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AN A_2-PROOF OF STRUCTURE THEOREMS FOR CHEVALLEY GROUPS OF TYPES E_6 AND E_7

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In the present paper we prove the main structure theorem for Chevalley groups $G = \mathrm{G}(\Phi, R)$ of types $\Phi = \mathrm{E}_6, \mathrm{E}_7$ over a commutative ring R. More precisely, we describe subgroups in G normalized by the elementary subgroup $\mathrm{E}(\Phi, R)$. This result is not new, since structure theorems are known for all Chevalley groups [25, 27, 28, 30], [38]–[40], and [58, 61] (see [42, 65, 34, 56] for further references). The gist of the present paper resides not in the results themselves, but rather in the method of their proof based on the *geometry of exceptional groups*. We believe that this method is novel and of *significant* interest. Actually the *Schwerpunkt* of the present paper abides in a new descent procedure, which enables reduction to groups of smaller rank. This procedure is both simpler and more powerful than *any* other method known today. Our results on the geometry of the 27-dimensional module for Chevalley groups of type E_6 and of the 56-dimensional module for Chevalley groups of type E_7 pave the way to much more general results such as description of subgroups normalized by *some of* elementary matrices. Groups of types E_8 and F_4 can be handled in essentially the same style, and we intend to return to these cases in our subsequent publications. However, from a technical viewpoint the proofs in these cases are noticeably more involved because these groups do not have microweight representations.

§1. Introduction

The proofs of structure theorems are based either on reduction of the dimension of the ground ring, or on reduction of the rank of the group, or eventually on a combination of both. The early proofs for *classical* groups were based on reduction of rank [71], [17]–[19], and [23, 24, 20, 3, 9]; the dimension reduction proofs appeared later [60], [62]–[64], and [44, 45, 33]. At the same time for *exceptional* groups the first published proofs leant upon reduction of dimension (see [25, 27, 28, 30, 58, 61]). The paper [30] by E. Abe and K. Suzuki was a *near miss* as far as the normal structure theorem for Chevalley groups is concerned (see also [25]). In fact, a somewhat weaker result was proved in [30], namely, a description of the normal subgroups of the *elementary* Chevalley group $\mathrm{E}(\Phi, R)$ over a class of rings subject to a certain finiteness condition. However, as was

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noticed in [7], this finiteness condition can easily be discarded. On the other hand, a pregnant result by Taddei [58] tells us that for all irreducible root systems Φ of rank at least 2, the elementary subgroup is normal in the Chevalley group itself. This result infers that the description of normal subgroups in $\mathrm{E}(\Phi, R)$ can be extended to subgroups of $\mathrm{G}(\Phi, R)$ normalized by $\mathrm{E}(\Phi, R)$. This was observed in a paper by Vaserstein (see [61]). One should not be misled by the relatively short proofs in [61]. As a matter of fact, apart from the Taddei theorem, they rest upon two fairly profound results: the Chevalley simplicity theorem, which describes normal subgroups in the field case, and the factorization modulo radical results by Abe and Suzuki [25, 30]. All the calculations in the above papers are *elementary* in the sense that they do not depend on anything except the *Steinberg relations* [21, 37, 54]. In [65, 69] one can find a detailed discussion of elementary calculations in Chevalley groups over rings as well as dozens of further references.

In turn, the proofs in the papers [58, 61] were based on a method originated by Quillen and Suslin [23] and baptized *localization and patching* by Vaserstein. This method wields partitions of 1 in the ground ring, and the resulting formulas depend on the nature of the ring rather than on the root system alone. Clearly, reduction of rank must show up at some stage as well. But it only occurs at the level of zero-dimensional rings, where it comes under disguise of Gauss decomposition, and goes unnoticed by a casual reader. Following this track, Abe [27, 28] obtained definitive results for all Chevalley groups of rank at least 3. Later, using novel matrix techniques, in their marvelous papers [38]–[40] D. Costa and G. Keller removed *all* remaining restrictions in the case of groups of rank 2.

About the same time, the first-named author, E. B. Plotkin, and Stepanov [15] worked up a radically different approach to the main structure theorems for Chevalley groups of all types. This approach was based on the geometry of minimal modules. The crux of this method, which was christened *decomposition of unipotents* in [15], can be expressed as follows. Reduction to groups of smaller rank requires two types of calculations: elementary calculations, as above, and *stable* calculations that focus upon one column or one row of a matrix in a Chevalley group in an appropriate representation at a time. Technology of stable calculations was initiated by Matsumoto and Stein in their foundational papers [47] and [55]. Since then, it has been further developed (see references in [49, 65, 67, 69]). An important tool thereby are *weight diagrams*, whose use goes back to E. B. Dynkin. In the cases that actually arise in the proofs of structure theorems these diagrams coincide with the *crystal graphs* of M. Kashiwara. In [49, 65, 67], a thorough discussion of weight diagrams and a lot of related references can be found. Decomposition of unipotents for classical groups *in vector representations* was expounded in the papers [22, 8, 56] by the first-named author and Stepanov. In turn, the papers [65, 67, 69] (*Fortsetzung folgt*) by the first-named author and Plotkin recount decomposition of unipotents for exceptional groups.

The proofs presented in these publications gave *explicit formulas*, which depend on the root system Φ alone, and not on the nature of the ground ring[1]. For classical groups the proofs were based on small rank embeddings such as A_2 in A_l, C_2 in B_l or C_l, and finally, D_3 in D_l. At the same time, the proofs for exceptional groups based on the method of decomposition of unipotents turned out to be technically much more demanding. Mainly, this is due to the fact that the proofs in these cases required an explicit choice of the signs of the action structure constants as well as the signs of equations on the highest weight orbit. Moreover, these proofs are based on fairly large rank embeddings. Thus, the proof in the 27-dimensional representation of the group of type E_6 depends on reduction to

[1] Provided it is commutative! For noncommutative rings no such formulas exist even for the case of GL_n; see [56].

D_5 by means of unipotents coming from the embedding $\mathrm{A}_5 \subseteq \mathrm{E}_6$. Similarly, the proof in the 56-dimensional representation of the group of type E_7 depends on reduction to E_6 by means of unipotents coming from the embedding $\mathrm{A}_7 \subseteq \mathrm{E}_7$. An outline of this proof was presented in [65], while in [67] one can find every possible detail. In the paper [67] we called these proofs an 'A_5-proof' and an 'A_7-proof', respectively. As a matter of comparison, we mention that the proofs involving *adjoint* representations of Chevalley groups obtained by the first-named author and E. B. Plotkin implement reduction of E_6, E_7, and E_8 to A_5, D_6, and E_7, respectively, with the help of D_5, D_6, and D_8. Thus, it would be only natural to call these proofs a 'D_5-proof', a 'D_6-proof', and a 'D_8-proof', respectively. In [65, 69, 56], a thorough description of the entire project can be found. We see that every time one uses the highest rank subsystem of type A_l or D_l among all such subsystems contained in Φ.

Of course, for the proofs of the structure theorems themselves this is not at all significant. However, one of our broader intentions while developing the method of decomposition of unipotents for exceptional groups was to generalize the results of the papers [3] and [8]–[13]. These papers were devoted to subgroups of classical groups normalized by *some* elementary matrices. In particular, the groups described there were normalized by elementary matrices of some regularly embedded semisimple subgroups (that were called 'subsystem subgroups' by Liebeck and Seitz; see, e.g., [46]). As can be seen from the preceding paragraph, the results that could be obtained in this direction for the groups of type E_l by the method of decomposition of unipotents were not quite comforting, *as compared* with the classical cases. Thus, working in the 27-dimensional and adjoint representation of E_6, one could at most hope to describe the subgroups of $\mathrm{G}(\mathrm{E}_6, R)$ normalized by $\mathrm{E}(\mathrm{A}_5 + \mathrm{A}_1, R)$ or $\mathrm{E}(\mathrm{D}_5, R)$, respectively. At the same time, for all classical groups similar results could be obtained for all subsystems $\Delta \subseteq \Phi$ such that in Φ there are no roots orthogonal to Δ and the ranks of all irreducible[2] components of Δ are at least 2.

In the present paper, which is an exposition of the *Diplomarbeit* of the second-named author under the supervision of the first-named author, we present a proof for the groups of type E_6 and E_7, which is based on the reduction to the types A_5 and D_5 or, respectively, to the types E_6 and D_6. Such reduction is carried out in the 27-dimensional or, respectively, in the 56-dimensional representation of the groups by using only elements that belong to subgroups of type A_2. Let us cite some of the most perceptible features of this proof as compared with the previously known ones:

- It only employs embeddings $\mathrm{A}_2 \subseteq \mathrm{E}_6, \mathrm{E}_7$.
- Like decomposition of unipotents, it does not invoke results in dimension 0, including those pertaining to the field case. In fact, among other things, this argument gives a new proof of the Chevalley simplicity theorem [21, 37] for the types E_6 and E_7.
- It does not depend on the results of Abe and Suzuki [25, 30] concerning factorization modulo radical.
- It makes no reference whatsoever to *any* information concerning the signs of action structure constants.
- It does not depend on any information related either to signs, or to the explicit form of the equations defining the highest weight orbit. In fact, we *only* refer to the following

[2] Recall that D_2 is not irreducible, and that is why we started with the embeddings of D_3 in D_l. On the other hand, in [8]–[13] the summands A_2 and A_3 in orthogonal groups were excluded only to avoid technical complications due to the exceptional behavior of D_4 and B_3 related to their twisting to G_2. In fact, at the cost of some additional exertion, the main results of those papers can be extended to the case where A_2 and A_3 occur as irreducible components of Δ.

obvious consequence of these equations: for any element x of the Lie algebra we have $x_{\lambda\mu} = 0$ for any two weights $\lambda \neq \mu$ whose difference is not a root.

• This proof complies with the very essence of the problem and can serve as a model for a vast variety of subsequent generalizations.

• It removes any trace of difference between the classical cases on one hand and the exceptional cases on the other, and amply demonstrates the power of explicit matrix calculations for exceptional groups.

Of course, the most important point is the fact that it is an A_2-*proof*. Among other things, this proof enables a description of the subgroups in $\mathrm{G}(\mathrm{E}_6, R)$ normalized by the elementary subgroups in relatively small subgroups, such as $\mathrm{E}(3\mathrm{A}_2, R)$, or maybe, though we have not checked it, $\mathrm{E}(2\mathrm{A}_2 + \mathrm{A}_1, R)$! This shows that the methods of the present paper (unlike any of the previously known ones!) make it possible to carry all results of the papers [3] and [8]–[13] over to the exceptional groups, *with the same estimate* 2 *for the ranks of irreducible components*! We plan to return to the proof of these results in a subsequent paper. Moreover, many of the subsidiary results in the present paper are stated in the form that covers also subgroups in $\mathrm{GL}(27, R)$ or $\mathrm{GL}(56, R)$ normalized by $\mathrm{E}(\mathrm{E}_6, R)$ or $\mathrm{E}(\mathrm{E}_7, R)$, respectively.

The composition of the present paper is rather unconventional. Instead of postponing the main new part of the standard description of subgroups in $\mathrm{G}(\mathrm{E}_6, R)$ and $\mathrm{G}(\mathrm{E}_7, R)$ to the end, we start with it. Namely, in §2 we reproduce the core part of the proof, which we call *the proof from the Book*. This part is based on several auxiliary statements all of which are *immediately obvious* to any expert familiar with the methods of [71], [17]–[19], and [30, 55, 3, 56, 65, 67]. The remaining part of the paper is a detailed commentary to §2. In §§3–6 we briefly recall the basic notation and machinery used in the proof and demonstrate some of the subsidiary results. In §§7–9 we explain to pedestrians *why* the subsidiary results stated in §2 are obvious. The proofs in these sections could be made somewhat more concise. However, we wished to make our presentation entirely self-contained and independent even of the auxiliary results of the papers [25]–[30] and [61]. Also, to slightly enliven the routine calculations in these sections, we prove somewhat more than what is immediately required for our purposes. Finally, in §10 we formulate some further related problems.

The present paper expounds mainly *technical*, rather than ideological aspects of the work. The background History and Philosophy, as well as further applications are discussed in much greater detail in the talk by the authors "Structure of Chevalley groups: the proof from the Book" [41].

§2. A_2-proof of the structure theorem

Starting with this section, we assume that H is a subgroup of $\mathrm{G}(\Phi, R)$ normalized by $\mathrm{E}(\Phi, R)$. For the most part we assume that $\Phi = \mathrm{E}_6, \mathrm{E}_7$, and to indicate that the answer depends on whether we are currently speaking of E_6 or E_7, we write 'A risp B'. Such a record should be interpreted as follows: A occurs in the case of E_6, while B occurs in the case of E_7. Throughout the present paper, starting with this section, we strictly observe the following alphabetic coding: $\alpha, \beta, \gamma, \delta \in \Phi$ denote roots; $\lambda, \mu, \nu, \rho, \sigma, \tau \in \Lambda$ denote weights of the module V; $\xi, \zeta, \eta \in R$ denote elements of the ground ring; x, y, z, g, h denote elements of the group $\mathrm{G}(\Phi, R)$; and finally, u, v denote vectors in V. The remaining notation is standard and will be recalled as needed in §§3–6.

Our goal in this paper is to present a new proof of the following result, first established in full generality in the work of Abe, Abe–Suzuki, and Vaserstein (see [25, 30, 27, 28, 61]).

Theorem. *For any commutative ring R, the standard description of subgroups normalized by the elementary subgroup $\mathrm{E}(\Phi, R)$ holds in Chevalley groups $\mathrm{G}(\Phi, R)$, $\Phi = \mathrm{E}_6, \mathrm{E}_7$.*

In other words, for any such subgroup H there exists a unique ideal $I \trianglelefteq R$ such that

$$\mathrm{E}(\Phi, R, I) \leq H \leq \mathrm{C}(\Phi, R, I).$$

We state several subsidiary results to be used in the proof of this theorem. As we have already mentioned, all of these results are immediately evident to an expert. However, for convenience of beginners and outsiders, in §§4–9 we reproduce detailed calculations. Namely, Proposition 1 is verified in §4, Propositions 2 and 3 in §5, Propositions 4 and 5 in §7, and finally Proposition 6 in §§6, 8, 9.

After the work of Bass [35, 1] the following argument, known as *level reduction*, has become the usual and *de facto* almost *unique* way to reduce the verification of the standard description to a technically much easier extraction of a single nontrivial root element. This approach is adopted in the overwhelming majority of papers dedicated to description of normal subgroups.[3]

Proposition 1 (Level reduction)**.** *Assume that for any pair (R, I), $I \trianglelefteq R$, the following condition is satisfied: if H is a noncentral subgroup of $\mathrm{G}(\Phi, R/I)$ normalized by $\mathrm{E}(\Phi, R/I)$, then H contains a nontrivial elementary root unipotent $x_\alpha(\zeta)$, $\zeta \in R/I$, $\zeta \neq 0$. Then for any ring R the standard description of subgroups normalized by $\mathrm{E}(\Phi, R)$ holds in $\mathrm{G}(\Phi, R)$.*

Observe that the proof of this proposition in §4 is *the only* spot in the entire paper where we materially refer to a profound external result. Namely, we invoke the fact that the standard commutator formulas are satisfied in $\mathrm{G}(\Phi, R)$ for all ideals $I \trianglelefteq R$. In the case of exceptional groups, this result was first proved in full generality by Taddei and Vaserstein [58, 61] (see also [15, 65, 67, 43]). In principle, this theorem as well could be demonstrated by the methods of the present paper. However, to this end one would reproduce *extensive* portions of [65, 67] concerning elements of root type and Whitehead type lemmas.

Proposition 2. *If $\lambda - \alpha, \rho + \alpha \in \Lambda$, then $d(\lambda, \rho) \geq 2$. In other words, $\lambda - \rho \notin \Phi \cup \{0\}$.*

The following banal but basic observation accounts for the exclusion of the case of E_8 from the present paper.

Proposition 3. *If g is a root element of the group $\mathrm{G}(\Phi, R)$ in a microweight representation V, then $g_{\lambda\mu} = 0$ for any two weights λ, μ of the representation V such that $d(\lambda, \mu) \geq 2$.*

To the group of type E_8 having no microweight representations, the above proposition cannot be applied as is, without a much more detailed analysis of root type elements.

Proposition 4. *If $[g, x_\alpha(1)] = e$, then g lies in a proper parabolic subgroup of type P_2 risp P_1.*

Proposition 5. *If the commutator $[g, x_\alpha(\xi)x_\beta(\zeta)]$ is central, it equals e.*

Now we approach another major ancillary result known as *extraction from a proper parabolic*. Conceptually, the following statement is overtly and painfully obvious, nevertheless on a technical plane about half of the following text is wasted in its verification! Moreover, to abate references to the results of [31] and [50]–[53] concerning internal Chevalley modules, we only check this proposition in the following two special cases, where we actually use it: parabolic with Abelian or extra-special unipotent radical.

[3] The work of Golubchik (see [17]–[19]) provides a remarkable counterexample to this claim. In fact, in these papers the standard description of normal subgroups was established for classical groups over noncommutative rings in some situations where the validity of the standard commutator formulas was not known!

Proposition 6 (Extraction from a proper parabolic)**.** *If H contains a noncentral element in a proper parabolic subgroup P, then H contains a nontrivial elementary root unipotent.*

We mention two important special cases of this proposition. Actually, we only use it in one of these forms.

Corollary 1. *If $x = x_\alpha(\xi)x_\beta(\zeta)$ stabilizes a column of a matrix $g \in H$ but does not commute with g, then H contains a nontrivial elementary root unipotent.*

Proof. By assumption, $z = [g, x] \in H$ lies in a proper parabolic of type P_1 risp P_7. Since x does not commute with g, by Proposition 5 the element z is noncentral and thus, by Proposition 6 (the case of an Abelian unipotent radical), the group H contains a nontrivial elementary root unipotent. □

In the present paper the chief reduction is performed to parabolic subgroups with Abelian unipotent radical. The case of an extra-special unipotent radical is only summoned in the following context.

Corollary 2. *If the commutator $[g, x_\alpha(1)]$ of a noncentral element $g \in H$ with a root element $x_\alpha(1)$ is noncentral for some root $\alpha \in \Phi$, then H contains a nontrivial elementary root unipotent.*

Proof. If the commutator $[g, x_\alpha(1)]$ is central, then it is equal to e by Proposition 5. By Proposition 4, this implies that $g \in P_2$ risp $g \in P_1$, and it only remains to refer to Proposition 6 once again (this time for the case of an extra-special unipotent radical). □

The following charming little argument is the primary novel turn of the present proof as compared with the proofs in the papers [15, 65, 67] based on decomposition of unipotents.

Principal Lemma. *Fix a weight $\lambda \in \Lambda$. If for a noncentral element $g \in H$ we have $g'_{\sigma\lambda} = 0$ for all $\sigma \in \Lambda$, $d(\lambda, \sigma) = 2$, then H contains a nontrivial elementary root unipotent.*

Proof. 1) Take any two distinct weights $\mu, \nu \in \Lambda$ such that $d(\lambda, \mu) = d(\lambda, \nu) = 1$, and consider the root element

$$x = x(\mu, \nu) = x_\alpha(g'_{\nu\lambda})x_\beta(-g'_{\mu\lambda}),$$

where $\alpha = \lambda - \mu$, $\beta = \lambda - \nu$. Next, form the commutator $z = z(\mu, \nu) = [g, x]$. Take any $\rho \neq \mu, \nu$. By Proposition 2, the fact that $\rho + \alpha \in \Lambda$ or $\rho + \beta \in \Lambda$ implies that $d(\lambda, \rho) \geq 2$. On the other hand, $g'_{\rho\lambda} = 0$ for all $\rho \in \Lambda$, $d(\lambda, \rho) = 2$. This shows that $(xg^{-1})_{*\lambda} = g^{-1}_{*\lambda}$ and thus $(gx_\alpha(1)g^{-1})_{*\lambda} = v^\lambda$. Since $\lambda - \alpha, \lambda - \beta \in \Lambda$, and V is a microweight representation, we can conclude that $\lambda + \alpha, \lambda + \beta \notin \Lambda$, whence $z_{*\lambda} = v^\lambda$. It follows that if the element z is noncentral, Corollary 1 implies that H contains a nontrivial elementary root unipotent.

2) Thus, it only remains to treat the case where *all* the elements $z = z(\mu, \nu)$ that we get by varying the weights μ, ν in the preceding argument are central. Indeed, if z is central, then by Proposition 4 it equals e, so that g (and, with it, also g^{-1}) commutes with all $x = x(\mu, \nu)$. Take any ρ such that $d(\rho, \lambda) = 1$, whereas $d(\rho, \mu), d(\rho, \nu) \leq 1$. By comparing the entries of the matrices xg^{-1} and $g^{-1}x$ at the positions (ρ, μ) and (ρ, ν), we see that $g'_{\rho\lambda}g'_{\mu\lambda}, g'_{\rho\lambda}g'_{\nu\lambda} = 0$. By varying ν, we conclude that $g'_{\rho\lambda}g'_{\mu\lambda} = 0$ for all ρ such that $d(\rho, \lambda) = d(\rho, \mu) = 1$.

3) Now, we take any weight τ at distance 1 from λ. For any weight ρ of the type considered in the preceding clause, we have $x_\gamma(g'_{\mu\lambda})g'_{*\lambda} = g'_{*\lambda}$, where $\gamma = \lambda - \rho$. By Corollary 1, either H contains a nontrivial root element, or else the matrix g^{-1} commutes with $x = x_\gamma(g'_{\mu\lambda})$. Comparing the entries of the matrices xg^{-1} and $g^{-1}x$ at the positions

(τ,μ), we see that $g'_{\tau\lambda}g'_{\mu\lambda}=0$ also in the case where $d(\tau,\mu)=2$. In other words, if A is the ideal of R generated by the entries $g'_{\mu\lambda}$, $\mu\neq\lambda$, then $A^2=0$. Now reduction modulo A shows that $g_{\mu\lambda}\in A$ for all $\mu\neq\lambda$.

4) It only remains to consider the commutator $z=[g,x_\alpha(1)]$. As in Clause 1), the column $v=x_\alpha(1)g^{-1}_{*\lambda}$ differs from $g^{-1}_{*\lambda}$ only in the component $v_\lambda=g'_{\lambda\lambda}+g'_{\mu\lambda}$; all other components coincide. By the content of Clauses 2) and 3), this means that $gxg^{-1}_{*\lambda}$ is proportional to the standard base column v^λ, so that z lies in a proper parabolic of type P_1 risp P_7. But then, by Corollary 1, either H contains a nontrivial root unipotent, or z is central. On the other hand, if z is central, then H also contains a nontrivial root unipotent, now by Corollary 2. $\square$

Remark. In the preceding argument, we only used elements $x(\mu,\nu)$ for $d(\mu,\nu)=1$. As a matter of fact, the calculation in Clause 1) works also when $d(\mu,\nu)=2$. Using this observation, we could omit Clause 3) by an additional reference to Lemma 5. This extra degree of freedom is redundant here, but it becomes instrumental in the analysis of the twisted Chevalley group of type ${}^2\,\mathrm{E}_6$.

Now we are all set to finish the proof of the standard description for subgroups normalized by $\mathrm{E}(\Phi,R)$.

The proof from the Book. By Proposition 1 it suffices to show that any noncentral subgroup H in $G=\mathrm{G}(\Phi,R)$, $\Phi=\mathrm{E}_6,\mathrm{E}_7$, normalized by the elementary subgroup $\mathrm{E}(\Phi,R)$, contains a nontrivial elementary root unipotent. Indeed, let $g\in H$ be a noncentral element. Take any root $\alpha\in\Phi$ and consider the commutator $[g,x_\alpha(1)]\in H$. If the above commutator $[g,x_\alpha(1)]$ is central for at least one root α, then H contains a nontrivial root unipotent by Corollary 2. Thus, replacing g by $[g,x_\alpha(1)]=(gx_\alpha(1)g^{-1})x_\alpha(-1)$ if necessary, we can assume from the very start that $g\in H$ is a noncentral element of the following shape: the product of a root element by an elementary root element.

The element g differs from a root element in at most 6 risp 12 columns. By Proposition 3, this means that at least 21 risp 44 columns of g satisfy conditions of the principal lemma. Thus, H contains a nontrivial root element as claimed. $\square$

§3. Basic notation

The present paper is enveloped by a certain general context, and there is not a faintest chance for us to recall all necessary background here. The references [4, 21, 37] may serve as excellent generic sources on Chevalley groups and their representations. All more technical definitions related to linear groups over rings can be found in [42, 56] and those related to Chevalley groups over rings can be found in [25]–[30], [47, 54, 55, 58, 61], [65]–[67], and [69, 70]. We refer the reader to the same sources also for comprehensive bibliography. In this section we merely summon basic notation used in the sequel. For the most part, our notation pertaining to Lie algebras, root systems, Weyl groups, weights and representations is modeled after Bourbaki [5, 6].

1. Chevalley groups. Let Φ be a reduced irreducible root system of rank l (for the most part, we assume that $\Phi=\mathrm{E}_6$ or E_7). Next, let P be a lattice lying between the root lattice $Q(\Phi)$ and the weight lattice $P(\Phi)$. We fix an order on Φ and denote by $\Pi=\{\alpha_1,\dots,\alpha_l\}$, Φ^+, and Φ^- the respective sets of fundamental, positive and negative roots. Our numbering of the fundamental roots follows [5]. By δ we denote the maximal root of Φ with respect to this order. Thus, for $\Phi=\mathrm{E}_6$ and E_7 we have $\delta=\genfrac{}{}{0pt}{}{12321}{2}$ and $\delta=\genfrac{}{}{0pt}{}{234321}{2}$, respectively. We denote by $P(\Phi)_{++}$ the set of dominant weights for this order and recall that it consists of all nonnegative integral linear combinations of the fundamental weights $\overline{\omega}_1, \dots, \overline{\omega}_l$. By $W = W(\Phi)$ we denote the Weyl group of Φ.

Further, let R be a commutative ring with 1. It is classically known that, starting with this set of data, we can construct a *Chevalley group* $G = G_P(\Phi, R)$. This is the group of R-points of a certain affine group scheme $G = G_P(\Phi, -)$, known as the *Chevalley–Demazure scheme*. For the type of problems discussed here, we can limit ourselves without loss of generality to the *simply connected* (alias, universal) groups, for which $P = P(\Phi)$. Usually, for a simply connected group we drop any mention of P and write $\mathrm{G}(\Phi, R)$. Sometimes, to stress that the group is simply connected, we write $G_{\mathrm{sc}}(\Phi, R)$. The adjoint group for which $P = Q(\Phi)$ is denoted by $G_{\mathrm{ad}}(\Phi, R)$.

All usual constructions of Chevalley groups start with the choice of a Chevalley base e_α, $\alpha \in \Phi$, h_i, $1 \leq i \leq l$, in the complex simple Lie algebra L of type Φ. Next, the Chevalley algebra L_R spanned by e_α, h_i over R is considered. For any $\alpha, \beta, \alpha + \beta \in \Phi$ we have $[e_\alpha, e_\beta] = N_{\alpha\beta} e_{\alpha+\beta}$, and for a Chevalley base all structure constants $N_{\alpha\beta}$ are integers. Henceforth we fix a certain *positive* Chevalley base. In other words, we assume that $N_{\alpha_i \beta} > 0$ provided that $\alpha_i + \beta \in \Phi^+$, and if $a_j + \gamma = \alpha_i + \beta$ for a fundamental root a_j and a positive root γ, then $j > i$.

Among other things, the choice of a Chevalley base determines a split maximal torus $\mathrm{T}(\Phi, R)$ in the Chevalley group $\mathrm{G}(\Phi, R)$ and the parametrization of root unipotent subgroups X_α, $\alpha \in \Phi$, with respect to this torus. We fix such a parametrization and let $x_\alpha(\xi)$ be the elementary root unipotent corresponding to $\alpha \in \Phi$, $\xi \in R$. In the sequel we systematically use the Steinberg relations (R1)–(R8) among root unipotents $x_\alpha(\xi)$, and in particular the Chevalley commutator formula [4, 21, 37] without any specific reference. The group $X_\alpha = \{x_\alpha(\xi), \xi \in R\}$ is called an *elementary root subgroup*. The group $\mathrm{E}(\Phi, R) = \langle X_\alpha, \alpha \in \Phi \rangle$ generated by all elementary root subgroups is called the (absolute) *elementary subgroup* of the Chevalley group $\mathrm{G}(\Phi, R)$. *A priori*, the subgroup $\mathrm{E}(\Phi, R)$ depends on the choice of a maximal torus $T = \mathrm{T}(\Phi, R)$. Actually, the main result of [58] asserts exactly that for $\mathrm{rk}(\Phi) \geq 2$ the elementary subgroup does not depend on the choice of T. For the classical groups this was established earlier by Suslin and Kopeiko [23, 24, 20]; see [15, 33, 34, 43, 56, 65, 67] for the history of this result, other proofs and generalizations.

2. Weyl modules. Usually, we consider a Chevalley group together with its action on the *Weyl module* $V = V(\omega)$ for a dominant root ω. Usually, we assume that the highest weight ω of the module V is fundamental, $\omega = \overline{\omega}_r$. Whenever we talk of Chevalley groups of type E_6 or E_7, these groups are considered in one of the two contragredient 27-dimensional modules $V(\overline{\omega}_1)$ or $V(\overline{\omega}_6)$ risp the 56-dimensional module $V(\overline{\omega}_7)$. Observe that these modules are *microweight*; see [6, 47, 55, 49, 65, 67] and references there. We denote by $\Lambda = \Lambda(\omega)$ the set of weights of the module $V = V(\omega)$ *with multiplicities*. For a microweight representation, all weights are extremal and thus have multiplicity 1, so that in this case Λ coincides with the Weyl orbit of the highest weight $\Lambda = W\omega$.

In the sequel we fix an *admissible* base v^λ, $\lambda \in \Lambda$, of the module V. Recall that a base is said to be admissible if it consists of weight vectors and $x_\alpha(\xi)v^\lambda$ can be expressed as an integral linear combination of the vectors v^μ, $\mu \in \lambda$. For a microweight representation, we can normalize an admissible base in such a way that $x_\alpha(\xi)v^\lambda = v^\lambda + c_{\lambda\alpha}\xi v^{\lambda+\alpha}$, where all action structure constants $c_{\lambda\alpha}$ are equal to ± 1 ('Matsumoto lemma'; see [47, 55]). As a matter of fact, we always choose a *crystal* base where all the structure constants $c_{\lambda\alpha}$ are equal to $+1$ for the fundamental and the negative fundamental roots, $c_{\lambda\alpha} = +1$ if $\alpha \in \pm\Pi$. The existence of such a base follows from general results of G. Lusztig and M. Kashiwara; elementary proofs were furnished in [67] and [4].

We conceive a vector $a \in V$, $a = \sum a_\lambda v^\lambda$, as a coordinate *column* $a = (a_\lambda)$, $\lambda \in \Lambda$. In this setting it is natural to think of an element b of the contragredient module V^* as a coordinate *row* $b = (b_\lambda)$, $\lambda \in \Lambda$. However, with respect to the weights Λ^* of the contragredient module V^* the picture is reversed: the elements of V^* should be represented by *columns* $b = (b_\lambda)$, $\lambda \in \Lambda^*$, while the elements of V by *rows* $a = (a_\lambda)$, $\lambda \in \Lambda^*$. It should be stressed that in the present paper we index both columns and rows by the weights of V itself — indices such as λ, μ, ν, etc. always belong to Λ. In other words, it is convenient for us to index the coordinates of a vector in V^* by the weights of the module V and visualize them as rows (usually they are indexed by the weights of the module V^* itself and visualized as columns). It is precisely this convention (quite usual in the elementary linear algebra but rather uncommon in representation theory) that explains why the formulas describing the action of elements of G look differently for rows and for columns.

One of the key technical issues is that the components of these rows and columns are not linearly ordered but rather partially ordered, in accordance with the order on Λ determined by the choice of the fundamental system Π. Namely, we set $\lambda \geq \mu$ if $\lambda - \mu = \sum m_i \alpha_i$, where $m_i \geq 0$. Under the above interpretation of the elements of V, it is natural to regard the elements of the Chevalley group as matrices $g = (g_{\lambda\mu})$, $\lambda, \mu \in \Lambda$, with respect to the base v^λ. As usual, the columns of such a matrix are the coordinate columns of the vectors gv^μ, $\mu \in \Lambda$, with respect to the base v^λ, $\lambda \in \Lambda$. The following piece of notation is used throughout: the μth column of the matrix g is denoted by $g_{*\mu}$, while the λth row of this matrix is denoted by $g_{\lambda *}$.

§4. The standard description

In the present section we recall the definitions of the subgroups that arise in the standard description, and prove Proposition 1.

1. Standard description. We are interested in the subgroups in $\mathrm{G}(\Phi, R)$ that are normalized by $\mathrm{E}(\Phi, R)$. With this goal in mind, we introduce relative subgroups. Let I be an ideal of the ring R. With this ideal, we associate the corresponding reduction homomorphism $\phi_I : \mathrm{G}(\Phi, R) \longrightarrow \mathrm{G}(\Phi, R/I)$, which takes a matrix $g = (g_{\lambda\mu})$ to its class $\overline{g} = (g_{\lambda\mu} + I)$ modulo I. The kernel of this homomorphism is denoted by $G_I = \mathrm{G}(\Phi, R, I)$ and is called the *principal congruence subgroup* in G of level I. The inverse image of the center of the group $\mathrm{G}(\Phi, R/I)$ under ϕ_I is denoted by $C_I = \mathrm{C}(\Phi, R, I)$ and is called the *full congruence subgroup* of level I.

By definition, the *relative elementary subgroup* $\mathrm{E}(\Phi, R, I)$ of level I is the normal subgroup of $\mathrm{E}(\Phi, R)$ generated by all elementary root unipotents of level I. In other words,

$$\mathrm{E}(\Phi, R, I) = \langle x_\alpha(\xi), \alpha \in \Phi, \xi \in I \rangle^{\mathrm{E}(\Phi, R)}.$$

From the Chevalley commutator formula we can easily deduce that $\mathrm{E}(\Phi, R, I)$ is generated *as a subgroup* by the elements $x_\alpha(\zeta) x_{-\alpha}(\xi) x_\alpha(-\zeta)$, where $\alpha \in \Phi$, $\xi \in I$, $\zeta \in R$ (this fact was noticed independently in 1976 by Vaserstein and Suslin [16] in the case of SL_n and by Tits [59] in the general case). In the sequel, we consider the subgroups

$$X_\alpha(I) = X_\alpha \cap \mathrm{G}(\Phi, R, I) = \{x_\alpha(\xi), \xi \in I\}.$$

It is well known that W acts transitively on the roots of a given length. It follows that if Φ is a simply laced irreducible root system, then for any root $\alpha \in \Phi$ we have $\mathrm{E}(\Phi, R, I) = X_\alpha(I)^{\mathrm{E}(\Phi, R)}$.

We make a substantial use of the following crucial result. For two elements x, y of a group G, we denote by $[x, y]$ their left-normed commutator $xyx^{-1}y^{-1}$. Next, for any two subgroups $H, F \leq G$, by $[H, F]$ we understand their mutual commutator subgroup generated by all commutators $[x, y]$, $x \in H$, $y \in F$. The following result plays a central role in the verification of level reduction.

Lemma 1. *Let Φ be a reduced irreducible root system of rank at least 2. Then for any ideal $I \trianglelefteq R$ the subgroup $\mathrm{E}(\Phi, R, I)$ is normal in $\mathrm{G}(\Phi, R)$. In fact, we have*

$$[\mathrm{E}(\Phi, R), \mathrm{C}(\Phi, R, I)] \le \mathrm{E}(\Phi, R, I).$$

Actually, for rank greater than or equal to 3, equality occurs in the above inclusion, whereas the only exceptions in rank 2 are Chevalley groups of types B_2 or G_2 over a ring R with a residue field $\mathbb{F}_2$. For exceptional groups, this result was first proved in [58] and [61] (in [15, 43, 65, 67] different proofs can be found for the cases of $\Phi = \mathrm{E}_l$ relevant for the present paper; for classical groups compare also [3, 9, 34, 44, 45, 56] and [60]–[64]). The relations

$$[\mathrm{G}(\Phi, R), \mathrm{E}(\Phi, R, I)] = [\mathrm{E}(\Phi, R), \mathrm{C}(\Phi, R, I)] = \mathrm{E}(\Phi, R, I)$$

are called the *standard commutator formulas*.

Definition. We say that the *standard description* of the subgroups in $\mathrm{G}(\Phi, R)$ normalized by $\mathrm{E}(\Phi, R)$ is true if for any such subgroup H there exists a unique ideal $I \trianglelefteq R$ such that

$$\mathrm{E}(\Phi, R, I) \le H \le \mathrm{C}(\Phi, R, I).$$

2. Level reduction. Usually, the ideal appearing in the above definition is referred to as the *level* of H. In conjunction with the standard commutator formulas, the level of a subgroup H is uniquely determined by the equation $[H, \mathrm{E}(\Phi, R)] = \mathrm{E}(\Phi, R, I)$. This is why many authors stipulate that the standard commutator formulas are satisfied in the definition of the standard description.

In general, the largest ideal I such that $\mathrm{E}(\Phi, R, I) \le R$ is called the *lower level* of H. Now we describe the lower level of a subgroup for the case of a simply laced system. Whenever the standard description holds, the lower level coincides with the level. We continue to assume that H is a subgroup of $\mathrm{G}(\Phi, R)$ normalized by $\mathrm{E}(\Phi, R)$.

Lemma 2. *Let Φ be a simply laced reduced irreducible root system of rank at least 2. Then for any root $\alpha \in \Phi$ the set*

$$I = I_\alpha = \{\xi \in R \mid x_\alpha(\xi) \in H\}$$

is an ideal in R. This ideal does not depend on α. Furthermore, $\mathrm{E}(\Phi, R, I) \le H$ and I is the largest ideal with this property.

Proof. From the additivity $x_\alpha(\xi + \eta) = x_\alpha(\xi)x_\alpha(\eta)$ it follows that I is closed under addition. Since by assumption Φ is simply laced and its rank is at least 2, every root α is contained in a subsystem of type A_2. In other words, there exists a root β at the angle of $\pi/3$ with α. In this case the Chevalley commutator formula takes the following form:

$$x_\beta(\pm\zeta\xi) = [x_{\beta-\alpha}(\zeta), x_\alpha(\xi)] \in H$$

for any $\zeta \in R$. Thus, $RI_\alpha \subseteq I_\beta$, and by symmetry also $RI_\beta \subseteq I_\alpha$. Comparing these inclusions, we see that $RI_\alpha = I_\alpha$, so that I_α is an ideal and, moreover, $I_\alpha = I_\beta$ for all α, β at an angle of $\pi/3$. Since any two nonorthogonal roots α, β lie in a subsystem of type A_2, it follows that $I_\alpha = I_\beta$ unless α and β are orthogonal. Finally, since Φ is irreducible, two arbitrary roots α, β can be joined by a chain in which any two *adjacent* roots are not orthogonal, so that $I = I_\alpha$ does not depend on the choice of α. Now the inclusion $\mathrm{E}(\Phi, R, I) \le H$ and the maximality of I among all ideals with this property follow from the fact that $\mathrm{E}(\Phi, R, I)$ coincides with the normal closure of $X_\alpha(I)$ in $\mathrm{E}(\Phi, R)$. $\square$

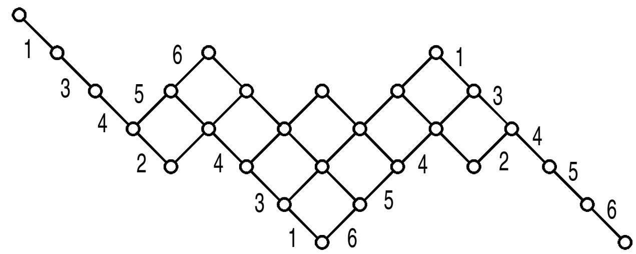

FIGURE 1. $(\mathrm{E}_6, \overline{\omega}_1)$.

A similar but somewhat subtler result holds also for multiply laced root systems. However, in general the sets I_α for a long root α, and I_β for a short root β, may fail to coincide; see [25, 27, 28, 30, 39, 40, 61] for details. Since, basically, we are interested in the root systems of type E_l, we do not discuss similar phenomena typical of a multiply laced root system in the case where 2 is not invertible.

Proof of Proposition 1. Let H be a subgroup in $\mathrm{G}(\Phi, R)$ normalized by $\mathrm{E}(\Phi, R)$, and let I be the lower level of the subgroup H. Consider the image $\overline{H} \leq \mathrm{G}(\Phi, R/I)$ of the subgroup H under reduction π_I modulo I. Clearly, $\overline{H}$ is normalized by the subgroup $\mathrm{E}(\Phi, R/I) = \pi_I(E(\Phi, R))$. By assumption, if $\overline{H}$ in noncentral, then it contains an elementary root unipotent $x_\alpha(\overline{\xi}) = x_\alpha(\xi + I)$ for an appropriate $\xi \notin I$. This means that H contains an element of the form $h = x_\alpha(\xi)g$ for some $g \in \mathrm{G}(\Phi, R, I)$. Let β be a root at an angle of $2\pi/3$ with α. Then

$$[h, x_\beta(1)] = {}^{x_\alpha(\xi)}[g, x_\beta(1)][x_\alpha(\xi), x_\beta(1)] \in H.$$

By the standard commutator formula, the first factor belongs to $\mathrm{E}(\Phi, R, I) \leq H$, so that the second factor $x_{a+\beta}(\pm\xi)$ also belongs to H. But $\xi \notin I$, and this contradicts the definition of the lower level. $\square$

§5. THE MINIMAL MODULE

In this section we introduce the principal tool used in the present paper, namely, the 27-dimensional module $V = V(\overline{\omega}_1)$ risp the 56-dimensional module $V = V(\overline{\omega}_7)$. Here we prove Propositions 2 and 3.

1. Weight diagrams. In Figures 1 and 2 we reproduce the *weight diagrams* (alias the *crystal graphs*) of these modules.

The Weyl group $W = W(\mathrm{E}_l)$ acts transitively on the set of weights $\Lambda = \Lambda(\omega)$ of the module V. On the other hand, the orbits of the Weyl group on the pairs of weights (λ, μ), $\lambda, \mu \in \Lambda$, are determined by a unique invariant, namely, by the distance $d(\lambda, \mu)$ in the *weight graph* (which displays the edges corresponding to *all* positive roots and not merely those corresponding to the simple ones). For E_6, this distance can take three values; it equals 0 if $\lambda = \mu$; it equals 1 if $\lambda - \mu \in \Phi$; finally, it equals 2 if $\lambda \neq \mu$ and the difference $\lambda - \mu$ is not a root. For E_7, the fourth value $d(\lambda, \mu) = 3$ is also possible, and for every weight λ there exists a *unique* weight μ at distance 3 from λ, namely $\mu = -\lambda$. Clearly, for every root there are exactly 16 risp 27 weights at distance 1, and 10 risp 27 weights at distance 2 from this weight. For example, if $\lambda = \omega$, then the weights at distance 2 from ω are exactly the weights to the right of exactly two edges marked by 1 risp 7.

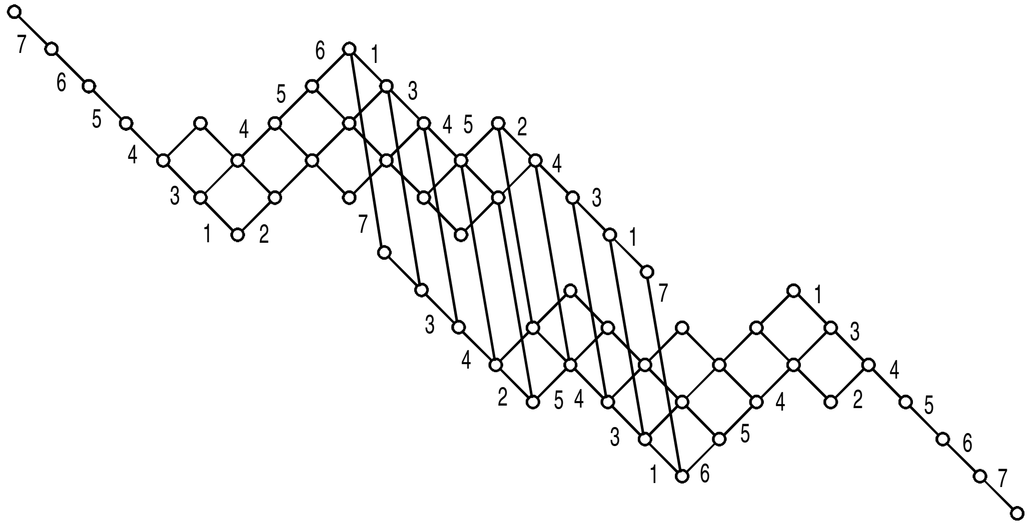

FIGURE 2. $(\mathrm{E}_7, \overline{\varpi}_7)$.

2. Lemmas on weights. We mention a handful of obvious statements concerning the weights of the module V.

Lemma 3. *If $d(\lambda,\mu)=1$, then there exists a root $\alpha\in\Phi$ such that $\lambda+\alpha\in\Lambda$, $\mu-\alpha\in\Lambda$.*

Proof. It suffices to take $\alpha=\mu-\lambda$. □

Note that, since V is a microweight representation, we automatically have $\mu+\alpha\notin\Lambda$.

Lemma 4. *If $d(\lambda,\mu)\geq 2$, then there exists at most one root $\alpha\in\Phi$ such that $\lambda+\alpha,\mu+\alpha\in\Lambda$. In particular, there exists a root $\beta\in\Phi$ with $\lambda+\beta\in\Lambda$, $\mu+\beta\notin\Lambda$.*

Proof. Indeed, any pair of weights at distance 2 can be transformed to the pair $(\omega,-\omega^*)$ risp $(\omega,-\omega+\alpha_7)$ by an element of the Weyl group. If $\omega+\alpha\in\Lambda$, then $\alpha\in\Phi^-$, and if $-\omega^*+\alpha\in\Lambda$ risp $-\omega+\alpha_7+\alpha\in\Lambda$, then $\alpha\in\Phi^+$ risp $\alpha\in\Phi^+$ or $a=-\alpha_7$. In the case of $\Phi=\mathrm{E}_7$, any pair of weights at distance 3 can be transformed to the pair $(\omega,-\omega)$, and the same argument shows that there is no root α such that $\lambda+\alpha,\mu+\alpha\in\Lambda$. Thus, out of 16 risp 27 roots $\alpha\in\Phi$ such that $\lambda+\alpha\in\Lambda$, at most 0 risp 1 root has the property that $\mu+\alpha\in\Lambda$. □

Lemma 5. *If $d(\lambda,\omega)=d(\lambda,\rho)=1$, then there exists a weight $\nu\neq\mu$ such that $d(\lambda,\nu)=1$ and $\rho+(\nu-\lambda)\notin\Lambda$.*

Proof. Since $d(\lambda,\rho)=1$, there are 10 risp 16 roots β such that $\lambda+\beta,\rho+\beta\in\Lambda$. Altogether, there are 16 risp 27 roots such that $\lambda+\alpha\in\Lambda$, and the condition $\mu\neq\lambda+\alpha$ prohibits at most one of then. Since $15>10$ risp $26>16$, there exists a root $\alpha\in\Phi$ such that $\nu=\lambda+\alpha\in\Lambda$, $\nu\neq\mu$, whereas $\rho+\alpha\notin\Lambda$, as claimed. □

Proof of Proposition 2. Since the Weyl group acts transitively on the pairs of weights at distance 1, there is no loss of generality in assuming that $\lambda=\omega$, and $\alpha=\alpha_1$ risp $\alpha=\alpha_7$. Thus, α occurs in the expression of $\lambda-\rho$ as a linear combination of the fundamental roots with coefficient 2. But this means that $\lambda-\rho$ cannot be a root because α occurs in the expression of the maximal root with coefficient 1. □

3. Equations determining the root elements. Recall that $g \in G$ is called a *root element* if it is conjugate to $x_\alpha(\xi)$. Since V is a microweight module, this is equivalent to g having the form $e + x$, where x is a root element of the Chevalley algebra L_R, or, in other words, x is conjugate to ξe_α. Indeed, the assumption that V is a microweight module amounts to the fact that $x^2 = 0$ for any root element $x \in L_R$. The following straightforward observation is the key to our proof.

Lemma 6. *Let $x \in L_R$. Then*
i) $x_{\lambda\mu} = 0$ *for all* $\lambda, \mu \in \Lambda$ *such that* $d(\lambda, \mu) \geq 2$;
ii) $x_{\lambda\mu} = \pm x_{\rho\sigma}$ *for all* $\lambda, \mu, \rho, \sigma \in \Lambda$ *such that* $d(\lambda, \mu) = d(\rho, \sigma) = 1$ *and* $\lambda - \mu = \rho - \sigma$.

Proof. Consider the expansion of x in the Chevalley base $x = \sum x_\alpha e_\alpha + \sum x_i h_i$, where the first sum is taken over all $\alpha \in \Phi$, and the second is taken over $i = 1, \dots, l$. With respect to an admissible base, the second sum is represented by a diagonal matrix and, thus, is irrelevant. In the sequel we look only at the first sum. Clearly, for $d(\lambda, \mu) = 2$ the matrix entries of all e_α at the position (λ, μ) are equal to 0; thus, i) is obvious. On the other hand, if $d(\lambda, \mu) = 1$, then the *only* elementary root element e_α for which the matrix entry at the position (λ, μ) is distinct from 0 is the element $e_{\lambda-\mu}$. Now the claim of ii) immediately follows from the fact that the base v^λ is admissible. □

Proof of Proposition 3. Indeed, V being a microweight module, we have $g - e \in L_R$ and a reference to the preceding lemma finishes the proof. □

§6. Parabolic subgroups

In the present section, we recall some basic facts about parabolic subgroups. These facts will be used to prove Propositions 4 and 6.

1. Parabolic subgroups of GL_n. The group $\mathrm{GL}(n, R)$ acts on the left on the free *right* R-module $V = R^n$ of rank n, and on the right on the free *left* R-module $V^* = {}^nR$ of rank n. As usual, we depict the elements of R^n as columns of height n over the ring R, and the elements of ${}^nR$ as rows of length n over R. Let $e_1, \dots, e_n$ and $e^1, \dots, e^n$, be the standard bases of R^n and ${}^nR$, respectively. In other words, let $e_i = e_{*i}$ be the ith column of the identity matrix, and $e^i = e_{i*}$ its ith row.

By P_i, $i = 1, \dots, n-1$, we denote the ith standard *maximal parabolic subgroup* in $G = \mathrm{GL}(n, R)$. From a geometric viewpoint, this is exactly the stabilizer of the submodule $V_i = \langle e_1, \dots, e_i \rangle \leq V$. In the matrix form, P_i is realized as the following group of upper block triangular matrices:

$$P_i = \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix}, x \in \mathrm{GL}(i, R), y \in M(i, n-i, R), z \in \mathrm{GL}(n-i, R) \right\}.$$

Any subgroup conjugate to P_i is called a *parabolic subgroup of type* P_i.

In the sequel we need also parabolic subgroups that are not maximal. Namely, if $1 \leq i_1 < i_2 < \dots < i_r \leq n-1$ is a set of indices, then

$$P_{i_1 \dots i_r} = P_{i_1} \cap \dots \cap P_{i_r}.$$

Thus, from the above geometric viewpoint, the group $P_{i_1 \dots i_r}$ is the stabilizer of the following flag of submodules:

$$0 < \langle e_1, \dots, e_{i_1} \rangle < \langle e_1, \dots, e_{i_2} \rangle < \dots < \langle e_1, \dots, e_{i_r} \rangle < V.$$

As a matter of fact, to avoid confusion with parabolic subgroups in $\mathrm{G}(\mathrm{E}_l, R)$, in the sequel it will be more convenient to describe the subgroup $P_{i_1 \dots i_r}$ by the sequence of codimensions in this flag, and we shall write

$$P_{i_1 \dots i_r} = P(i_1, i_2 - i_1, \dots, i_r - i_{r-1}, n - i_r)$$

and call $P_{i_1\dots i_r} = P(\nu)$ a parabolic subgroup of type

$$\nu = (n_1, \dots, n_{r+1}) = (i_1, i_2 - i_1, \dots, i_r - i_{r-1}, n - i_r).$$

Clearly, ν is a partition of n or, in other words, $n = n_1 + \dots + n_{r+1}$. Observe that the order of the summands n_i of ν is essential: after a permutation of the summands we can get a subgroup not conjugate to the initial one.

Any parabolic subgroup P admits Levi decomposition or, in other words, can be presented as a semidirect product $P = L_P \leftthreetimes U_P$ of its unipotent radical U_P and some (or, for that matter, any) Levi subgroup L_P. For example, for a maximal parabolic subgroup P_i, Levi decomposition takes the form $P_i = L_i \leftthreetimes U_i$, where

$$U_i = \left\{ \begin{pmatrix} e & y \\ 0 & e \end{pmatrix}, y \in M(i, n-i, R) \right\},$$

and as L_i we can take, *for example*,

$$L_i = \left\{ \begin{pmatrix} x & 0 \\ 0 & z \end{pmatrix}, x \in \mathrm{GL}(i, R), z \in \mathrm{GL}(n-i, R) \right\}.$$

Together with the subgroup P_i, we consider also the *opposite* subgroup P_i^-, which stabilizes the submodule in V generated by $e_{i+1}, \dots, e_n$. In particular, P_i^- is a subgroup of type P_{n-i} rather than of type P_i. In matrices, P_i^- has the form

$$P_i^- = \left\{ \begin{pmatrix} x & 0 \\ y & z \end{pmatrix}, x \in \mathrm{GL}(i, R), y \in M(n-i, i, R), z \in \mathrm{GL}(n-i, R) \right\}$$

or, in other words, $P_i^- = P_i^t$. The unipotent radical U_i^- of the group P_i^- equals U_i^t.

2. Parabolic subgroups of a Chevalley group. The parabolic subgroups in $\mathrm{G}(\mathrm{E}_l, R)$ are intersections of *some* parabolic subgroups of $\mathrm{GL}(n, R)$ with $\mathrm{G}(\mathrm{E}_l, R)$. For rank reduction we *only* need maximal parabolic subgroups in $\mathrm{G}(\mathrm{E}_l, R)$ of the following two types:

- parabolic subgroups with Abelian unipotent radical;
- parabolic subgroups with extra-special unipotent radical.

The structure of such groups is studied thoroughly; see, for example, [31] and [50]–[53]. However, to minimize the reference to these papers, we do everything by hand. In fact, in the proof of Proposition 7 (which in turn is a key step in the proof of Proposition 6) we refer to Theorem 2 of [31]. This is done only to slightly shorten the paper. In §8 we explain how we could completely avoid any such reference, replacing it by a couple of pages of routine calculations.

To see what a maximal parabolic subgroup P_i of the group $\mathrm{G}(\mathrm{E}_l, R)$ looks like in matrix form, it suffices to consider the weight diagram $(\mathrm{E}_6, \overline{\omega}_1)$ risp $(\mathrm{E}_7, \overline{\omega}_7)$ and to remove all the edges marked by i thereof. The two above types of maximal parabolic subgroups can be described as follows.

The extra-special case: these are the *normalizers of the root subgroups* X_α. They can be characterized as the maximal parabolic subgroups P_i obtained by deleting the root α_i joined to the maximal root of Φ. In other words, P_i equals P_2 risp P_1.

The Abelian case: these are the *stabilizers of the weight subspaces* V^λ of the minimal module V. They can be characterized as the maximal parabolic subgroups P_i obtained by deleting the root α_i for the same i as that of the highest weight $\overline{\omega}_i$ of the module V. In other words, P_i equals P_1 risp P_7.

In matrices, these groups can be described as follows:

$$P_2 = \mathrm{G}(\mathrm{E}_6, R) \cap P(6, 15, 6), \qquad P_1 = \mathrm{G}(\mathrm{E}_6, R) \cap P(1, 16, 10),$$
$$P_1 = \mathrm{G}(\mathrm{E}_7, R) \cap P(12, 32, 12), \quad P_7 = \mathrm{G}(\mathrm{E}_7, R) \cap P(1, 27, 27, 1).$$

Remark. Clearly, since we consider the intersections of these groups with $\mathrm{G}(\mathrm{E}_l, R)$, further equations on the entries of these matrices arise, apart from the fact that some of these entries are equal to 0. For example, the Levi subgroup L_2 of the parabolic subgroup $P_2 \le \mathrm{G}(\mathrm{E}_6, R)$ is isomorphic to $\mathrm{GL}(6, R)$ in the representation $g \mapsto \mathrm{diag}(g, \bigwedge^4(g), g)$. However, in the sequel we *could* completely ignore these additional restrictions and do all calculations inside the Levi subgroup of the corresponding parabolic subgroup of $\mathrm{GL}(n, R)$. In fact, the *only* reference to these further equations is buried somewhere in the proof of Proposition 7.

Let Δ_j be the subsystem of Φ generated by $\Pi \setminus \{\alpha_j\}$, $\Sigma_j = \Phi^+ \setminus \Delta_j$. Clearly,

$$\Delta_j = \Big\{\alpha = \sum m_i\alpha_i, m_j = 0\Big\}, \quad \Sigma_j = \Big\{\alpha = \sum m_i\alpha_i, m_j > 0\Big\}.$$

Let $U_j(h)$ denote the hth term of the lower central series of the group U_j. Obviously, in our case $U_j(h) = \prod X_\alpha$, where $\alpha \in \Sigma_j(h) = \{\alpha = \sum m_i\alpha_i, m_j \ge h\}$. In the Abelian case we have $U_j(2) = e$, and in the extra-special case $U_j(2) = [U_j, U_j] = \mathrm{Cent}(U_j) = X_\delta$ and $U_j(3) = e$.

§7. The centralizer of a root element

In the present section we calculate the centralizer of an elementary root element $x_\alpha(\xi)$. The following fact can be verified by a direct calculation.

Lemma 7. *For all $g \in \mathrm{GL}(n, R)$, $\alpha \in \Phi$, and $\xi \in R$, the following formulas are satisfied*:

$$(x_\alpha(\xi)g)_{\lambda\mu} = g_{\lambda\mu} \pm \xi g_{\lambda-\alpha,\mu}, \qquad (gx_\alpha(\xi))_{\lambda\mu} = g_{\lambda\mu} \pm \xi g_{\lambda,\mu+\alpha}.$$

One could be more specific about the signs in these formulas; see [67]–[69] and [14] on this matter. But an absolutely remarkable feature of the *proof from the Book* as compared with the decomposition of unipotents is that now *this is not necessary*!

Remark. In the first formula, $x_\alpha(\xi)$ acts in the direction of α, and in the second it acts in the direction of $-\alpha$. Recall that this is due to the fact that we *agreed* to index both rows and columns of the matrix g by the weights of Λ. In reality, the weights of Λ index only the rows and the components of columns of the matrix g, whereas the columns and the components of rows should be indexed by the weights of the *contragredient* module.

As an illustration, we adduce the explicit form of the element $x_\delta(\xi)$ in the case where $\alpha = \delta$ is the maximal root. For an appropriate numbering of weights, $x_\delta(1)$ has the form

$$x_\delta(1) = \begin{pmatrix} e_6 & 0 & e_6 \\ 0 & e_{15} & 0 \\ 0 & 0 & e_6 \end{pmatrix}, \qquad \begin{pmatrix} e_{12} & 0 & e_{12} \\ 0 & e_{32} & 0 \\ 0 & 0 & e_{12} \end{pmatrix}.$$

In Proposition 1.1 of [67] it was noted that, for a crystal base, the element $x_\delta(1)$ exercises *all* additions with the sign $+1$.[4]

Remark. Since *all* calculations in the present paper are performed by using Figures 1 and 2 rather than matrices, an explicit choice of a linear order on weights does not really matter. Nevertheless, for a reader not familiar with the use of weight diagrams to have some chance to develop a feeling of how to pass from pictures to matrices, let us mention that the above numbering can be designed as follows. We present the set Λ as the disjoint union $\Lambda = \Theta \sqcup \Xi \sqcup \Theta^*$, with the 6 elements of Θ ordered by descending height,

$$\Theta = \Big\{\omega,\ \omega - {\textstyle\genfrac{}{}{0pt}{}{10000}{0}},\ \omega - {\textstyle\genfrac{}{}{0pt}{}{11000}{0}},\ \omega - {\textstyle\genfrac{}{}{0pt}{}{11100}{0}},\ \omega - {\textstyle\genfrac{}{}{0pt}{}{11110}{0}},\ \omega - {\textstyle\genfrac{}{}{0pt}{}{11111}{0}}\Big\},$$

[4]This is a peculiar property of the representations we consider. It does not necessarily occur even for microweight representations. However, for microweight representations these signs are all equal; see [14, Theorem 4].

risp 12 elements of Θ in the following order:

$$\Theta = \Big\{\omega,\ \omega - \begin{smallmatrix}000001\\0\end{smallmatrix},\ \omega - \begin{smallmatrix}000011\\0\end{smallmatrix},\ \omega - \begin{smallmatrix}000111\\0\end{smallmatrix},\ \omega - \begin{smallmatrix}001111\\0\end{smallmatrix},\ \omega - \begin{smallmatrix}001111\\1\end{smallmatrix},$$
$$\omega - \begin{smallmatrix}011111\\0\end{smallmatrix},\ \omega - \begin{smallmatrix}011111\\1\end{smallmatrix},\ \omega - \begin{smallmatrix}012111\\1\end{smallmatrix},\ \omega - \begin{smallmatrix}012211\\1\end{smallmatrix},\ \omega - \begin{smallmatrix}012221\\1\end{smallmatrix},\ \omega - \begin{smallmatrix}012222\\1\end{smallmatrix}\Big\}.$$

Next we put 15 elements of Ξ risp 32 weights of Ξ in an *arbitrary* order respecting height. Finally, we put the 6 elements of Θ^* ordered by descending height,

$$\Theta^* = \Big\{\omega - \begin{smallmatrix}12321\\2\end{smallmatrix},\ \omega - \begin{smallmatrix}22321\\2\end{smallmatrix},\ \omega - \begin{smallmatrix}23321\\2\end{smallmatrix},\ \omega - \begin{smallmatrix}23421\\2\end{smallmatrix},\ \omega - \begin{smallmatrix}23431\\2\end{smallmatrix},\ \omega - \begin{smallmatrix}23432\\2\end{smallmatrix} = -\omega^*\Big\},$$

risp 12 elements of Θ^* in the following order:

$$\Theta^* = \Big\{\omega - \begin{smallmatrix}234321\\2\end{smallmatrix},\ \omega - \begin{smallmatrix}234322\\2\end{smallmatrix},\ \omega - \begin{smallmatrix}234332\\2\end{smallmatrix},\ \omega - \begin{smallmatrix}234432\\2\end{smallmatrix},\ \omega - \begin{smallmatrix}235432\\2\end{smallmatrix},\ \omega - \begin{smallmatrix}2355432\\3\end{smallmatrix},$$
$$\omega - \begin{smallmatrix}245432\\2\end{smallmatrix},\ \omega - \begin{smallmatrix}245432\\3\end{smallmatrix},\ \omega - \begin{smallmatrix}246432\\3\end{smallmatrix},\ \omega - \begin{smallmatrix}246532\\3\end{smallmatrix},\ \omega - \begin{smallmatrix}246542\\3\end{smallmatrix},\ \omega - \begin{smallmatrix}246543\\3\end{smallmatrix}\Big\}.$$

Usually, that sort of painstaking precision is required *only* at the moment when one chooses to *actually* calculate with elements of the group $\mathrm{G}(\mathrm{E}_l, R)$ as matrices of degree 27 risp 56 (or 248 for the group of type E_8). For example, this is needed if we wish to determine the order of this element, its eigenvalues, etc. However, only *very* seldom do we something of the sort by hand. In all usual calculations, including *all* calculations of the present paper, the weight diagrams $(\mathrm{E}_6, \overline{\omega}_1)$ risp $(\mathrm{E}_7, \overline{\omega}_7)$ are an expedient substitute of matrices.

Lemma 8. *If an element $g \in \mathrm{GL}(n, R)$ commutes with a root element $x_\alpha(\xi)$ for a root $\alpha \in \Phi$, then*

1) $\xi g_{\lambda\mu} = 0$ *if* $\lambda + \alpha \in \Lambda$, *but* $\mu + \alpha \notin \Lambda$;
2) $\xi g_{\lambda\mu} = 0$ *if* $\mu - \alpha \in \Lambda$, *but* $\lambda - \alpha \notin \Lambda$;
3) $\xi(g_{\lambda\mu} - g_{\lambda+\alpha,\mu+\alpha}) = 0$ *if* $\lambda + \alpha, \mu + \alpha \in \Lambda$.

Proof. It suffices to compare the entries of $x_\delta(\xi)g$ and $gx_\delta(\xi)$ at the position (λ, μ), by using the formulas of Lemma 7. For formula 3), it should be observed that, moreover, both $g_{\lambda\mu}$ and $g_{\lambda+\alpha,\mu+\alpha}$ are added to $g_{\lambda+\alpha,\mu}$ by *the same* matrix entry of the root element $x_\alpha(\xi)$, namely, by the element at the position $(\lambda + \alpha, \mu)$. □

Proof of Proposition 4. Actually, we shall prove a refinement of Proposition 4 in §2. Namely, if an element $g \in \mathrm{GL}(n, R)$, $n = 27$ risp 56, commutes with a root element $x_\alpha(1)$ for a root $\alpha \in \Phi$, then it lies in a parabolic subgroup of type $(6, 15, 6)$ risp $(12, 32, 12)$. Without loss of generality, we may assume that $\alpha = \delta$ is the maximal root. By the preceding lemma we conclude that $g_{\lambda\mu} = 0$ if $\lambda - \delta \notin \Lambda$ and $\mu - \delta \in \Lambda$ or if $\lambda + \delta \in \Lambda$ and $\mu + \delta \notin \Lambda$. Thus, for the same weight numbering as above, the matrix g has the form

$$g = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix},$$

where the first and third diagonal blocks are of degree 6 risp 12, while the second is of degree 15 risp 32. It only remains to refer to the explicit matrix form of the parabolic subgroups of types P_2 risp P_1, described in §6. □

Proof of Proposition 5. As usual, we refine Proposition 5 somewhat. Namely, let $g \in \mathrm{GL}(n, R)$, where $n = 27$ risp 56. We claim that if the commutator $[g, x_\alpha(\xi)x_\beta(\zeta)]$ is central, then it equals 1. Consider the relation $x_\alpha(1)g = \varepsilon g x_\alpha(1)$ for some $\varepsilon \in R^*$. Using the same weight order as above, we see that

$$(x_\alpha(\xi)x_\beta(\zeta)g)_{\lambda\mu} = g_{\lambda\mu} = (gx_\alpha(\xi)x_\beta(\zeta))_{\lambda\mu}$$

for all $\lambda, \mu \in \Lambda$ such that $\lambda - \alpha, \lambda - \beta \notin \Lambda$, $\mu + \alpha, \mu + \beta \notin \Lambda$. Thus, $g_{\lambda\mu} = \varepsilon g_{\lambda\mu}$ for all such λ and μ. It only remains to observe that there are at most 12 risp 24 weights λ and μ that fail to have this property. This means that such $g_{\lambda\mu}$ generate the unit ideal in R. Indeed, if they had not, they would all belong to some maximal ideal M. The image of g in $\mathrm{GL}(n, R/M)$ would then have a zero block of size 15×15 risp 32×32 and, thus, would be degenerate (since $15 > 27/2$ risp $32 > 56/2$). It follows that $\varepsilon = 1$. □

§8. The centralizers of some unipotent subgroups

In this section we proceed with the preliminaries towards the proof of the second major reduction result, Proposition 5. The overall setting of the subsequent analysis is as follows: let a matrix $g \in \mathrm{GL}(n, R)$, $n = 27$ risp 56, probably subject to some further restrictions, commute with a fair portion of root unipotents $x_\alpha(1)$. Then g is a scalar matrix. We start with the following easy observation.

Lemma 9. *Let $\Sigma \subseteq \Phi$ be a special set of roots such that $-\Sigma \cup \Sigma$ generates Φ. Then for any matrix $g \in \mathrm{GL}(n, R)$ that commutes with all root unipotents $x_\alpha(1)$, $\alpha \in \Sigma$, all diagonal elements are equal.*

Proof. Lemma 8 implies that for any root $\alpha \in \Sigma$ and any weight λ such that $\lambda + \alpha \in \Lambda$ we have $g_{\lambda\lambda} = g_{\lambda+\alpha,\lambda+\alpha}$. By reading this identity in the opposite direction, we conclude that $g_{\lambda\lambda} = g_{\lambda-\alpha,\lambda-\alpha}$ for all $\alpha \in \Sigma$. Since the difference of any two weights is a sum of roots, and by assumption any root is a sum of $\alpha, -\alpha \in \Sigma$, it follows that $g_{\lambda\lambda} = g_{\mu\mu}$ for any two weights λ, μ. □

If the ground ring R is a field, the following claim immediately follows from the fact that the representation of G with the highest weight $\overline{\omega}_1$ risp $\overline{\omega}_7$ is absolutely irreducible. For an arbitrary ring, the following lemma was first proved in [29], but since it is our policy to eliminate the dependence on external facts entirely, we force it by a brute calculation.

Lemma 10. *If a matrix $g \in \mathrm{GL}(n, R)$ commutes with all elements $x_\alpha(1)$, $\alpha \in \Phi$, then it must be scalar, $g = \varepsilon e$ for some $\varepsilon \in R^*$.*

Proof. First, observe that g must be diagonal. Indeed, by Lemma 8 we have $g_{\lambda\mu} = 0$, provided there exists a root α such that $\lambda + \alpha \in \Lambda$, but $\mu + \alpha \notin \Lambda$. On the other hand, Proposition 2 and Lemma 4 assert precisely that for any $\lambda \neq \mu$ such a root α does exist. Thus, $g_{\lambda\mu} = 0$ for all $\lambda \neq \mu$. To conclude that g is scalar it only remains to apply Lemma 9 with $\Sigma = \Phi^+$. □

In particular, it follows that the centralizer of the elementary subgroup $\mathrm{E}(\mathrm{E}_l, R)$ in the Chevalley group $\mathrm{G}(\mathrm{E}_l, R)$, $l = 6, 7$, coincides with the center of $\mathrm{G}(\mathrm{E}_6, R)$ (see [29]), so that it is isomorphic to the group $\{\varepsilon \in R^*, \varepsilon^3 = 1\}$. The following obvious fact follows immediately from the Chevalley commutator formula.

Lemma 11. *For any i, $1 \leq i \leq l$, the group $\mathrm{E}(\Phi, R)$ is generated by the unipotent radicals U_i and U_i^- of two opposite parabolic subgroups.*

Now we are in a position to prove the following important subsidiary result.

Proposition 7. *If an element $z \in L_i$ commutes with all $x_\alpha(1)$, $\alpha \in \Sigma_i$, then it is central.*

Proof. Indeed, if $g \in L_i$ commutes with all $x_\alpha(1)$, $\alpha \in \Sigma_i$, then g^{-1} also commutes with them. Since the action of L_i on $U_i^-/U_i^-(2)$ is dual to the action of L_i on $U_i/U_i(2)$ (see, for example, [31, Theorem 2]), the element g commutes with all $x_{-\alpha}(1)$, $\alpha \in \Sigma_i$, as well. By the preceding lemma, U_i and U_i^- generate $\mathrm{E}(\Phi, R)$, and it remains to refer to Lemma 10 to conclude that g is central. □

Digression. We could avoid any reference to the results of [31] on the structure of U_i as an L_i-module. Instead, we could prove the fact we need — or even more general facts! — in a downright fashion, on the basis of Lemma 8 exclusively. To give some idea of how it works, we check that even in $\mathrm{GL}(27, R)$ there are no noncentral block diagonal matrices of type $(6, 15, 6)$ that commute elementwise with all $x_\alpha(1)$, $\alpha \in \Sigma_2$.

It suffices to prove that the matrix g is diagonal, because then the claim follows immediately from Lemma 9 applied to $\Sigma = \Sigma_2$. Let $\lambda \neq \mu$ be two distinct weights both lying in one of the following sets: Θ, Θ^*, $\Xi = \Lambda \setminus (\Theta \cup \Theta^*)$. We wish to prove that $g_{\lambda\mu} = 0$. First, let $\lambda, \mu \in \Theta$. Set $\mathrm{A}_5 = \langle \alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \rangle$. Since the Weyl group $W(\mathrm{A}_5)$ is doubly transitive on Θ, without loss of generality we can assume that $\lambda = \omega$, $\mu = \omega - \alpha_1$. In this case $\alpha = \substack{11100\\1} \in \Sigma_2$ is a root such that $\lambda + \alpha \in \Lambda$ and $\mu + \alpha \notin \Lambda$. But then $g_{\lambda\mu} = 0$ by item 1) of Lemma 8. The case where $\lambda, \mu \in \Theta^*$ is treated similarly with the use of item 2) of the same lemma. On the other hand, $W(\mathrm{A}_5)$ has two orbits on the set of pairs (λ, μ), $\lambda, \mu \in \Xi$, $\lambda \neq \mu$. One of them consists of all pairs for which the distance $d(\lambda, \mu)$ equals 1; the other consists of all pairs for which this distance equals 2. As a representative of the first orbit, we can take the pair (λ, μ), where $\lambda = \omega^* + \substack{00111\\1}$, $\mu = \omega^* + \substack{01111\\1}$. We readily see that $\lambda + \alpha \in \Lambda$ for the root $\alpha = \alpha_2 + \alpha_4$, $\mu + \alpha \notin \Lambda$, and thus again $g_{\lambda\mu} = 0$. On the other hand, as a representative of the second orbit we could take the pair (λ, μ), where $\lambda = \omega^* + \substack{00111\\1}$, $\mu = \omega^* + \substack{12211\\1}$. For the root $\alpha = \alpha_2 + \alpha_4$, we have $\lambda + \alpha \in \Lambda$ and $\mu + \alpha \notin \Lambda$, which shows that $g_{\lambda\mu} = 0$ in this case as well.

Obviously, similar facts could be verified for all other cases in the same style; this would be nothing more than an exercise in patience.

§9. Extraction in parabolic subgroups

In this section we assume that H is a subgroup in the Chevalley group $G = \mathrm{G}(\Phi, R)$, $\Phi = \mathrm{E}_6, \mathrm{E}_7$, normalized by the elementary subgroup $\mathrm{E}(\Phi, R)$. We say that an element $g \in H$ is *noncentral* if it does not belong to the center of the group G. Our purpose in this section is the proof of Proposition 6. As a matter of fact, we prove it only for the four cases in which we actually use it, even though it is true as stated for all parabolic subgroups. We start with a proof of the following subsidiary result.

Lemma 12. *If H contains a noncentral element $g \in P_i$, then H contains a noncentral element of the form εu, where $\varepsilon \in R^*$ is an invertible scalar, while $u \in U_i$, $u \neq e$, is a nontrivial element in the unipotent radical of P_i.*

Proof. If the element g itself has this form, there is nothing to prove. Thus, we can assume that $g = zv$, where $z \in L_i$, $v \in U_i$, and $z \neq \varepsilon e$. We fix a root $\alpha \in \Sigma_i$ and consider the commutator

$$[g, x_\alpha(1)] = [zv, x_\alpha(1)] = {}^z[v, x_\alpha(1)][z, x_\alpha(1)].$$

Since U_i is either Abelian or extra-special, the first commutator on the right-hand side equals $x_\delta(*)$. However, by Proposition 7, it cannot occur that all the commutators $[z, x_\alpha(1)]$ are *simultaneously* of the form $x_\delta(*)$, because otherwise the element z would have been central. □

Proof of Proposition 6. By the preceding lemma, from the very start we can assume that $g = \varepsilon u$, where $u \in U_i$. In the case where U_i is extra-special, the proof is immediate. If g is of the form $g = e x_\delta(\xi)$ and is also noncentral, then $\xi \neq 0$, so that the commutator $[g, x_{-\alpha_i}(1)] = x_{\delta-\alpha_i}(\pm\xi) \in H$ is a nontrivial root element, as required. Otherwise, we present the element u occurring in $g = \varepsilon u$ as the product $u = \prod x_\alpha(u_\alpha)$, $\alpha \in \Sigma_i$, $u_\alpha \in R$.

By assumption, at least one of the coefficients u_α, $\alpha \neq \delta$, is distinct from 0. Let, say, $u_\beta \neq 0$. Then $[g, x_{\delta-\beta}(1)] = x_\delta(\pm u_\beta) \in H$ is the required root unipotent.

The easiest way to prove the proposition in the Abelian unipotent radical case is to look at the weight diagrams $(\mathrm{E}_6, \overline{\varpi}_1)$ risp $(\mathrm{E}_7, \overline{\varpi}_7)$. As above, let $u = \prod x_\alpha(u_\alpha)$, $\alpha \in \Sigma_i$, $u_\alpha \in R$, where at least one of the coefficients u_α is distinct from 0. Since the Weyl group $W(\Delta_i)$ acts transitively on Σ_i, in the case of E_6 we can assume that $u_{\alpha_1} \neq 0$, whereas in the case of E_7 we can assume that $u_{\alpha_7} \neq 0$. If $\Phi = E_6$, we set $\gamma = \begin{smallmatrix}0&1&2&2&1\\&&1&&\end{smallmatrix}$. Then

$$x_{\gamma+\alpha_1+\alpha_3}(\pm u_{\alpha_1}) = [[g, x_\gamma(1)], x_{\alpha_3}(1)] \in H$$

is the required root unipotent. If $\Phi = E_7$, we set $\gamma = \begin{smallmatrix}1&2&3&2&1&0\\&&2&&&\end{smallmatrix}$. Then again

$$x_{\gamma+\alpha_1+\alpha_7}(\pm u_{\alpha_7}) = [[g, x_\gamma(1)], x_{\alpha_1}(1)] \in H$$

is a nontrivial root unipotent in H. $\square$

In §2 we already illustrated the use of Proposition 6. As another illustration, we prove an analog of Lemma 8 of [56]. Of course, [56] dealt with subgroups of GL_n, and for the commutator of g with $x_\alpha(1)$ to fall into a proper parabolic subgroup, *one* zero sufficed in the matrix g. Now we need *six* risp *twelve* zeros in one column, and moreover, in specified positions.

Corollary 3. *Let a weight $\lambda \in \Lambda$ and a root $\alpha \in \Phi$ be such that $\lambda - \alpha \notin \Lambda$. If for a noncentral element $g \in H$ we have $g'_{\mu\lambda} = 0$ for all $\mu \in \Lambda$ such that $\mu + \alpha \in \Lambda$, then H contains a nontrivial elementary root unipotent.*

Proof. Consider the commutator $z = [g, x_\alpha(1)]$. By Corollary 2, we can assume that z is noncentral. On the other hand, obviously $(x_\alpha(1)g^{-1})_{*\lambda} = (g^{-1})_{*\lambda}$. Thus, $(gx_\alpha(1)g^{-1})_{*\lambda} = v^\lambda$, and since $\lambda - \alpha \notin \Lambda$, we have $z_{*\lambda} = 0$. In other words, z is contained in P_1 risp P_7. It only remains to use Proposition 6. $\square$

§10. Concluding remarks

In conclusion, we mention several unsolved problems. Unless explicitly stated otherwise, we continue to assume that $\Phi = \mathrm{E}_6, \mathrm{E}_7$.

The following problem offers itself as the most urgent, and we are determined to address it in a subsequent paper.[5]

Problem 1. *Devise an A_2-proof for Chevalley groups of types E_8 and F_4.*

Unlike the cases considered in the present paper, the groups $\mathrm{G}(\mathrm{E}_8, R)$ and $\mathrm{G}(\mathrm{F}_4, R)$ have no microweight representations. The most natural representation to look at for E_8 is the 248-dimensional adjoint representation. In this case, apart from the summand belonging to the Lie algebra itself, a quadratic summand occurs in a root element. It is this quadratic summand that bans a literal translation of the arguments used in the present paper to the case of E_8. Yet, we are convinced that, in this case too, root elements are so special that with some ingenuity one could still contrive a similar argument. As for the case of F_4, there are serious grounds to believe that it is much more convenient to work in the *reducible* 27-dimensional representation (the one considered in the present

[5] After the publication of the Russian original, this problem for F_4 and ${}^2\mathrm{E}_6$ was completely solved by the first-named author and Sergei Nikolenko. These groups are indeed treated as twisted groups of type E_6 in the same 27-dimensional representation. Though based on the same ideas, the proof for these cases requires a *distinctly* more delicate analysis due to the fact that now we are allowed to use only one third of the unipotents we employ in the present paper. These unipotents still suffice to prove the principal lemma, but barely. The case of E_8 remains open as of this writing.

paper!) rather than in the 26-dimensional short-root representation or the 52-dimensional adjoint representation.

We mention another extremely interesting problem in the same spirit whose solution seems quite realistic with the methods developed in the present paper.

Problem 2. *Prove the standard description of the normal subgroups in twisted groups of types* E_6 *and* E_7 *over an arbitrary commutative ring* R, *provided they contain a split subgroup of type* A_2.

Perturbation theory is based on the principle stating that, to see the difference of two things which coincide for an accidental reason, we must introduce additional tension. Passage from split groups to isotropic groups creates much additional technical tension. Furthermore, this tension reveals dramatic advantages of *the proof from the Book* as compared with the decomposition of unipotents. For exceptional types, the decomposition of unipotents depends upon the existence of *huge* split subgroups of type A_l or D_l and was fit essentially only to calculate in Chevalley groups. The proof from the Book works for groups of rank at least 2 as well. Such a restriction seems to be perfectly reasonable. Recall that a description of the normal structure of anisotropic groups is universally recognized as a *hopeless* problem even in the case of an arbitrary field. On the other hand, description of the normal structure of groups of rank 1 is an *extremely* difficult problem for zero-dimensional rings and becomes a *hopeless* problem in the case of rings of dimension at least 1. Sceptics could try to approach something as plain as $\mathrm{SL}(2,\mathbb{Z})$.

For classical groups, an analog of the following problem has been completely solved in a series of recent papers by the first-named author and V. A. Petrov.

Problem 3. *Describe the subgroups in* $\mathrm{GL}(27,R)$ *normalized by* $\mathrm{E}(\mathrm{E}_6,R)$ *and the subgroups in* $\mathrm{GL}(56,R)$ *normalized by* $\mathrm{E}(\mathrm{E}_7,R)$.

As the first step in this direction, one could describe subgroups of $\mathrm{GL}(n,R)$ containing $\mathrm{E}(\Phi,R)$. We guess that to completely solve this problem one will have to use a blend of the methods similar to those of the present paper with localization methods.[6]

It is well known (see, for example, [71]) that a solution of the following problem would be the main step in description of the subnormal subgroups of $\mathrm{G}(\Phi,R)$.

Problem 4. *Describe the subgroups of* $\mathrm{G}(\Phi,R)$ *normalized by* $\mathrm{E}(\Phi,R,A)$ *for some ideal* $A\trianglelefteq R$.

The standard answer to this problem can be stated as follows. For any such subgroup H there exists an ideal $A\trianglelefteq R$ such that

$$\mathrm{E}(\Phi,R,A^mI)\le H\le \mathrm{C}(\Phi,R,I)$$

for some m. Such an ideal I is unique up to a certain natural equivalence relation $\Diamond_A$. The most important technical aspect of this problem consists in finding the *smallest* possible m such that these inclusions are true for all subgroups H. As an example, we could mention that for $\mathrm{GL}_{n\ge 3}$ this terrain was completely explored in the years 1973–1989. For this case, such an m took consecutive values: $m=7$ (J. S. Wilson), $m=6$ (L. N. Vaserstein), $m=5$ (N. A. Vavilov) $m=4$ (L. N. Vaserstein). In this matter,

[6] After the publication of the Russian original, A. Yu. Luzgarev made important progress towards the solution of this problem. His work has exhibited a *dramatic* difference between the cases of E_6 and E_7. For E_6, there is a workable approach towards an answer which is very similar to that for classical cases. At the same time, overgroups of $\mathrm{E}(\mathrm{E}_7,R)$ have a much fancier description. First, they fall into families depending on two ideals corresponding to the roots of $\mathrm{GL}(n,R)$ and the short roots of $\mathrm{Sp}(56,R)$. On top of that, another parameter occurs corresponding to the fact that there are various symplectic overgroups of $\mathrm{E}(\mathrm{E}_7,R)$. The resulting description seems to be too technical to work up in detail.

"less" is definitively "more": the less commutations are used in a proof, the better bound it gives. In this perspective the proof in the present paper should give a better bound for the groups of types E_6 and E_7 than any other proof known today.

Now, let $\Delta \subseteq \Phi$ be a root subsystem in Φ. Consider the embedding $G(\Delta, R) \leq G(\Phi, R)$. A complete solution of the following problem would furnish a generalisation to exceptional groups of results by Borewicz and Vavilov [3] and [8]–[13] on overgroups of subsystem subgroups.

Problem 5. *Describe the subgroups in* $G(\Phi, R)$ *normalized by* $E(\Delta, R)$ *under the assumption that* $\Delta^{\perp} = \varnothing$ *and all irreducible components of* Δ *except, maybe, one have rank at least* 2.

The standard answer to this problem is stated in terms of certain functions $\sigma : \alpha \mapsto \sigma_\alpha$ from Φ to the set of ideals of R. We limit ourselves to the statement of the standard answer for subgroups *containing* $E(\Delta, R)$. The above function is called a Δ-net provided $\sigma_\alpha \sigma_\beta \subseteq \sigma_{\alpha+\beta}$ whenever $\alpha + \beta \in \Phi$ and, moreover, $\sigma_\alpha = R$ for all $\alpha \in \Delta$. With every Δ-net σ, we can associate a standard subgroup $E(\sigma)$ generated by root unipotents $x_\alpha(\xi)$, $\xi \in \sigma_\alpha$. In the case of standard description, $E(\sigma) \trianglelefteq H$ for a unique Δ-net σ. To solve this problem, it would be useful to perform the following exercise.

Problem 6. *Write down an* A_2*-proof of the standard commutator formulas in* $G(\Phi, R)$.

Of course, as we saw in §4, for the usual elementary groups $E(\Phi, R, I)$ the commutator formulas themselves are known. However, it would be desirable to have their proof in terms of the subsystem Δ rather than the system Φ itself. This would be vital to yield a generalization to all subgroups $E(\sigma)$.

Obviously, there are a number of further major problems both settled and wide open, which can be treated successfully by the methods of the present paper. We could mention, for example, some key results such as description of the automorphisms of $G(\Phi, R)$, centrality[7] of K_2, etc.

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