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RIEMANNIAN MANIFOLDS WITH CURVATURE AT MOST κ: GLUING WITH RAMIFICATION

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Abstract. Under some necessary conditions, the result of attaching several Riemannian manifolds along some isometry of their boundaries has curvature at most κ in Aleksandrov's sense.

§1. Introduction

In the paper [ABB], a necessary and sufficient condition was found for a Riemannian manifold with boundary to have curvature not exceeding κ. In particular, there it was proved that any Riemannian manifold with boundary has upper bounded curvature. In the case of attaching two Riemannian manifolds along some isometry of their boundaries, a necessary and sufficient condition for the resulting manifold to have curvature not exceeding κ was stated in the author's earlier paper [K2].

In the present paper we attach a finite number of manifolds (sheets) along some isometries of their boundaries. More than two boundary components may be identified, which results in *ramifications*. As in [K2], we obtain a precise upper bound for curvature; this bound depends on the sectional curvatures of the boundary (see Theorem 1). On the contrary, a sharp lower bound for the curvature (if it exists, i.e., only two manifolds are attached and the sum of the second fundamental forms of the boundaries is positive semidefinite) depends *only* on the sectional curvatures of the manifolds (see [K1]).

(Unfortunately, there was a disappointing misprint in the statement of Theorem 1.1 in [K2]. The condition that the form $\mathrm{B}_0 + \mathrm{B}_1$ is negative semidefinite must be imposed; cf. Theorem 1 below or the proof in [K2].)

We consider finitely many n-dimensional Riemannian manifolds $\{M_\alpha \mid \alpha \in I\}$ with boundaries Γ_α. Assuming that the boundaries are isometric to one another, we fix some isometries of one of the Γ_α to the other boundaries. Now we can identify all Γ_α with some space Γ. Thus, we can assume that the second fundamental forms B_α of Γ_α (in M_α) with respect to inward normals are defined on Γ; therefore, $\mathrm{B}_\alpha + \mathrm{B}_\beta$ is well defined. Let M denote the space with length metric obtained by attaching all M_α (along the isometries fixed above).

Theorem 1. *In the above notation, suppose that the sectional curvatures of all M_α are less than or equal to κ, and that for all $\alpha \neq \beta$ the form $\mathrm{B}_\alpha + \mathrm{B}_\beta$ is negative semidefinite. Consider the 2-directions σ of $T\Gamma$ such that the restrictions of all B_α to σ are negative definite. Suppose that the sectional curvatures of Γ in all these 2-directions are at most κ.*

Then M has curvature bounded above by κ.

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It is easily seen that the assumptions of Theorem 1 are also necessary. Theorem 1 generalizes known results about the curvature of Riemannian manifolds with boundary [ABB] and about the curvature of spaces obtained by attaching two Riemannian manifolds with boundary [K2]. Nevertheless, the proof of Theorem 1 leans upon both these results. However, we need only the following two facts from the paper [K2].

(1) We can C^2-slightly change the metrics on M_α (without changing it on Γ) in such a way that all second fundamental forms of Γ become smaller (see Lemma 5.1 in [K2]).
(2) The result of attaching two Riemannian manifolds along some isometry of their boundaries has upper bounded curvature if the sum of the second forms is negative semidefinite. This statement is easy to prove by using a Sobolev mollifier (see Lemmas 2.2 and 3.1 in [K2]).

Observe that if all forms B_α are negative semidefinite, then Theorem 1 follows easily from well-known results. Indeed, in this case every M_α has curvature not exceeding κ (see [ABB]), and their boundaries are convex subsets. Thus, in this case Theorem 1 is a consequence of the Reshetnyak theorem (see [R]).

It should be noted that if we glue more than two manifolds, then the condition of Theorem 1 is slightly weaker than the condition that for all $\alpha \neq \beta$ the spaces $M_\alpha \cup M_\beta$ have curvature at most κ.

§2. κ-convexity of Jacobi fields

One of the main results of the paper [ABB] is that for any Riemannian manifold M with (smooth) boundary the following three statements are equivalent.

(1) M has curvature not exceeding κ.
(2) In M all normal Jacobi fields are κ-convex.
(3) The sectional curvatures of the interior of M and the "outward" sectional curvatures of the boundary of M do not exceed κ.

Here an outward sectional curvature of the boundary is the curvature corresponding to a tangent section such that the restriction of the second fundamental form to it is negative definite. As in [ABB], by the normal Jacobi fields and the κ-convexity we mean natural generalizations of the usual normal Jacobi fields and of the differential condition $f'' + \kappa f \geq 0$. All this definitions can be found in [ABB], but we repeat them below.

Let M be a Riemannian manifold with boundary (as it was in [ABB]) or, more generally, a space satisfying the conditions of Theorem 1.

A vector field J along a geodesic γ in M is called a *Jacobi field* if there is a sequence of geodesics γ_i converging to γ in the uniform topology, and also a sequence of positive numbers u_i approaching 0 such that $\|J(t)\| = \lim u_i^{-1} d(\gamma(t), \gamma_i(t))$ for all t, and the unit vector in the direction of $J(t)$ is the limit of the initial unit vectors of the minimizing geodesics from $\gamma(t)$ to $\gamma_i(t)$. As usual, we say that a Jacobi field is *normal* if it is orthogonal to γ.

A Jacobi field J is $\tilde{\kappa}$-*convex* if it satisfies the differential inequality

$$\|J\|'' \geq -\tilde{\kappa} v^2 \|J\|$$

(where v is the speed of γ) in the following (barrier) sense. If $\tilde{\kappa} > 0$, this inequality means that on any parameter subinterval of length less than $\frac{\pi}{v\sqrt{\tilde{\kappa}}}$, the function $a \sin(\sqrt{\tilde{\kappa}} vt - b)$ that coincides with $\|J\|$ at the endpoints is an upper bound for $\|J\|$. If $\tilde{\kappa} \leq 0$, we use an appropriate linear function or a hyperbolic sinusoid instead, with no restriction on the subinterval. In other words, on appropriate subintervals, the length of $\|J\|$ must be dominated by the length of a normal Jacobi field in the $\tilde{\kappa}$-*plane* (i.e., the simply connected surface of constant curvature $\tilde{\kappa}$). On the other hand, this differential inequality can be equivalently understood in the sense of distributions: the functional $\|J\|'' + \tilde{\kappa}v^2\|J\|$ must be nonnegative on the nonnegative C_0^∞-functions.

In [ABB], the equivalence of statements (1) and (2) and the equivalence of statements (2) and (3) were proved separately.

Actually, in Proposition 1 of [ABB] it was proved more than claimed, i.e., more than the mere equivalence of statements (2) and (3).

Proposition 1. *Suppose that a geodesic γ satisfies the following conditions*:

(1) *the sectional curvatures of the manifold itself do not exceed $\tilde{\kappa}$ at the points of γ*;

(2) *the same is true for certain sectional curvatures of the boundary at the inner points of γ; specifically, this concerns the sectional curvatures that correspond to any* 2*-direction σ containing the direction of γ and such that the restriction of the second fundamental form to σ is negative definite.*

Then all normal Jacobi fields along γ are $\tilde{\kappa}$-convex.

Repeating the arguments of Proposition 2 in [ABB] (the equivalence of statements (1) and (2)), we can obtain the following.

Proposition 2. *For any space M satisfying the conditions of Theorem* 1, *the following two statements are equivalent*:

(1′) *M has curvature not exceeding $\tilde{\kappa}$*;

(2′) *all normal Jacobi fields in M are $\tilde{\kappa}$-convex.*

However, in order to make arguments run, we must verify the following assertions.

A. Any point of M has a neighborhood in which any geodesic variation with Lipschitz endpoint curves is Lipschitz itself.

B. The first variation formula for geodesics holds true in M.

C. Any sequence of geodesics γ_i uniformly converging to γ admits a subsequence that corresponds to some Jacobi field along γ.

Item A is true because M is a space of upper bounded curvature (see Lemma 4.3 below). Items B and C follow from Lemma 2 (see below) and similar statements for Riemannian manifolds with boundary.

From Proposition 2 it follows that, in order to prove Theorem 1, it suffices to show that all normal Jacobi fields in M are κ-convex.

§3. An outline of the proof of Theorem 1

From now on, we shall assume that the conditions of Theorem 1 are fulfilled.

Our arguments involve the additional condition that for all $\alpha \neq \beta \in I$ the form $\mathrm{B}_\alpha + \mathrm{B}_\beta$ is negative definite and bounded away from zero. This means that there exists a number $c > 0$ such that for all $\alpha \neq \beta \in I$ the form $\mathrm{B}_\alpha + \mathrm{B}_\beta + c\|\cdot\|^2$ is negative semidefinite. Here the norm $\|\cdot\|$ corresponds to the scalar product on Γ. The general case can be reduced to this one much as this was done in [K2] in the case of two manifolds. Namely, we can C^2-slightly change the metrics on M_α (with preservation of the metric on Γ) in such a way that all second fundamental forms of Γ become smaller (see Lemma 5.1 in [K2]).

The paper is organized as follows.

In §4 we consider properties of a geodesic variation lying on one sheet (i.e., M_α).

In §5 local systems of coordinates in M are introduced, near-geodesics are defined, nonvertical pairs of points are introduced, and a vector $\overline{pq}$ is put in correspondence to any nonvertical pair of points (p, q). Lemma 2 shows that, locally, near-geodesics pass through at most 3 sheets; also in that lemma, we choose a small neighborhood U to which we restrict all subsequent considerations.

In §6, we use Lemma 4 (proved in §7) to show that any minimizing geodesic is a near-geodesic (see Corollary 4.2) and that M has curvature bounded above by some constant $\tilde{\kappa}$ (see Corollary 4.3)

In Lemma 4 we prove the *angle comparison condition* (we mean comparison with the κ-plane) for triangles with shortest near-geodesic sides. (We remind the reader that the *κ-plane* is a simply connected surface of constant curvature κ.) The *angle comparison condition* for the triangle $\triangle abc$ means that $\angle abc \leq \angle a_0b_0c_0$, $\angle bca \leq \angle b_0c_0a_0$, and $\angle cab \leq \angle c_0a_0b_0$, where the triangle $\triangle b_0a_0c_0$ lies in the κ-plane and has the same side lengths as those of $\triangle abc$. (Note also that a space with length metric is of curvature at most κ if and only if the angle comparison condition is fulfilled locally for any triangles.)

In §8 we prove Theorem 1 by using Jacobi fields. This proof is similar to the last step in the proof of Theorem 1.1 in [K2].

§4. Behavior of geodesics in one sheet

We recall that the conditions of Theorem 1 are assumed to be satisfied.

Lemma 1. *Suppose $p \in \Gamma$ and a 2-direction $\sigma \subset T_p\Gamma$ contains a vector $v \neq 0$ such that $\mathrm{B}_\alpha(v) \leq 0$ for any $\alpha \in I$.*

Then the sectional curvature of Γ in the direction σ does not exceed κ.

Proof. If the restriction of B_α to σ is negative definite for all $\alpha \in I$, then the assertion is one of the assumptions of Theorem 1.

Otherwise, there exist $\alpha_0 \in I$ and linearly independent vectors $u_+, u_- \in \sigma$ such that $\mathrm{B}_{\alpha_0}(u_+) \geq 0$ and $\mathrm{B}_{\alpha_0}(u_-) \leq 0$. Then the required inequality follows from the Gauss theorem and the condition on the sectional curvatures of M_{α_0}. Indeed, suppose $k_{\alpha_0}(u_+, u_-)$ and $k_\Gamma(u_+, u_-)$ are the Riemann curvatures of M_{α_0} and Γ (respectively), and let the bilinear form $\mathrm{B}_{\alpha_0}(u_+, u_-)$ correspond to the quadratic form B_{α_0}. Then

$$\begin{aligned} k_\Gamma(u_+, u_-) &= k_{\alpha_0}(u_+, u_-) + \mathrm{B}_{\alpha_0}(u_+)\,\mathrm{B}_{\alpha_0}(u_-) - \mathrm{B}^2_{\alpha_0}(u_+, u_-) \leq k_{\alpha_0}(u_+, u_-) \\ &\leq \kappa \|u_+ \wedge u_-\|^2. \end{aligned}$$

□

The next statement is an easy consequence of the results of [ABB].

Corollary 1.1. *Suppose that a normal Jacobi field J along a geodesic γ in the space M corresponds to a sequence of geodesics γ_n in the space M, and that all γ_n lie on one and the same sheet M_α. Then J is κ-convex.*

Proof. First, we observe that if an open subinterval of γ lies on Γ, then $\mathrm{B}_\alpha(\gamma') \leq 0$ on this subinterval. Indeed, otherwise, if t is such that $\mathrm{B}_\alpha(\gamma'(t)) > 0$, the length of γ can be reduced near t by moving $\gamma(t-\varepsilon, t+\varepsilon)$ to the interior of M_α. But γ is a geodesic. This is a contradiction.

Note that we consider a Jacobi field in M_{α_0}. Moreover, if $\gamma(t) \in \Gamma$, then $\mathrm{B}_\alpha(\gamma'(t)) \leq 0$ for all $\alpha \in I$. Thus, the corollary immediately follows from Proposition 1 and Lemma 1. □

§5. Choice of a small neighborhood. Local behavior of near-geodesics

Since the assertion of Theorem 1 is local, it suffices to prove it in a small neighborhood U of a point $x^\star \in \Gamma$. In this neighborhood, the sectional curvatures of Γ do not exceed some constant. Therefore (see [ABB]), all spaces $U \cap M_\alpha$ for $\alpha \in I$ are spaces of upper bounded curvature.

Let U be so small that any two points of $U \cap \Gamma$ (or $U \cap M_\alpha$) are joined by a unique minimizing geodesic in $U' \cap \Gamma$ (respectively, in $U' \cap M_\alpha$) for some larger neighborhood U'.

We fix a special "local system of coordinates" on M. First, we choose an arbitrary coordinate system on Γ. Then we extend these coordinates in such a way that the nth coordinate x^n of $x \in M_\alpha$ be the distance from x to Γ, and the other coordinates be those of the point of Γ nearest to x. (The quotation marks have been used because these coordinates determine a unique point only if we know *a priori* to which M_α this point belongs.)

A curve $\gamma : [a; b] \to M$ is said to be *elementary* if there exists a partition $a = s_0 < s_1 < \cdots < s_N = b$ such that every subcurve $\gamma|_{[s_l, s_{l+1}]}$ lies in one M_{α_l} and is C^1-smooth there, and no breaks occur at the points $\gamma(s_l) \in \Gamma$, $l = 1, \dots, N-1$. The latter means that the following condition is fulfilled for the coordinates of the left and right derivatives of γ: $(\gamma')^i(s_l - 0) = (\gamma')^i(s_l + 0)$ for $i < n$ and $(\gamma')^n(s_l - 0) = -(\gamma')^n(s_l + 0)$.

An elementary curve $\gamma : [a; b] \to M$ is called a *near-geodesic* if any subcurve $\gamma|_{[s_l, s_{l+1}]}$ (see the preceding definition) is a minimizing geodesic in the corresponding space M_{α_l}. It should be noted that such a curve is not always a geodesic. Indeed, there may exist a curve that lies on Γ and is a minimizing geodesic in some M_α and is not a geodesic in M. Observe also that any two points in M are joined by a near-geodesic. Actually, the shortest curve joining these points and switching from one sheet to another at most once is a near-geodesic.

To each elementary curve $\gamma : [a, b] \to M$, we assign a map $\bar\gamma : [a, b] \to \mathbb{R}^n$ such that $\gamma^i(t) = \bar\gamma^i(t)$ for $i < n$ and $t \in [a, b]$, and $\gamma^n(t) = (-1)^l \bar\gamma^n(t)$ for $t \in (s_l, s_{l+1}]$. The latter means that the nth coordinate changes its sign when γ changes a sheet. It is readily seen that $\bar\gamma$ is C^1-smooth.

For any elementary curve $\gamma : [a, b] \to M$, we denote by $\gamma_\Gamma(t) : [a, b] \to \Gamma$ the projection of γ to Γ. In coordinates, this is "forgetting" the last coordinate of a point.

A pair (p, q) of points is said to be *nonvertical* if the projections of p and q to Γ do not coincide. The set of all nonvertical pairs is open. To each nonvertical pair (p, q), we assign the vector $T_{x^\star}\Gamma$ with the coordinates $(q_1 - p_1, \dots, q_{n-1} - p_{n-1})$. We normalize this vector and denote by $\overline{pq}$ the unit vector obtained. Note that this correspondence is continuous.

Lemma 2. *A neighborhood U of the point $x^\star$ can be chosen in such a way that for any distinct points $p \in M_\alpha \cap U$ and $q \in M_\beta \cap U$ and each near-geodesic $\gamma : [0, 1] \to U$ that joins p to q the following statement is fulfilled.*

If the intersection of γ and Γ is neither a segment nor a point, then the pair (p, q) is nonvertical and there exists $\delta \in I$ such that $\mathrm{B}_\delta(\overline{pq}) > -\frac{c}{2}$; here the constant $c > 0$ is such that for all $\alpha \neq \beta \in I$ the form $\mathrm{B}_\alpha + \mathrm{B}_\beta + c\|\cdot\|^2$ is negative semidefinite. Moreover, in this case γ can be split in three subintervals $\gamma|_{[0, t^\alpha)}$, $\gamma|_{[t^\alpha, t^\beta]}$, and $\gamma|_{(t^\beta, 1]}$ in such a way that the first and the third (possibly empty) subintervals lie in the interiors of M_α and M_β, respectively, and the second lies in the (closed) sheet M_δ.

Proof of Lemma 2. Let the neighborhood U be so small that if the coordinates of two vectors $v \in T_p\Gamma$ and $v' \in T'_p\Gamma$ are sufficiently close (with arbitrary p, p' within $U \cap \Gamma$), then $|\,\mathrm{B}_\sigma(v) - \mathrm{B}_\sigma(v')| < \frac{c}{2}$ for any $\sigma \in I$.

Suppose the near-geodesic γ is parameterized proportionally to arc length. Then the coordinates of $\gamma|_{[s_l, s_{l+1}]}$ satisfy the generalizations of the usual equations for geodesics. We talk of "generalizations" because we consider Riemannian manifolds with boundary (see [ABB] for the details). By using these equations (precisely as in the Riemannian case), it is easy to show that in every M_α we have

$$|\bar{\gamma}''| \leq C_\alpha |\bar{\gamma}'|^2$$

for some constant C_α. Therefore, there exists a constant C such that

$$(1) \qquad |\bar{\gamma}''| \leq C |\bar{\gamma}'|^2.$$

Hence (reducing U if necessary) we may assume that the rotation of $\bar{\gamma}$ is small. Indeed, if γ is small, then from (1) it follows that the rotation of $\bar{\gamma}$ is small. Otherwise, γ leaves the small neighborhood U. Therefore, the directions of all vectors $(\bar{\gamma}_1(t_1) - \bar{\gamma}_1(t_0), \dots, \bar{\gamma}_n(t_1) - \bar{\gamma}_n(t_0))$ (for $0 \leq t_0 < t_1 \leq 1$) are close to the direction of $(\bar{\gamma}_1(1) - \bar{\gamma}_1(0), \dots, \bar{\gamma}_n(1) - \bar{\gamma}_n(0))$.

Suppose the intersection of γ and Γ is neither a segment nor a point. The condition on the sum of the second forms implies that there is at most one index $\delta \in I$ such that $\mathrm{B}_\delta(\overline{pq}) > -\frac{c}{2}$. Therefore, it suffices to prove that $\mathrm{B}_\sigma(\overline{pq}) > -\frac{c}{2}$ whenever there exists a subinterval $\gamma|_{[t_0,t_1]} \subset M_\sigma$ such that $\gamma(t_0), \gamma(t_1) \in \Gamma$ and $\gamma|_{[t_0,t_1]} \not\subset \Gamma$. We verify this. The directions of $(\bar{\gamma}_1(t_1) - \bar{\gamma}_1(t_0), \dots, \bar{\gamma}_{n-1}(t_1) - \bar{\gamma}_{n-1}(t_0), 0)$ and $(\bar{\gamma}_1(1) - \bar{\gamma}_1(0), \dots, \bar{\gamma}_n(1) - \bar{\gamma}_n(0)) = (q_1 - p_1, \dots, q_{n-1} - p_{n-1}, \pm q_n - p_n)$ are close to each other. Therefore, the pair (p, q) is nonvertical, and the directions of the vectors $(\bar{\gamma}_1(t_1) - \bar{\gamma}_1(t_0), \dots, \bar{\gamma}_{n-1}(t_1) - \bar{\gamma}_{n-1}(t_0))$ and $(q_1 - p_1, \dots, q_{n-1} - p_{n-1})$ are also close to each other.

Consider a minimizing geodesic s (parameterized by arc length) in Γ that joins $\gamma(t_0)$ to $\gamma(t_1)$. Arguing as above, we see that the directions of the vectors $\bar{s}'$ are close to the direction of $(\bar{\gamma}_1(t_1) - \bar{\gamma}_1(t_0), \dots, \bar{\gamma}_{n-1}(t_1) - \bar{\gamma}_{n-1}(t_0))$; hence, they are close to $\overline{pq}$.

Suppose $\mathrm{B}_\sigma(\overline{pq}) \leq -\frac{c}{2}$; then $\mathrm{B}_\sigma(s') < 0$. Therefore, the curve s is a geodesic of the space M_σ. Thus, we have found a geodesic digon $s \cup \gamma|_{[t_0,t_1]}$ in M_σ, which is impossible. □

Corollary 2.1. *For every pair of distinct points $p \in M_\alpha \cap U$ and $q \in M_\beta \cap U$, each near-geodesic $\gamma : [0,1] \to M$ that joins p to q passes through at most three interiors of sheets.*

Furthermore, either γ lies in the interior of one sheet $M_\alpha = M_\beta$, or γ can be split in three subintervals $\gamma|_{[0,t^\alpha)}$, $\gamma|_{[t^\alpha,t^\beta]}$, and $\gamma|_{(t^\beta,1]}$ in such a way that the first and third (*possibly empty*) *subintervals lie in the interiors of M_α and M_β, respectively, and the second interval lies in some* (*closed*) *sheet M_δ.*

§6. Near-geodesics and minimizing geodesics

Lemma 3. *Locally, for any two distinct points a and b in U there exists a minimizing geodesic that joins a to b and is a near-geodesic.*

Proof. Let γ_k be the shortest among the curves that join a to b and switch from one sheet to another at most k times. By definition, the distance between two points is the limit (as $k \to \infty$) of the length of γ_k. On the other hand, if γ_k lies in U, then, in fact, there are at most two switches (see Lemma 2). Therefore, all γ_k for $k > 2$ have equal lengths and are minimizing geodesics. □

We shall use the term *shortest near-geodesic* instead of "minimizing geodesic that is a near-geodesic". (From Lemma 3 it follows that there exists a shortest curve among all near-geodesic curves that join a to b; moreover, it is actually a shortest curve among all curves that join a to b.)

Now, we choose a neighborhood $V \subset U$ in such a way that certain "natural" constructions that "start" in V should remain within U. Namely, consider the following procedure. For 8 points in V, take 4 shortest near-geodesics that join some 4 pairs of our 8 points. Now, choose 4 points on these shortest near-geodesics and repeat the procedure for these 4 points, i.e., take 2 shortest near-geodesics that join pairs of this points and pick a point on each of these shortest near-geodesics. Finally, join these points by a shortest geodesic. We take V so small that for any 8 point in V the construction just described is accomplished within U.

Lemma 4. *Suppose that, for any $\alpha \neq \beta$ and any triangle $\triangle pqr$ in the space $(M_\alpha \cup M_\beta) \cap U$ with shortest near-geodesic sides, the angle comparison condition (with the $\tilde{\kappa}$-plane) holds true.*

Then the angle comparison condition (with respect to the $\tilde{\kappa}$-plane) is true for any triangle $\triangle pqr \subset V$ with shortest near-geodesic sides.

Since $M_\alpha \cup M_\beta$ has upper bounded curvature (see [K2]), the hypothesis of Lemma 4 is fulfilled locally, for some $\tilde{\kappa}$.

Before passing to the proof of Lemma 4, we state and prove three corollaries of it.

Corollary 4.1. *Locally, two points in V can be joined by a unique shortest near-geodesic in U.*

Proof. Suppose distinct points p and q are joined by two distinct shortest near-geodesics. Then on these near-geodesics there exist two distinct points a and b such that $|pa| = |pb|$. We join a and b by a shortest near-geodesic. Consider the triangles $\triangle pab$ and $\triangle qab$. The sum of their angles at a (as well as at b) is at least π. Therefore, the same is true for the comparison triangles $\triangle \bar{p}\bar{a}\bar{b}$ and $\triangle \bar{q}\bar{a}\bar{b}$ in the $\tilde{\kappa}$-plane (we assume that these triangles lie in different half-planes with respect to their common side $[\bar{a}\bar{b}]$). However, this is impossible if $|\bar{p}\bar{a}| + |\bar{a}\bar{q}| + |\bar{q}\bar{b}| + |\bar{b}\bar{p}| < 2\pi/\sqrt{\tilde{\kappa}}$. □

Corollary 4.2. *Any minimizing geodesic that lies in V is a near-geodesic.*

Remark. This is proved much as the standard fact that a shortest curve that joins any two points is smooth.

Proof. Suppose $\tilde{\gamma}$ is a minimizing geodesic that joins a and b, and γ is a shortest near-geodesic that joins these points. Since the statement of the corollary is local, we may assume that a and b are joined by a unique shortest near-geodesic (see Corollary 4.1) and they can be joined by a shortest near-geodesic to every point on $\tilde{\gamma}$ (see Lemma 4).

If $\tilde{\gamma}$ and γ do not coincide, then there exists a point $c \in \tilde{\gamma}$ that does not belong to γ. Let γ_a and γ_b be shortest near-geodesics that join c to a and b, respectively. Then the sum of the lengths of γ_a and γ_b is equal to the length of γ. Suppose the curve $\gamma_a \cup \gamma_b$ has a break at c. Then the first variation formula implies the existence of a curve shorter than $\gamma_a \cup \gamma_b$ (or γ) that joins a an b. This contradicts the fact that γ is a minimizing geodesic. Therefore, $\gamma_a \cup \gamma_b$ has no break at c and is a near-geodesic. This contradicts the fact that a and b are joined by a unique shortest near-geodesic. □

Now, yet another corollary to Lemma 4 becomes obvious.

Corollary 4.3. *If for any $\alpha \neq \beta$ the spaces $M_\alpha \cup M_\beta$ have curvature bounded above by $\tilde{\kappa}$, then M has curvature bounded above by $\tilde{\kappa}$.*

Note that the hypothesis of Corollary 4.3 is satisfied locally for some $\tilde{\kappa}$ (see [K2]).

§7. Proof of Lemma 4

In the course of this proof, we shall use the standard arguments showing that if the angle comparison condition is fulfilled for two parts of a triangle, then it is fulfilled for the entire triangle. Specifically, suppose a triangle whose sides are minimizing geodesics (and whose perimeter is less than $2\pi/\sqrt{\tilde{\kappa}}$ for $\tilde{\kappa} > 0$) is divided by a minimizing geodesic into two triangles for which the angle comparison condition (with respect to the $\tilde{\kappa}$-plane) is fulfilled. Then the angle comparison condition is fulfilled for the initial triangle. The proof of this fact is omitted. It is based on the triangle inequality for angles in M (which is obviously true) and easy geometry of the $\tilde{\kappa}$-plane.

1. We may assume that the side $[pq]$ lies in one sheet. Indeed, from Corollary 2.1 it follows that there exist points p_1 and q_1 such that every segment among $[pp_1]$, $[p_1q_1]$, and $[q_1q]$ lies in one sheet. By standard arguments, it is easy to check that if the angle comparison condition is fulfilled for $\triangle pp_1r$, $\triangle p_1q_1r$, and $\triangle q_1qr$, then it is fulfilled for $\triangle pqr$.

2. Thus, we assume that there exist α and β such that $[pq] \subset M_\alpha$ and $r \in M_\beta$. Yet another condition can be imposed, however: we may assume that there exist $\delta \in I$ such that $\triangle pqr = [pq] \cup [qr] \cup [rp] \subset M_\alpha \cup M_\beta \cup M_\delta$. Indeed, we shall show that if $\triangle pqr$ does not satisfy this condition, then there exist $x \in [pq]$ and a shortest near-geodesic $[xr] \subset M_\alpha \cup M_\beta$. Then the triangles $\triangle pxr$ and $\triangle xqr$ satisfy the above condition. But if the angle comparison condition is fulfilled for $\triangle pxr$ and $\triangle xqr$, then it is fulfilled for $\triangle pqr$.

Now, we prove that the point x and the shortest near-geodesic $[xr]$ just mentioned do exist. Assume the contrary. Then Lemma 2 implies that for any point $x \in [pq]$ the pair (x, r) is nonvertical and there exists $\delta_x \in I$ such that $\mathrm{B}_{\delta_x}(\overline{xr}) > -\frac{c}{2}$. For each $\delta \in I$ the set of all points x such that $\mathrm{B}_\delta(\overline{xr}) > -\frac{c}{2}$ is open. Next, these sets are disjoint, and for $\delta_p \neq \delta_q$ they are nonempty. This contradicts the fact that the segment $[pq]$ is connected.

3. Henceforth, we shall consider the space $M_\alpha \cup M_\beta \cup M_\delta$ instead of M. (We recall that $\triangle pqr = [pq] \cup [qr] \cup [rp]$ lies in this space.) Note that if some indices among α, β, and δ coincide, then the angle comparison condition is fulfilled by assumption.

Let $\gamma_1, \gamma_2 : [0,1] \to M_\alpha \cup M_\beta \cup M_\delta$ be parameterizations of the sides $[pr]$ and $[qr]$, respectively. From Corollary 2.1 it follows that for $s = 1, 2$ there exist $0 \le t_s^\alpha \le t_s^\beta \le 1$ such that

$$\gamma_s([0, t_s^\alpha]) \subset M_\alpha, \quad \gamma_s([t_s^\alpha, t_s^\beta]) \subset M_\delta, \quad \gamma_s([t_s^\beta, 1]) \subset M_\beta.$$

We shall prove that there exist numbers t_1, t_2 and a shortest near-geodesic joining $\gamma_1(t_1)$ and $\gamma_2(t_2)$ such that the triangle

$$[r\gamma_1(t_1)] \cup [\gamma_1(t_1)\gamma_2(t_2)] \cup [\gamma_2(t_2)r]$$

and the quadrangle

$$[p\gamma_1(t_1)] \cup [\gamma_1(t_1)\gamma_2(t_2)] \cup [\gamma_2(t_2)q] \cup [qp]$$

lie in the union of only two sheets (i.e., they lie in spaces in which the angle comparison condition holds true). In this case, we join $\gamma_1(t_1)$ and q by a shortest near-geodesic in the *corresponding* space. Then the angle comparison condition is fulfilled for the triangles $\triangle r\gamma_1(t_1)\gamma_2(t_2)$, $\triangle p\gamma_1(t_1)q$, and $\triangle q\gamma_1(t_1)\gamma_2(t_2)$. Therefore, it is also fulfilled for $\triangle pqr$.

We show that the numbers t_1, t_2 and the shortest near-geodesic described above do exist. Note that if there exist numbers t_1, t_2 and a shortest near-geodesic joining $\gamma(t_1)$ to $\gamma(t_2)$ that lies in M_δ, then the quadrangle $p\gamma_1(t_1)\gamma_2(t_2)q$ lies in $M_\alpha \cup M_\delta$, and the triangle $\triangle r\gamma_1(t_1)\gamma_2(t_2)$ lies in $M_\delta \cup M_\beta$. Suppose there is no such shortest geodesics. Then, for any pair $(t_1, t_2) \in [t_1^\alpha, t_1^\beta] \times [t_2^\alpha, t_2^\beta]$, the pair $(\gamma_1(t_1), \gamma_2(t_2))$ is nonvertical and no near-geodesic joining these points intersects Γ in a segment or in a point.

For $\sigma = \alpha$, β, let A_σ denote the set of all pairs $(t_1, t_2) \in [t_1^\alpha, t_1^\beta] \times [t_2^\alpha, t_2^\beta]$ such that the pair $(\gamma_1(t_1), \gamma_2(t_2))$ is nonvertical and $\mathrm{B}_\sigma(\overline{\gamma_1(t_1)\gamma_2(t_2)}) > -\frac{c}{2}$.

By Lemma 2, the sets A_α and A_β are open and disjoint; their union is $[t_1^\alpha, t_1^\beta] \times [t_2^\alpha, t_2^\beta]$ (we recall that the space $M_\alpha \cup M_\beta \cup M_\delta$ is considered). Therefore, one of the sets A_α and A_β coincides with $[t_1^\alpha, t_1^\beta] \times [t_2^\alpha, t_2^\beta]$ (and the other is empty). For definiteness, suppose that

$A_\alpha = [t_1^\alpha, t_1^\beta] \times [t_2^\alpha, t_2^\beta]$. Since $(t_1^\beta, t_2^\beta) \in A_\alpha$, we have $\mathrm{B}_\alpha(\overline{\gamma_1(t_1^\beta)\gamma_2(t_2^\beta)}) > -\frac{c}{2}$. However, since $\gamma_1(t_1^\beta), \gamma_2(t_2^\beta) \in \Gamma$, Lemma 2 shows that any shortest near-geodesic joining these points lies in M_α. Thus, the quadrangle $pq\gamma_1(t_1^\beta)\gamma_2(t_2^\beta)q$ lies in $M_\alpha \cup M_\delta$, and the triangle $\triangle r\gamma_1(t_1^\beta)\gamma_2(t_2^\beta)$ lies in $M_\alpha \cup M_\beta$. □

§8. Proof of Theorem 1

Since the assertion of Theorem 1 is local, it suffices to prove it in the small neighborhood V obtained earlier. The proof employs Jacobi fields. We shall show that all normal Jacobi fields of any minimizing geodesic $\gamma : [0,1] \to M$ are κ-convex. Since it is known that M has curvature not exceeding some constant $\tilde{\kappa}$ (see Corollary 4.3 and the remark after it), we conclude that all normal Jacobi fields are $\tilde{\kappa}$-convex (see Proposition 2). Therefore, we must verify κ-convexity near all but finitely many points.

The definition of a Jacobi field involves a sequence of geodesics γ_k converging to γ. For every k, either γ_k lies in the interior of one sheet, or there exist indices $\alpha_k, \beta_k, \delta_k \in I$ and numbers $0 \le t_k^\alpha \le t_k^\beta \le 1$ such that the restrictions of γ to the segments $[0, t_k^\alpha]$, $[t_k^\alpha, t_k^\beta]$, and $[t_k^\beta, 1]$ lie in M_{α_k}, M_{δ_k}, and M_{β_k}, respectively. Taking a subsequence, we may assume that either all γ_k lie in the interior of one sheet, or there exist $\alpha, \beta, \delta \in I$ and $0 \le t_k^\alpha \le t_k^\beta \le 1$ such that for any k the restrictions of γ to $[0, t_k^\alpha]$, $[t_k^\alpha, t_k^\beta]$, and $[t_k^\beta, 1]$ lie in M_α, M_δ, and M_β, respectively. In the first case, κ-convexity follows from the standard formulas of Riemannian geometry. In the second case we may assume (possibly, after passing to a subsequence) that the sequences t_k^α and t_k^β tend to t_0^α and t_0^β, respectively. Then the κ-convexity of the normal Jacobi field on the segments $[0, t_0^\alpha]$, $[t_0^\alpha, t_0^\beta]$, and $[t_0^\beta, 1]$ follows from Corollary 1.1, in which normal Jacobi fields in Riemannian manifolds with boundary were considered. □

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