ON HIGHER SPIN $U_q(\mathrm{sl}_2)$-INVARIANT R-MATRICES

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Dedicated to Professor L. D. Faddeev on the occasion of his 70th birthday

ABSTRACT. The spectral decomposition of regular $U_q(\mathrm{sl}_2)$-invariant solutions of the Yang–Baxter equation is studied. An algorithm for the search of all possible spin s solutions is developed, also allowing reconstruction of the R-matrix by a given nearest neighbor spin chain Hamiltonian. The algorithm is based on reduction of the Yang–Baxter equation to certain subspaces. As an application, a complete list of nonequivalent regular $U_q(\mathrm{sl}_2)$-invariant R-matrices is obtained for generic q and spins $s \leq \frac{3}{2}$. Some further results about spectral decompositions for higher spins are also proved. In particular, it is shown that certain types of regular sl_2-invariant R-matrices have no $U_q(\mathrm{sl}_2)$-invariant counterparts.

§1. PRELIMINARIES

The quantum Lie algebra $U_q(\mathrm{sl}_2)$ is defined as a universal enveloping algebra over $\mathbb{C}$ with identity element e and with generators $S^{\pm}$, S^z that obey the following defining relations [1, 2]:

$$(1) \qquad [S^+, S^-] = [2S^z]_q, \quad [S^z, S^{\pm}] = \pm S^{\pm},$$

where $[t]_q = (q^t - q^{-t})/(q - q^{-1})$. The algebra $U_q(\mathrm{sl}_2)$ can be equipped with a Hopf algebra structure [3, 4, 5]. In particular, the comultiplication (a coassociative linear homomorphism) is defined by

$$(2) \qquad \Delta(S^{\pm}) = S^{\pm} \otimes q^{-S^z} + q^{S^z} \otimes S^{\pm}, \quad \Delta(S^z) = S^z \otimes e + e \otimes S^z.$$

For generic q, the algebra (1), (2) has the same structure of representations as the undeformed algebra sl_2 [6]. In particular, the irreducible highest weight representations V_s are parameterized by a nonnegative integer or half-integer s (referred to as the *spin*) and are $(2s+1)$-dimensional. We use the standard notation $|k\rangle$, $k = -s, \ldots, s$, for the basis vectors of V_s such that $S^z|k\rangle = k|k\rangle$, $\langle k'|k\rangle = \delta_{kk'}$.

Let $\mathbb{E}$ denote the identity operator on $V_s^{\otimes 2}$. Consider operator-valued functions $R(\lambda)$: $\mathbb{C} \mapsto \mathrm{End}\, V_s^{\otimes 2}$ that have the following properties:

$$(3) \qquad \text{regularity:} \qquad R(0) = \mathbb{E},$$

$$(4) \qquad \text{unitary property:} \quad R(\lambda)R(-\lambda) = \mathbb{E},$$

$$(5) \qquad \text{spectral decomposition:} \qquad R(\lambda) = \sum_{j=0}^{2s} r_j(\lambda) P^j,$$

$$(6) \qquad \text{normalization:} \qquad r_{2s}(\lambda) = 1.$$

Here P^j is the projection onto the spin j subspace V_j in $V_s^{\otimes 2}$, and $r_j(\lambda)$ is a scalar function. Property (5) is equivalent to the requirement of $U_q(\mathrm{sl}_2)$-*invariance*, i.e.,

$$[R(\lambda), \Delta(\xi)] = 0, \quad \xi \in U_q(\mathrm{sl}_2). \tag{7}$$

In order that (3)–(4) be true, the coefficients $r_j(\lambda)$ must satisfy the relations

$$r_j(0) = 1, \quad r_j(\lambda) r_j(-\lambda) = 1. \tag{8}$$

In what follows it will be assumed that the $r_j(\lambda)$ are analytic in some neighborhood of $\lambda = 0$. The normalization condition (6) is imposed in order to eliminate the inessential freedom of rescaling $R(\lambda)$ by an arbitrary analytic function preserving conditions (8).

For a function $R(\lambda)$ satisfying (3) and (4), we define the *Yang–Baxter* (YB) *operator* $Y(\lambda, \mu) : \mathbb{C}^2 \mapsto \mathrm{End}\, V_s^{\otimes 3}$ as follows:

$$Y(\lambda, \mu) = R_{12}(\lambda) R_{23}(\lambda + \mu) R_{12}(\mu) - R_{23}(\mu) R_{12}(\lambda + \mu) R_{23}(\lambda). \tag{9}$$

Here and below we use the standard notation—the subscripts specify the components of the tensor product $V_s^{\otimes 3}$. We say that $R(\lambda)$ is a ($U_q(\mathrm{sl}_2)$-invariant) *R-matrix* if the corresponding YB operator vanishes on $V_s^{\otimes 3}$,

$$Y(\lambda, \mu) = 0. \tag{10}$$

An advantage of treating the YB *equation* (10) as a condition of vanishing for the YB operator is that, as will be shown below, conditions of vanishing of the YB operator on some *subspaces* of $V_s^{\otimes 3}$ involve fewer coefficients $r_j(\lambda)$. Moreover, the $r_j(\lambda)$ found by resolving such a condition for a given subspace can further be used in order to write down and solve conditions of vanishing of the YB operator on other subspaces. A recursive procedure of this type will be presented in the next section.

Remark 1. Equations (8) and (10) are preserved under rescaling of the spectral parameter,

$$\lambda \to \gamma \lambda, \tag{11}$$

by an arbitrary finite constant γ. R-matrices related by such a transformation with a finite nonzero γ will be regarded as equivalent.

Remark 2. Conditions (3) and (6) along with the YB equation ensure the unitary property of an R-matrix (see Appendix A).

For $q = 1$, four different types of sl_2-invariant R-matrices are known [7, 8, 9, 10, 11]:

$$R(\lambda) = (1 - \lambda)^{-1}(\mathbb{E} - \lambda \mathbb{P}), \tag{12}$$

$$R(\lambda) = P^{2s} + \sum_{j=0}^{2s-1} \Big(\prod_{k=j+1}^{2s} \frac{k + \lambda}{k - \lambda} \Big) P^j, \tag{13}$$

$$R(\lambda) = (1 - \lambda)^{-1} \Big(\mathbb{E} - \lambda \mathbb{P} + \frac{\beta \lambda}{\lambda - \alpha} P^0 \Big), \tag{14}$$
$$\alpha = s + \tfrac{1}{2} + (-1)^{2s+1}, \quad \beta = (2s+1)(-1)^{2s+1},$$

$$R(\lambda) = \mathbb{E} + (b^2 + 1) \frac{1 - e^\lambda}{e^\lambda - b^2} P^0, \quad b + b^{-1} = 2s + 1, \tag{15}$$

where

$$\mathbb{P} = \sum_{j=0}^{2s} (-1)^{2s-j} P^j \tag{16}$$

is the permutation operator in $V_s \otimes V_s$. Observe that for all but the last type of solutions we have

$$(17) \qquad\qquad\qquad\qquad R(\pm\infty) = \mathbb{P}.$$

For $s = \frac{1}{2}$, the R-matrices (13) and (14) degenerate into (12), and the fourth solution (15) is absent. For $s = 1$, the R-matrices (13) and (14) are equivalent. For $q = 1$ and $s = 3$, there is an additional solution, which is not of the form (12)–(15). It is given (see [12]) by the formula

$$(18) \qquad R(\lambda) = P^6 + \frac{1+\lambda}{1-\lambda}P^5 + P^4 + \frac{4+\lambda}{4-\lambda}P^3 + P^2 + \frac{1+\lambda}{1-\lambda}P^1 + \frac{1+\lambda}{1-\lambda}\frac{6+\lambda}{6-\lambda}P^0.$$

Numerical, computer-based checks [12] suggest that equations (12)–(15) and (18) exhaust the list of sl_2-invariant R-matrices, but no classification theorem has yet been proved.

For $q \neq 1$, counterparts of (13) and (15) are given by

$$(19) \qquad\qquad R(\lambda) = P^{2s} + \sum_{j=0}^{2s-1}\Big(\prod_{k=j+1}^{2s}\frac{[k+\lambda]_q}{[k-\lambda]_q}\Big)P^j,$$

$$(20) \qquad\qquad R(\lambda) = \mathbb{E} + (b^2+1)\frac{1-e^\lambda}{e^\lambda - b^2}P^0, \quad b + b^{-1} = [2s+1]_q$$

(see [13, 11]).

Our aim in the present paper is to study $U_q(\mathrm{sl}_2)$-invariant solutions of the Yang–Baxter equation for a generic q (i.e., q is not a root of unity and $q \neq 0, \infty$) and to develop a systematic method of finding all possible sets of $r_j(\lambda)$ for a given spin s. In particular, we shall prove that (12), (14), and (18) have no regular $U_q(\mathrm{sl}_2)$-invariant counterparts. Our approach is based on the fact that the $U_q(\mathrm{sl}_2)$-invariance of an R-matrix implies that the corresponding YB operator commutes with the action of $U_q(\mathrm{sl}_2)$ on $V_s^{\otimes 3}$. This action is defined as follows:

$$(21) \qquad S_{123}^z = (\Delta \otimes \mathrm{id})\Delta S^z = S_1^z + S_2^z + S_3^z,$$

$$(22) \qquad S_{123}^\pm = (\Delta \otimes \mathrm{id})\Delta S^\pm = S_{12}^\pm q^{-S_3^z} + q^{S_{12}^z}S_3^\pm = q^{S_1^z}S_{23}^\pm + S_1^\pm q^{-S_{23}^z}.$$

The assertion

$$(23) \qquad\qquad\qquad [Y(\lambda,\mu), S_{123}^z] = [Y(\lambda,\mu), S_{123}^\pm] = 0$$

follows from the fact that the P^j are functions of ΔC, where C is the Casimir element of $U_q(\mathrm{sl}_2)$. From (21)–(22) it is clear that the P_l^j, $l = \{12\}, \{23\}$ commute with S_{123}^z and $S_{123}^\pm$.

§2. Reduced Yang–Baxter equations

2.1. Hecke–Temperley–Lieb algebra in YB. Interrelations between Hecke algebras, braid groups, and constant (independent of the spectral parameter λ) solutions of the YB equation are well known. In the case of nontrivial spectral parameter dependence, a construction of an R-matrix employing the Temperley–Lieb algebra [14] generators was introduced by Baxter in [11]. For the purposes of the present paper, we shall need the following slightly more general version of this construction.

Lemma 1. *Consider an associative algebra over $\mathbb{C}$ with unit element $\mathbb{E}$ and generators U_l labeled by $l = \{12\}, \{23\}$ and obeying the following Hecke-type relations (where η_0 and η_1 are scalar constants, $\mathrm{Re}\,\eta_0 \geq 0$):*

$$(24) \qquad\qquad\qquad\qquad U_l^2 = \eta_0 U_l + \eta_1 \mathbb{E},$$

$$(25) \qquad\qquad U_{12}U_{23}U_{12} - U_{23}U_{12}U_{23} = U_{12} - U_{23}.$$

Let $g(\lambda)$ be a function analytic in a neighborhood of $\lambda = 0$ and satisfying the condition $g(0) = 0$. Then $R_l(\lambda) = \mathbb{E} + g(\lambda)U_l$ satisfies the YB equation (10) *if and only if*

(26)
$$g(\lambda) = \begin{cases} \frac{2\gamma\lambda}{1-\gamma\lambda} & \text{if } \eta_0 = 2; \\ b\frac{1-e^{\gamma\lambda}}{e^{\gamma\lambda}-b^2}, & b+b^{-1} = \eta_0 \quad \text{if } \eta_0 \neq 2. \end{cases}$$

Here γ is an arbitrary finite constant.

Remark 3. The R-matrices (12) and (15) provide two examples where the U_l are elements of $\operatorname{End} V_s^{\otimes 3}$ given by

(27)
$$U_{12} = U \otimes e, \quad U_{23} = e \otimes U,$$

where $U = \mathbb{E} + \mathbb{P}$, $\eta_0 = 2$ and $U = P^0$, $\eta_0 = 2s+1$, respectively, and e is the unit element on V_s. However, we emphasize that, in general, the hypotheses of Lemma 1 *do not* require that the U_l be of the form (27) with $U \in \operatorname{End} V^{\otimes 2}$. In fact, below we shall apply Lemma 1 in the cases where the underlying linear space is not of the form $V^{\otimes 3}$.

Proof. For completeness of exposition, we give the proof of the lemma. Substituting $R_l(\lambda)$ in (10) and employing (24), (25), we reduce the YB equation to the form $(\cdots)(U_{12} - U_{23}) = 0$, where $(\cdots)$ is a scalar factor. Therefore, the YB equation is satisfied if and only if this factor vanishes, which is equivalent to the requirement that $g(\lambda)$ satisfy the functional relation

(28) $$g(\lambda - \mu)g(\lambda)g(\mu) + \eta_0 g(\lambda - \mu)g(\mu) + g(\lambda - \mu) - g(\lambda) + g(\mu) = 0.$$

Differentiating (28) with respect to μ, setting $\mu = \lambda$, and taking the condition $g(0) = 0$ into account, we derive the differential equation

(29)
$$g'(\lambda) = g'(0)\big((g(\lambda))^2 + \eta_0 g(\lambda) + 1\big).$$

Its solution is given by (26), and it is easily verified that this solution does satisfy (28). $\quad\square$

Remark 4. The analysis of (28) changes if we drop the condition $g(0) = 0$. In this case, by setting $\mu = \lambda$, equation (28) is reduced to an algebraic equation, namely,

(30)
$$(g(\lambda))^2 + \eta_0 g(\lambda) + 1 = 0,$$

which implies that $g(\lambda)$ is a constant function. The possible values of the constant, i.e., the roots of (30), are given by formula (26) in the limit as $\gamma\lambda \to \pm\infty$.

Now, we apply Lemma 1 in order to obtain some information about the spectral resolution of a regular $U_q(\mathrm{sl}_2)$-invariant R-matrix.

Proposition 1. *Suppose a $U_q(\mathrm{sl}_2)$-invariant solution $R(\lambda)$ of the YB equation* (10) *on $V_s^{\otimes 3}$ satisfies* (6) *and* (8). *Then the second highest coefficient in its spectral resolution is given by*

(31)
$$r_{2s-1}(\lambda) = \begin{cases} \frac{1+\gamma\lambda}{1-\gamma\lambda} & \text{if } q = 1; \\ \frac{[2s+\gamma\lambda]_q}{[2s-\gamma\lambda]_q} & \text{if } q \neq 1, \end{cases}$$

where γ is an arbitrary finite constant.

Proof. Let $\tilde{W}_1$ denote the subspace of $V_s^{\otimes 3}$ that is the linear span of the vectors

(32) $$|1\rangle_{123} = |s-1\rangle_1 |s\rangle_2 |s\rangle_3, \quad |2\rangle_{123} = |s\rangle_1 |s-1\rangle_2 |s\rangle_3, \quad |3\rangle_{123} = |s\rangle_1 |s\rangle_2 |s-1\rangle_3.$$

From (23) and the Clebsch–Gordan (CG) decomposition of $V_s^{\otimes 2}$,

(33)
$$|2s, 2s-1\rangle = \alpha_s |s\rangle|s-1\rangle + \beta_s |s-1\rangle|s\rangle,$$

(34)
$$|2s-1, 2s-1\rangle = \beta_s |s\rangle|s-1\rangle - \alpha_s |s-1\rangle|s\rangle,$$

(35)
$$\alpha_s = q^s (q^{2s} + q^{-2s})^{-\frac{1}{2}}, \quad \beta_s = q^{-s}(q^{2s} + q^{-2s})^{-\frac{1}{2}}$$

(see [15, 16] for an explicit form of the CG coefficients), we infer that $\tilde{W}_1$ is an invariant subspace of the YB operator for the R-matrix under consideration. Notice that the restrictions of P_l^j, $l = \{12\}, \{23\}$, to $\tilde{W}_1$ vanish if $j < 2s - 1$. Thus,

$$(36) \qquad R_l(\lambda)\big|_{\tilde{W}_1} = P_l^{2s} + r_{2s-1}(\lambda) P_l^{2s-1}.$$

Furthermore, taking (33) and (34) into account and introducing $\tilde{g}(\lambda) = r_{2s-1}(\lambda) - 1$, we observe that (36) can be rewritten in the form

$$(37) \qquad R_l(\lambda)\big|_{\tilde{W}_1} = \mathbb{E} + \tilde{g}(\lambda)\pi_l,$$

where the π_l are projections, $\pi_l^2 = \pi_l$, given in the basis (32) by

$$(38) \qquad \pi_{12} = \alpha_s^2 |1\rangle\langle 1| - \alpha_s \beta_s |1\rangle\langle 2| - \alpha_s \beta_s |2\rangle\langle 1| + \beta_s^2 |2\rangle\langle 2|,$$

$$(39) \qquad \pi_{23} = \alpha_s^2 |2\rangle\langle 2| - \alpha_s \beta_s |2\rangle\langle 3| - \alpha_s \beta_s |3\rangle\langle 2| + \beta_s^2 |3\rangle\langle 3|.$$

Now, observing that

$$(40) \qquad \pi_{12}\pi_{23}\pi_{12} = (\alpha_s \beta_s)^2 \pi_{12}, \quad \pi_{23}\pi_{12}\pi_{23} = (\alpha_s \beta_s)^2 \pi_{23},$$

we see that the conditions of Lemma 1 are fulfilled for (37) upon identification $U_l = (\alpha_s \beta_s)^{-1}\pi_l$, $g(\lambda) = \alpha_s \beta_s \tilde{g}(\lambda)$, and $\eta_0 = (\alpha_s \beta_s)^{-1}$. Substituting this value of η_0 in (26) and recalling that $\tilde{g}(\lambda) = r_{2s-1}(\lambda) - 1$, we obtain (31), where we have replaced $e^{\gamma\lambda}$ with $q^{2\gamma\lambda}$ for the sake of convenience of comparison with the $q = 1$ limit. Since we require that $R(\lambda)$ be regular, the constant γ must be finite. $\qquad \square$

2.2. Invariant subspaces. The proof of Proposition 1 demonstrates that the reduction of the YB operator to some invariant subspace facilitates finding the coefficients $r_j(\lambda)$ of the R-matrix. In what follows we shall develop this approach further, exploiting available knowledge about the CG decomposition of tensor products of representations of $U_q(\mathrm{sl}_2)$. Along the way, we shall derive systems of coupled functional equations similar to (28) and show that the corresponding necessary conditions are provided by a set of coupled algebraic equations.

In Proposition 1 we used the fact that the YB operator (9) commutes with S_{123}^z. Now we are going to use the fact that $Y(\lambda, \mu)$ commutes with $S_{123}^\pm$ as well.

Let $\lfloor t \rfloor$ denote the integral part of t.

For $n = 0, 1, \ldots, \lfloor 3s \rfloor$, we define a subspace $W_n^{(s)} \subset V_s^{\otimes 3}$ as the linear span of the highest weight vectors of spin $3s - n$, i.e.,

$$(41) \qquad W_n^{(s)} = \{\psi \in V_s^{\otimes 3} \mid S_{123}^+\psi = 0, S_{123}^z\psi = (3s - n)\psi\}.$$

Consider the following two orthonormal bases in $W_n^{(s)}$ (here and below, $\begin{bmatrix} \vdots \vdots \vdots \end{bmatrix}_q$ and $\begin{Bmatrix} \vdots \vdots \vdots \end{Bmatrix}_q$ stand, respectively, for the CG coefficients and $6j$-symbols of $U_q(\mathrm{sl}_2)$):

$$(42) \qquad |n; k\rangle_{123} = \sum_m |m\rangle_1 |2s - k, 3s - n - m\rangle_{23} \begin{bmatrix} s & 2s - k & 3s - n \\ m & 3s - n - m & 3s - n \end{bmatrix}_q,$$

$$(43) \qquad |n; k\rangle'_{123} = \sum_m |2s - k, 3s - n - m\rangle_{12} |m\rangle_3 \begin{bmatrix} 2s - k & s & 3s - n \\ 3s - n - m & m & 3s - n \end{bmatrix}_q.$$

The basis vectors of $W_n^{(s)}$ are enumerated by $k \in I_n^{(s)}$, where

$$(44) \qquad I_n^{(s)} = \begin{cases} 0 \le k \le n & \text{for } 0 \le n \le 2s; \\ n - 2s \le k \le 4s - n & \text{for } 2s \le n \le \lfloor 3s \rfloor. \end{cases}$$

Summation in (42)–(43) runs over those m for which the CG coefficients on the right-hand side of (42), (43) do not vanish, i.e., $s - n + k \le m \le \min(s, 5s - n - k)$.

Let $A^{(s,n)}$ be the transition matrix from the basis (42) to the basis (43), i.e., the orthogonal matrix whose entries are the scalar products

$$A^{(s,n)}_{kk'} = \langle n; k \mid n; k' \rangle'. \tag{45}$$

The transition matrix is q-dependent, but for short we shall not write the argument q explicitly unless this is required by the context.

Proposition 2.

   i) *The entries of $A^{(s,n)}$ are expressed in terms of $6j$-symbols of $U_q(\mathrm{sl}_2)$ as follows:*

$$A^{(s,n)}_{kk'} = (-1)^{2s-n}\sqrt{[4s-2k+1]_q[4s-2k'+1]_q}\left\{\begin{matrix} s & s & 2s-k \\ s & 3s-n & 2s-k' \end{matrix}\right\}_q. \tag{46}$$

   ii) $A^{(s,n)}$ *is self-dual in q,*

$$A^{(s,n)}_q = A^{(s,n)}_{q^{-1}}. \tag{47}$$

   iii) $A^{(s,n)}$ *is orthogonal, symmetric, and coincides with its inverse (t denotes the matrix transposition):*

$$A^{(s,n)} = \left(A^{(s,n)}\right)^t = \left(A^{(s,n)}\right)^{-1}. \tag{48}$$

As a consequence, the only eigenvalues of $A^{(s,n)}$ are ± 1.

   iv) *The transition matrices enjoy the following "spin-level duality" relations:*

$$A^{(s,1)}_q = A^{(\frac{1}{2},1)}_{q^{2s}}, \tag{49}$$

$$A^{(s,n)}_q = A^{(2s-\frac{n}{2},6s-2n)}_q, \tag{50}$$

where $n \leq 2s$.

Explicit formulas for the entries of the matrix $A^{(s,n)}$ and the proof of its properties listed above are given in Appendices B and C.

Remark 5. From iii) and iv) it follows that $\frac{1}{2}\left(\mathbb{E} \pm A^{(s,n)}\right)$ are projections of ranks $n_\pm$. In particular, for $n \leq 2s$ we have $n_+ = \lfloor \frac{n}{2} + 1 \rfloor$, $n_- = \lfloor \frac{n+1}{2} \rfloor$.

The properties of the transition matrix given in Proposition 2 make it an efficient tool for dealing with restrictions of $U_q(\mathrm{sl}_2)$-invariant operators to subspaces $W^{(s)}_n$. As a simple example, we prove the following well-known statement.

Lemma 2. *The following identity is fulfilled on $V^{\otimes 3}_s$:*

$$P^0_{23} P^j_{12} P^0_{23} = \frac{[2j+1]_q}{[2s+1]^2_q} P^0_{23}. \tag{51}$$

Proof. Observe that $P^0_{12}\big|_{W^{(s)}_n}$ and $P^0_{23}\big|_{W^{(s)}_n}$ vanish for all n except $n = 2s$. Thus, it suffices to prove (51) when it is restricted to $W^{(s)}_{2s}$. Denote $p^j = P^j_{23}\big|_{W^{(s)}_{2s}}$. In the basis (42), we have $p^j_{ab} = \delta_{a,2s-j}\delta_{b,2s-j}$. Therefore, in this basis, the left-hand side of (51) takes the form

$$p^0 A^{(s,2s)} p^j A^{(s,2s)} p^0 = \left(A^{(s,2s)}_{2s-j,2s}\right)^2 p^0. \tag{52}$$

The value of $A^{(s,2s)}_{2s-j,2s}$ is easily computable (see formula (97) in Appendix B), and its square yields the scalar coefficient on the right-hand side of (51).

2.3. Reduced Yang–Baxter equations. Equations (23) imply that $W_n^{(s)}$ is an invariant subspace for the YB operator (9). We introduce the *reduced* YB operator: $Y_n(\lambda,\mu) = Y(\lambda,\mu)|_{W_n^{(s)}}$ (the restriction of $Y(\lambda,\mu)$ to $W_n^{(s)}$). Notice that the restrictions of P_l^j to $W_n^{(s)}$ are diagonal in the bases (43) and (42) for $l = \{12\}$ and $l = \{23\}$, respectively. Moreover, they vanish unless

$$(53) \qquad |2s - n| \leq j \leq \min(2s, 4s - n).$$

Therefore, in the basis (43), $R_l(\lambda)|_{W_n^{(s)}}$ are represented as

$$(54) \qquad R_{12}(\lambda)|_{W_n^{(s)}} = A^{(s,n)} D(\lambda) \big(A^{(s,n)}\big)^{-1}, \quad R_{23}(\lambda)|_{W_n^{(s)}} = D(\lambda),$$

$$(55) \qquad D_{kk'}(\lambda) = \delta_{kk'} r_{2s-k}(\lambda),$$

where $k \in I_n^{(s)}$, as specified in (44). Therefore, taking (48) into account, we conclude that $Y_n(\lambda,\mu)$ takes the following form in the basis (43):

$$(56) \qquad \begin{aligned} Y_n(\lambda,\mu) &= A^{(s,n)} D(\lambda - \mu) A^{(s,n)} D(\lambda) A^{(s,n)} D(\mu) \\ &\quad - D(\mu) A^{(s,n)} D(\lambda) A^{(s,n)} D(\lambda - \mu) A^{(s,n)}. \end{aligned}$$

The corresponding *reduced* YB equation reads

$$(57) \qquad A^{(s,n)} D(\lambda - \mu) A^{(s,n)} D(\lambda) A^{(s,n)} D(\mu) = D(\mu) A^{(s,n)} D(\lambda) A^{(s,n)} D(\lambda - \mu) A^{(s,n)}.$$

This is the vanishing condition for the YB operator (9) on $W_n^{(s)}$. Observe that (48) implies that the reduced YB operator is antisymmetric, $\big(Y_n(\lambda,\mu)\big)^t = -Y_n(\lambda,\mu)$. Therefore, (57) contains the following independent relations:

$$(58) \qquad \begin{aligned} \sum_{i,j \in I_n^{(s)}} & r_{2s-i}(\lambda - \mu) r_{2s-j}(\lambda) A_{ij}^{(s,n)} \\ &\times \Big(r_{2s-a}(\mu) A_{aj}^{(s,n)} A_{ib}^{(s,n)} - r_{2s-b}(\mu) A_{ai}^{(s,n)} A_{jb}^{(s,n)} \Big) = 0, \quad a < b, \ a,b \in I_n^{(s)}. \end{aligned}$$

We emphasize that, since the YB operator commutes with S_{123}^-, equations (58) at the level n ensure the vanishing of the YB operator not only on the subspace $W_n^{(s)}$ but also on the larger subspace that is spanned by *all* vectors obtained by the action of $(S_{123}^-)^m$, $m = 0, \ldots, 6s - 2n$ on $W_n^{(s)}$. (This picture resembles closely the structure of eigenvectors in the algebraic Bethe *Ansatz*; see the survey [18].) Thus, the set of reduced YB equations (58), $n = 1, \ldots, \lfloor 3s \rfloor$, is less overdetermined than the initial YB equation (10) containing $\dim V_s^{\otimes 3} = (2s+1)^3$ functional equations (although, in general, some of them are not independent). However, even this set of equations remains overdetermined. Indeed, (58) at the level $n \leq 2s$ involves $r_j(\lambda)$ with $j = 2s - n, \ldots, 2s$. Therefore, it suffices to solve (58) for $n = 1, \ldots, 2s$ to determine all coefficients $r_j(\lambda)$ of the R-matrix. But these coefficients must also satisfy the remaining reduced YB equations for $n = 2s+1, \ldots, \lfloor 3s \rfloor$.

Remark 6. For $s = \frac{1}{2}$ we have $2s = \lfloor 3s \rfloor = 1$, and therefore the corresponding set of reduced YB equations is not overdetermined. Indeed, in this case (58) contains only one independent equation. A slightly less trivial remark is that the set of reduced YB equations is not overdetermined for $s = 1$ as well (see the proof of Proposition 5).

Now, as an immediate application of the reduced YB equation technique, we prove the following statement.

Proposition 3. *Let $R(\lambda)$ be a $U_q(\mathrm{sl}_2)$-invariant solution of the YB equation (10) on $V_s^{\otimes 3}$ for a spin $s \geq \frac{n}{2}$, $n \in \mathbb{Z}_+$, satisfying (6) and (8). Suppose that the n highest coefficients*

in its spectral resolution coincide, $r_{2s}(\lambda) = r_{2s-1}(\lambda) = \cdots = r_{2s-n+1}(\lambda) = 1$. Then

(59) $$r_{2s-n}(\lambda) = 1 + \eta_0 g(\lambda),$$

where $g(\lambda)$ is given by (26) with

(60) $$\eta_0 = \frac{[2s-n]!}{[2s]!} \frac{[4s-n+1]!}{[4s-2n+1]!}.$$

Here the q-factorial is defined as follows: $[n]! = \prod_{k=1}^{n} [k]_q$ for $n \in \mathbb{Z}_+$ and $[0]! = 1$.

Proof. The corresponding reduced YB equation (with n in (57) being the same as in (59)) multiplied from the left by $A^{(s,n)}$ can be regarded as the YB equation (10) for $R_{12}(\lambda) = D(\lambda)$ and $R_{23}(\lambda) = A^{(s,n)}D(\lambda)A^{(s,n)}$. Furthermore, we notice that

(61) $$D(\lambda) = \mathbb{E} + \tilde{g}(\lambda)\pi, \quad A^{(s,n)}D(\lambda)A^{(s,n)} = \mathbb{E} + \tilde{g}(\lambda)\pi',$$

where $\tilde{g}(\lambda) = r_{2s-n}(\lambda) - 1$, π is a matrix such that $\pi_{ab} = \delta_{an}\delta_{bn}$, $a, b = 0, \ldots, n$, and $\pi' = A^{(s,n)}\pi A^{(s,n)}$. Obviously, π and π' are projections of rank one. Moreover, a computation similar to (52) shows that

(62) $$\pi\pi'\pi = \eta_0^{-2}\pi, \quad \pi'\pi\pi' = \eta_0^{-2}\pi', \quad \eta_0 = \left|A_{nn}^{(s,n)}\right|^{-1}.$$

Hence, (59) follows by invoking Lemma 1 upon the identification $U_{12} = \eta_0\pi$, $U_{23} = \eta_0\pi'$, and $\tilde{g}(\lambda) = \eta_0 g(\lambda)$. The explicit form of η_0 given in (60) is easily obtained from (97). □

This proposition generalizes both Lemma 1 and Proposition 1. For $n = 1$, formula (60) yields $\eta_0 = q^{2s} + q^{-2s}$, and we recover the case of Proposition 1. For $n = 2s$, (60) yields $\eta_0 = [2s+1]_q$; the corresponding R-matrix is given by (20), which is a particular example covered by Lemma 1 (cf. Remark 3). It is not clear whether an example of an R-matrix with coefficients $r_j(\lambda)$ as described in Proposition 3 exists for $n \neq 1$ and $n \neq 2s$. Nevertheless, this proposition is useful for the analysis of solutions of the YB equation (see the next section). Another statement useful for this analysis reads as follows.

Proposition 4. *Let $R(\lambda)$ be a $U_q(\mathrm{sl}_2)$-invariant solution of the YB equation (10) on $V_s^{\otimes 3}$ for a half-integral spin $s \geq \frac{3}{2}$ satisfying (6) and (8). Then the coefficients $r_{s-\frac{1}{2}}(\lambda)$ and $r_{s+\frac{1}{2}}(\lambda)$ in its spectral decomposition are related as follows:*

(63) $$\frac{r_{s-\frac{1}{2}}(\lambda)}{r_{s+\frac{1}{2}}(\lambda)} = \begin{cases} \frac{1+\gamma\lambda}{1-\gamma\lambda} & \text{if } q = 1; \\ \frac{[s+\frac{1}{2}+\gamma\lambda]_q}{[s+\frac{1}{2}-\gamma\lambda]_q} & \text{if } q \neq 1, \end{cases}$$

where γ is an arbitrary finite constant.

Proof. The matrix $D(\lambda)$ in the reduced YB equation can be multiplied by an arbitrary function $\varphi(\lambda)$ analytic in a neighborhood of $\lambda = 0$ and satisfying $\varphi(\lambda)\varphi(-\lambda) = 1$. Therefore, in (57) for $n = 3s - \frac{1}{2}$, we can choose $D(\lambda) = \mathrm{diag}(1, g(\lambda))$, where $g(\lambda) = \frac{r_{s-\frac{1}{2}}(\lambda)}{r_{s+\frac{1}{2}}(\lambda)}$. Next, by the duality relation (50), we have $A^{(s,3s-\frac{1}{2})} = A^{(\frac{s}{2}+\frac{1}{4},1)}$ for half-integral spins $s \geq \frac{3}{2}$. Applying Proposition 1, we conclude that $g(\lambda) = r_{2s'-1}(\lambda)$, where $s' = \frac{s}{2} + \frac{1}{4}$. □

2.4. Necessary conditions. Differentiating (58) with respect to μ, setting $\mu = \lambda$, and taking the regularity condition $D(0) = \mathbb{E}$ into account, we derive the following system of equations (the prime denotes the derivative with respect to the spectral parameter):

(64)
$$\sum_{i,j \in I_n^{(s)}} r'_{2s-i}(0) r_{2s-j}(\lambda) A_{ij}^{(s,n)} \left(r_{2s-a}(\lambda) A_{aj}^{(s,n)} A_{ib}^{(s,n)} - r_{2s-b}(\lambda) A_{ai}^{(s,n)} A_{jb}^{(s,n)} \right)$$
$$= A_{ab}^{(s,n)} \left(r'_{2s-a}(\lambda) r_{2s-b}(\lambda) - r'_{2s-b}(\lambda) r_{2s-a}(\lambda) \right), \quad a < b, \ a, b \in I_n^{(s)}.$$

Here we have carried out summation on the right-hand side and used the relation $\left(A^{(s,n)}A^{(s,n)}\right)_{ab} = \delta_{ab}$. It is important to note that, although equations (64) contain derivatives, they are actually linear *algebraic* equations for $r_j(\lambda)$ with $j \neq a, b$.

It is easy to check that system (64) is satisfied trivially for $\lambda = 0$. Therefore, we look at the higher-order terms in the expansion of (64) about $\lambda = 0$. Denote $r'_{2s-a}(0) \equiv \xi_a$. In the first order in λ, summation over i, j can be carried out, leading to the conditions

$$(65) \qquad A^{(s,n)}_{ab}\left(r''_{2s-a}(0) - r''_{2s-b}(0)\right) = A^{(s,n)}_{ab}\left(\xi_a^2 - \xi_b^2\right),$$

which are always satisfied, because the unitary property (4) implies that

$$(66) \qquad\qquad\qquad r''_{2s-a}(0) = \xi_a^2.$$

In the second order in λ, equations (64) turn into a system of algebraic equations

$$(67) \qquad \begin{aligned} &\sum_{i,j \in I_n^{(s)}} \xi_i \xi_j^2 A^{(s,n)}_{ij}\left(A^{(s,n)}_{aj} A^{(s,n)}_{ib} - A^{(s,n)}_{ai} A^{(s,n)}_{jb}\right) \\ &\quad + \left(\xi_a - \xi_b\right) \sum_{i,j \in I_n^{(s)}} \xi_i \xi_j A^{(s,n)}_{ij}\left(A^{(s,n)}_{aj} A^{(s,n)}_{ib} + A^{(s,n)}_{ai} A^{(s,n)}_{jb}\right) \\ &\quad = A^{(s,n)}_{ab}\left(r'''_{2s-a}(0) - r'''_{2s-b}(0) - \xi_a^3 + \xi_b^3 + \xi_a^2 \xi_b - \xi_b^2 \xi_a\right), \quad a < b, \ a, b \in I_n^{(s)}. \end{aligned}$$

Remarkably, equations (64) and (67) can be solved explicitly by recursion. We provide the corresponding algorithm. We start with the level $n = 1$, where we have $r_{2s}(\lambda) = 1$ and $r_{2s-1}(\lambda)$ is given by (31). Now, suppose we have found the $r_j(\lambda)$, $j = 2s - n, \ldots, 2s$, that solve (64) for a level $n < 2s$. Then (64) and (67) at the level $n+1$ allow us to express $r_{2s-n-1}(\lambda)$ *algebraically* in terms of the $r_j(\lambda)$ found previously. Indeed, since we already know ξ_j for $j = 0, \ldots, n$, equations (67) for $2s - n \leq a < b \leq n$ turn into quadratic equations with respect to ξ_{n+1}. Solving them and substituting the resulting values of ξ_{n+1} in (64) for $2s - n \leq a < b \leq n$, we obtain a system of linear equations for $r_{2s-n-1}(\lambda)$. Finding all possible solutions of this system completes the $(n+1)$st step of recursion. Continuing this procedure up to $n = 2s$, we shall obtain all possible solutions for all $r_j(\lambda)$ and thus construct *all* possible *Ansätze* for the regular $U_q(\mathrm{sl}_2)$-invariant R-matrices of spin s. Next, since formula (64) provides necessary but not sufficient conditions, we need to check which of these *Ansätze* indeed satisfy the YB equation (10) or, alternatively, the reduced YB equations (58) for all n up to $\lfloor 3s \rfloor$.

2.5. Spin chain Hamiltonians and reconstruction of R-matrices.
The utmost importance of the YB equation in the quantum inverse scattering method (see the surveys [17, 18]) is due to the fact that its solutions can be used to construct families of quantum integrals of motion in involution. In particular, regular solutions of the YB equation make it possible to construct *local* integrals of motion for lattice models. For the R-matrix of type (5), the first of these integrals,

$$(68) \qquad \mathcal{H} = \sum_k H_{k,k+1}, \quad H = \partial_\lambda R(\lambda)\big|_{\lambda=0} = \sum_{j=0}^{2s-1} \xi_{2s-j} P^j,$$

is usually regarded as a Hamiltonian of a spin s magnetic chain with the nearest neighbor interaction. Here $H \in \mathrm{End}\, V_s^{\otimes 2}$ and $\mathcal{H} \in \mathrm{End}\, V_s^{\otimes L}$, where L is the number of lattice sites. Notice that in (68) we took the normalization condition (6) into account, which implies $\xi_0 = 0$ (this fixes the choice of the additive constant in the Hamiltonian). R-matrices equivalent in the sense of transformation (11) yield Hamiltonians related simply by rescaling $H \to \gamma H$; we regard such Hamiltonians as equivalent.

It is important to remark that, as was observed in [12], for regular solutions of the YB equation different Hamiltonians correspond to nonequivalent R-matrices. In the present context this statement can be formulated as follows.

Lemma 3. *Let $R^{(1)}(\lambda)$ and $R^{(2)}(\lambda)$ be two solutions of the YB equation* (10) *on $V_s^{\otimes 3}$ satisfying* (5), (6), *and* (8). *The corresponding Hamiltonians given by* (68) *coincide, $H^{(1)} = H^{(2)}$, if and only if $R^{(1)}(\lambda) = R^{(2)}(\lambda)$.*

Proof. The "if" part is obvious. Next, Theorem 3 in [12] asserts that if the Hamiltonians corresponding to two regular R-matrices analytic in a neighborhood of $\lambda = 0$ coincide, then $R^{(1)}(\lambda) = \varphi(\lambda)R^{(2)}(\lambda)$, where the scalar function $\varphi(\lambda)$ is analytic in a neighborhood of $\lambda = 0$ and satisfies $\varphi(0) = 1$. In the case under consideration, the analyticity of $r_j(\lambda)$ along with condition (6) implies that $\varphi(\lambda) = 1$. $\square$

Remark 7. The algorithm described at the end of the preceding subsection complements this lemma with a constructive procedure that allows us to *recover* the R-matrix by a given Hamiltonian. Indeed, if we know a Hamiltonian in the form (68), i.e., we know all ξ_j, then we can solve (64) recursively starting with $r_{2s}(\lambda) = 1$, thus recovering all the coefficients $r_j(\lambda)$ of the corresponding regular $U_q(\mathrm{sl}_2)$-invariant R-matrix. In contrast to the general situation, Lemma 3 guarantees that the resulting set of $r_j(\lambda)$ will be unique.

§3. Analysis of reduced Yang–Baxter equations

3.1. Asymptotic solutions. We remark that the technique described above applies in the limit as $\lambda \to \infty$ as well. In this limit, assuming that $\check{R}^{\pm 1} = \lim_{\lambda \to \pm\infty} R(\lambda)$ exists, the YB equation (10) turns into

$$(69) \qquad\qquad \check{R}_{12}\check{R}_{23}\check{R}_{12} = \check{R}_{23}\check{R}_{12}\check{R}_{23}.$$

Denote $d_j = \lim_{\lambda \to +\infty} r_j(\lambda)$, so that $\check{R} = \sum_{j=0}^{2s} d_j P^j$. Condition (6) implies that $d_{2s} = 1$. Taking the limit as $\lambda \to \infty$ in (57), we obtain a set of algebraic equations for the coefficients d_j,

$$(70) \qquad\qquad A^{(s,n)}DA^{(s,n)}DA^{(s,n)}D = DA^{(s,n)}DA^{(s,n)}DA^{(s,n)}.$$

Here $n = 1, \ldots, \lfloor 3s \rfloor$, and $D_{kk'} = \delta_{kk'}d_{2s-k}$, where $k \in I_n^{(s)}$. As in the spectral dependent case, the independent equations contained in (70) are

$$(71) \qquad (d_a - d_b) \sum_{i,j \in I_n^{(s)}} d_i d_j A_{ij}^{(s,n)} A_{ai}^{(s,n)} A_{jb}^{(s,n)} = 0, \quad a < b, \ a, b \in I_n^{(s)}.$$

This system of equations can be solved in a recursive way with the help of the algorithm described at the end of Subsection 2.4. In this context, it is worth noticing that one particular solution is known *a priori*, namely

$$(72) \qquad\qquad d_j = (-1)^{2s-j} q^{2s(2s+1)-j(j+1)},$$

which corresponds to (19) in the limit as $q^\lambda \to \infty$.

3.2. Reduced YB for $n = 1$ and $n = 2$. The explicit form of the transition matrix for $n = 1$ is

$$(73) \qquad\qquad A^{(s,1)} = \frac{1}{\{2s\}_q} \begin{pmatrix} 1 & \sqrt{1 + \{4s\}_q} \\ \sqrt{1 + \{4s\}_q} & -1 \end{pmatrix},$$

where we denote $\{t\}_q = q^t + q^{-t}$. In this case (64) contains only one equation. Taking the fact that $r_{2s}(\lambda) = 1$ into account and making the substitution $r_{2s-1}(\lambda) = 1 + \{2s\}_q g(\lambda)$, it is easy to show that this equation coincides with (29). Hence, we recover the same

expression for $r_{2s-1}(\lambda)$ as in Proposition 1. Accordingly, equation (71) is either satisfied trivially if $d_{2s-1} = 1$, or it represents a quadratic equation with roots $d_{2s-1} = -q^{\pm 4s}$.

The explicit form of the transition matrix for $n = 2$ is

$$(74) \quad A^{(s,2)} = \begin{pmatrix} \frac{[2s-1]_q}{\{2s\}_q[4s-1]_q} & \rho_s\sqrt{\frac{[2]_q[6s-1]_q}{\{2s\}_q[4s-1]_q}} & \rho_s\frac{\sqrt{[6s-1]_q[6s-2]_q}}{[4s-1]_q} \\ \rho_s\sqrt{\frac{[2]_q[6s-1]_q}{\{2s\}_q[4s-1]_q}} & \{4s-1\}_q(\rho_s)^2 & -\rho_s\sqrt{\frac{[2]_q[6s-2]_q}{\{2s-1\}_q[4s-1]_q}} \\ \rho_s\frac{\sqrt{[6s-1]_q[6s-2]_q}}{[4s-1]_q} & -\rho_s\sqrt{\frac{[2]_q[6s-2]_q}{\{2s-1\}_q[4s-1]_q}} & \frac{[2s]_q}{\{2s-1\}_q[4s-1]_q} \end{pmatrix},$$

where $\rho_s = \left(\{2s-1\}_q\{2s\}_q\right)^{-\frac{1}{2}}$. For computations, it is useful to observe the following identities:

$$(75) \qquad A^{(s,2)}_{01}A^{(s,2)}_{12} = A^{(s,2)}_{02}\left(A^{(s,2)}_{11} - 1\right), \quad A^{(s,2)}_{11} = 1 - [2]_q(\rho_s)^2.$$

The analysis of (58) splits into two cases: $r_{2s-1}(\lambda) = 1$ and $r_{2s-1}(\lambda) \neq 1$. The former case is covered by Proposition 3, which yields

$$(76) \qquad r_{2s-2}(\lambda) = \frac{b^2 e^\lambda - 1}{b^2 - e^\lambda}, \quad b + b^{-1} = \frac{[4s-1]_q\{2s-1\}_q}{[2s]_q}.$$

In the latter case, without loss of generality we can choose $\gamma = 1$ in (31), which corresponds to

$$(77) \qquad r_{2s-1}(\lambda) = \frac{[2s+\lambda]_q}{[2s-\lambda]_q}, \quad d_{2s-1} = -q^{4s}, \quad \xi_1 = \varkappa_q\frac{\{2s\}_q}{[2s]_q}, \quad \varkappa_q \equiv \frac{2\log q}{q - q^{-1}}.$$

In this case, it is easily seen that the three equations contained in (71) for $n = 2$ have only one common root, namely

$$(78) \qquad d_{2s-2} = q^{8s-2}.$$

Substituting $\xi_0 = 0$ and ξ_1 given by (77) in (67), we obtain a quadratic equation for ξ_2 with the roots

$$(79) \qquad \xi_2 = \frac{2\varkappa_q[4s-1]_q}{[2s-1]_q[2s]_q}, \quad \xi_2 = \varkappa_q(q - q^{-1})^2\frac{[4s-1]_q}{\{4s-1\}_q}.$$

The corresponding solutions of (64) are given by

$$(80) \qquad r_{2s-2}(\lambda) = \frac{[2s+\lambda]_q}{[2s-\lambda]_q}\frac{[2s-1+\lambda]_q}{[2s-1-\lambda]_q}, \quad r_{2s-2}(\lambda) = \frac{\{4s-1+\lambda\}_q}{\{4s-1-\lambda\}_q}.$$

It is straightforward to verify that both these solutions satisfy the $n = 2$ level reduced YB equation (58).

Proposition 5. *For a generic q and spin $s = 1$, the nonequivalent regular $U_q(\mathrm{sl}_2)$-invariant solutions of the YB equation (10) satisfying condition (6) are exhausted by the following three types:*

$$(81) \qquad R(\lambda) = P^2 + P^1 + \frac{b^2 e^\lambda - 1}{b^2 - e^\lambda}P^0, \quad b + b^{-1} = [3]_q,$$

$$(82) \qquad R(\lambda) = P^2 + \frac{[2+\lambda]_q}{[2-\lambda]_q}P^1 + \frac{[2+\lambda]_q}{[2-\lambda]_q}\frac{[1+\lambda]_q}{[1-\lambda]_q}P^0,$$

$$(83) \qquad R(\lambda) = P^2 + \frac{[2+\lambda]_q}{[2-\lambda]_q}P^1 + \frac{\{3+\lambda\}_q}{\{3-\lambda\}_q}P^0.$$

Proof. For $n \leq 2s$ we have $\dim W_n^{(s)} = (n+1)$. As has already been mentioned, the reduced YB equation (57) ensures that the YB operator vanishes at all vectors of the form $(S_{123}^-)^m W_n^{(s)}$, i.e., on a subspace of dimension $\Delta_n^{(s)} = (6s - 2n + 1) \dim W_n^{(s)}$. In particular, we have $\Delta_0^{(1)} + \Delta_1^{(1)} + \Delta_2^{(1)} = 26$, which means that the level $n = 3$ reduced YB equation is satisfied automatically, because the corresponding subspace $W_3^{(1)}$ is one-dimensional. Therefore, for $s = 1$, the $n = 1, 2$ reduced YB equations provide not only necessary but also sufficient conditions. As we have shown above in this subsection, the solutions of these equations are exhausted by (76) and (80), which for $s = 1$ yield (81)–(83). $\qquad\square$

Thus, for spin $s = 1$, all three inequivalent sl$_2$-invariant R-matrices (15), (13), and (12) have $U_q(\text{sl}_2)$-invariant counterparts. Two of them belong to the well-known types (19) and (20). The last one, (83), appears to be rather an exceptional case; it was found previously [19, 20] by means of Baxterization of the Birman–Wenzl–Murakami algebra.

3.3. Reduced YB for $n = 3$. For $n = 3$ the transition matrix has 12 entries given by (97)–(98), and the remaining four entries are

$$(84) \qquad A_{11}^{(s,3)} = \frac{[2]_q [2s-1]_q [6s-2]_q - ([2s-2]_q)^2}{\{2s-1\}_q [4s-3]_q [4s]_q},$$

$$A_{12}^{(s,3)} = A_{21}^{(s,3)}$$

$$(85) \qquad\qquad = \frac{[2s-2]_q}{[4s-2]_q} \big([6s-2]_q - [2]_q [2s-1]_q\big) \sqrt{\frac{[2s-1]_q [6s-3]_q}{[4s-4]_q [4s-3]_q [4s-1]_q [4s]_q}},$$

$$(86) \qquad A_{22}^{(s,3)} = \frac{[2s-2]_q - [2]_q [6s-3]_q}{\{2s-2\}_q \{2s-1\}_q [4s-1]_q}.$$

Proposition 6. *For a generic q and spin $s = \frac{3}{2}$, the nonequivalent regular $U_q(\text{sl}_2)$-invariant solutions of the YB equation* (10) *satisfying condition* (6) *are exhausted by the two types given by* (19) *and* (20).

Proof. We analyze the spectral resolution of possible spin $\frac{3}{2}$ solutions to the reduced YB equations for $n = 1, 2, 3$. The first possibility is $r_2(\lambda) = r_1(\lambda) = 1$, in which case $r_0(\lambda)$ is determined by Proposition 3; the corresponding R-matrix is given by (20). Next, the case where $r_2(\lambda) = 1$, $r_1(\lambda) \neq 1$ is covered by the same proposition for $n = 2$, and $r_1(\lambda)$ is given by (76). However, this case is ruled out, because (76) for $s = 1$ is incompatible with the statement of Proposition 4, which requires $b = q^2$. In the remaining case, $r_2(\lambda) \neq 1$, without loss of generality we can choose $\gamma = 1$ in (31), which yields $r_2(\lambda) = \frac{[3+\lambda]_q}{[3-\lambda]_q}$, and, in accordance with the analysis carried out in the preceding subsection, $r_1(\lambda)$ is given by one of the expressions in (80). However, the second form in (80) is ruled out again, being incompatible with Proposition 4. Thus, we are left with

$$(87) \qquad r_2(\lambda) = \frac{[3+\lambda]_q}{[3-\lambda]_q} \frac{[2+\lambda]_q}{[2-\lambda]_q}, \quad \xi_0 = 0, \quad \xi_1 = \varkappa_q \frac{\{3\}_q}{[3]_q}, \quad \xi_2 = 2\varkappa_q \frac{[5]_q}{[2]_q [3]_q}.$$

Substituting these values in (67) for $n = 3$ and $s = \frac{3}{2}$, we obtain a system of three quadratic equations for ξ_3. A direct computation using (97)–(98) and (84)–(86) shows that these equations have only one common root given by

$$(88) \qquad\qquad \xi_3 = \varkappa_q \frac{\{2\}_q (5 + 3\{2\}_q)}{[2]_q [3]_q},$$

which is the value corresponding to (19) for $s = \frac{3}{2}$. By Lemma 3, an R-matrix determined by (87) and (88) is unique; therefore, it is the one given by (19). $\qquad\square$

The proposition proved above shows that, unlike the case of spin $s = 1$, only two out of four sl_2-invariant R-matrices (12)–(15) have $U_q(\mathrm{sl}_2)$-counterparts for spin $s = \frac{3}{2}$. Actually, analyzing the $n = 3$ reduced YB equations, we can extend this observation to higher spins as well.

Proposition 7. *Let $R(\lambda)$ be a $U_q(\mathrm{sl}_2)$-invariant solution of the YB equation (10) on $V_s^{\otimes 3}$ for a spin $s \geq 2$ satisfying (6) and (8). If $r_{2s-1}(\lambda) = \frac{[2s+\lambda]_q}{[2s-\lambda]_q}$, then*

$$(89) \qquad r_{2s-2}(\lambda) = \frac{[2s+\lambda]_q}{[2s-\lambda]_q} \frac{[2s-1+\lambda]_q}{[2s-1-\lambda]_q}.$$

As a consequence, for $s \geq 2$, there exist no $U_q(\mathrm{sl}_2)$-invariant regular R-matrices whose limit as $q \to 1$ coincides with (12) or (14).

Proof. Let $q = 1 + h$, $h \ll 1$, so that $[t]_q = t + t(t-1)h^2/3 + O(h^3)$. Since $A_q^{(s,n)}$ depends on q smoothly, we have $A_q^{(s,n)} = A_{q=1}^{(s,n)} + O(h^2)$. Consider the $n = 3$ reduced YB equations (67), where $\xi_0 = 0$, ξ_1 is as in (77), and ξ_2 is given by the second expression in (79). Using (97)–(98) and (84)–(86), we find the following h-expansions of these equations for $(a, b) = (0, 1)$, $(0, 2)$, and $(1, 3)$, respectively:

$$(90)$$
$$0 = (5s^2 - 3s)\xi_3^2 + (3 - 6s)\xi_3 + 1$$
$$\quad - \tfrac{2}{3}h^2\big((25s^4 - 32s^3 + 9s^2)\xi_3^2 + (78s^3 - 81s^2 + 21s)\xi_3 - 47s^2 + 35s - 3\big) + O(h^3),$$

$$(91)$$
$$0 = h^2\big((7s^2 - 3s)\xi_3^2 + (3 - 10s)\xi_3 + 3\big) + O(h^3),$$

$$(92)$$
$$0 = (5s^2 - 3s)\xi_3^2 + (3 - 6s)\xi_3 + 1$$
$$\quad + \tfrac{4}{3}h^2\big((19s^4 - 35s^3 + 17s^2 - 3s)\xi_3^2 + (138s^3 - 201s^2 + 96s - 15)\xi_3 - 85s^2 + 90s - 20\big)$$
$$\quad + O(h^3).$$

We see that, in the zeroth order in h, (91) is satisfied trivially, while (90) and (92) yield one and the same quadratic equation, which has the following roots:

$$(93) \qquad\qquad \xi_3 = \tfrac{1}{s}, \quad \xi_3 = \tfrac{1}{5s-3}.$$

Thus, for $q = 1$, equations (90)–(92) are compatible (in particular, the first value in (93) corresponds to solutions of type (12) and (14)).

In the second order in h, (91) has roots $\xi_3 = \frac{1}{s}$ and $\xi_3 = \frac{1}{7s-3}$. But, for (90) and (92), the h^2 corrections to the first value in (93) are

$$(94) \quad \xi_3 = \tfrac{1}{s} + h^2\big(\tfrac{28}{3}s - 6\big) + O(h^3), \quad \xi_3 = \tfrac{1}{s} + h^2\big(\tfrac{46}{3} - \tfrac{4}{s} - 12s\big) + O(h^3).$$

Therefore, already in the second order in h, compatibility of (90)–(92) is lost. This implies that the second expression in (80) for $r_{2s-2}(\lambda)$ is ruled out, and, as follows from the analysis in the preceding subsection, the only possible form of $r_{2s-2}(\lambda)$ is the first expression in (80). An R-matrix with such a spectral coefficient cannot be a q-deformation of (12) or (14) for $s \geq 2$, because the corresponding value of ξ_2 does not vanish in the limit as $q \to 1$. $\qquad\square$

Remark 8. Notice that the coefficient $r_3(\lambda)$ of the exceptional solution (18) corresponds (after rescaling $\lambda \to \lambda/6$) to the second value in (93). As we saw in the proof of Proposition 7, this value is not a root of (91) for $h \neq 0$. Therefore, we conclude that (18) has no regular $U_q(\mathrm{sl}_2)$-invariant counterpart.

Appendix A

Lemma 4. *Let $R(\lambda)$ be a $U_q(\mathrm{sl}_2)$-invariant solution of the YB equation* (10) *on $V_s^{\otimes 3}$ satisfying conditions* (3) *and* (6). *Then $R(\lambda)$ is unitary, i.e., it satisfies* (4) *as well.*

Proof. By (5), $R(\lambda)$ commutes with $R(\mu)$. We introduce $X^\lambda = R(\lambda)R(-\lambda)$. Then the YB equation for $\mu = -\lambda$ implies that $X_{12}^\lambda = X_{23}^\lambda$. Applying tr_{23} and tr_3 to this equation (along the lines of [12], where a less trivial equation $X_{12} - X_{23} = Z_{123}$ was considered), we infer that $X^\lambda = c\mathbb{E}$, with a scalar constant c. On the other hand, we have $X^\lambda = P^{2s} + \cdots$ by (6). Hence, $c = 1$ and $X^\lambda = \mathbb{E}$. $\qquad\qquad\square$

Appendix B

The comultiplication (2) determines the structure of the Clebsch–Gordan (CG) decomposition of tensor products of irreducible representations. The corresponding CG coefficients and $6j$-symbols were derived and studied in [15, 16]. The particular $6j$-symbol that appeared in (46) is given by

(95)
$$
\left\{ \begin{matrix} s & s & 2s-k \\ s & 3s-n & 2s-k' \end{matrix} \right\}_q
$$
$$
= F_k^s F_{k'}^s \sum_l (-1)^l [l+1]! \Big([l-4s+k]![l-4s+k']!
$$
$$
\times [l-6s+n+k]![l-6s+n+k']!
$$
$$
\times [6s-n-l]![6s-k-k'-l]![8s-n-k-k'-l]! \Big)^{-1},
$$

where

(96)
$$
F_k^s = [2s-k]! \Big(\frac{[k]![n-k]![2s-n+k]![4s-n-k]!}{[4s-k+1]![6s-n-k+1]!} \Big)^{\frac{1}{2}},
$$

and the q-factorial is defined as follows: $[n]! = \prod_{k=1}^n [k]_q$ for $n \in \mathbb{Z}_+$ and $[0]! = 1$. Summation in (95) runs over all l for which the arguments of the q-factorials are nonnegative.

For $n \leq 2s$ and $k' = 0$ or $k' = n$, the sum on the right-hand side of (95) contains only one term ($l = 6s - n$ or $l = 6s - n - k$, respectively), and we obtain

(97)
$$
A_{k,n}^{(s,n)} = \frac{(-1)^k \sqrt{[4s-2k+1]_q}}{[2s-n]!}
$$
$$
\times \Big(\frac{[n]![2s]![2s-n+k]![4s-n-k]![4s-2n+1]![6s-n-k+1]!}{[k]![n-k]![4s-k+1]![4s-n+1]![6s-2n+1]!} \Big)^{\frac{1}{2}},
$$

(98)
$$
A_{0,k}^{(s,n)} = [2s]! \sqrt{[4s-2k+1]_q}
$$
$$
\times \Big(\frac{[n]![4s-n]![4s-n-k]![6s-n+1]!}{[k]![n-k]![2s-n+k]![2s-n]![4s]![4s-k+1]![6s-n-k+1]!} \Big)^{\frac{1}{2}}.
$$

Appendix C

Proof of Proposition 2. i) Applying the CG decomposition to the $\{23\}$ and $\{12\}$ components in (42) and (43), respectively, and using the orthonormality of the basis of V_s, $\langle p|p' \rangle = \delta_{pp'}$, we find straightforwardly that the scalar product in (45) is given by

(99)
$$
A_{kk'}^{(s,n)} = \sum_{m,m'} \left[\begin{matrix} s & 2s-k & 3s-n \\ m & 3s-n-m & 3s-n \end{matrix} \right]_q \left[\begin{matrix} 2s-k' & s & 3s-n \\ 3s-n-m' & m & 3s-n \end{matrix} \right]_q
$$
$$
\times \left[\begin{matrix} s & s & 2s-k \\ 3s-n-m-m' & m' & 3s-n-m \end{matrix} \right]_q \left[\begin{matrix} s & s & 2s-k' \\ m & 3s-n-m-m' & 3s-n-m' \end{matrix} \right]_q.
$$

In order to carry out the summation over m, we invoke the following identity (see [15, 16]):

$$(100) \quad \sum_m \begin{bmatrix} a & b & e \\ m & m'-m & m' \end{bmatrix}_q \begin{bmatrix} b & d & f \\ m'-m & m'' & m''+m'-m \end{bmatrix}_q \begin{bmatrix} a & f & c \\ m & m''+m'-m & m''+m' \end{bmatrix}_q$$
$$= (-1)^{a+b+c+d}\sqrt{[2e+1]_q[2f+1]_q}\begin{bmatrix} e & d & c \\ m' & m'' & m''+m' \end{bmatrix}_q \begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}_q.$$

After this, summation over m' reduces to

$$(101) \qquad \sum_{m'} \begin{bmatrix} 2s-k' & s & 3s-n \\ 3s-n-m' & m' & 3s-n \end{bmatrix}_q^2 = \langle n;k'|n;k'\rangle = 1.$$

The remaining factors in (100) yield the right-hand side of (46).

ii) The self-duality of the transition matrix with respect to the replacement $q \mapsto q^{-1}$ follows from the fact that $6j$-symbols are invariant with respect to this operation (because, unlike the CG coefficients, they are expressed entirely in terms of q-numbers [15, 16]).

iii) The obvious invariance of (95) with respect to the replacement $k \leftrightarrow k'$ implies that the transition matrix is symmetric. Since $A_q^{(s,n)}$ is orthogonal by construction, we conclude that $A_q^{(s,n)}$ coincides with its inverse.

iv) Formula (49) is obvious from (73). In terms of matrix entries, the duality relation (50) looks like this:

$$(102) \qquad A_{k,k'}^{(s,n)} = A_{\tilde k,\tilde k'}^{(\tilde s,\tilde n)},$$

$$(103) \qquad \tilde s = 2s - \tfrac{n}{2}, \quad \tilde n = 6s - 2n, \quad \tilde k = k - n + 2s, \quad \tilde k' = k' - n + 2s,$$

where $0 \le k, k' \le n$. The shifts in $\tilde k$, $\tilde k'$ are necessary in order to ensure (44) (notice that $2s - n = \tilde n - 2\tilde s \ge 0$). Equation (102) is checked straightforwardly by making the change of variables (103) in (46) and using the explicit expressions (95)–(96). This completes the proof. □

Acknowledgments

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