', C', $\mathcal{D}'$ in K' in such a way that (15) is fulfilled with $\beta = \beta(a)$ (see (25)). Clearly, $\beta(a) = o(a\sqrt{\varepsilon(a)})$ as $a \to 0$ (see (28)).

Therefore, for sufficiently small $a(\beta)$, the inequality $0 < a < a(\beta)$ implies (16) and (17), so that the quintuplets $q = (K, A, B, C, D)$ and $q' = (K', A', B', C', D')$ satisfy the assumptions of Lemma 9, and Lemma 10 yields (22) for $\zeta \in O \setminus (K \cup K')$:

$$\begin{aligned} \|\psi_q - \psi_{q'}\|_{\infty, O\setminus(K\cup K')} &\le c(T)\Big[\frac{|l(\beta(a))|}{a\sqrt{\varepsilon(a)}} + \frac{|l(a)|\beta(a)}{a^2\varepsilon(a)} + 1\Big] \\ &< 3c(T) \quad \text{if } 0 < a < a(\beta), \end{aligned} \tag{31}$$

because by (29) the first summand in (31) is smaller than the second, and the second is equal to 1 (see (28)).

On the other hand, $\psi_q \in \operatorname{Hol}(\widehat{\mathbb{C}} \setminus K)$, $\psi_q(\infty) = 0$, and for $\zeta \in \mathbb{C} \setminus K$ we have

$$|\psi_q(\zeta)| \ge |\psi_q^*(\zeta)| - c'(T)$$

by Lemma 7(ii). Letting $\zeta \in \mathbb{C} \setminus K$ tend to A, we obtain

$$\begin{aligned} |\psi_q^*(\zeta)| &\ge \log\frac{|C-\zeta|}{|B-\zeta|} - \frac{|\zeta - A|}{|B-A|}\log\frac{|B-\zeta|}{|A-\zeta|} - \frac{|\zeta - C|}{|C-D|}\log\frac{|\zeta - D|}{|\zeta - C|} \\ &\xrightarrow[\substack{\zeta\to A\\ \zeta\notin K}]{} \log\frac{|C-A|}{|B-A|} - \frac{|A-C|}{|C-D|}\log\frac{|A-D|}{|C-A|}, \end{aligned}$$

so that

$$\overline{\lim}_A|\Psi_q| \ge \log\frac{1}{\sqrt{\varepsilon(a)}} - (\log 2)/2 - c'(T) > \frac{1}{4}\log\frac{1}{\varepsilon(a)}$$

if $0 < a < a(\beta, T)$. It remains to apply Lemma 4 and obtain (30). □

3.3. Some pairs for which separation fails. Again, we consider relatively closed subsets S, S' of O with $S' \cap S = \varnothing$. Suppose that β satisfies (28). Invoking the discussion at the end of Subsection 1.5, we construct two families $(K_\gamma)_{\gamma\in\Gamma}$ and $(K'_\gamma)_{\gamma\in\Gamma}$ of continua $K_\gamma \subset g_\gamma \cap S$, $K'_\gamma \subset S'$, where $(g_\gamma)_{\gamma\in\Gamma}$ is a family of cells in O centered at $A_\gamma \in \partial K_\gamma$, $\gamma \in \Gamma$.

We put

$$a_\gamma := \frac{1}{2}\max\{|\zeta - A_\gamma| : \zeta \in K_\gamma\}$$

(cf. (24)). The next result follows from Theorem 1 and Lemma 5 (see Subsection 1.5).

Theorem 2. *Suppose that for some $T > 1$ and $r > 0$ the following conditions are fulfilled*:

(i) *for every $\gamma \in \Gamma$ the two continua K_γ and K'_γ are T-proper, and $\rho_{g_\gamma}(A_\gamma) \ge r$;*

(ii) *for every $\gamma \in \Gamma$ the pair (K, K'), where $K := K_\gamma$, $K' = K'_\gamma$, satisfies the assumptions of Theorem 1 with $g_i = g_\gamma$, $A := A_\gamma$. Then*

$$b(S, S', O) = +\infty,$$

so the pair (S, S') does not admit separation in O.

3.4. Examples. In this subsection, we present specific applications of Theorem 2.

3.4.1. Let k be a positive function defined on $(0, b)$ and satisfying

$$k(y) = o(y/|\log y|), \quad u \to 0. \tag{32}$$

For the role of the domain O, we take the upper half-plane $\mathbb{C}_+ = \{\operatorname{Im} z > 0\}$. Consider real Lipschitz functions φ_1, φ_2 on $[0, b]$ with

$$-k(y) < \varphi_1(y) < \varphi_2(y) < k(y), \quad y \in [0, b], \tag{33}$$

and their graphs (related to the y-axis): $S_j := \{z_j(y) : 0 < y \le b\}$, $j = 1, 2$, where $z_j(y) := \varphi_j(y) + iy$ (see Figure 2).

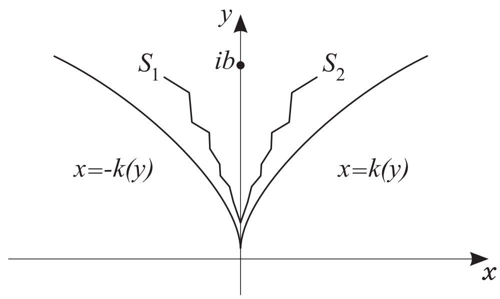

FIGURE 2.

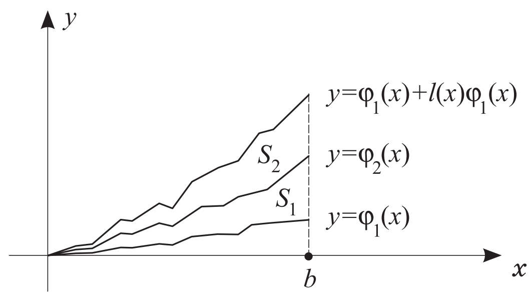

FIGURE 3.

Theorem 3. *The pair (S_1, S_2) does not admit separation in $\mathbb{C}_+$.*

Remark. In [11] it was shown that if φ_1, $\varphi_2 \in C^{1+\varepsilon}([0, b])$, then the claim remains true under the condition $k(y) = o(y)$ $(y \to 0)$.

Proof. We put $K_y := S_1 \cap \{y \leq \operatorname{Im} z \leq 2y\}$, $K'_y := S_2 \cap \{y \leq \operatorname{Im} z \leq 2y\}$, and $g_y = (-y, y) \times (0, 3y)$, $A_y := \varphi_1(y) + iy$ for $y \in (0, b]$. Clearly,

$$y \leq a_y \leq c(L)y, \quad 0 < y \leq b, \tag{34}$$

where $c(L)$ depends only on the Lipschitz constant for φ_1. If $\eta \in [y, 2y]$, then by (33) we have

$$|z_1(\eta) - z_2(\eta)| = \varphi_2(\eta) - \varphi_1(\eta) \leq 2k(\eta) = o(y/|\log y|) = o(a_y/|\log a_y|), \quad y \to 0$$

(the last estimate follows from (34)), so the continua $K := K_y$ and $K' := K'_y$ satisfy (25), and the function β satisfies (27). The continua K_y and K'_y are T-proper, where T depends only on the Lipschitz constant for φ_1 and φ_2 (by Lemma 6); the cells g_y are uniformly rotund. It remains to apply Theorem 2 to the families $(K_y)_{0<y\leq b}$ and $(K'_y)_{0<y\leq b}$. □

3.4.2. This time, our Lipschitz functions φ_1 and φ_2 (defined on $[0, b]$ as before) satisfy the conditions

$$0 < \varphi_1(x) < \varphi_2(x), \;\; x \in (0, b]; \quad \varphi_1(0) = \varphi_2(0) = 0.$$

We put $z_j(x) = x + i\varphi_j(x)$, $x \in [0, b]$, $S_j := \{z_j(x) : 0 < x \leq b\}$, $j = 1, 2$.

Figure 3 represents the most interesting case where the curves S_1 and S_2 have a common tangent (the ray $[0, +\infty)$) at the origin, though the next result is applicable also in the case where $\varphi'_1(0) = \varphi'_2(0) > 0$.

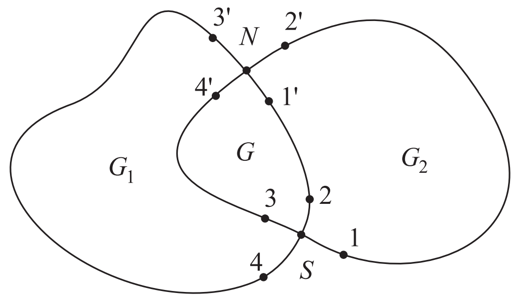

FIGURE 4.

Theorem 4. *Suppose*

$$\varphi_2(x) - \varphi_1(x) \le \varphi_1(x) l(x), \quad x \in (0, b], \tag{35}$$

where $l(x) = o(1/|\log \varphi_1(x)|)$, $x \to 0$. *Then the pair* (S_1, S_2) *does not admit separation in* $\mathbb{C}_+$.

In [11] it was shown that if φ_1, $\varphi_2 \in C^{1+\varepsilon}([0, b])$, then Theorem 4 remains true under the condition $l(x) = o(1)$ as $x \to 0$.

Proof. We put $\theta = 1/2L$, where L is the Lipschitz constant of φ_1, and

$$\begin{aligned} I_x &:= [x - \theta\varphi_1(x), x + \theta\varphi_1(x)], \\ K_x &:= S_1 \cap \{\operatorname{Re} z \in I_x\}, \\ K'_x &:= S_2 \cap \{\operatorname{Re} z \in I_x\}, \\ g_x &:= (x - 2\theta\varphi_1(x), x + 2\theta\varphi_1(x)) \times (0, 2\varphi_1(x)), \\ A_x &:= x + i\varphi_1(x), \quad x \in (0, b]. \end{aligned}$$

Clearly, $K_x \subset g_x$, because for $\zeta \in I_x$ we have

$$\varphi_1(\zeta) \le \varphi_1(x) + L\theta\varphi_1(x) = 3\varphi_1(x)/2.$$

But

$$a_x^2 = \max_{\xi \in I_x} [(x - \xi)^2 + (\varphi_1(x) - \varphi_1(\xi))^2] \le 5(\varphi_1(x))^2/4,$$

so that

$$\theta\varphi_1(x) \le a_x \le 10\varphi_1(x), \quad x \in (0, b]. \tag{36}$$

If $\xi \in I_x$, then

$$|z_1(\xi) - z_2(\xi)| = \varphi_2(\xi) - \varphi_1(\xi) \le \varphi_1(\xi) l(\xi) \le 2\varphi_1(x) l(\xi) \le 2La_x l(\xi),$$

and

$$l(\xi) = \frac{\varepsilon(\xi)}{|\log \varphi_1(\xi)|} \le \frac{\sup\{\varepsilon(\xi) : 0 < \xi \le x + \theta\varphi_1(x)\}}{|\log(\varphi_1(x)/2)|} = o\Big(\frac{1}{|\log \varphi_1(x)|}\Big) = o\Big(\frac{1}{|\log a_x|}\Big),$$

where $\varepsilon(x) = o(1)$ as $x \to 0$ (we have used the estimate $\varphi_1(\xi) \ge \varphi_1(x) - L\theta\varphi_1(x) = \varphi_1(x)/2$ in I_x, and also inequality (36)). The continua K_x, K'_x under study are uniformly proper (Lemma 6), and the cells g_x are uniformly rotund. It remains to apply Theorem 2. $\square$

3.4.3. Consider two Jordan domains G_1 and G_2 depicted in Figure 4, and their intersection G.

Suppose the curves ∂G_1 and ∂G_2 are piecewise C^1-smooth and meet *transversally* at the points S and N, i.e., every two among the four arcs 1, 2, 3, 4 (respectively, $1'$, $2'$, $3'$, $4'$) form a nonzero angle at S (respectively, at N). In [10] it was shown (see Example 4.1 in Subsection 4.6 therein) that for every $f \in H^\infty(G)$ there exist $f_j \in H^\infty(G_j)$, $j = 1, 2$, such that

$$f = f_1 + f_2 \quad \text{in } G. \tag{37}$$

Theorems 3 and 4 imply that the transversality condition cannot be dropped here (the deduction of this statement from Theorems 3 and 4 can be found in [11]).

II. SEPARATION FOR TANGENT PAIRS

The main result of this part of the paper is Theorem 5 in § II.2, which describes some pairs admitting separation in $\mathbb{C}_+$. In that theorem we shall deal with pairs of smooth arcs in $\mathbb{C}_+$ for which $\mathbb{R}$ is a common tangent at the origin. As a positive statement, Theorem 5 opposes Theorem 4 in the first part.

The first section of Part II is devoted to technical preparations to the proof of Theorem 5. At the end of §II.2, we give some examples illustrating Theorem 5.

We need the following notation: for a path $\gamma : I \to \mathbb{C}$, where $I \subset \mathbb{R}$ is an interval, we put

$$C^F_\gamma(\zeta) := \frac{1}{2\pi i} \int_\gamma \frac{F(z)\,dz}{z - \zeta}, \quad \zeta \in \mathbb{C} \setminus \gamma(I).$$

Here we assume that γ is absolutely continuous, the domain of the complex function F includes the trajectory $\gamma(J)$, $F \circ \gamma$ is Lebesgue measurable, and

$$\int_I |F \circ \gamma|\,|\gamma'| < +\infty.$$

The principal value of the integral $(2\pi i)^{-1} \int_\gamma (F(z)/(z - \zeta))dz$, where $\zeta \in \gamma(I)$, will be denoted by $\not{C}^F_\gamma(\zeta)$; by definition, $\not{C}^F_\gamma(\zeta) = C^F_\gamma(\zeta)$ for $\zeta \in \mathbb{C} \setminus \gamma(I)$.

The symbol γ_φ will denote the graph of a real function φ defined on a subset E of $\mathbb{R}$. We treat γ_φ as a *mapping*: $\gamma_\varphi(x) = x + i\varphi(x)$, $x \in E$ (i.e., γ_φ is viewed as a *path* if φ is continuous and E is a segment). However, sometimes we perceive γ_φ as the *set* $\gamma_\varphi(E)$.

§II.1. Pushing singularities to an auxiliary arc

1.1. The quad of arcs γ_j, $j = -1, 0, 1, 2$. Let φ_1, φ_2 be nonnegative functions defined on $[0, b]$ with $b > 0$ and such that

$$\varphi_1(0) = \varphi_2(0) = 0, \;\; \varphi_j(x) > 0 \quad \text{for } x \in (0, b],\ j = 1, 2.$$

We shall assume that for some $\mu > 0$ we have

$$\varphi_2(x) \ge (1 + \mu)\varphi_1(x), \quad 0 \le x \le b. \tag{38}$$

This means that the hyperbolic distance (relative to $\mathbb{C}_+$) between $\gamma_{\varphi_1}(x)$ and $\gamma_{\varphi_2}(x)$ is bounded away from zero uniformly in $x \in (0, b]$. (It should be noted that precisely this condition was violated in Theorem 4.) Furthermore, we cannot do without the following smoothness conditions:

$$\varphi_1 \in C^{1+\varepsilon}([0, b]) \quad \text{for some } \varepsilon > 0,\ \varphi_1'(0) = 0,\ \varphi_2 \in C^1([0, b]). \tag{39}$$

Put $\gamma_j := \gamma_{\varphi_j}$ $(j = 1, 2)$, $\gamma_0 := [-b, b]$, $\gamma_{-1} := \overline{\gamma_1}$.

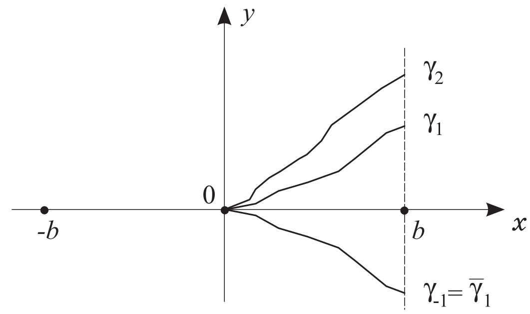

FIGURE 5.

1.2. The main lemma: the statement and the beginning of the proof. We denote by A' the complement $\mathbb{C} \setminus A$ of a set $A \subset \mathbb{C}$.

Lemma 11. *Let* $f \in H^\infty((\gamma_0 \cup \gamma_1 \cup \gamma_2)')$, *where the paths* γ_1 *and* γ_2 *satisfy* (38) *and* (39). *Then there exist functions* $f_j \in H^\infty((\gamma_0 \cup \gamma_j \cup \gamma_{-1})')$, $j = 1, 2$, *such that*

$$f = f_1 + f_2 \quad \textit{in } (\gamma_{-1} \cup \gamma_0 \cup \gamma_1 \cup \gamma_2)'.$$

Proof. We may assume that $f(\infty) = 0$, so that

$$f = C^F_{\gamma_0} + C^F_{\gamma_1} + C^F_{\gamma_2},$$

where $F \in L^\infty(\gamma_0 \cup \gamma_1 \cup \gamma_2, s)$ (s stands for length; see Lemma 4.1 in [10]). We extend F to γ_{-1}:

$$F(x - i\varphi_1(x)) := F(x + i\varphi_1(x)), \quad 0 < x \le b,$$

and put $C_j := C^F_{\gamma_j}$, $j = -1, 0, 1, 2$; the principal values $\not{C}_j$ are defined similarly (we recall that $\not{C}_j = C_j$ in γ_j'). Finally, we put

$$f_1 := C_1 - C_{-1} \text{ in } (\gamma_{-1} \cup \gamma_1)', \quad f_2 := C_{-1} + C_0 + C_2 \text{ in } (\gamma_{-1} \cup \gamma_0 \cup \gamma_2)', \tag{40}$$

so that $f = f_1 + f_2$ in $(\bigcup_{j=-1}^{2} \gamma_j)'$. It remains to show that

$$f_2 \in H^\infty((\gamma_{-1} \cup \gamma_0 \cup \gamma_2)'). \tag{41}$$

The boundedness of f_1 in $(\gamma_{-1} \cup \gamma_1)'$ follows from (41) and the boundedness of f. $\square$

1.3. A digression. The method outlined in Subsection 1.2 is parallel to the proof of Theorems 4.1 and 5.1 in [10]. First, we split the Cauchy potential $C^F_{\gamma_0 \cup \gamma_1 \cup \gamma_2} = f$ crudely by the formula

$$f = C_1 + [C_0 + C_2],$$

which separates singularities but destroys boundedness. To restore the latter, we introduce the auxiliary arc γ_{-1} lying off $\mathbb{C}_+$, and try to ensure the boundedness of f_1 and f_2 by subtraction of C_{-1} from C_1 and addition of C_{-1} to $C_0 + C_2$. This procedure does not create new singularities in $\mathbb{C}_+$ (they arise only in $\mathbb{C}'_+$), and the boundedness of $f_2 = C_0 + C_2 + C_{-1}$ becomes quite plausible. Indeed, the charge on γ_{-1} giving rise to the potential C_{-1} is a twin copy of the initial charge on γ_1 generating C_1. But the sum $C_0 + C_2 + C_1$ *was* bounded, and the deviations of the points $\gamma_1(x)$ and $\gamma_{-1}(x)$ from $\gamma_0(x)$ and $\gamma_2(x)$, respectively, are comparable (and are roughly equal to $\varphi_1(x)$) by (39). In other words, C_{-1} has no smaller capability of compensating for the growth of $C_0 + C_2$ than C_1 are because the charges that lie on γ_1 and γ_{-1} are identical, and the distances $|\gamma_1(x) - \gamma_0(x)|$ and $|\gamma_{-1}(x) - \gamma_0(x)|$ (respectively, $|\gamma_2(x) - \gamma_1(x)|$ and $|\gamma_2(x) - \gamma_{-1}(x)|$) are comparable. The proof presented below justifies these heuristic arguments. As in Subsection 5 of [10], the proof will involve the maximum of the modulus principle for

Cauchy potentials, but the estimates will differ and, unfortunately, will require an additional smoothness condition on φ_1 (see (39)), much stronger than in the "transversal" setting treated in [10].

1.4. Continuation of the proof of Lemma 11. In order to prove (41), we apply Lemma 5.1 and the material of Subsection 5.2 in [10]. We show the existence of a constant M such that

$$|\not{C}_{-1}(z_0)+\not{C}_0(z_0)+\not{C}_2(z_0)|\le M \quad \text{for } s\text{-almost every } z_0\in\gamma_0\cup\gamma_{-1}\cup\gamma_2. \tag{42}$$

The Sokhotskiĭ–Privalov formulas for the boundary values of Cauchy potentials together with the boundedness of the density F ensure the boundedness a.e. of the nontangential boundary values of $f_2=C_{-1}+C_0+C_2$ on $\gamma_0\cup\gamma_{-1}\cup\gamma_2$; in accordance with [10, Subsection 5.2], we arrive at (41).

1.5. In order to prove (42), we estimate the difference $\not{C}_{-1}(z_0)-\not{C}_1(z_0)=:\Delta(z_0)$ at an arbitrary point $z_0=x_0+iy_0$, $x_0\in[-b,b]$, $y_0\in\mathbb{R}$. Putting $F_j(x)=F(x+i\varphi_j(x))$, we obtain

$$\begin{aligned}\Delta(z_0)&=\frac{1}{2\pi i}\fint_0^b F_1(x)\Big(\frac{\overline{\gamma_1'(x)}}{\overline{\gamma_1(x)}-z_0}-\frac{\gamma_1'(x)}{\gamma_1(x)-z_0}\Big)dx\\&=\frac{1}{\pi}\fint_0^b\frac{\operatorname{Im}(\overline{\gamma_1'(x)}\gamma_1(x))+z_0\operatorname{Im}\gamma_1'(x)}{(\overline{\gamma_1(x)}-z_0)(\gamma_1(x)-z_0)}dx\\&=\frac{1}{\pi}\fint_0^b F_1(x)\frac{-\varphi_1'(x)(x-x_0)+\varphi_1(x)+iy_0\varphi_1'(x)}{[(x-x_0)-i(\varphi_1(x)+y_0)][(x-x_0)+i(\varphi_1(x)-y_0)]}dx.\end{aligned}\tag{43}$$

In the integrand's numerator, we add and subtract $\varphi_1(x_0)$. Under the agreement that $\varphi_1(x_0)=0$ for $x_0\in[-b,0]$, we get

$$|\Delta(z_0)|\le\|F\|_\infty(a(z_0)+B(z_0)+C(z_0)),\tag{44}$$

where

$$\begin{aligned}B(z_0)&=\varphi_1(x_0)\int_0^b\frac{dx}{[|x-x_0|+|\varphi_1(x)+y_0|][|x-x_0|+|\varphi_1(x)-y_0|]},\\C(z_0)&=y_0\int_0^b\frac{\varphi_1'(x)\,dx}{[|x-x_0|+|\varphi_1(x)+y_0|][|x-x_0|+|\varphi_1(x)-y_0|]}.\end{aligned}\tag{45}$$

The quantity $A(z_0)$ can be defined and estimated as follows:

$$\begin{aligned}A(z_0)&:=\int_0^b\frac{|\varphi_1(x)-\varphi_1(x_0)-\varphi_1'(x)\cdot(x-x_0)|}{(x-x_0)^2}\,dx\\&=\int_0^b\frac{|\varphi_1'(c(x,x_0))-\varphi_1'(x)|}{|x-x_0|}\,dx\le\int_0^b\frac{k\cdot|x-x_0|^\varepsilon}{|x-x_0|}\,dx\end{aligned}\tag{46}$$

($c(x,x_0)$ is a point between x and x_0). The last integral is bounded uniformly in $x_0\in[-b,b]$ (we have used condition (39)).

Turning to (42), we consider the following particular cases:

$$\text{(I)}\quad z_0=x_0\in\gamma_0;\quad\text{(II)}\quad z_0\in\overline{\gamma_1}=\gamma_{-1};\quad\text{(III)}\quad z_0\in\gamma_2.\tag{47}$$

1.6. Case (I). For $x_0 \in \gamma_0$ we have

$$\not{C}_0(x_0) + C_{-1}(x_0) + C_2(x_0) = [\not{C}_0(x_0) + C_1(x_0) + C_2(x_0)] + \Delta(x_0);$$

as a function of x_0, the expression in square brackets belongs to $L^\infty([-b, b])$ by the boundedness of f and F and the Privalov–Sokhotskiĭ formulas. Estimate (44) reduces to

$$|\Delta(x_0)| \le (A(x_0) + B(x_0))\|F\|_\infty,$$

because $y_0 = 0$ and $C(x_0) = 0$. By (46), $A \in L^\infty([0, b])$. In order to estimate $B(x_0)$ for $0 < x_0 < b$, in $(0, x_0)$ we take the greatest solution x_1 of the equation $\varphi_1(x_1) = \frac{1}{2}\varphi_1(x_0)$. We have

$$\varphi_1(x_0)/2 = \varphi_1(x_0) - \varphi_1(x_1) = \varphi_1'(c)(x_0 - x_1)$$

for some $c \in (x_1, x_0)$; $\varphi_1(x) \ge \varphi_1(x_0)/2$ for $x \in (x_1, b)$. Thus,

$$\begin{aligned} B(x_0) &\le \varphi_1(x_0) \int_{-\infty}^{x_1} \frac{dx}{(x_0 - x)^2} + \varphi_1(x_0) \int_{-\infty}^{\infty} \frac{dx}{(|x - x_0| + \varphi_1(x_0)/2)^2} \\ &= \frac{\varphi_1(x_0)}{x_0 - x_1} + \varphi_1(x_0) \cdot 2 \int_0^\infty \frac{du}{(u + \varphi_1(x_0)/2)^2} \le \max_{[0,b]} |\varphi_1'| + 4. \end{aligned}$$

If $x \in [-b, 0]$, then $B(x_0) = 0$.

1.7. Case (II). Let $z_0 = x_0 - i\varphi_1(x_0) \in \gamma_{-1}$, $x_0 \in [0, b]$. We have

$$\begin{gathered} C_0(z_0) + \not{C}_{-1}(z_0) + C_2(z_0) = [C_0(\bar{z}_0) + \not{C}_1(\bar{z}_0) + C_2(\bar{z}_0)] + \alpha_0 + \alpha_1 + \alpha_2; \\ \alpha_j := C_j(z_0) - C_j(\bar{z}_0), \;\; j = 0, 2, \quad \alpha_1 := \not{C}_{-1}(z_0) - C_1(\bar{z}_0). \end{gathered} \tag{48}$$

As a function of z_0, the expression in square brackets in (48) belongs to $L^\infty(\gamma_{-1}, s)$ (because f and F are bounded). It remains to estimate α_0, α_1, and α_2. But

$$\begin{aligned} |\alpha_0| &= \frac{1}{\pi}\left| \int_0^b F(x) \frac{\operatorname{Im} z_0}{|x - z_0|^2}\, dx \right| \le \|F\|_\infty; \\ |\alpha_1| &= \left| \frac{1}{2\pi i} \fint_0^b F_1(x) \left[\frac{\gamma_1'(x)}{\gamma_1(x) - \bar{z}_0} - \frac{\overline{\gamma_1'(x)}}{\overline{\gamma_1(x)} - z_0} \right] dx \right| \\ &= \frac{1}{\pi}\left| \fint_0^b F_1(x) \frac{\operatorname{Im}[\gamma_1'(x)(\overline{\gamma_1(x)} - z_0)]}{|\gamma_1(x) - z_0|^2}\, dx \right| \\ &= \frac{1}{\pi}\left| \int_0^b F_1(x) \frac{\varphi_1'(x)(x - x_0) - (\varphi_1(x) - \varphi_1(x_0))}{(x - x_0)^2 + (\varphi_1(x) - \varphi_1(x_0))^2}\, dx \right| \\ &\le \|F_1\|_\infty \int_0^b \frac{|\varphi_1'(x)(x - x_0) - (\varphi_1(x) - \varphi_1(x_0))|}{(x - x_0)^2}\, dx; \end{aligned}$$

the uniform boundedness (in $x_0 \in [0, b]$) of the last integral was proved earlier (see (46)).

Next,

$$\begin{aligned} |\alpha_2| &= \frac{1}{2\pi}\left| \int_0^b F_2(x)\gamma_2'(x) \left[\frac{1}{\gamma_2(x) - z_0} - \frac{1}{\gamma_2(x) - \bar{z}_0} \right] dx \right| \\ &\le \|F\|_\infty \|\gamma_2'\|_\infty \int_0^b \frac{|\operatorname{Im} z_0|}{|\gamma_2(x) - z_0||\gamma_2(x) - \bar{z}_0|}\, dx \\ &\le 2\|F\|_\infty \|\gamma_2'\|_\infty \varphi_1(x_0) \int_0^b \frac{dx}{(|x - x_0| + \varphi_2(x) + \varphi_1(x_0))(|x - x_0| + |\varphi_2(x) - \varphi_1(x_0)|)}. \end{aligned}$$

We use the inequality

$$\frac{1}{\alpha + \beta} \le \frac{R}{R\alpha + \beta}, \quad \alpha, \beta > 0, \;\; R > 1.$$

The result is

$$|\alpha_2| \le K\varphi_1(x_0)\int_0^b \frac{1}{|x-x_0|+\varphi_1(x_0)}\cdot\frac{R\,dx}{R|x-x_0|+|\varphi_2(x)-\varphi(x_0)|},$$

where $K := 2\|F\|_\infty\|\gamma_2'\|_\infty$. But, by (38), we have

$$|\varphi_2(x)-\varphi_1(x_0)| \ge |\varphi_2(x_0)-\varphi_1(x_0)| - |\varphi_2(x)-\varphi_2(x_0)| \ge \mu\varphi_1(x_0) - \|\varphi_2'\|_\infty|x-x_0|.$$

Taking $R = \|\varphi_2'\|_\infty + 1$, we obtain

$$\begin{aligned}|\alpha_2| &\le RK\varphi_1(x_0)\int_0^b \frac{1}{|x-x_0|+\varphi_1(x_0)}\cdot\frac{1}{|x-x_0|+\mu\varphi_1(x_0)}\,dx\\ &\le RK\varphi_1(x_0)\int_{-\infty}^{+\infty}\frac{dx}{(x-x_0)^2+\widetilde{\mu}^2\varphi_1^2(x_0)} = RK\pi/\widetilde{\mu},\end{aligned}$$

where $\widetilde{\mu} := \min(\mu, 1)$. Therefore, Case (II) is exhausted.

1.8. Case (III). If $z_0 \in \gamma_2$, $z_0 = x_0 + i\varphi_2(x_0)$, $x_0 \in (0, b]$, then

$$C_0(z_0) + C_{-1}(z_0) + \cancel{C}_2(z_0) = [C_0(z_0) + C_1(z_0) + \cancel{C}_2(z_0)] + \Delta(z_0)$$

(see (44) and (45)). In the variable z_0, the expression in square brackets is a function of class $L^\infty(\gamma_2, s)$. Applying (44) and (45) with $y_0 = \varphi_2(x_0)$, we obtain

$$\begin{aligned}B(z_0) &= \varphi_1(x_0)\int_0^b \frac{dx}{(|x-x_0|+\varphi_1(x)+\varphi_2(x_0))(|x-x_0|+|\varphi_1(x)-\varphi_2(x_0)|)}\\ &= \varphi_1(x_0)C(z_0)/\varphi_2(x_0).\end{aligned}$$

We recall that the quantity $A(z_0)$ is uniformly bounded and $\varphi_2(x_0) < \varphi_1(x_0)$. Therefore, it remains to prove that $C(z_0)$ is bounded uniformly in $z_0 \in \gamma_2$. As in Case (II) for $R > 1$, we get

$$C(z_0) \le \varphi_2(x_0)\int_0^b \frac{R\,dx}{(|x-x_0|+\varphi_2(x_0))(R|x-x_0|+|\varphi_1(x)-\varphi_2(x_0)|)}.$$

But $|\varphi_1(x)-\varphi_2(x_0)| \ge (\varphi_2(x_0)-\varphi_1(x_0)) - \|\varphi_1'\|_\infty|x-x_0| = \varphi_2(x_0)(1-\varphi_1(x_0)/\varphi_2(x_0)) - \|\varphi_1'\|_\infty|x-x_0| \ge \varphi_2(x_0)\mu/(1+\mu) - \|\varphi_1'\|_\infty|x-x_0|$ (we have used (38) once again). Putting $R = 1 + \|\varphi_1'\|_\infty$, we obtain

$$C(z_0) \le R\varphi_2(x_0)\int_0^b \frac{dx}{(|x-x_0|+\widetilde{\mu}\varphi_2(x_0))^2} \le R\pi/\widetilde{\mu}, \quad \widetilde{\mu} = \mu/(1+\mu).$$

Lemma 11 is proved. □

§II.2. Tangent pairs admitting separation: Sufficient conditions and examples

2.1. Theorem 5. *Let $\gamma_j = \gamma_{\varphi_j}$, $j = 1, 2$, be the same arcs as in Lemma* 11. *Then the pair (γ_1, γ_2) admits separation in $\mathbb{C}_+$.*

Proof. The arguments are based on the following observation: for every $h \in H^\infty(\mathbb{C}_+ \setminus (\gamma_1 \cup \gamma_2))$ there exist $f_0 \in H^\infty(\mathbb{C}_+)$ and $f^0 \in H^\infty((\gamma_0 \cup \gamma_1 \cup \gamma_2)')$ such that

$$h = f_0 + f^0 \quad \text{in } \mathbb{C}_+ \setminus (\gamma_1 \cup \gamma_2). \tag{49}$$

Applying Lemma 11 to f^0, from (49) we deduce that

$$h = (f_0 + f_1) + f_2 \quad \text{in } \mathbb{C}_+ \setminus (\gamma_1 \cup \gamma_2),$$

where $f_1 \in H^\infty((\gamma_0 \cup \gamma_1)')$ and $f_2 \in H^\infty((\gamma_0 \cup \gamma_2)')$, so $(f_0 + f_1)|(\mathbb{C}_+ \setminus \gamma_1) \in H^\infty(\mathbb{C}_+ \setminus \gamma_1)$ and $f_2|(\mathbb{C}_+ \setminus \gamma_2) \in H^\infty(\mathbb{C}_+ \setminus \gamma_2)$. This completes the proof of the theorem. □

In order to verify (49), consider a linear-fractional mapping Φ satisfying $\Phi(\mathbb{D}) = \mathbb{C}_+$ ($\mathbb{D}$ is the unit disk), $\Phi(1) = 0$, and put $\Gamma := \Phi^{-1}[-b, b]$. Let $\widetilde{\Gamma} \subset \Gamma$ be an arc open relative to the unit circle $\mathbb{T}$ and containing 1. We put $k := \Phi^{-1}(\gamma_1 \cup \gamma_2)$, $g := h \circ \Phi$ in $\mathbb{D} \setminus k$, $g := 0$ in $\mathbb{C} \setminus (\mathbb{D} \cup \mathbb{T})$, so that $g \in H^\infty((\mathbb{T} \cup k)')$. But $\mathbb{T} \cup k = (\Gamma \cup k) \cup (\mathbb{T} \setminus \widetilde{\Gamma})$. The pair $(\Gamma \cup k, \mathbb{T} \setminus \widetilde{\Gamma})$ satisfies the assumptions of the "preseparation theorem" and its Corollary 3.3 in [10], because $\mathbb{T} \setminus \Gamma$ and $\widetilde{\Gamma}$ are at a positive distance from each other. Therefore,

$$g = g_1 + g_2, \quad g_1 \in H^\infty((\Gamma \cup k)'), \ \ g_2 \in H^\infty((\mathbb{T} \setminus \widetilde{\Gamma})'),$$

and (49) is fulfilled with $f_0 = (g_2 \circ \Phi^{-1})|\mathbb{C}_+$, $f^0 = g \circ \Phi^{-1}$. □

2.2. A generalization. We begin with a simple observation. Let O, $\widetilde{O}$ be domains, and let K_1, K_2, $\widetilde{K_1}$, $\widetilde{K_2}$ be compact subsets of $\widehat{\mathbb{C}}$. The triples (O, K_1, K_2) and $(\widetilde{O}, \widetilde{K}_1, \widetilde{K}_2)$ are said to be (*conformally*) *equivalent* if there exists a conformal homeomorphism Φ of O onto $\widetilde{O}$ that takes $K_j \cap O$ onto $\widetilde{K}_j \cap \widetilde{O}$, $j = 1, 2$. Clearly, the pair (K_1, K_2) admits separation in O if and only if so does the pair $(\widetilde{K}_1, \widetilde{K}_2)$ in $\widetilde{O}$. (As in [10], in the case where either K_1 or K_2 is not necessarily a subset of O, we say that the pair (K_1, K_2) admits separation in O if so does the pair $(K_1 \cap O, K_2 \cap O)$).

Suppose that $K_1 \cap K_2 = \{0\}$, and let v be a neighborhood of the origin. We put $\varkappa_j := \operatorname{Clos}(K_j \cap v)$, $j = 1, 2$. Let ω be a domain in $\widehat{\mathbb{C}}$.

Corollary (to Theorem 5). *Suppose the triples* $(\omega, \varkappa_1, \varkappa_2)$ *and* $(\mathbb{C}_+, \gamma_1, \gamma_2)$ *are equivalent* (γ_1 *and* γ_2 *are the arcs occurring in Lemma* 11). *Then the pair* (K_1, K_2) *admits separation in* ω.

Proof. This follows from Corollary 3.2 in [10] and Lemma 11, combined with the remark at the beginning of this subsection (about the separation property for the pair $(\varkappa_1, \varkappa_2)$ in ω). □

2.3. A "two-sided" version of Theorem 5. This version will be needed for construction of examples.

Theorem 5′. *Suppose that* φ_1 *and* φ_2 *are functions defined on the segment* $[-b, b]$, $\varphi_1 \in C^{1+\varepsilon}([-b, b])$, $\varphi_2 \in C^1([-b, b])$, $\varphi_1(0) = \varphi_2(0) = 0$, $\varphi_j(x) > 0$ *for* $x \neq 0$ (*so that* $\varphi_j'(0) = 0$), $j = 1, 2$. *If condition* (38) *is fulfilled for every* $x \in [-b, b]$, *then the pair* $(\gamma_{\varphi_1}, \gamma_{\varphi_2})$ *of graphs admits separation in* $\mathbb{C}_+$.

We could have inspected the proof of Theorems 5 to see that it would work under the new assumptions. Instead, we prefer a formal reduction of Theorem 5′ to Theorem 5.

Proof. Putting $\varphi_j^+ := \varphi_j|[0, b]$, $\varphi_j^- := \varphi_j|[-b, 0]$, $\gamma_j := \gamma_{\varphi_j}$, $\gamma_j^\pm := \gamma_{\varphi_j^\pm}$, $j = 1, 2$, we show that

$$\text{the pair } (\gamma_1^+ \cup \gamma_2^+, \gamma_1^- \cup \gamma_2^-) \text{ admits separation in } \mathbb{C}_+. \tag{50}$$

Having proved (50), we shall be able to apply Theorem 5 to the pairs (γ_1^+, γ_2^+) and (γ_1^-, γ_2^-) separately. In order to verify (50), we consider a function $h \in H^\infty(\mathbb{C}_+ \setminus (\gamma_1 \cup \gamma_2))$ and use the representation $h = f_0 + f^0$ in $\mathbb{C}_+ \setminus (\gamma_1 \cup \gamma_2)$, where $f_0 \in H^\infty(\mathbb{C}_+)$, $f^0 \in H^\infty((\gamma_0 \cup \gamma_1 \cup \gamma_2)')$, $\gamma_0 := [-b, b]$, as we did in the proof of Theorem 5 in Subsection 2.1; we apply that proof to the new pair (γ_1, γ_2). Then we consider a compact triangle $T \subset \mathbb{C} \setminus \mathbb{C}_+$ with apex at the origin and symmetric with respect to the imaginary axis. By [10, Theorem 6.2], we have

$$f^0 = \psi^+ + \psi^- \quad \text{in } \mathbb{C} \setminus (\gamma_1 \cup \gamma_2 \cup T),$$

where $\psi^\pm \in H^\infty(\mathbb{C} \setminus (\gamma_1^\pm \cup \gamma_2^\pm \cup T))$. Thus, $h = (f_0 + \psi^+) + \psi^-$ in $\mathbb{C}_+ \setminus (\gamma_1 \cup \gamma_2)$, where $f_0 + \psi^+ \in H^\infty(\mathbb{C}_+ \setminus (\gamma_1^+ \cup \gamma_2^+))$ and $\psi^- \in H^\infty(\mathbb{C}_+ \setminus (\gamma_1^- \cup \gamma_2^-))$. □

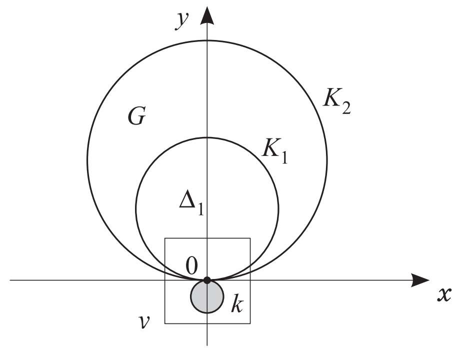

FIGURE 6.

The corollary of Theorem 5 in Subsection 2.2 admits an obvious counterpart in the setting of Theorem 5′.

In conclusion of this subsection, we turn once again to Theorem 5′. Suppose that $\varphi_j \in C^2([-b,b])$, $\varphi_j(0) = \varphi_j'(0) = 0$, $\varphi_j(x) > 0$ for $x \neq 0$, $j = 1,2$. If, moreover, $\varphi_2''(0) > \varphi_1''(0)$, then condition (38) is fulfilled for every $x \in [-b,b]$. Thus, if the *curvatures* of the arcs γ_1 and γ_2 at the origin are different, then the pair (γ_1,γ_2) admits separation in $\mathbb{C}_+$ (by Theorem 5′).

2.3. Some applications of Theorem 5′.

Example 1. Let K_1, K_2 be two (distinct) circles in $\mathbb{C}_+ \cup \{0\}$ passing through the origin and centered at imaginary points (Figure 6).

The pair (K_1,K_2) *admits separation* in $\mathbb{C}_+$. To see this, we consider a small square v centered at the origin and apply the corollary of Theorem 5′ and the discussion at the end of Subsection 2.2 about the separability in $\mathbb{C}_+$ of arcs with different curvatures at the origin.

Example 2. For $j = 1,2$, let Δ_j denote an open disk with boundary K_j. We put $\varkappa := \{|z+i\varepsilon| \le \varepsilon\}$, $\varepsilon > 0$ (see Figure 6). *The pair* (K_1,K_2) *admits separation in the domain* $\mathbb{C} \setminus \varkappa$.

Recall that in $\mathbb{C}$ (and even in an arbitrary open disk centered at the origin), separation fails for (K_1,K_2). Moreover, there exists a function $f \in H^\infty(G)$, $G := \Delta_2 \setminus (\Delta_1 \cup K_1)$, nonrepresentable in the form

$$f = f_1 + f_2 \quad \text{in } G \tag{51}$$

with $f_1 \in H^\infty((\Delta_1 \cup K_1)')$, $f_2 \in H^\infty(\Delta_2)$ (see [10, Subsection 2.3]). But an *arbitrary* function $f \in H^\infty(G)$ is representable as in (51) with $f_1 \in H^\infty((\Delta_1 \cup K_1 \cup \varkappa)')$, $f_2 \in H^\infty(\Delta_2)$.

Proof. The triple $(\widehat{\mathbb{C}} \setminus k, K_1, K_2)$ is equivalent to $(\mathbb{C}_+, \tilde{K}_1, \tilde{K}_2)$ (we use a linear-fractional function that maps $\widehat{\mathbb{C}} \setminus k$ onto $\mathbb{C}_+$); the $\tilde{K}_j$ are circles in $\mathbb{C}_+ \cup \{0\}$ passing through the origin and centered at imaginary points. ☐

Example 3. Let $G = \{0 < \operatorname{Im} z < 1\}$. We fix a positive number L. *An arbitrary function* $f \in H^\infty(G)$ *can be represented as in* (51) with

$$f_1 \in H^\infty(\mathbb{C}_+), \quad f_2 \in H^\infty(\{-L < \operatorname{Im} z < 1\}).$$

We recall that, generally speaking, $f_2 \notin H^\infty(\mathbb{C}_- + i)$; see [10, Subsection 2.3].

Proof. The pair of lines $\mathbb{R}$, $\mathbb{R}+i$ admits separation in $\mathbb{C}_+ - Li$ since the triple $(\mathbb{R}, \mathbb{R}+i, \mathbb{C}_+ - Li)$ is equivalent to the triple in Example 2. $\square$

Example 4 ("the Poincaré pair"; see the Introduction)**.** Separation in $\mathbb{C}$ fails for the pair $(\mathbb{R}_-, \mathbb{R}_+)$, where $\mathbb{R}_- = (-\infty, 0]$, $\mathbb{R}_+ = [0, +\infty)$ (see [10]). However, an arbitrary function $f \in H^\infty(\mathbb{C}_+)$ coincides in $\mathbb{C}_+$ with the sum $f_+ + f_-$, where $f_\pm \in H^\infty(\mathbb{C} \setminus \mathbb{R}_\pm)$.

Proof. Put $\varphi(w) := f(\exp w)$, $w \in S_\pi := \{0 < \operatorname{Im} w < \pi\}$. By Example 3, for every $k > 0$ we have a representation

$$\varphi = \varphi_+ + \varphi_- \quad \text{in } S_\pi,$$

where $\varphi_+ \in H^\infty(\mathbb{C}_+)$, $\varphi_- \in H^\infty(\{\pi - 2k\pi < \operatorname{Im} w < \pi\})$. Therefore,

$$f(z) = \varphi_+(l_+(z)) + \varphi_-(l_-(z)), \quad z \in \mathbb{C}_+,$$

where $l_\pm$ denotes the branch of the logarithm analytic in $\mathbb{C}_\pm$ and such that $0 < \operatorname{Im} l_\pm < \pi$ in $\mathbb{C}_+$. Clearly, $\varphi_\pm \circ l_\pm \in H^\infty(\mathbb{C} \setminus \mathbb{R}_\pm)$. Moreover, the function $\varphi_+ \circ l_+$ admits analytic continuation from $\mathbb{C} \setminus \mathbb{R}_+$ to the "upper half" of the Riemann surface of the logarithm, i.e., to the union of the sheets

$$\mathcal{L}_j = \{2\pi j < \arg z \le 2\pi(j+1)\}, \quad j = 0, 1, 2, \dots,$$

and the function $\varphi_- \circ l_-$ admits analytic continuation "downward" to any *finite* union of sheets

$$\mathcal{L}'_j = \{-\pi + 2\pi j \le \arg z < \pi + 2\pi j\}, \quad j = 0, 1, \dots, k. \qquad \square$$

References

[1] N. Aronszajn, *Sur les décompositions des fonctions analytiques uniformes et sur leurs applications*, Acta Math. **65** (1935), 1–156.

[2] C. A. Berenstein and R. Gay, *Complex variables. An introduction*, Grad. Texts in Math., vol. 125, Springer-Verlag, New York, 1991. MR1107514 (92f:30001)

[3] M. Fréchet, *Sur certaines décompositions de la fonction complexe uniforme la plus générale*, Acta Math. **54** (1930), 37–79.

[4] D. Gaier, *Lectures on complex approximation*, Birkhäuser Boston, Inc., Boston, MA, 1987. MR0894920 (88i:30059b)

[5] ———, *Remarks on Alice Roth's fusion lemma*, J. Approx. Theory **37** (1983), 246–250. MR0693011 (84h:41032)

[6] T. Gamelin, *Uniform algebras*, Prentice-Hall, Inc., Englewood Cliffs, NJ, 1969. MR0410387 (53:14137)

[7] P. M. Gauthier, *Mittag-Leffler theorems on Riemann surfaces and Riemannian manifolds*, Canad. J. Math. **50** (1998), 547–562. MR1629823 (99h:30046)

[8] V. P. Khavin, *The separation of singularities of analytic functions*, Dokl. Akad. Nauk SSSR **121** (1958), no. 2, 239–242. (Russian) MR0098168 (20:4630)

[9] ———, *Golubev series and the analyticity on a continuum*, Linear and Complex Analysis Problem Book. 199 Research Problems (V. P. Khavin, S. V. Khrushchev, N. K. Nikolskiĭ, eds.), Lecture Notes in Math., vol. 1043, Springer-Verlag, Berlin etc., 1984, pp. 670–673. MR0734178 (85k:46001)

[10] V. P. Havin and A. H. Nersessian, *Bounded separation of singularities of analytic functions*, Entire Functions in Modern Analysis (Tel-Aviv, 1997), Israel Math. Conf. Proc., vol. 15, Bar-Ilan Univ., Ramat Gan, 2001, pp. 149–171. MR1890536 (2002m:31018)

[11] V. P. Havin, A. H. Nersessian, and J. Ortega Cerdà, *Uniform estimates in the Poincaré–Aronszajn theorem on the separation of singularities of analytic functions*, Preprint, 2003.

[12] B. S. Mityagin and G. M. Khenkin, *Linear problems of complex analysis*, Uspekhi Mat. Nauk **26** (1971), no. 4, 93–152; English transl. in Russian Math. Surveys **26** (1971), no. 4. MR0287297 (44:4504)

[13] H. Poincaré, *Sur les fonctions à espaces lacunaires*, Amer. J. Math. **14** (1892), 201–221.

[14] P. L. Polyakov and G. M. Khenkin, *Integral formulas for solution of the $\overline{\partial}$-equation, and interpolation problems in analytic polyhedra*, Trudy Moskov. Mat. Obshch. **53** (1990), 130–170; English transl., Trans. Moscow Math. Soc. **1991**, 135–175. MR097995 (92b:32008)

[15] P. L. Polyakov, *Continuation of bounded holomorphic functions from an analytic curve in general position into the polydisc*, Funktsional. Anal. i Prilozhen. **17** (1983), no. 3, 87–88; English transl., Funct. Anal. Appl. **17** (1983), no. 3, 237–239. MR0714234 (84k:32019)

[16] G. Valiron, *Fonctions analytiques*, Presses Univ. France, Paris, 1954. MR0061658 (15:861a)

[17] I. I. Privalov, *Boundary properties of analytic functions*, 2nd ed., GITTL, Moscow–Leningrad, 1950 (Russian); German transl., *Randeigenschaften analytischer Funktionen*, Deutscher Verlag des Wissenschaften, Berlin, 1956. MR0047765 (13:926h)

[18] L. Hörmander, *An introduction to complex analysis in several variables*, D. Van Nostrand Co., Inc., Princeton, NJ etc., 1966. MR0203075 (34:2933)

[19] A. G. Vitushkin, *Analytic capacity of sets in problems of approximation theory*, Uspekhi Mat. Nauk **22** (1967), no. 6, 141–199; English transl., Russian Math. Surveys **22** (1967), 130–200. MR0229838 (37:5404)

[20] A. V. Kozlov and V. P. Khavin, *Separation of singularities of analytic functions with the preservation of continuity up to the boundary* (in preparation). (Russian)

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