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# A KINK IN A FUNNY PLACE

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ABSTRACT. When the 3-dimensional gravitational Chern–Simons term is reduced to two dimensions, a dilation-like gravity theory emerges. Its solutions involve kinks, which therefore describe 3-dimensional, conformally flat spaces.

Two topics—among many—which I have discussed with Ludwig Faddeev over the twenty-five years that I have known him are first, gravity theory, and second, topological entities, both in mathematical settings such as characteristic classes, and in physical realizations such as kink profiles on a line. So, on the occasion of his significant birthday I present to Ludwig an investigation which unites these diverse elements.

Let me begin with lineal kinks. Consider the field equation

$$\Box\varphi - C\varphi + \varphi^3 = 0, \tag{1}$$

where $C$ is a positive constant. When we look for a lineal kink solution, we take $\varphi$ to depend on a single spatial variable. Then (1) reduces to

$$-\varphi'' - C\varphi + \varphi^3 = 0 \tag{2}$$

and has the well-known kink solution,

$$\varphi_k(x) = \sqrt{C}\tanh\sqrt{\frac{C}{2}}x, \tag{3}$$

which interpolates between the "vacuum" solutions $\varphi_0 = \pm\sqrt{C}$ (see, e.g., [1]). The kink has interesting roles in condensed matter physics, where it triggers fermion fractionization [2]. Other kinks in other models give rise to completely solvable field theories, both in classical and quantal frameworks. These stories do not belong here. But I shall return to the above kink later.

Next let me consider the Chern–Simons characteristic class. In non-Abelian gauge theory it is constructed from a matrix gauge connection $(A_\alpha)^\mu_\nu$ as [3]

$$W(A) = \frac{1}{4\pi^2}\int d^3x\,\varepsilon^{\alpha\beta\gamma}\mathrm{tr}\Big(\frac{1}{2}A_\alpha\partial_\beta A_\gamma + \frac{1}{3}A_\alpha A_\beta A_\gamma\Big). \tag{4}$$

This gauge-theoretic entity finds physical application in the quantum Hall regime, perhaps also in high $T$ superconductivity. When added with strength $m$ to the usual Yang–Mills action, the Chern–Simons term gives rise to massive, yet gauge-invariant excitations in (2 + 1)-dimensional space-time. Also, for consistency in the quantized version of a non-Abelian theory, $m$ must be an integer multiple of $2\pi$. This is a precise field-theoretic analog of Dirac's celebrated quantization of magnetic monopole strength. Finally, in an important mathematical application, the Chern–Simons term gives a functional integral formula for knot invariants.

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The gauge-theoretic Chern–Simons term (4) can be translated into a 3-dimensional geometric quantity by replacing the matrix gauge connection $(A_\alpha)^\mu_\nu$ with the Christoffel connection $\Gamma^\mu_{\alpha\nu}$ [3],

$$W(\Gamma) = \frac{1}{4\pi^2}\int d^3x\,\varepsilon^{\alpha\beta\gamma}\Big(\frac{1}{2}\Gamma^\rho_{\alpha\sigma}\partial_\beta\Gamma^\sigma_{\gamma\rho} + \frac{1}{3}\Gamma^\rho_{\alpha\sigma}\Gamma^\sigma_{\beta\tau}\Gamma^\tau_{\gamma\rho}\Big). \tag{5}$$

But it is to be remembered that $\Gamma^\mu_{\alpha\nu}$ is constructed in the usual way from the metric tensor, $g_{\mu\nu}$, which is taken as the fundamental, independent variable. When $W(\Gamma)$ is varied with respect to $g_{\mu\nu}$, there emerges the Cotton tensor, which has an important role in 3-dimensional geometry:

$$\delta W(\Gamma) = -\frac{1}{4\pi^2}\int d^3x\,\delta g_{\mu\nu}\sqrt{g}C^{\mu\nu}, \tag{6}$$

$$C^{\mu\nu} \equiv \frac{1}{2\sqrt{g}}\Big(\varepsilon^{\mu\alpha\beta}D_\alpha R^\nu_\beta + \varepsilon^{\nu\alpha\beta}D_\alpha R^\mu_\beta\Big). \tag{7}$$

(In (7) one may freely replace the Ricci tensor $R^\mu_\nu$ by the Einstein tensor $G^\mu_\nu \equiv R^\mu_\nu - \frac{1}{2}\delta^\mu_\nu R^\alpha_\alpha$.) The Cotton tensor is like the covariant curl of the Ricci or Einstein tensor. It is manifestly symmetric; it is covariantly conserved and traceless because it is the variation of the diffeomorphism and conformally invariant $W(\Gamma)$. Furthermore, the Cotton tensor replaces the Weyl tensor, which is absent in three dimensions, as a template for conformal flatness: $C^{\mu\nu}$ vanishes if and only if the space is conformally flat, i.e.,

$$C^{\mu\nu} = 0 \iff \text{conformally flat space}. \tag{8}$$

The absence of the 3-dimensional Weyl tensor has the consequence that 3-dimensional geometries satisfying Einstein's equation carry nonvanishing curvature only in regions where there are sources. Therefore, there are no propagating excitations. However, upon extending Einstein's gravity equation by adding $\frac{1}{m}C^{\mu\nu}$ to the Einstein tensor (equivalently, adding $\frac{4\pi^2}{m}W(\Gamma)$ to the Einstein–Hilbert action) the theory acquires a propagating mode with mass $m$, all the while preserving diffeomorphism invariance! Here we have another perspective on the absence of propagating modes in 3-dimensional Einstein theory: to regain the Einstein equations from the modified equations, we must pass $m$ to infinity, whereupon the supermassive propagating mode decouples.

This is a well-known story, with which I do not concern myself now. Rather, I consider the opposite limit of the extended theory, where only the Cotton tensor survives, and the equation that I shall examine demands its vanishing, i.e., equation (8). But as indicated previously, that equation is not sufficiently restrictive to be interesting: any conformally flat space-time (coordinates $(t, x, y)$) is a solution. So I shall place a further restriction: the solution that I seek should be independent of the $y$-coordinate in a Kaluza–Klein dimensional reduction from (2+1) to (1+1) dimensions of the gravitational Chern–Simons term $W(\Gamma)$ in (5) and of the Cotton tensor $C^{\mu\nu}$ in (7) (see [4]).

To effect the dimensional reduction, we begin by making a Kaluza–Klein *Ansatz* for the 3-dimensional metric tensor. It is taken in the form

$$\text{3-d metric tensor} = \varphi\begin{pmatrix} g_{\alpha\beta} - a_\alpha a_\beta, & -a_\alpha \\ -a_\beta, & -1 \end{pmatrix}, \tag{9}$$

where the 2-dimensional metric tensor $g_{\alpha\beta}$, vector $a_\alpha$, and scalar $\varphi$ depend only on $t$ and $x$. (Henceforth, Greek letters from the beginning of the alphabet $(\alpha, \beta, \gamma, \dots)$ index 2-dimensional $(t, x)$-dependent geometric entities, which are written with lower case letters; in three space-time dimensions geometric entities are capitalized (save the metric tensor) and are indexed by middle Greek alphabet letters $(\mu, \nu, \rho, \dots)$.)

It is easy to show that under infinitesimal diffeomorphisms that leave the $y$-coordinate unchanged, $g_{\alpha\beta}$, $a_\alpha$ and $\varphi$ transform as a 2-dimensional coordinate tensor, a vector and a scalar, respectively, and, moreover, $a_\alpha$ undergoes a gauge transformation.

With the above *Ansatz* for the 3-dimensional metric, the Chern–Simons action becomes

$$CS = -\frac{1}{8\pi^2}\int d^2x\,\sqrt{-g}(fr + f^3). \tag{10}$$

Here $g = \det g_{\alpha\beta}$, $\partial_\alpha a_\beta - \partial_\beta a_\alpha \equiv f_{\alpha\beta} \equiv \sqrt{-g}\varepsilon_{\alpha\beta} f$, and $r$ is the 2-dimensional scalar curvature. The absence of $\varphi$-dependence is a consequence of the conformal invariance of the gravitational Chern–Simons term, and this also ensures that the Cotton tensor is traceless. Henceforth we set $\varphi$ to 1.

The above expressions look like they are describing 2-dimensional dilation gravity, with $f$ taking the role of a dilation field [5]. However, in fact $f$ is not a fundamental field; rather it is the curl of the vector potential $a_\alpha$. Alternative expressions for the action (10) are $\int da\,(r + f^2)$ (where $da$ is a 2-form; this exposes the topological character of our theory), and $\int d^2x\,\Theta\varepsilon^{\alpha\beta} f_{\alpha\beta}, \Theta \equiv r + f^2$ (which highlights an axion-like interaction in 2-dimensional space-time).

Variation of $a_\alpha$ and $g_{\alpha\beta}$ produces the equations

$$0 = \varepsilon^{\alpha\beta}\partial_\beta(r + 3f^2), \tag{11}$$

$$0 = g_{\alpha\beta}(D^2 f - f^3 - \tfrac{1}{2} r f) - D_\alpha D_\beta f. \tag{12}$$

The first is solved by

$$r + 3f^2 = \text{constant} \equiv C. \tag{13}$$

Eliminating $r$ in the second equation, and decomposing it into the trace and trace-free parts leaves

$$0 = D^2 f - Cf + f^3, \tag{14}$$

$$0 = D_\alpha D_\beta f - \frac{1}{2} g_{\alpha\beta} D^2 f. \tag{15}$$

Note that the equations are invariant against changing the sign of $f$ (the action then also changes sign).

A homogenous solution that respects the $f \leftrightarrow -f$ symmetry is

$$f = 0, \quad r = C. \tag{16}$$

However, there is also is a "symmetry breaking" solution,

$$f = \pm\sqrt{C}, \;\; r = -2C, \quad C > 0. \tag{17}$$

Forms for $g_{\alpha\beta}$ and $a_\alpha$ that lead to the above results are

$$\text{(a)}\quad f = 0, r = C > 0: \qquad g_{\alpha\beta} = \tfrac{2}{Ct^2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \;\; a_\alpha = (0, 0), \tag{18}$$

$$\text{(b)}\quad f = 0, r = C < 0: \qquad g_{\alpha\beta} = \tfrac{2}{|C|x^2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \;\; a_\alpha = (0, 0), \tag{19}$$

$$\text{(c)}\quad f = \pm\sqrt{C}, r = -2c < 0: \;\; g_{\alpha\beta} = \tfrac{1}{Cx^2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \;\; a_\alpha = \left(\tfrac{\pm 1}{\sqrt{C}x}, 0\right). \tag{20}$$

In the first case, the 2-dimensional space-time is deSitter; in the last two, it is anti-deSitter.

The 3-dimensional scalar curvature $R$, with metric tensor as in our *Ansatz* (9) at $\varphi = 1$, is related to the 2-dimensional curvature $r$ by

$$R = r + \frac{1}{2} f^2. \tag{21}$$

Hence for the three cases, the 3-dimensional curvature and line element read

$$\text{(a)} \qquad R = C > 0, \quad (ds)^2 = \frac{2}{C}\Big[\Big(\frac{dt}{t}\Big)^2 - \Big(\frac{dx}{t}\Big)^2\Big] - (dy)^2, \tag{22}$$

$$\text{(b)} \qquad R = C < 0, \quad (ds)^2 = \frac{2}{|C|}\Big[\Big(\frac{dt}{x}\Big)^2 - \Big(\frac{dx}{x}\Big)^2\Big] - (dy)^2, \tag{23}$$

$$\text{(c)} \qquad R = -\frac{3}{2} C < 0, \quad (ds)^2 = \pm \frac{2}{\sqrt{C} x} dt dy - \Big(\frac{dx}{\sqrt{C} x}\Big)^2 - (dy)^2. \tag{24}$$

Although all three solutions carry constant 3-dimensional curvature, the "symmetry breaking" solution, (c) above, possesses greater geometrical symmetry in three dimensions: it supports six Killing vectors that span $SO(2.1) \times SO(2.1) = SO(2.2)$, the isometry of 3-dimensional anti-deSitter space. Moreover, one verifies that, as expected, $R^\mu_\nu = \frac{1}{3}\delta^\mu_\nu R = -\frac{1}{2}\delta^\mu_\nu C$—the 3-dimensional space-time is maximally symmetric. The "symmetry preserving" solutions, (a) and (b) above, admit only four Killing vectors that span $SO(2.1) \times SO(2)$.

Since the Cotton tensor vanishes, we expect that the above space-times are (locally) conformally flat. This can be seen explicitly for the "symmetry preserving" solutions. In (22), set $T = t \cosh \sqrt{\frac{C}{2}} y$, $Y = t \sinh \sqrt{\frac{C}{2}} y$, and $X = x$ to find

$$(ds)^2 = \frac{2}{C(T^2 - Y^2)}\big((dT)^2 - (dX)^2 - (dY)^2\big), \tag{25}$$

while in (23) the coordinate transformation

$$X = x \cos \sqrt{\frac{|C|}{2}} y, \quad Y = x \sin \sqrt{\frac{|C|}{2}} y, \quad T = t$$

gives the line element

$$(ds)^2 = \frac{2}{|C|(X^2 + Y^2)}\big((dT)^2 - (dX)^2 - (dY)^2\big). \tag{26}$$

We have not found the relevant coordinate transformation for the "symmetry breaking" solution (c), but we know that the 3-dimensional space-time is indeed conformally flat since it is anti-deSitter.

Equations (14), (15) also possess a kink solution, which interpolates between the "symmetry breaking" solutions (17). One can verify that

$$f(x) = \sqrt{C} \tanh \frac{\sqrt{C}}{2} x, \tag{27}$$

with

$$g_{\alpha\beta} = \begin{pmatrix} 1/\cosh^4 \frac{\sqrt{C}}{2} x & 0 \\ 0 & -1 \end{pmatrix}, \tag{28}$$

satisfies the relevant equations. That the solution depends only on one variable (only $x$, not both $t, x$) is a general property (provided coordinates are selected properly). Thus in (14), (15) one is dealing with a system of second-order ordinary (not partial) differential equations, whose solution involves two integration constants. One integration constant is the trivial origin of the $x$-coordinate (taken to be $x = 0$ in (27), (28)). The other involves choosing an integration constant in a first integral, so that one achieves a kink: a profile that interpolates between $\pm\sqrt{C}$ as $x \to \pm\infty$. (Other choices for this second constant lead to the same local geometry, but to different global properties. This has been thoroughly explained by Grumiller and Kummer [6].)

The 2-dimensional curvature corresponding to (28) is

$$r = -2C + \frac{3C}{\cosh^2(\frac{\sqrt{C}}{2}x)}. \tag{29}$$

Also the 3-dimensional line element for (27), (28) reads

$$(ds)^2 = -(dx)^2 - \frac{2dtdy}{\cosh^2 \frac{\sqrt{C}}{2}x} - (dy)^2, \tag{30}$$

and the 3-dimensional scalar curvature is according to (21), (27) and (28),

$$R = -\frac{3C}{2} + \frac{5C}{2\cosh^2 \frac{\sqrt{C}}{2}x}. \tag{31}$$

Again, because (30) ensures that the Cotton tensor vanishes, there should exist a coordinate transformation to conformally flat, 3-dimensional coordinates. We have not found it.

This then is the kink in a "funny place"—in conformally flat (2+1)-dimensional space-time. A question remains: can one understand *a priori* that such a kink should exist in that geometry? This question may be posed in a more general setting.

Observe that the flat space kink in equations (1)–(3) possesses the same profile as (27), except for a change in scale. In fact this is a general feature. The following can be proved. If the nonlinear equation in flat space-time

$$\Box\varphi + V'(\varphi) = 0 \tag{32}$$

possesses a kink solution $\varphi_k(x) = k(x)$, then the curved (1+1)-dimensional space-time equations

$$D^2 f + V'(f) = 0, \tag{33}$$

$$D_\alpha D_\beta f - \frac{1}{2} g_{\alpha\beta} D^2 f = 0 \tag{34}$$

are solved by

$$f(x) = k(x/\sqrt{2}), \tag{35}$$

with 2-dimensional line element

$$(ds)^2 = V(f)(dt)^2 - (dx)^2, \tag{36}$$

leading to a 2-dimensional curvature

$$r = -V''(F). \tag{37}$$

Perhaps Ludwig Faddeev can illuminate the deeper geometric reasons behind this coincidence.

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