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HOMOGENIZATION OF AN ELLIPTIC SYSTEM UNDER CONDENSING PERFORATION OF THE DOMAIN

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Abstract. Homogenization of a system of second-order differential equations is performed in the case of a nonuniformly perforated rectangle where the sizes of the holes and the distances between pairs of them decrease as the distance from one of the bases of the rectangle increases. The Neumann conditions are assumed on the boundaries of the holes. The formal asymptotics of the solution is constructed, which involves the usual *Ansatz* of homogenization theory and also some *Ansätze* typical of solutions of boundary-value problems in thin domains, in particular, exponential boundary layers. Justification of the asymptotics is done with the help of the Korn inequality, which is proved for the perforated domain $\Omega(h)$. Depending on the properties of the right-hand side, the norm of the difference between the true and the approximate solutions in the Sobolev space $H^1(\Omega(h))$ is estimated by the quantity $ch^\varkappa$ with $\varkappa \in (0, 1/2]$.

§1. Setting of the problem and description of results

1. Geometry of the domain. We start with introducing a fractal-type perforated domain for which the sizes of the holes and the distances between pairs of holes decrease when moving away from the base. For this, we consider periodicity cells of different sizes. Let $\Xi = (0, b_1) \times (0, b_2)$ be a rectangle. Also, let $\omega \in \mathbb{R}^2$ be a domain having smooth boundary $\partial\omega$ and such that the closure $\overline{\omega} = \omega \cup \partial\omega$ lies in the interior of Ξ. We denote by $\Xi^{(j)}$ the result of contraction of Ξ along the x_1-axis with the coefficient q^{-j} and split the rectangle $\Xi^{(j)}$, which turns out to be thin for large j, into q^j equal rectangles Ξ_j of size $b_1 q^{-j} \times b_2 q^{-j}$; it is assumed that $q, j \in \mathbb{N} := \{1, 2, \dots\}$ and the number $q > 1$ is fixed. Here and in the sequel, a lower index corresponds to contraction in two directions, and an upper index symbolizes one-directional contraction (along the x_1-axis). Replacing each of the small rectangles Ξ_j with the cell $S_j = \Xi_j \setminus \overline{\omega_j}$, we obtain a cell $S^{(j)}$ that consists of q^j equal cells S_j and is a thin rectangle with small holes $\overline{\omega_j}$ (cf. Figure 1).

Now we describe the perforated domain itself. We consider another rectangle $Q = [0, a_1] \times [0, a_2]$ and introduce a small parameter $h > 0$ so that $a_i = N_i b_i h$, where the N_i are large integers, $i = 1, 2$. The segments $\Gamma_j = \{x = (x_1, x_2) : x_1 \in (0, a_1), x_2 = \gamma_j\}$, where $\gamma_j = jhb_2$, split Q into strips $\Pi_1(h), \dots, \Pi_{N_2}(h)$ of width hb_2. Each of these strips is divided into congruent closed rectangles $Q_p^{(j)}(h) \simeq Q^{(j)}(h)$ of size $hq^{-j}b_1 \times hb_2$; here $q \in \mathbb{N}$ and $p = 1, \dots, q^j N_1$. If $q = 1$, then all rectangles are equal, but for $q \geq 2$ the bases of the rectangles become shorter as j grows, and the partition acquires an anisotropic fractal structure (Figure 1).

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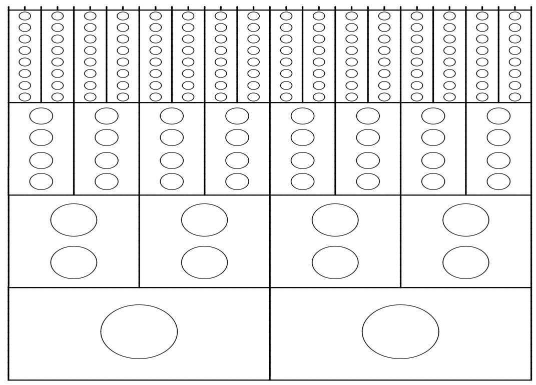

FIGURE 1. The case where $q = 2, N_1 = 2, h = 1$.

We replace each of the rectangles $Q_p^{(j)}(h) \subset \overline{\Pi_j(h)}$ with the cell $S^{(j)}(h)$ obtained from $S^{(j)}$ by contraction with the coefficient h^{-1}. The union of the resulting cells is the set $\Omega(h) \cup \partial Q$. The open set $\Omega(h)$ is a perforated domain, and the sizes of the holes and the number of holes in a row depend on their distance from the base $\Gamma_0 = \{x = (x_1, x_2) : x_2 = 0\}$ of the rectangle Q. Such geometric structures occur, e.g., in diastatic tissues of plants. We say that a function Ψ is *periodic with q-branching in the direction* ξ_2 if the following conditions with $m = 0$ are satisfied:

$$\begin{aligned} \Psi(0, \xi_2) &= \Psi(b_1, \xi_2), \quad \xi_2 \in (0, b_2), \\ q^{-m}\Psi(q\xi_1, 0) &= \Psi(\xi_1, b_2), \quad \xi_1 \in (0, b_1). \end{aligned} \tag{1.1}$$

Of course, by the right-hand side of $(1.1)_2$ we mean the b_1-periodic extension of Ψ relative to the variable y_1 (in accordance with the requirement $(1.1)_1$). Note that for $m = 0$ and $q = 1$ relations (1.1) mean the usual periodicity, but $q > 1$ in what follows throughout.

If (1.1) is fulfilled for $m \in \mathbb{N}$, we say that Ψ has *q-branching with coefficient q^{-m}*. In this case, the derivative $\partial^m \Psi / \partial y_1^m$ is periodic with q-branching, because each differentiation brings an additional factor of q in the coefficient.

Apart from the "slow" variables $x = (x_1, x_2)$, we consider the "fast" variables

$$\eta^{(j)} = q^j h^{-1} x_1, \quad \zeta = h^{-1} x_2, \tag{1.2}$$

which correspond to the fact that the rectangles $Q^{(j)}(h)$ become thinner as j grows, and also the fast variables

$$y^{(j)} = (y_1^{(j)}, y_2^{(j)}) = (q^j h^{-1} x_1, q^j h^{-1} x_2), \tag{1.3}$$

which correspond to uniform dilation of the small cell $S_j(h)$. The coordinates $(\eta^{(j)}, y)$ will be used for functions that q-branch in the direction ζ, and the coordinates $(y_1^{(j)}, y_2^{(j)})$ for functions periodic in $y_1^{(j)}$ and in $y_2^{(j)}$.

Next, for functions periodic in $\eta^{(j)}$ and $y^{(j)}$, by η and y we shall mean the variables that coincide with $\eta^{(j)}$ and $y^{(j)}$ in the strips $\Pi_j(h), j = 1, \dots, N_2$. With such a function Φ we associate the function $\Phi^{[m]}$ defined on $\Omega(h)$ by the rule

$$\overline{\Pi_j(h)} \ni x \mapsto \Phi^{[m]}(x, y, \eta, \zeta) = q^{-mj}\Phi(x, y^{(j)}, \eta^{(j)}, \zeta), \quad j = 1, \dots, N_2. \tag{1.4}$$

2. Setting of the problem and Korn's inequality. Let $D(\nabla_x)$ be a $(k \times n)$-matrix composed of homogeneous first-order differential operators with constant coefficients. We assume that $D(\nabla_x)$ satisfies the following *algebraic completeness* condition [2]: for any row $P = (P_1, \dots, P_n)$ of homogeneous polynomials of degree $\rho \geq \rho_D$, there is a row

$Q = (Q_1, \dots, Q_k)$ of polynomials such that $P(t_1, t_2) = Q(t_1, t_2)D(t_1, t_2)$. In particular, the subspace $\mathcal{P}$ of vector polynomials annihilated by the differential operator $D(\nabla_x)$ is finite-dimensional; we choose a basis $\{p^1, \dots, p^J\}$ in this subspace. We note that, by [2], the above condition is equivalent to the fact that the matrix $D(t_1, t_2)$ is of maximal rank for any $(t_1, t_2) \in \mathbb{R}^2 \setminus 0$. Also, we have $k \geq n$.

Consider the following system of second-order differential equations:

$$\mathcal{L}(h, x, \nabla_x)u^h(x) := D(-\nabla_x)^\top A^{[0]}(x, \eta, \zeta)D(\nabla_x)u^h(x) = f^h(x), \quad x \in \Omega(h), \tag{1.5}$$

where $u^h = (u_1^h, \dots, u_n^h)^\top$, $f^h = (f_1^h, \dots, f_n^h)^\top$, $\top$ means transposition, and $A(x, \xi_1, \xi_2)$ is a matrix of size $k \times k$. The elements of the matrix A are smooth real-valued functions of variables $x \in \overline{\Omega(h)}$ and $\xi_1 \in [0, b_1]$, $\xi_2 \in [0, b_2]$; these functions, together with their derivatives with respect to x and ξ_2, are a_1-periodic in x_1 and periodic in (ξ_1, ξ_2) with q-branching in the direction ξ_2. Finally, the derivatives $\nabla_x^\alpha \partial_{\xi_1}^m \partial_{\xi_2}^p A(x, \xi_1, \xi_2)$ possess q-branching with coefficient q^{-m}. If the dependence on η and ζ is determined by formula (1.4), then the matrix-valued function $x \mapsto A^{[0]}(x, \eta, \zeta)$ is smooth on the set $\overline{\Omega(h)}$.

We assume that the homogeneous Neumann conditions

$$\mathcal{N}(h, x, \nabla_x)u^h(x) := D(\nu(x))^\top A^{[0]}(x, \eta, \zeta)D(\nabla_x)u^h(x) = 0, \quad x \in \partial\Omega(h) \setminus \partial Q, \tag{1.6}$$

are fulfilled on the boundaries of the holes $\omega_{jp}(h)$; here ν is the unit normal to these boundaries that looks outwards with respect to $\Omega(h)$. On the sides of the rectangle Q we impose the Dirichlet and the periodicity conditions

$$\begin{aligned} &u^h(x_1, 0) = 0, \quad u^h(x_1, a_2) = 0, \quad x_1 \in (0, a_1); \\ &u^h(0, x_2) = u^h(a_1, x_2), \quad \partial_{x_1} u^h(0, x_2) = \partial_{x_1} u^h(a_1, x_2), \quad x_2 \in (0, a_2). \end{aligned} \tag{1.7}$$

The matrix A is assumed to be symmetric and positive definite for all $x \in \overline{Q}$ and all $(\eta, \zeta) \in \overline{S}$. Then, due to algebraic completeness, the operator $\mathcal{L}$ turns out to be *elliptic and formally positive* [2], and also possesses the *polynomial property* [3, 4].

We introduce the subspace $\mathring{H}^1_{\mathrm{reg}}(\Omega(h))$ of functions in $H^1(\Omega(h))$ that satisfy $(1.7)_1$ and $(1.7)_2$, and we denote by $H^{-1}_{\mathrm{reg}}(\Omega(h)) = \mathring{H}^1_{\mathrm{reg}}(\Omega(h))^*$ the dual space.

Lemma 1.1. *Under the above assumptions, the map*

$$\mathcal{L} : \mathring{H}^1_{\mathrm{reg}}(\Omega(h))^n \to H^{-1}_{\mathrm{reg}}(\Omega(h))^n$$

is an isomorphism, and the (*unique*) *weak solution* $u^h \in \mathring{H}^1_{\mathrm{reg}}(\Omega(h))^n$ *of problem* (1.5)–(1.7) *satisfies the inequality*

$$\|u^h; H^1(\Omega(h))\| \leq c\|f^h; H^{-1}_{\mathrm{reg}}(\Omega(h))\|, \tag{1.8}$$

where c is a constant independent of the parameter h and of the right-hand side $f^h \in H^{-1}_{\mathrm{reg}}(\Omega(h))^n$.

Proof. By the general results of [1, 2], it suffices to check that the constant c in the Korn inequality

$$\|u^h; H^1(\Omega(h))\| \leq c\|Du^h; L_2(\Omega(h))\|, \quad u \in \mathring{H}^1(\Omega(h)), \tag{1.9}$$

is independent of the parameter h. On each cell $S_{pj}(h)$ with one hole $\omega_{pj}(h)$ (the domains S and ω contract with coefficient $h^{-1}q^j$ in two directions), we represent the vector-valued function u in the form

$$u(x) = u^\perp(x) + p(x)a, \tag{1.10}$$

where $p = (p^1, \dots, p^J)$ is an $(n \times J)$-matrix of vector polynomials, and a is the column $(a_1, \dots, a_J)^\top$ defined by the formula

$$a = \left\{ \int_{S_{pj}(h)} p(x)^\top p(x)\, dx \right\}^{-1} \int_{S_{pj}(h)} p(x)^\top u(x)\, dx. \tag{1.11}$$

In the braces we have a Gram matrix, symmetric and positive definite. Relations (1.10) and (1.11) imply the orthogonality conditions

$$\int_{S_{pj}(h)} p(x)^\top u(x)^\perp\, dx = 0 \in \mathbb{R}^J,$$

which, by [2], ensure the Korn inequality

$$\begin{aligned} \left|\!\left|\!\left| u^\perp; S_{pj} \right|\!\right|\!\right|^2_{hq^{-j}} &:= \|\nabla_x u^\perp; L_2(S_{pj})\|^2 + h^{-2} q^{2j} \|u^\perp; L_2(S_{pj})\|^2 \\ &\le C \|D(\nabla_x) u^\perp; L_2(S_{pj})\|^2 \\ &= C \|D(\nabla_x)(u - pa); L_2(S_{pj})\|^2 = C \|D(\nabla_x) u; L_2(S_{pj})\|^2 \end{aligned}$$

with a constant C independent of h, j and, of course, of the vector-valued function u (independence of the parameter is verified by dilation of the cell with the coefficient $h^{-1} q^j$). Since the boundary of the hole is smooth, there exists a continuous extension operator

$$H^1(S) \ni v \mapsto \widehat{v} \in H^1(\Xi).$$

Keeping the notation "$\widehat{\ \ }$" for the extension operator corresponding to the cell S_{pj}, we obtain

$$\left|\!\left|\!\left| \widehat{u^\perp}; S_{pj} \right|\!\right|\!\right|^2_{hq^{-j}} \le C \|u^\perp; L_2(S_{pj})\|^2 \le C \|D(\nabla_x) u; L_2(S_{pj})\|^2. \tag{1.12}$$

Let $\hat{u}$ denote the vector-valued function that coincides with u on $\Omega(h)$ and with the sum $\widehat{u^\perp_{(pj)}} + p a_{(pj)}$ on the holes $\omega_{pj}(h)$; here $\widehat{u^\perp_{(pj)}}$ and $a_{(pj)}$ are the elements of the decomposition (1.10) on the cell S_{pj}. Summing the inequalities (1.12) and recalling that $D(\nabla_x) p(x) = \mathbb{O}_{k \times J}$ (= the zero matrix of size $k \times J$), we see that

$$\|D(\nabla_x) \widehat{u}; L_2(Q)\|^2 \le C \|D(\nabla_x) u; L_2(\Omega(h))\|^2. \tag{1.13}$$

Since u satisfies the Dirichlet conditions (1.7), the results of [2] yield the inequality

$$\|\widehat{u}; H^1(Q)\|^2 \le C \|D(\nabla_x) \widehat{u}; L_2(Q)\|^2. \tag{1.14}$$

It remains to observe that, by the construction of the extension $\widehat{u}$, the left-hand side of (1.14) dominates the square of the norm $\|u; H^1(\Omega(h))\|$, and then to estimate the right-hand side with the help of (1.13). □

3. Content of the paper. Homogenization of boundary-value problems in perforated domains has long been drawing the attention of specialists in mathematics and mechanics, and results in this direction are, undoubtedly, important for applications. In the pioneer publications by Marchenko and Khruslov (see the book [5]), convergence of Green functions and proximity of solutions of boundary-value problems for scalar second-order equations were studied in the case of the so-called domains with fine-grained boundary. The shape and the distribution of small holes was to a large extent arbitrary (and even random), and the conditions ensuring the passage to the limit were expressed in geometric and capacity terms. The first results have been developed in numerous investigations (see, e.g., [6]–[10]). Among many generalizations, the direction of study related to periodic or locally periodic disposition of the holes was distinguished. In this case, it is possible not only to prove convergence theorems, but also to construct asymptotic formulas for solutions and deduce more or less strong estimates for the remainder terms.

The number of publications on this topic is huge; the corresponding surveys can be found in the books mentioned above.

Another regular geometric perforation structure is presented by fractals (the Cantor set, the Sierpiński carpet, etc.), which, somewhat loosely, can be thought of as "periodic in the logarithmic scale". The subtle results obtained for boundary-value problems on such sets (see [11, 12] and others) do not pertain, however, to asymptotic analysis as this term is customarily understood. The reason is that application of the usual methods of homogenization meets an insurmountable obstruction: the setting of the problem suggests that by the periodicity cell we should mean the fractal set itself.

In the present paper we consider perforated domains with a simplest anisotropic fractal structure. As is seen in Figure 1, the rectangle Q is split into rows of congruent rectangles whose bases decrease as the number of a row increases, and whose heights remain the same. As j grows, the rectangles become thinner, and the number of holes in them grows. This allows us to investigate the problem on a cell with the help of asymptotic *Ansätze* suitable for solutions of elliptic problems in thin domains (see [13, 3, 14] and others). Thus, in the asymptotic *Ansätze* common with homogenization theory, the corrector is found only approximately, but the homogenized system of differential equations on the rectangle Q is determined completely.

This asymptotic procedure is presented in §2. It was announced in the authors' paper [15], where the boundary-value problem was studied for a system of equations (similar to (1.5)) with rapidly oscillating and q-branching coefficients on the entire rectangle Q. Compared to [15], in this paper we need to overcome an additional difficulty: the formal asymptotics found in §2 does not allow us to construct a continuous global approximation. Namely, the resulting asymptotic correctors have jumps on the lines that separate rows with holes of different sizes; to eliminate these jumps, in §3 we construct boundary layers that solve the limiting problem in a perforated strip (Figure 2). To smooth the jumps, we could employ some specially chosen cut functions, but exponentially decaying boundary layers are preferable, because they exhibit correctly the asymptotic structure of the solution. In the final §4 we justify the asymptotics obtained; this is done with the help of inequalities (1.9) and (1.8).

We turn the reader's attention to the new points in the arguments, caused by the specific character of the problem in question as well as by the method of proof. First, our approximation to the asymptotic corrector gives rise to a discrepancy $O(h^{-1}q^{-j})$ in the equations; this discrepancy is small only if the number j of the strip $\Pi_j(h)$ is large. Nevertheless, the difference between the true and the approximate solutions of (1.5)–(1.7) in the Sobolev norm of the space $H^1(\Omega(h))^n$ can be estimated by repeated applications of some versions of the Hardy inequality, with the use of the fact that $q^{-j}x_2 \leq ch$ in the strip $\Pi_j(h)$. Second, the differentiability properties of the right-hand side f^h relative to the slow variables x are described in terms of the Sobolev–Slobodetskiĭ space $H^\varkappa$ with $\varkappa \in (0, 1/2]$, and the right-hand sides of the error estimates for asymptotic approximations (see formulas (4.36) and (4.37) below) involve the factor $h^\varkappa$. In earlier investigations, pertaining to the purely periodic case, it was assumed that $f^h \in H^1(\Omega(h))^n$, and the majorants in the estimates involved the factor $h^{1/2}$, so that they were not sharp in the sense of the smoothness exponent. Third, in Lemma 4.2, which ensures the sharpness mentioned above, the L_∞-norms of functions of fast variables occur; this forces us to construct three terms of the asymptotics and two terms of the boundary layer that approximate the solution of the boundary-value problem in the thin domain $S_j(1) \subset \Xi^{(j)}$ respectively far from and near the small sides of the rectangle $\Xi^{(j)}$. Only this method allows us to ensure that the discrepancy arising in auxiliary problems on cells be *pointwise* small. Finally, unlike the situation in [15], the asymptotic solution

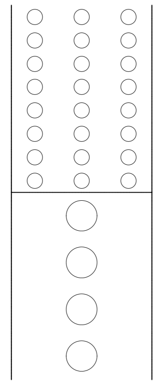

FIGURE 2. Domain G. The case $q = 3$.

leaves discrepancies in the boundary conditions on the boundaries of the holes; in the corresponding integral identity, these discrepancies can always be counterbalanced with the help of an appropriate integration by parts. In other words, the results of the present paper can be adapted easily to the case of the entire rectangle Q, for which they are also new. The reason is that, in [15], the Hardy inequality "with logarithm" was employed, which led to the extra factor $1 + |\log h|$ in the error estimate for $\varkappa = 1/2$. Below, in Subsection 3 of §4, we avoid using such an inequality, thus refining the result of [15].

§2. Formal asymptotics (without boundary layer)

1. Asymptotic *Ansätze*. Since the cells $S^{(j)}(h)$ become narrow as the number j of the row grows, the standard *Ansatz* of homogenization theory (see, e.g., [6, 7]) needs modification. Namely, the asymptotic corrector $\mathcal{V}$ in the usual presentation of the solution u^h of problem (1.5)–(1.7),

$$u^h(x) = v(x) + h\mathcal{V}^j(x, y^{(j)}, \zeta)D(\nabla_x)v(x) + \cdots \tag{2.1}$$

depends on $j = 1, \dots, N_2$ and is sought in the form

$$\mathcal{V}^j(x, y^{(j)}, \zeta) = w(x, \zeta) + q^{-j}W(x, y^{(j)}, \zeta) + q^{-2j}U(x, y^{(j)}, \zeta) + \cdots, \tag{2.2}$$

where the $y^{(j)}$ are the coordinates (1.3) (the variable $y_1^{(j)}$ coincides with $\eta^{(j)}$; see (1.2)). We recall that v is a column of height n, and $\mathcal{V}^j$, w, and W are $(n \times k)$-matrices periodic in the variables $y^{(j)}$. Formula (2.2) can be viewed as the asymptotic *Ansatz* for a thin perforated domain (cf. [8, 13]); ζ is a slow variable, and $y^{(j)}$ represents fast variables.

We substitute (2.1) and (2.2) in system (1.5) and the boundary conditions (1.6) and use the chain rule:

$$\begin{aligned} &D(\nabla_x)\Phi(x, h^{-1}q^j x, h^{-1}x_2) \\ &\qquad = \Big(D(\partial_x) + h^{-1}D(e_2\partial_\zeta) + h^{-1}q^j D(\partial_y)\Big)\Phi(x, y, \zeta)\Big|_{\zeta = h^{-1}x_2,\, y = h^{-1}q^j x}, \end{aligned} \tag{2.3}$$

where e_i is the unit vector of the axis x_i. In formula (2.3) and below, it is convenient to discriminate between the full and the partial derivatives, which are united in the gradients ∇_x and $\partial_x = (\partial/\partial x_1, \partial/\partial x_2)$, $\partial_y = (\partial/\partial y_1, \partial/\partial y_2)$, respectively. We transform relations (1.5) and (1.6) in the way indicated above, separate the leading terms with respect to h, and equate them to zero, obtaining the boundary-value problem

$$
\begin{aligned}
&- D(\partial_y)^\top A(x,y_1,\zeta) D(\partial_y) W(x,y,\zeta) \\
&\qquad = D(\partial_y)^\top A(x,y_1,\zeta) \\
&\qquad\quad + D(\partial_y)^\top A(x,y_1,\zeta) D(e_2\partial_\zeta) w(x,\zeta), \quad y \in S; \\
&- D(\nu(y))^\top A(x,y_1,\zeta) D(\partial_y) W(x,y,\zeta) \\
&\qquad = D(\nu(y))^\top A(x,y_1,\zeta) \\
&\qquad\quad + D(\nu(y))^\top A(x,y_1,\zeta) D(e_2\partial_\zeta) w(x,\zeta), \quad y \in \partial\omega,
\end{aligned}
\tag{2.4}
$$

which will be supplemented with the periodicity conditions on opposite sides of the cell S:

$$
\begin{aligned}
W(x,0,y_2,\zeta) &= W(x,b_1,y_2,\zeta), \\
D(\partial_y) W(x,0,y_2,\zeta) &= D(\partial_y) W(x,b_1,y_2,\zeta); \\
W(x,y_1,0,\zeta) &= W(x,y_1,b_2,\zeta), \\
D(\partial_y) W(x,y_1,0,\zeta) &= D(\partial_y) W(x,y_1,b_2,\zeta).
\end{aligned}
\tag{2.5}
$$

Conditions $(2.5)_1$ arise because of periodicity in the variable $\eta^{(j)} = y_1^{(j)}$, required in the decomposition (2.1) of the asymptotic corrector, and conditions $(2.5)_2$ are typical of the *Ansatz* (2.2) in thin domains.

2. Homogenization of the problem on a cell. By the algebraic completeness of the matrix D and the polynomial property of the operator $\mathcal{L}$, on the cell $S = \Xi \setminus \overline{\varpi}$ the model problem

$$
\begin{aligned}
D(-\partial_y)^\top A(x,y_1,\zeta) D(\partial_y) \mathbf{w}(x,y,\zeta) &= \mathbf{f}(x,y,\zeta), \quad y \in S, \\
D(\nu(y))^\top A(x,y_1,\zeta) D(\partial_y) \mathbf{w}(x,y,\zeta) &= \mathbf{g}(x,y,\zeta), \quad y \in \partial\omega,
\end{aligned}
$$

with the periodicity conditions (2.5) has a solution $\mathbf{w} \in H^2_{\mathrm{reg}}(S)^n$ for right-hand sides $\mathbf{f} \in L_2(S)^n$ and $\mathbf{g} \in H^{1/2}(\partial\omega)^n$ if and only if the orthogonality conditions are fulfilled:

$$
\int_S \mathbf{f}(x,y,\zeta)\,dy + \int_{\partial\omega} \mathbf{g}(x,y,\zeta)\,ds_y = 0 \in \mathbb{R}^n.
\tag{2.6}
$$

A solution is unique up to a constant summand belonging to $\mathbb{R}^n$, and the solution with zero mean over S becomes unique, inherits smoothness in x and ζ from the right-hand sides, and satisfies the estimate

$$
\|\mathbf{w}; H^2_{\mathrm{reg}}(S)\| \le c \left(\|\mathbf{f}; L_2(S)\| + \|\mathbf{g}; H^{1/2}(\partial\omega)\| \right).
$$

For problem (2.4), (2.5), the solution W and the right-hand sides are $(n \times k)$-matrices, i.e., the matrix W is formed by k column solutions. Matrix calculations show that the solvability conditions (2.6) for problem (2.4), (2.5) are satisfied and look like this:

$$
\left\{ \int_S D(\partial_y)^\top A(x,y_1,\zeta)\,dy - \int_{\partial\omega} D(\nu(y))^\top A(x,y_1,\zeta)\,ds_y \right\} \big(\mathbb{I}_k + D(e_2\partial_\zeta) w(x,\zeta) \big) = \mathbb{O}_{n\times k}.
\tag{2.7}
$$

Here and in what follows, $\mathbb{I}_k$ is the unit $(k \times k)$-matrix, and $\mathbb{O}_{n\times k}$ is the zero $(n \times k)$-matrix. Thus, problem (2.4), (2.5) admits a solution W representable in the form

$$
\begin{aligned}
W(x,y,\zeta) &= \mathcal{W}(x,y,\zeta) \big(\mathbb{I}_k + D(e_2\partial_\zeta) w(x,\zeta) \big), \\
\int_S \mathcal{W}(x,y,\zeta)\,dy &= \mathbb{O}_{n\times k},
\end{aligned}
\tag{2.8}
$$

where $\mathcal{W}$ is the periodic solution with zero mean of the following boundary-value problem on the cell S:

$$
\begin{aligned}
-D(\partial_y)^\top A(x,y_1,\zeta)D(\partial_y)\mathcal{W}(x,y,\zeta) &= D(\partial_y)^\top A(x,y_1,\zeta), \qquad y\in S,\\
-D(\nu(y))^\top A(x,y_1,\zeta)D(\partial_y)\mathcal{W}(x,y,\zeta) &= D(\nu(y))^\top A(x,y_1,\zeta), \quad y\in\partial\omega.
\end{aligned} \tag{2.9}
$$

An easy calculation shows that the last term U in (2.2) solves a problem similar to (2.9):

$$
\begin{aligned}
-D(\partial_y)^\top A(x,y_1,\zeta)D(\partial_y)U(x,y,\zeta) &= \mathcal{F}(x,y,\zeta), \quad y\in S,\\
-D(\nu(y))^\top A(x,y_1,\zeta)D(\partial_y)U(x,y,\zeta) &= \mathcal{G}(x,y,\zeta), \quad y\in\partial\omega,
\end{aligned} \tag{2.10}
$$

with the right-hand sides

$$
\begin{aligned}
\mathcal{F} &= D(e_2\partial_\zeta)^\top A + D(e_2\partial_\zeta)^\top AD(\partial_y)W\\
&\quad + D(\partial_y)^\top AD(e_2\partial_\zeta)W + D(e_2\partial_\zeta)^\top AD(e_2\partial_\zeta)w,\\
\mathcal{G} &= -D(\nu(y))^\top AD(e_2\partial_\zeta)W.
\end{aligned} \tag{2.11}
$$

Substituting the vector-valued functions $(2.11)_1$ and $(2.11)_2$ for $\mathbf{f}$ and $\mathbf{g}$, we reshape (2.6) to a system of equations for the matrix function w:

$$
\begin{aligned}
0 &= D(e_2\partial_\zeta)^\top \int_S \big(AD(\partial_y)\mathcal{W} + A\big)\,dy + D(e_2\partial_\zeta)^\top\\
&\quad + D(e_2\partial_\zeta)^\top \int_S \big(AD(\partial_y)\mathcal{W} + A\big)\,dy D(e_2\partial_\zeta)w\\
&=: D(e_2\partial_\zeta)^\top \mathcal{A} + D(e_2\partial_\zeta)^\top \mathcal{A}D(e_2\partial_\zeta)w.
\end{aligned} \tag{2.12}
$$

We show that the $(k\times k)$-matrix $\mathcal{A}$ that arose in (2.12) is symmetric and positive definite. Integrating by parts and using (2.4) and (2.5), we make transformations with an arbitrary column $X\in\mathbb{R}^k$:

$$
\begin{aligned}
X^\top\mathcal{A}X &= X^\top\bigg\{\int_S \big(A + AD(\partial_y)\mathcal{W} + (D(\partial_y)^\top A)\mathcal{W} + AD(\partial_y)^\top\mathcal{W}\big)\,dy\\
&\qquad\qquad - \int_{\partial\omega} D(\nu)^\top A\mathcal{W}\,ds_y\bigg\}X\\
&= X^\top\bigg\{\int_S \Big[A + AD(\partial_y)\mathcal{W} + \big(D(\partial_y)^\top A\big)W + AD(\partial_y)^\top\mathcal{W}\\
&\qquad + D(\partial_y)^\top\mathcal{W}AD(\partial_y)\mathcal{W} + \big(D(\partial_y)^\top AD(\partial_y)\mathcal{W}\big)\mathcal{W}\Big]\,dy\bigg\}X\\
&= \int_S \big(X + D(\partial_y)\mathcal{W}X\big)^\top A\big(X + D(\partial_y)\mathcal{W}X\big)\,dy.
\end{aligned}
$$

The last-written integral is nonnegative; it remains to check that it does not vanish for $X\neq 0$. Since the matrix A is positive definite, vanishing is possible only if $D(\partial_y)\ \mathcal{W}(x,y,\zeta)X = -X$ for all $y\in S$. Since the matrix D is algebraically complete, for any row of homogeneous polynomials of degree $\rho\ge\max(2,\rho_D)$ there is a row $Q = (Q_1,\dots,Q_k)$ of polynomials such that $P(\partial_y)\mathcal{W}X = Q(\partial_y)D(\partial_y)\mathcal{W}X = Q(\partial_y)X = 0$, whence we see that $\mathcal{W}X$ is a vector polynomial. Being periodic, this polynomial is constant, and it is equal to zero by the second condition in (2.8).

So, the term U in (2.2) can be determined if the leading term w in (2.2) satisfies the following system of ordinary differential equations:

$$
D(e_2\partial_\zeta)^\top\mathcal{A}D(e_2\partial_\zeta)w = -D(e_2\partial_\zeta)^\top\mathcal{A}, \quad \zeta\in(0,b_2). \tag{2.13}
$$

Since the matrix function w is independent of y, the q-periodicity of w means the usual periodicity:

$$w(x,0) = w(x,b_2), \quad D(e_2\partial_\zeta)w(x,0) = D(e_2\partial_\zeta)w(x,b_2). \tag{2.14}$$

Solving system (2.13), we obtain

$$D(e_2)^\top \mathcal{A}(x,\zeta)D(e_2)\partial_\zeta w(x,\zeta) = -D(e_2)^\top \mathcal{A}(x,\zeta) + \mathcal{C}(x), \tag{2.15}$$

where $\mathcal{C}(x)$ is a matrix of size $n \times k$. The symmetric matrix $D(e_2)^\top \mathcal{A} D(e_2)$ occurring in (2.15) is positive definite, because so is $\mathcal{A}$ and $\operatorname{rank} D(e_2) = n$. Again by (2.15), we have

$$\begin{aligned} D(e_2\partial_\zeta)w = &-D(e_2)[D(e_2)^\top \mathcal{A} D(e_2)]^{-1} D(e_2)\mathcal{A} \\ &+ D(e_2)[D(e_2)^\top \mathcal{A} D(e_2)]^{-1}\mathcal{C}(x). \end{aligned} \tag{2.16}$$

The integral of the expression in (2.16) over the interval $(0,b_2)$ must vanish because of the periodicity conditions (2.14); therefore,

$$\mathcal{C} = \left\langle [D(e_2)^\top \mathcal{A} D(e_2)]^{-1} \right\rangle^{-1} \left\langle [D(e_2)^\top \mathcal{A} D(e_2)]^{-1} D(e_2)^\top \mathcal{A} \right\rangle. \tag{2.17}$$

Here $\langle \Phi \rangle (x) = {b_2}^{-1} \int_0^{b_2} \Phi(x,\zeta)\, d\zeta$ is the mean of a function Φ over the interval $(0,b_2)$.

We have finished the processing of the *Ansatz* (2.2): an explicit formula for the leading term was found, the second term (2.8) was indicated, and the existence of the third term, the form of which will not be needed in the sequel, was ensured. We emphasize that the second term in (2.2) has jumps on the bases Γ_j of the strips, because this term has different periods in the strips $\Pi_{j-1}(h)$ and $\Pi_j(h)$. To eliminate this drawback, in §3 we shall construct boundary layers.

3. Calculation of the intermediate discrepancy and the homogenized equation. We unite the *Ansätze* (2.1) and (2.2). For this, we plug

(2.18)

$$u^h = v(x) + h\Big\{ w(x,\zeta) + q^{-j}W(x,y,\zeta) + q^{-2j}U(x,y,\zeta)\Big\} D(\partial_x)v(x) + \cdots, \quad x \in \Pi_j(h),$$

in the system (1.5) restricted to $\Pi_j(h)$ and in the boundary condition (1.6) on the boundaries of the holes located within the same strip $\Pi_j(h)$. By (2.3), the discrepancy obtained in (1.5) can be written in the form

$$\begin{aligned} \mathfrak{F} = {} & h^{-1}q^j D(\partial_y)^\top \mathfrak{A}^{[0]} D(\partial_x)v + h^{-1}\Big\{ D(\partial_y)^\top \mathfrak{B}^{[0]} + D(e_2\partial_\zeta)^\top \mathfrak{A}^{[0]} \Big\} D(\partial_x)v \\ & + h^0 \Big\{ f + D(\partial_x)^\top \mathfrak{A}^{[0]} D(\partial_x)v \Big\} + h^{-1}q^{-j} D(e_2\partial_\zeta)^\top \mathfrak{B}^{[0]} D(\partial_x)v \\ & + h^0 q^{-j} D(\partial_x)^\top \mathfrak{B}^{[0]} D(\partial_x)v + h^0 D(\nabla_x)^\top A D(e_2\partial_\zeta) U^{[2]} D(\partial_x)v \\ & + h^1 D(\nabla_x)^\top A D(\partial_x)\Big\{ w^{[0]} + W^{[1]} + U^{[2]} \Big\} D(\partial_x)v, \end{aligned} \tag{2.19}$$

where the abbreviated notation

$$\mathfrak{A} = A + A(D(e_2\partial_\zeta)w + D(\partial_y)W), \quad \mathfrak{B} = A(D(e_2\partial_\zeta)W + D(\partial_y)U)$$

is used, and the matrix functions $\mathfrak{A}^{[0]}$ and $\mathfrak{B}^{[0]}$ are defined by the rule (1.4).

The discrepancy in the boundary condition (1.6) is similar to (2.19). Namely,

$$\begin{aligned} \mathfrak{G} = {} & -h^0 D(\nu)^\top \mathfrak{A}^{[0]} D(\partial_x)v - h^0 q^{-j} D(\nu)^\top \mathfrak{B}^{[0]} D(\partial_x)v \\ & - h^0 D(\nu)^\top A D(e_2\partial_\zeta) U^{[0]} D(\partial_x)v \\ & - h^1 D(\nu)^\top A D(\partial_x)\Big\{ w^{[0]} + W^{[1]} + U^{[2]} \Big\} D(\partial_x)v. \end{aligned} \tag{2.20}$$

The terms on the right in (2.19) that involve the factors $h^{-1}q^j$ and h^{-1} vanish, in accordance with identities $(2.4)_1$ and $(2.10)_1$, respectively. We require that the coefficient of h^0 in (2.19), which is a function of $x \in \Omega(h)$, $y \in S$, and $\zeta \in (0, b_2)$, have zero mean over the set $S \times (0, b_2)$, i.e.,

$$\overline{D(\partial_x)^\top \mathfrak{A} D(\partial_x) v} + \overline{f} = 0 \quad \text{on } Q. \tag{2.21}$$

We have extended identity (2.21) to the entire Q, because only functions of the variable x survived in the expression on the left-hand side. In the same expression, the bar denotes taking the mean value, i.e.,

$$\overline{\mathcal{T}}(x) = \Big\langle (\operatorname{meas}_2 S)^{-1} b_2^{-1} \int_S \mathcal{T}(x, y, \zeta)\, dy d\zeta \Big\rangle.$$

By (2.8) and (2.12), we have

$$\int_S \big\{ A + A(D(e_2 \partial_\zeta) w + D(\partial_y) W) \big\}\, dy = \mathcal{A} + \mathcal{A} D(e_2 \partial_\zeta) w.$$

Therefore, using (2.16) and (2.17), we can rewrite (2.21) as a system of equations for an unknown vector-valued function v:

$$\mathbf{L}(x, \nabla_x) v(x) := D(-\nabla_x)^\top \mathbf{A}(x) D(\nabla_x) v(x) = \mathbf{F}(x), \quad x \in Q. \tag{2.22}$$

If the right-hand side of (1.5) has the form

$$f^h(x) = F^{[0]}(x, y, \zeta), \tag{2.23}$$

where F is a vector-valued function of the variables $x \in Q, y \in S$, and $\zeta \in (0, b_2)$, then the right-hand side $\mathbf{F}$ in (2.22) can be found by the formula

$$\mathbf{F}(x) = \int_S \int_0^{b_2} F(x, y, \zeta)\, dy d\zeta. \tag{2.24}$$

The matrix $\mathbf{A}$ of size $k \times k$ occurring in (2.22) looks like this:

$$\begin{aligned} \mathbf{A} = \langle \mathcal{A} \rangle &- \big\langle \mathcal{A} D(e_2) [D(e_2)^\top \mathcal{A} D(e_2)]^{-1} D(e_2)^\top \mathcal{A} \big\rangle \\ &+ \big\langle \mathcal{A} D(e_2) [D(e_2)^\top \mathcal{A} D(e_2)]^{-1} \big\rangle \big\langle [D(e_2)^\top \mathcal{A} D(e_2)]^{-1} \big\rangle^{-1} \\ &\times \big\langle [D(e_2)^\top \mathcal{A} D(e_2)]^{-1} D(e_2)^\top \mathcal{A} \big\rangle. \end{aligned}$$

Clearly, this matrix is symmetric; we prove that it is positive definite. Consider the auxiliary matrix

$$\begin{aligned} \mathcal{B} = &-D(e_2) [D(e_2)^\top \mathcal{A} D(e_2)]^{-1} D(e_2)^\top \mathcal{A} \\ &+ D(e_2) [D(e_2)^\top \mathcal{A} D(e_2)]^{-1} \big\langle [D(e_2)^\top \mathcal{A} D(e_2)]^{-1} \big\rangle^{-1} \\ &\times \big\langle [D(e_2)^\top \mathcal{A} D(e_2)]^{-1} D(e_2)^\top \mathcal{A} \big\rangle, \end{aligned}$$

where $\langle \ \rangle$ is the operation introduced after formula (2.17). A direct calculation yields the relation

$$\mathbf{A} = \big\langle \mathcal{A} + \mathcal{B}^\top \mathcal{A} + \mathcal{A}\mathcal{B} + \mathcal{B}^\top \mathcal{A} \mathcal{B} \big\rangle = \big\langle (\mathbb{I}_k + \mathcal{B})^\top \mathcal{A} (\mathbb{I}_k + \mathcal{B}) \big\rangle.$$

Since the matrix $\mathcal{A}$ is positive definite, the above relation implies that $X\mathbf{A}(x)X$ can be zero for some $x \in \overline{Q}$ and $X \in \mathbb{R}^n \setminus \{0\}$ only if $(\mathbb{I}_k + \mathcal{B}(x, \zeta))X = 0$ for all $\zeta \in (0, b_2)$ and the same x and X. However, in this case $X = \langle X \rangle = -\langle \mathcal{B} X \rangle = X \langle \mathcal{B} \rangle = 0$, because $\langle \mathcal{B} \rangle = 0$. This contradiction establishes the following statement.

Theorem 2.1. *The homogenized matrix operator* $\mathbf{L}(x, \nabla_x)$ *on the left-hand side in* (2.22) *is elliptic and formally positive*; *hence, it possesses the polynomial property.*

§3. Boundary layer

1. Boundary layer problem. Let G denote the domain obtained by removing two infinite collections of holes from the strip $\{\mathfrak{y} \in \mathbb{R}^2 : \mathfrak{y}_1 \in (0, b_1)\}$. Namely, the part $G_- = \{\mathfrak{y} \in G : \mathfrak{y}_2 < 0\}$ of G is formed by the cells $S = \Xi \setminus \overline{\varpi}$, and the part $G_+ = \{\mathfrak{y} \in G : \mathfrak{y}_2 > 0\}$ by the cells $S_1 = \Xi_1 \setminus \overline{\varpi}_1$ contracted with coefficient q (see Figure 2, and also the notation in Subsection 1 of §1 and Figure 1). We consider the following problem with the Neumann conditions on the boundaries of the holes and the periodicity conditions on the sides of the strip:

$$
\begin{aligned}
L(\mathfrak{y}, \nabla_{\mathfrak{y}})\mathcal{Z}(\mathfrak{y}) &:= D(-\nabla_{\mathfrak{y}})^\top A(\mathfrak{y}) D(\nabla_{\mathfrak{y}})\mathcal{Z}(\mathfrak{y}) = \mathcal{F}(\mathfrak{y}), \quad \mathfrak{y} \in G;\\
N(\mathfrak{y}, \nabla_{\mathfrak{y}})\mathcal{Z}(\mathfrak{y}) &:= D(\nu(\mathfrak{y}))^\top A(\mathfrak{y}) D(\nabla_{\mathfrak{y}})\mathcal{Z}(\mathfrak{y}) = \mathcal{G}(\mathfrak{y}),\\
&\quad \mathfrak{y} \in \partial G^0 = \{\mathfrak{y} \in \partial G : \mathfrak{y}_1 \in (0, b_1)\};\\
&\quad \mathcal{Z}(0, \mathfrak{y}_2) = \mathcal{Z}(b_1, \mathfrak{y}_2),\\
D(\nabla_{\mathfrak{y}})\mathcal{Z}(0, \mathfrak{y}_2) &= D(\nabla_{\mathfrak{y}})\mathcal{Z}(b_1, \mathfrak{y}_2), \quad \mathfrak{y}_2 \in \mathbb{R}.
\end{aligned}
\tag{3.1}
$$

Here ν is the outward unit normal to the boundary ∂G of the domain G. The data of the problem depend on the parameter $x \in Q$, but this dependence is not indicated explicitly; in particular,

$$
A(\mathfrak{y}) = \begin{cases} A(x, q^{-1}\mathfrak{y}_1, b_2) & \text{if } \mathfrak{y}_2 > 0,\\ A(x, \mathfrak{y}_1, 0) & \text{if } \mathfrak{y}_2 < 0. \end{cases}
\tag{3.2}
$$

The requirements imposed on the matrix A in Subsection 2 of §1 imply that the matrix function $\mathfrak{y} \mapsto A\mathfrak{y}$ defined by (3.2) is smooth on the closure $\overline{G}$ and b_1-periodic in the variable $\mathfrak{y}_1$.

We introduce some function spaces for which, by the results of [16, 17], problem (3.1) turns out to be Fredholm. For $\beta \in \mathbb{R}$ and $l \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, by $W^l_{\beta,\mathrm{reg}}(G)$ we shall mean the completion of the linear space $C^\infty_{0,\mathrm{reg}}(\overline{G})$ of smooth functions with compact support in the norm

$$
\|\mathcal{Z}; W^l_{\beta,\mathrm{reg}}(G)\| = \|\exp(\beta\sqrt{1+\mathfrak{y}_2^2})\mathcal{Z}; H^l(G)\|.
$$

The index "reg" corresponds to the periodicity of the elements of a space with respect to $\mathfrak{y}_1$. Accordingly, for $l \in \mathbb{N}$, $W^{l-1/2}_\beta(\partial G^0)$ is the trace space with the natural norm

$$
\|z; W^{l-1/2}_\beta(\partial G^0)\| = \inf\Big\{\|\mathcal{Z} \in W^l_{\beta,\mathrm{reg}}(G)\| \;\Big|\; z = \mathcal{Z} \text{ on } \partial G^0\Big\}.
$$

The next statements are specifications of some general results of [18, 3, 19]; they are based on the polynomial property of the operator $\mathcal{L}$ [3]; see Subsection 2 of §1.

Proposition 3.1. *If a solution $\mathcal{Z}$ of the homogeneous problem* (3.1) *grows at infinity at most polynomially, then*

$$
\mathcal{Z} = c^0 + c^1_1\mathcal{Z}^1 + \cdots + c^1_n\mathcal{Z}^n,
\tag{3.3}
$$

where c^0, $c^1 = (c^1_1, \dots, c^1_n)^\top \in \mathbb{R}^n$, and the $\mathcal{Z}^p$ are special solutions of (3.1) *that admit the representations*

$$
\begin{aligned}
\mathcal{Z}^p(\mathfrak{y}) &= \mathcal{Y}^p_+(\mathfrak{y}) + C^p + o(e^{-\delta\mathfrak{y}_2}), \quad \mathfrak{y}_2 \to +\infty,\\
\mathcal{Z}^p(\mathfrak{y}) &= \mathcal{Y}^p_-(\mathfrak{y}) + o(e^{\delta\mathfrak{y}_2}), \quad \mathfrak{y}_2 \to -\infty.
\end{aligned}
\tag{3.4}
$$

Here

$$
\mathcal{Y}^p_+(\mathfrak{y}) = \mathfrak{y}_2\mathbf{e}_p + \mathcal{W}(q^{-1}\mathfrak{y})D(e_2)\mathbf{e}_p, \quad \mathcal{Y}^p_-(\mathfrak{y}) = \mathfrak{y}_2\mathbf{e}_p + \mathcal{W}(\mathfrak{y})D(e_2)\mathbf{e}_p,
\tag{3.5}
$$

the $\mathbf{e}_p = (\delta_{p,1}, \dots, \delta_{p,n})^\top$ *are unit vectors in the Euclidean space* $\mathbb{R}^n$, δ *is a small positive number, the* C^p *are constant vectors depending on the matrix* A *and on the shape* ω *of the holes, and* $\mathcal{W}(\mathfrak{y}) = \mathcal{W}(x, 0, \mathfrak{y})$ *is the zero mean solution of problem* (2.5), (2.9) *on the cell* S.

Proposition 3.2. *There exists a number* $\beta_0 > 0$ *with the following property: for any exponent* $\beta \in (0, \beta_0)$, *problem* (3.1) *with right-hand sides* $\mathcal{F} \in W^{l-1}_{\beta,\mathrm{reg}}(G)^n$ *and* $\mathcal{G} \in W^{l-1/2}_\beta(\partial G^0)^n$ *admits a solution* $\mathcal{Z} \in W^{l+1}_{\beta,\mathrm{reg}}(G)^n$ *decaying at infinity if and only if the following* $2n$ *solvability conditions are satisfied*:

$$\begin{gathered} \int_G \mathcal{F}\, d\mathfrak{y} + \int_{\partial G^0} \mathcal{G}\, ds_{\mathfrak{y}} = 0 \in \mathbb{R}^n, \\ \int_G \mathcal{F}^\top \mathcal{Z}^p\, d\mathfrak{y} + \int_{\partial G^0} \mathcal{G}^\top \mathcal{Z}^p\, ds_{\mathfrak{y}} = 0 \in \mathbb{R}^n, \quad p = 1, \dots, n. \end{gathered} \tag{3.6}$$

Such a solution is unique. If only condition $(3.6)_1$ *is fulfilled, then there is a unique solution* $\widehat{\mathcal{Z}}$ *representable as*

$$\widehat{\mathcal{Z}}(\mathfrak{y}) = \mathfrak{x}(\mathfrak{y}_2)\mathfrak{C} + \widetilde{\mathcal{Z}}(\mathfrak{y}), \quad \mathfrak{C} \in \mathbb{R}^n, \;\; \widetilde{\mathcal{Z}} \in W^{l+1}_{\beta,\mathrm{reg}}(G)^n, \tag{3.7}$$

and satisfying

$$\|\widetilde{\mathcal{Z}}; W^{l+1}_{\beta,\mathrm{reg}}(G)\| + |\mathfrak{C}| \le c(\|\mathcal{F}; W^{l-1}_{\beta,\mathrm{reg}}(G)\| + \|\mathcal{G}; W^{l-1/2}_\beta(\partial G^0)\|).$$

Here $\mathfrak{x} \in \mathbb{C}^\infty(\mathbb{R}^1)$ *is a cut-off function*,

$$\mathfrak{x}(t) = 0 \text{ for } t < 1/3; \quad \mathfrak{x}(t) = 1 \text{ for } t > 2/3. \tag{3.8}$$

Condition $(3.6)_2$ *means that in* (3.7) *the column* $\mathfrak{C}$ *is zero.*

With the help of Proposition 3.2, it is not hard to construct the special solutions $\mathcal{Z}$ mentioned in Proposition 3.1. For this, the first pairs of terms on the right in $(3.4)_1$ and $(3.4)_2$, which satisfy the homogeneous equations in the perforated domains G^+ and G^- and the boundary conditions on the parts $\{\mathfrak{y} \in \partial G^+ : \mathfrak{y}_1 > 0, \mathfrak{y}_2 \in (0, b_2)\}$ and $\{\mathfrak{y} \in \partial G^- : \mathfrak{y}_1 < 0, \mathfrak{y}_2 \in (0, b_2)\}$ of the boundaries of G^+ and G^-, are multiplied by the cut-off functions $\mathfrak{x}(\eta_1)$ and $\mathfrak{x}(-\eta_1)$. Next, it can be checked that the resulting discrepancies $\mathfrak{F}^\pm$ and $\mathfrak{G}^\pm$ satisfy the orthogonality conditions $(3.6)_2$. To the sum of the products mentioned above, we add the solution $\widehat{\mathcal{Z}}^p$ of problem (3.1) with the right-hand sides $\mathcal{F} = \mathfrak{F}^+ + \mathfrak{F}^- \in C_0^\infty(\overline{G})^n$ and $\mathcal{G} = \mathfrak{G}^+ + \mathfrak{G}^- \in C_0^\infty(\partial \overline{G})^n$. By (3.7), the solutions constructed in this way admit representations as in (3.4).

We note that the solvability conditions (3.6) can be deduced by the formal substitution in the Green formula for the domain G of the solution $\mathcal{Z} \in W^{l+1}_{\beta,\mathrm{reg}}(G)^n$ and the solution $\mathbf{e}_1, \dots, \mathbf{e}_n$ and $\mathcal{Z}^1, \dots, \mathcal{Z}^n$ of the homogeneous problem.

Proposition 3.3. *There exist bounded solutions* $\mathbf{Z}^s$ *of problem* (3.1) *with the right-hand sides*

$$\mathcal{F}^s(\mathfrak{y}) = D(\nabla_\eta)^\top A^s(\mathfrak{y}), \quad \mathcal{G}^s(\mathfrak{y}) = -D(\nu(\eta))^\top A^s(\mathfrak{y}), \tag{3.9}$$

where A^s *is the* s*th column of the matrix* (3.2), $s = 1, \dots, k$. *These solutions can be written as*

$$\begin{aligned} \mathbf{Z}^s(\mathfrak{y}) &= \mathcal{W}^s(q^{-1}\mathfrak{y}) + \mathbf{C}^s + o(e^{-\delta \mathfrak{y}_2}), && \mathfrak{y}_2 \to +\infty, \\ \mathbf{Z}^s(\mathfrak{y}) &= \mathcal{W}^s(\mathfrak{y}) + o(e^{\delta \mathfrak{y}_2}), && \mathfrak{y}_2 \to -\infty, \end{aligned} \tag{3.10}$$

where $\mathbf{C}^s \in \mathbb{R}^n$ *is a constant vector, and* $\mathcal{W}^1, \dots, \mathcal{W}^k$ *are the columns of the matrix* $\mathcal{W}$.

Proof. Despite the fact that the right-hand sides $\mathcal{F}$ and $\mathcal{G}$ defined by (3.9) do not belong to the space $W^{l-1}_{\beta,\mathrm{reg}}(G)^n \times W^{l-1/2}_{\beta,\mathrm{reg}}(G)^n$, the solution turns out to be bounded because the expression in braces in (2.7) vanishes. Using the formulas

$$\mathbf{Z}^s(\mathfrak{y}) = \mathfrak{x}(\mathfrak{y}_2)\mathcal{W}^s(q^{-1}\mathfrak{y}) + \mathfrak{x}(-\mathfrak{y}_2)\mathcal{W}^s(\mathfrak{y}) + \widehat{\mathbf{Z}}^s, \tag{3.11}$$

we see that the functions $\widehat{\mathbf{Z}}^s$ satisfy problem (3.1) with the new right-hand sides

$$\begin{aligned}\widehat{\mathcal{F}}^s(\mathfrak{y}) &= (1-\mathfrak{x}(\mathfrak{y}_2)-\mathfrak{x}(-\mathfrak{y}_2))D(\nabla_\eta)^\top A^s(\mathfrak{y})\\ &\quad - [L(\mathfrak{y},\nabla_\mathfrak{y}),\mathfrak{x}(\mathfrak{y}_2)]\mathcal{W}^s(q^{-1}\mathfrak{y}) + [L(\mathfrak{y},\nabla_\mathfrak{y}),\mathfrak{x}(-\mathfrak{y}_2)]\mathcal{W}^s(\mathfrak{y}),\\ \widehat{\mathcal{G}}^s(\mathfrak{y}) &= -(1-\mathfrak{x}(\mathfrak{y}_2)-\mathfrak{x}(-\mathfrak{y}_2))D(\nu(\eta))^\top A^s(\mathfrak{y})\\ &\quad + [N(\mathfrak{y},\nabla_\mathfrak{y}),\mathfrak{x}(\mathfrak{y}_2)]\mathcal{W}^s(q^{-1}\mathfrak{y}) - [N(\mathfrak{y},\nabla_\mathfrak{y}),\mathfrak{x}(-\mathfrak{y}_2)]\mathcal{W}^s(\mathfrak{y}),\end{aligned}$$

which are smooth and have compact support. Here and in what follows $[\mathbf{P},\mathbf{Q}]$ denotes the commutator of the operators $\mathbf{P}$ and $\mathbf{Q}$. The calculation

$$\begin{aligned}\int_G \widehat{\mathcal{F}}^s\,d\mathfrak{y} + \int_{\partial G^0}\widehat{\mathcal{G}}^s\,ds_\mathfrak{y} &= \sum_{k=0}^{q-1}\int_{q^{-1}b_1k}^{q^{-1}b_1(k+1)} D(e_2)A(q^{-1}\mathfrak{y}_1)D(\nabla)\mathcal{W}^s(q^{-1}\mathfrak{y}_1,-b_2)\,d\mathfrak{y}_1\\ &\quad - \int_0^{b_1} D(e_2)A(\mathfrak{y}_1)D(\nabla)\mathcal{W}^s(\mathfrak{y}_1,-b_2)\,d\mathfrak{y}_1 = 0\end{aligned}$$

verifies the orthogonality conditions $(3.6)_1$ and employs the q-branching with coefficient q^{-1} of the matrix $\mathcal{W}^s$. The required properties of the solutions follow from Proposition 3.2. □

A further application of the solutions of (3.1) described in Propositions 3.1 and 3.3 is based on the following observation. If we expand the first two terms on the right in (2.2) in the Taylor series at the point $\zeta = 0$ and make the substitution $\zeta \mapsto y_2^j = q^j\zeta$, then, by (2.2), the leading terms of the corrector $\mathcal{V}^j(x, y^{(j)}, \zeta)$ will take the form

$$w(x,0) + q^{-j}\left\{y_2^j\partial_\zeta w(x,0) + W(x,y,0)\right\}. \tag{3.12}$$

If we put $c^0 = w(x,0)$ and $c^1 = \partial_\zeta w(x,0)q^{-j}$, then expression (3.12) will turn into a term of the asymptotics as $\mathfrak{y}_2 \to +\infty$ of the linear combination (3.3). It can be checked that a term of the asymptotics as $\mathfrak{y}_2 \to +\infty$ of the same linear combination can be obtained by the rule described above from the expansion of the corrector $\mathcal{V}^{j-1}(x, y^{(j)}, \zeta)$ near the point $\zeta = b_2$. Thus, combinations of the special solutions of (3.1) allow us to "smoothly pass" from $\mathcal{V}^{j-1}$ to $\mathcal{V}^j$, eliminating the jumps of the corrector. The above heuristic arguments imitate the method of matched expansions. In what follows we use the method of compound expansions to construct a boundary layer that decays exponentially at infinity and is obtained from the linear combination (3.3) by subtracting the terms separated in (3.4) after multiplication by some cut-off functions.

§4. Justification of the asymptotics

1. Properties of terms of the asymptotic *Ansatz* and auxiliary inequalities. We denote by $H^\varkappa_{\mathrm{reg}}(Q)$ the Sobolev–Slobodetskiĭ space of functions periodic in the variable x_1 (if $\varkappa < 1/2$, the periodicity condition can be lifted because $H^\varkappa_{\mathrm{reg}}(Q) = H^\varkappa(Q)$). The following statement is a consequence of the ellipticity of the operator $\mathbf{L}$, ensured by the algebraic completeness of the matrix D and the symmetry property and positive definiteness of the matrix $\mathbf{A}$ (see, e.g., [3, 4]).

Lemma 4.1. *Suppose* $\mathbf{F} \in H^{\varkappa}_{\mathrm{reg}}(Q)^n$ *and* $\varkappa \in (0,1/2]$. *Then system* (2.22) *admits a unique solution* $v \in H^{2+\varkappa}_{\mathrm{reg}}(Q)^n$ *satisfying the Dirichlet conditions*

$$v(x_1,0)=0,\quad v(x_1,a_2)=0,\quad x_1\in(0,a_1), \tag{4.1}$$

and

$$\|v;H^{2+\varkappa}(Q)\|\le c_\varkappa\|\mathbf{F};H^{\varkappa}(Q)\|. \tag{4.2}$$

To justify the asymptotics (2.18) of the solution of problem (1.5)–(1.7), we revise the definition of the right-hand side f^h in (1.5). Suppose that the function f^h is defined by formula (2.23) and that, for some $\varkappa \in (0,1/2]$, we have

$$F\in H^{\varkappa}_{\mathrm{reg}}(Q\to L_\infty(S\times(0,b_2)))^n, \tag{4.3}$$

where $H^{\varkappa}_{\mathrm{reg}}(Q\to L_\infty(S\times(0,b_2)))$ is the Sobolev space of abstract functions periodic in the variable $x_1\in(0,a_1]$ and having finite norm

$$\begin{aligned}&\|F;H^{\varkappa}(Q\to L_\infty(S\times(0,b_2)))\|\\&\quad=\bigg(\int_Q\sup_{(y,\zeta)\in S\times(0,b_2)}|F(x,y)|^2\,dx\\&\qquad+\int_Q\int_Q\sup_{(y,\zeta)\in S\times(0,b_2)}|F(x,y,\zeta)-F(\mathbf{x},y,\zeta)|^2\,\frac{dx d\mathbf{x}}{|x-\mathbf{x}|^{2(1+\varkappa)}}\bigg)^{1/2}.\end{aligned}$$

Let $\mathbf{N}$ denote the norm of the function (4.3). By the definition (2.24), we have $\|\mathbf{F};H^{\varkappa}(Q)\|\le c\mathbf{N}$. The quantity $\mathbf{N}$ will be involved in all majorants that will appear below in estimates for discrepancies.

The next statement (Lemma 4.2) was, in essence, established in [14, Lemma 6.4.6]. However, it needs a proof, because the domain we deal with is perforated nonuniformly.

Lemma 4.2. *Suppose* $R\in \mathring{H}^1_{\mathrm{reg}}(Q)$ *and* $\mathcal{T}\in H^{\varkappa}(Q\to L_\infty(S\times(0,b_2)))$. *Then*

$$\begin{aligned}&\bigg|\int_{\Omega(h)}\Big(\mathcal{T}^{[0]}(x,y,\zeta)-\overline{\mathcal{T}}(x)\Big)R(x)\,dx\bigg|\\&\qquad\le ch^{\varkappa}\|\mathcal{T};H^{\varkappa}(\Omega(h)\to L_\infty(S\times(0,b_2)))\|\,\|R;H^1(\Omega(h))\|.\end{aligned} \tag{4.4}$$

Proof. For each cell $S_j(h)$ obtained from the cell S_j by contraction with coefficient h, we introduce the quantity

$$R^j=(\mathrm{meas}_2\,S_j(h))^{-1}\int_{S_j(h)}R(x)\,dx. \tag{4.5}$$

The Poincaré and the Cauchy–Bunyakovskii inequalities yield

$$\begin{aligned}\|R-R^j;L_2(S_j(h))\|^2&\le ch^2q^{-2j}\|\nabla_xR;L_2(S_j(h))\|^2,\\|R^j|^2&\le ch^{-2}q^{-2j}\|R;L_2(S_j(h))\|^2.\end{aligned} \tag{4.6}$$

We partition the domain $\Omega(h)$ into the cells $S_p^{(j)}(h)$ corresponding to the rectangles $Q_p^{(j)}(h)$, and then split the cells $S_p^{(j)}(h)$ into q^j cells $S^{(j)}_{pm}(h)$ congruent to the standard cell $S_j(h)$; for $S^{(j)}_{pm}(h)$ we introduce the quantity $R^j_{pm}(h)$ by a formula similar to (4.5).

For the left-hand side of (4.4) we write

$$\begin{aligned}
&\left|\int_\Omega \left(\mathcal{T}^{[0]}(x,y^{(j)},\zeta)-\overline{\mathcal{T}}(x)\right)R(x)\,dx\right| \\
&\quad\le \sum_{j=1}^{N_2}\sum_{p=1}^{N_1}\sum_{m=1}^{q^j}\Bigg\{\int_{S^{(j)}_{pm}(h)}\left|\mathcal{T}(x,h^{-1}q^jx,h^{-1}x_2)-\overline{\mathcal{T}}(x)\right|\left|R(x)-R^p_j\right|dx \\
&\qquad\qquad + \left|R^p_j\right|\left|\int_{S^{(j)}_p(h)}(\mathcal{T}(x,h^{-1}q^jx,h^{-1}x_2)-\overline{\mathcal{T}}(x))\,dx\right|\Bigg\} \\
&\quad =: \sum_{j=1}^{N_2}\sum_{p=1}^{N_1}\sum_{m=1}^{q^j}(I^{pm}_j+J^{pm}_j).
\end{aligned}$$

By the Poincaré inequality $(4.6)_1$, for any $\varepsilon>0$ we have

$$\begin{aligned}
I^{pm}_j &\le chq^j\left(\|\mathcal{T};L_2(S^{(j)}_{pm}(h))\|+\|\overline{\mathcal{T}};L_2(S^{(j)}_{pm}(h))\|\right)\|\nabla R;L_2(S^{(j)}_{pm}(h))\| \\
&\le chq^j\left(\varepsilon\|\mathcal{T};L_2(S^{(j)}_{pm}(h)\to L_\infty(S))\|^2+\varepsilon^{-1}\|\nabla R;L_2(S^{(j)}_{pm}(h))\|\right).
\end{aligned} \tag{4.7}$$

Finally, applying $(4.6)_2$, we see that

$$\begin{aligned}
J^{pm}_j &\le ch^{-1}q^{-j}\|R;L_2(S^{(j)}_{pm}(h))\|\int_{S^{(j)}_{pm}(h)}\left|\mathcal{T}(x,h^{-1}q^jx,h^{-1}x_2)-\overline{\mathcal{T}}(x)\right|dx \\
&= ch^{-1}q^{-j}\|R;L_2(S^{(j)}_{pm}(h))\|(\operatorname{meas}_2 S^{(j)}_{pm}(h))^{-1} \\
&\quad\times\int_{S^{(j)}_{pm}(h)}\int_{S^{(j)}_{pm}(h)}\Big|\mathcal{T}(x,h^{-1}q^jx,h^{-1}x_2)-\overline{\mathcal{T}}(x) \\
&\qquad\qquad -\mathcal{T}(\mathbf{x},h^{-1}q^jx,h^{-1}x_2)+\overline{\mathcal{T}}(\mathbf{x})\Big|\,dxd\mathbf{x} \\
&\le ch^{-1}q^{-j}\|R;L_2(S^{(j)}_{pm}(h))\|q^{-j(1+\varkappa)}h^{1+\varkappa} \\
&\quad\times\Bigg(\int_{S^{(j)}_{pm}(h)}\int_{S^{(j)}_{pm}(h)}\Bigg|\sup\Big\{\left|\mathcal{T}^{[0]}(x,y,\zeta)-\overline{\mathcal{T}}(x)-\mathcal{T}^{[0]}(\mathbf{x},y,\zeta)+\overline{\mathcal{T}}(\mathbf{x})\right|; \\
&\qquad\qquad (y,\zeta)\in S\times(0,b_2)\Big\}\Bigg|^2|x-\mathbf{x}|^{-2-2\varkappa}\,dxd\mathbf{x}\Bigg)^{1/2}.
\end{aligned}$$

Observe that $|x-y|\le(h^2a_2^2+h^2q^{-2j}a_1^2)^{1/2}\le ch$ for all $x,y\in S^{(j)}_{pm}(h)$; therefore, the introduction of the factor $|x-y|^{-2-2\varkappa}$ is compensated by the coefficient $h^{1+\varkappa}q^{-j(1+\varkappa)}$ in front of the parenthesis. So,

$$J^p_j\le ch^{\varkappa}q^{-2j-\varkappa}(\varepsilon\|\mathcal{T}:H^{\varkappa}(\Omega\to L_\infty(S))\|^2+\varepsilon^{-1}\|R;L_2(S^p_j(h))\|^2). \tag{4.8}$$

We discard the irrelevant factors $q^{-j}<1$ and $q^{-2j-\varkappa}<1$ on the right-hand sides of (4.7) and (4.8) and sum the resulting inequalities over all cells. It remains to use the following simple algebraic fact: if $c_k^2\le\varepsilon a_k^2+\varepsilon^{-1}b_k^2$ for any $\varepsilon>0$, then

$$\sum_{k=1}^K c_k^2\le 2\Bigg(\sum_{k=1}^K a_k^2\Bigg)^{1/2}\Bigg(\sum_{k=1}^K b_k^2\Bigg)^{1/2}. \tag{4.9}$$

Indeed, if $a_1=\cdots=a_K=0$, then the inequality is obvious, because $c_1=\cdots=c_K=0$ in this case since ε is arbitrary; otherwise, we choose

$$\varepsilon=\left(\sum a_k^2\right)^{-1/2}\left(\sum b_k^2\right)^{1/2}$$

and add the given inequalities for the c_k^2. $\square$

We put $\rho(x) = x_2(a_2 - x_2)$; this is a weight function positive inside of Q. Near the bases Γ_0 and $\Gamma_{N_2} = \{x = (x_1, x_2) : x_1 \in (0, a_1), x_2 = a_2\}$ of the rectangle Q, the quantity $\rho(x)$ is equivalent to the distance to these bases. To justify the asymptotics (2.18), we shall need the following consequence of the one-dimensional Hardy inequality:

$$\|\rho^{-1}u; L_2(\Omega(h))\| \le c\|\nabla_x u; H^1(\Omega(h))\|, \quad u \in \mathring{H}^1_{\mathrm{reg}}\Omega(h)). \tag{4.10}$$

Also, we shall employ the version

$$\|v; L_2(P^h_d)\| \le c_d h^{1/2}\|v; H^1(\Omega(h))\|, \quad v \in H^1(\Omega(h)), \tag{4.11}$$

of the same inequality (see, e.g., [14, Lemma 1.2.4]); here P^h_d denotes the union of two closed strips of width hd adjacent to Γ_0 and Γ_{N_2}. We emphasize that inequalities (4.10) and (4.11) are stated for the perforated domain, but the method used in the proof of Lemma 1.1 allows us to pass to the entire rectangle and to apply the corresponding known results.

2. Calculation of the discrepancy. Let d_ω denote the minimal distance between the points of $\overline{\omega}$ and the bases $\{(y_1, y_2) : 0 \le y_1 \le b_1, y_2 = 0\}$ and $\{(y_1, y_2) : 0 \le y_2 \le b_2, y_1 = 0\}$ of the rectangle Ξ. Using the standard cut-off function $\mathfrak{x}$ defined by (3.8), we introduce several cut-off functions to be employed in what follows:

$$\begin{gathered}
X^h(x_2) = \mathfrak{x}(h^{-1}b_2^{-1}x_2)\mathfrak{x}(h^{-1}b_2^{-1}(a_2 - x_2)),\\
\chi^h_{j,+}(x_2) = \mathfrak{x}(h^{-1}q^j d_\omega^{-1}(x_2 - \gamma_j)), \quad \chi^h_{j,-}(x_2) = \mathfrak{x}(h^{-1}q^{j-1}d_\omega^{-1}(\gamma_j - x_2)),\\
\chi^h(x_2) = \prod_{j=1}^{N_2-1} \left\{\chi^h_{j,+}(x_2) + \chi^h_{j,-}(x_2)\right\}, \quad \mathcal{X}^h_j(x_2) = 1 - \chi^h_{j,+}(x_2) - \chi^h_{j,-}(x_2),\\
\mathfrak{X}^h_j(x_2) = 1 - \mathfrak{x}(h^{-1}b_2^{-1}(x_2 - \gamma_j)) - \mathfrak{x}(h^{-1}b_2^{-1}(\gamma_j - x_2)).
\end{gathered} \tag{4.12}$$

Multiplication by the cut-off function X^h, which is equal to zero near the bases of Q, will ensure the Dirichlet conditions for the asymptotic approximation, and the cut-off function χ^h, which is equal to zero near the segments Γ_j, will help us to make this approximation smooth. The functions $\mathcal{X}^h_j$ and $\mathfrak{X}^h_j$ are designed for the boundary layer; they are equal to 1 at the distances $O(h^1 q^{-j})$ and $O(h^1)$ from Γ_j, respectively.

Our global asymptotic approximation $\mathcal{U}^h$ to the solution u^h of problem (1.5)–(1.7) will look like this:

$$\mathcal{U}^h(x) = v(x) + \chi^h(x_2)\mathfrak{W}^h(x) + \sum_{j=1}^{N_2-1} \mathfrak{Z}^h_j(x). \tag{4.13}$$

We explain the notation. First of all, $v \in H^{2+\varkappa}_{\mathrm{reg}}(Q)^n$ is the solution of problem (2.22), (4.1). The second term

$$\chi^h(x_2)\mathfrak{W}^h(x) = \chi^h(x_2)hX^h(x_2)\mathcal{V}^{[0]}(h, x)D(\nabla_x)v(x)$$

involves the matrix function $\mathcal{V}$ defined on the strips $\Pi_j(h)$ by formula (2.2), the cut-off function X^h that ensures the Dirichlet conditions $(1.7)_1$, and also the saw-like cut-off function χ^h that eliminates the jumps of $\mathcal{V}^{[0]}$ at the segments Γ_j.

The boundary layer term near Γ_j is of the form

$$\begin{gathered}
\mathfrak{Z}^h_j(x) = h\left(\mathcal{X}^h_j(x_2)w_0(x) + \mathfrak{X}^h_j(x_2)\boldsymbol{\mathcal{Z}}^{(j)}(\mathfrak{y}^{(j)})\right)D(\partial_x)v(x),\\
\boldsymbol{\mathcal{Z}}^{(j)}(\mathfrak{y}^{(j)}) = q^{-j}\left(\widehat{Z}(\mathfrak{y}^{(j)}) + \widehat{\mathbf{Z}}(\mathfrak{y}^{(j)})\right),
\end{gathered} \tag{4.14}$$

where $w_0(x) = w(x, 0)$ and $\mathfrak{y}^{(j)} = q^j h^{-1}(x_2 - \gamma_j)$. The matrix functions $\widehat{Z}^{(j)}$ and $\widehat{\mathbf{Z}}^{(j)}$ are defined as follows. In the expression (3.3), we put $c^0 = 0$ and $c^1 = q^{-j}\partial_\zeta w^s(x, 0)$, where $w^1, \dots, w^k$ are the columns of the matrix w; then we introduce the $(n \times k)$-matrix Z with the columns

$$Z^s(\mathfrak{y}) = q^{-j} \left\{ \partial_\zeta w^s \Big(\mathfrak{x}(\mathfrak{y}_2) \mathcal{Y}_+(\mathfrak{y}) + \mathfrak{x}(-\mathfrak{y}_2) \mathcal{Y}_-(\mathfrak{y}) \Big) + \widehat{Z}^s \right\},$$

where the $\mathcal{Y}_\pm$ are the $(n \times n)$-matrices with the columns $\mathcal{Y}^p_\pm$ defined by (3.5), and the column $\widehat{Z}^s$ admits representation (3.7). The matrix $\widehat{\mathbf{Z}}$ on the right in $(4.14)_1$ is formed by the columns $\widehat{\mathbf{Z}}^s$ given by (3.11). Thus, by (3.7), (3.10), and (3.11), the function $\boldsymbol{\mathcal{Z}}^{(j)}$ can be written as

$$\boldsymbol{\mathcal{Z}}^{(j)}(\mathfrak{y}^{(j)}) = q^{-j} \big(\chi^h_{j,+} \boldsymbol{\mathcal{C}}^{(j)} + \widetilde{\boldsymbol{\mathcal{Z}}}^{(j)}(\mathfrak{y}^{(j)}) \big),$$

where $\boldsymbol{\mathcal{C}}^{(j)}$ is a scalar $(n \times k)$-matrix, and $\widetilde{\boldsymbol{\mathcal{Z}}}^{(j)} \in W^{l+1}_{\beta,\mathrm{reg}}(G)^{n\times k}$ is a matrix function that decays exponentially at infinity in the perforated strip.

The difference $\mathcal{R}^h = u^h - \mathcal{U}^h$ between the true solution and the approximate one satisfies the system of equations

$$\mathcal{L}\mathcal{R}^h = \mathcal{L}u^h - \mathcal{L}\mathcal{U}^h = f - \mathcal{L}\mathcal{U}^h =: \mathfrak{F}, \quad x \in \Omega(h), \tag{4.15}$$

with the Neumann boundary conditions

$$\mathcal{N}\mathcal{R}^h(x) = -\mathcal{N}\mathcal{U}^h =: \mathfrak{G}, \quad x \in \partial\Omega(h) \setminus \partial Q; \tag{4.16}$$

also, by construction, this difference satisfies the homogeneous Dirichlet conditions $(1)_1$ and the periodicity conditions $(1.7)_{2,3}$.

It is not hard to check that $\mathcal{R}^h \in \mathring{H}^1_{\mathrm{reg}}(\Omega(h))$, by Lemma 1.1 and the definition of the cut-off function X^h. We take the scalar product of (4.15) by $\mathcal{R}^h$, integrate by parts over the domain $\Omega(h)$, and use conditions (1.7) and (4.16). Recalling the Korn inequality (1.9), we obtain

$$\begin{aligned} &(f^h - \mathcal{L}\mathcal{U}^h, \mathcal{R}^h)_{\Omega(h)} - (\mathcal{N}\mathcal{U}^h, \mathcal{R}^h)_{\partial\Omega(h)\setminus\partial Q} \\ &\qquad = (AD(\nabla_x)\mathcal{R}^h, D(\nabla_x)\mathcal{R}^h)_{\Omega(h)} \geq c\|\mathcal{R}^h; H^1(\Omega(h))\|^2. \end{aligned} \tag{4.17}$$

Here and in the sequel, $(\ ,\ )_\Upsilon$ is the scalar product in the space $L_2(\Upsilon)$; this space itself may be scalar or vector.

So, to estimate the norm of $\mathcal{R}^h$ in the space $H^1(\Omega(h))^n$, it remains to handle the left-hand side of (4.17). This will be done in two steps; the boundary layer will be treated separately. The transformation of the expressions will be accompanied by an estimation of their fragments.

3. Estimation of the discrepancy: the first step. The terms formed on the left-hand side in (4.17) by the first two summands on the right in (4.13) can be written as

$$(f^h - \mathcal{L}v - \chi^h \mathcal{L}\mathfrak{W}^h, \mathcal{R}^h)_{\Omega(h)} - (\chi^h \mathcal{N}\mathfrak{W}^h, \mathcal{R}^h)_{\partial\Omega(h)\setminus\partial Q} + ([\mathcal{L}, \chi^h]\mathfrak{W}^h, \mathcal{R}^h)_{\Omega(h)}. \tag{4.18}$$

At this step, we estimate the first pair of scalar products in (4.18). The third summand will be treated together with the discrepancies related to the boundary layer.

We turn to formulas (2.19) and (2.20), in which, by construction, the coefficients of $h^{-1}q^j$, h^{-1} and h^0, $h^0 q^{-j}$, respectively, are zero. We plug the remaining terms in the scalar products occurring in (4.18) and integrate by parts in the first of these products. The arising surface integrals and the second scalar product cancel. As a result, the first pair of terms in (4.18) turns into the following sum:

$$
\begin{aligned}
&- h\left(\chi^h A^{[0]}\left\{X^h D(\partial_x)\mathcal{V}^{[0]} + \chi^h A^{[0]} D(e_2\partial_\zeta) X^h U^{[2]}\right\} D(\partial_x)v, D(\nabla_x)\mathcal{R}^h\right)_{\Omega(h)}\\
&- h\left(\chi^h A^{[0]}\left\{w D(e_2\partial_\zeta)X^h + W^{[1]} D(e_2\partial_\zeta)X^h\right\} D(\partial_x)v, D(\nabla_x)\mathcal{R}^h\right)_{\Omega(h)}\\
&- \left((D(\nabla_x)\chi^h)^\top A^{[0]} X^h \left\{h D(\partial_x)\mathcal{V}^{[0]} + D(e_2\partial_\zeta)U^{[2]}\right\} D(\partial_x)v, \mathcal{R}^h\right)_{\Omega(h)}\\
&+ \left(\chi^h D(\partial_x)^\top A^{[0]}(1-X^h)\left\{D(e_2\partial_\zeta)w + D(\partial_y)W^{[0]}\right\} D(\partial_x)v, \mathcal{R}^h\right)_{\Omega(h)}\\
&+ h^{-1}\left(D(e_2\partial_\zeta)^\top A^{[0]}\mathfrak{U} D(\partial_x)v, \mathcal{R}^h\right)_{\Omega(h)} + \left(D(\partial_x)^\top A^{[0]}\mathfrak{U} D(\partial_x)v, \mathcal{R}^h\right)_{\Omega(h)}\\
&+ h^{-1}\left(\chi^h (D(e_2\partial_\zeta)X^h)^\top A^{[0]} D(\partial_y) W^{[0]} D(\partial_x)v, \mathcal{R}^h\right)_{\Omega(h)}\\
&+ \left(D(\partial_x)^\top \mathfrak{A}^{[0]} D(\partial_x)v, (1-\chi^h)\mathcal{R}^h\right)_{\Omega(h)} + \left(f + D(\partial_x)^\top \mathfrak{A}^{[0]} D(\partial_x)v, \mathcal{R}^h\right)_{\Omega(h)}\\
&\qquad =: -h\mathbf{I}_1 - h\mathbf{I}_2 - \mathbf{I}_3 + \mathbf{I}_4 + h^{-1}\mathbf{I}_5 + \mathbf{I}_6 + h^{-1}\mathbf{I}_7 + \mathbf{I}_8 + \mathbf{I}_9.
\end{aligned}
\tag{4.19}
$$

Here

$$\mathfrak{U} = \chi^h X^h \{D(e_2\partial_\zeta)W^{[1]} + D(\partial_y)U^{[1]}\}, \tag{4.20}$$

and we have used the relations

$$D(\nabla_x)X^h = h^{-1}D(e_2\partial_\zeta)X^h, \quad D(\nabla_x)\chi^h = h^{-1}q^j D(\partial_y)\chi^h, \tag{4.21}$$

which follow from the definitions $(4.12)_1$ and $(4.12)_2$. We have

$$|D(e_2\partial_\zeta)X^h| \le ch^0, \quad |D(\partial_y)\chi^h| \le ch^0, \quad x \in \Omega(h). \tag{4.22}$$

Finally, the set $T_X^h = \operatorname{supp}(1 - X_h)$, which includes, in particular, the support of the derivative $\partial_\zeta X^h$, lies inside the union of the strips $P_{d_\omega}^h$. Consequently, inequality (4.11) leads to the estimate

$$\|D(\partial_x)v; L_2(T_X^h)\| \le c_{d_\omega} h^{1/2} \|v; H^2(\Omega(h))\|. \tag{4.23}$$

We shall treat the sum (4.19) termwise, starting with $h\mathbf{I}_1$. In accordance with (4.20), this scalar product is the sum of two terms. For the first term we have

$$
\begin{aligned}
&h\left|\left(\chi^h A^{[0]} X^h \big(D(\partial_x)\mathcal{V}^{[0]}\big) D(\partial_x)v, D(\nabla_x)\mathcal{R}^h\right)_{\Omega(h)}\right|\\
&\quad \le ch\|\nabla_x^2 v; L_2(\Omega(h))\|\,\|D(\nabla_x)\mathcal{R}^h; L_2(\Omega(h))\|\\
&\quad \le ch^{1/2}\mathbf{N}\|D(\nabla_x)\mathcal{R}^h; L_2(\Omega(h))\|.
\end{aligned}
\tag{4.24}
$$

First, we applied the inequality

$$\{|\mathcal{V}^{[0]}(h,x)| + |D(\partial_x)\mathcal{V}^{[0]}(h,x)|\} \le ch^0, \quad x \in \Omega(h) \cap \Sigma_\chi^h, \tag{4.25}$$

where Σ_χ^h is the support of χ^h, and then we employed estimate (4.2) of Lemma 4.1. The following agreement was used in (4.24) and will be used repeatedly in what follows: the expression $(D(\partial_x)\mathcal{V}^{[0]})D(\nabla_x)$ is a product of two matrices, i.e., the operator $D(\partial_x)$ is applied to the matrix $\mathcal{V}^{[0]}$ only, which is indicated by the extra parentheses.

The second term in $\mathbf{I}_1$ satisfies the estimate

$$
\begin{aligned}
&\left|\left(\chi^h A^{[0]} D(e_2\partial_\zeta) X^h U^{[2]} D(\partial_x)v, D(\nabla_x)\mathcal{R}^h\right)_{\Omega(h)}\right|\\
&\quad \le c\left(\|\nabla_x v; L_2(T_X^h)\| + \|D(e_2\partial_\zeta)U^{[2]} D(\partial_x)v; L_2(\Omega(h))\|\right)\\
&\qquad \times \|D(\nabla_x)\mathcal{R}^h; L_2(\Omega(h))\|.
\end{aligned}
\tag{4.26}
$$

Since $(j-1)hb_2 < x_2 \le jhb_2$ for $x \in \Pi_j(h)$, we have

$$q^{-j}x_2 \le ch, \quad x \in \Pi_j(h). \tag{4.27}$$

Therefore, the Hardy inequality together with the obvious estimate

$$\left|U^{[2]}(x, y^{(j)}, \zeta)\right| \le ch^0, \quad x \in \Omega(h) \cap \Sigma_\chi^h,$$

shows that

$$\begin{aligned}&\|A^{[0]}X^h D(e_2\partial_\zeta)U^{[2]}D(\partial_x)v; L_2(\Omega(h))\|\\ &\qquad \le ch\|x_2^{-1}X^h D(\partial_x)v; L_2(\Omega(h))\| \le ch\|X^h D(\partial_x)v; H^1(\Omega(h))\|.\end{aligned}$$

Calculating the last-written norm and continuing estimate (4.26), we obtain

$$\begin{aligned}&\left|\left(\chi^h A^{[0]}D(e_2\partial_\zeta)X^h U^{[2]}D(\partial_x)v, D(\nabla_x)\mathcal{R}^h\right)_{\Omega(h)}\right|\\ &\quad \le c\left(\|\nabla_x v; L_2(T_X^h)\| + h\|v; H^2(\Omega(h))\|\right)\|D(\nabla_x)\mathcal{R}^h; L_2(\Omega(h))\|\\ &\quad \le ch^{1/2}\mathbf{N}\|D(\nabla_x)\mathcal{R}^h; L_2(\Omega(h))\|.\end{aligned}$$

The second inequality is ensured by (4.24) and (4.2).

To estimate the summand $\mathbf{I}_2$, which involves the function $D(e_2\partial_\zeta)X^h$ with support inside of the set T_X^h, we apply (4.22)–(4.25):

$$\begin{aligned}&\left|h^1\left(\chi^h A^{[0]}\{w + W^{[1]}\}D(e_2\partial_\zeta)X^h D(\partial_x)v, D(\nabla_x)\mathcal{R}^h\right)_{\Omega(h)}\right|\\ &\quad \le c\|\nabla_x v; L_2(T_X^h)\|\,\|D(\nabla_x)\mathcal{R}^h; L_2(\Omega(h))\|\\ &\quad \le ch^{1/2}\mathbf{N}\|D(\nabla_x)\mathcal{R}^h; L_2(\Omega(h))\|.\end{aligned}$$

Consider the third scalar product $\mathbf{I}_3$ on the right in (4.19). We have

$$\begin{aligned}\mathbf{I}_3 &= \mathbf{I}_{30} + \mathbf{I}_{31}\\ &:= h\Big((D(\nabla_x)\chi^h)^\top A^{[0]}X^h\big(D(\partial_x)\mathcal{V}^{[0]}\big)D(\partial_x)v, \mathcal{R}^h\Big)_{\Omega(h)}\\ &\quad + \Big((D(\nabla_x)\chi^h)^\top A^{[0]}X^h D(e_2\partial_\zeta)U^{[2]}D(\partial_x)v, \mathcal{R}^h\Big)_{\Omega(h)}.\end{aligned}$$

The embedding theorem $H^{\varkappa+2} \subset C^1$ yields

$$\|D(\partial_x)v; C(\overline{Q})\| \le c\|v; H^{\varkappa+2}(Q)\|. \tag{4.28}$$

Using (4.28), (4.21), and (4.27), we obtain

$$|\mathbf{I}_{31}| \le c\|v; H^{\varkappa+2}(Q)\|\,\|x_2^{-1}\mathcal{R}^h; L_2(\Omega(h))\|\sum_{j=1}^{N_2-1}\left(\operatorname{meas}_2(\sigma_j^h)\right)^{1/2}.$$

Here $\sigma_j^h = \sigma_{j,+}^h \cup \sigma_{j,-}^h$, and $\sigma_{j,\pm}^h$ is the support of $\nabla_x\chi_{j,\pm}^h$. Since

$$\operatorname{meas}_2(\sigma_j^h) \le chq^{-j}, \tag{4.29}$$

inequalities (4.2) and (4.10) imply

$$|\mathbf{I}_{31}| \le ch^{1/2}\mathbf{N}\|D(\nabla)\mathcal{R}^h; L_2(\Omega(h))\|.$$

We turn to $\mathbf{I}_{30}$. First of all we observe that relations (4.28), (4.21), (4.29), and (4.2) entail the chain of inequalities

$$\begin{aligned}(4.30)\quad &\Big|h\Big((D(\nabla_x)\chi^h)^\top A^{[0]}X^h\big\{D(\partial_x)\mathcal{V}^{[0]}-D(\partial_x)w\big\}D(\partial_x)v,\mathcal{R}^h\Big)_{\Omega(h)}\Big|\\ &\le c\|v;H^{\varkappa+2}(Q)\|\,\|\mathcal{R}^h;L_2(\Omega(h))\|\sum_{j=1}^{N_2-1}\big(\mathrm{meas}_2(\sigma_j^h)\big)^{1/2}\\ &\le ch^{1/2}\mathbf{N}\|\mathcal{R}^h;L_2(\Omega(h))\|.\end{aligned}$$

Consider the part of the expression for $\mathbf{I}_{30}$ that is not treated in (4.30). We use the estimates employed in the derivation of (4.30), integrate by parts, and take into account the fact that the support of $\mathcal{X}_j^h$ has measure $O(hq^{-j})$, obtaining finally

$$\begin{aligned}&h\Big|\Big((D(\nabla_x)\chi^h)^\top A^{[0]}X^h\big(D(\partial_x)w\big)D(\partial_x)v,\mathcal{R}^h\Big)_{\Omega(h)}\Big|\\ &=h\Big|-\sum_{j=1}^{N_2-1}\Big(\mathcal{X}_j^hD(\nabla_x)\big\{A^{[0]}X^h\big(D(\partial_x)w\big)D(\partial_x)v\big\},\mathcal{R}^h\Big)_{\Omega(h)}\\ &\qquad-\sum_{j=1}^{N_2-1}\Big(\mathcal{X}_j^hA^{[0]}X^h\big(D(\partial_x)w\big)D(\partial_x)v,D(\nabla_x)\mathcal{R}^h\Big)_{\Omega(h)}\Big|\\ &\le c\Big\{\|v;H^{\varkappa+2}(Q)\|\,\|\mathcal{R}^h;L_2(\Omega(h))\|\sum_{j=1}^{N_2-1}\big(\mathrm{meas}_2(\mathrm{supp}\,\mathcal{X}_j^h)\big)^{1/2}\\ &\qquad+h^1\|v;H^2(\Omega(h))\|\,\|\mathcal{R}^h;L_2(\Omega(h))\|+h^1\mathbf{N}\|D(\nabla_x)\mathcal{R}^h;L_2(\Omega(h))\|\Big\}\\ &\le ch^{1/2}\mathbf{N}\|\mathcal{R}^h;L_2(\Omega(h))\|.\end{aligned}$$

Since $|\partial_\zeta\omega(x,\zeta)|+|\partial_yW^{[0]}(x,y^{(j)},\zeta)|\le ch^0$ for $x\in\Omega(h)$, the term $\mathbf{I}_4$ in (4.19) satisfies

$$\begin{aligned}|\mathbf{I}_4|&\le c\|\rho D(\partial_x)(1-X^h)(w+D(\partial_y)W^{[0]})D(\partial_x)v;L_2(T_X^h)\|\,\|\rho^{-1}\mathcal{R}^h;L_2(\Omega(h))\|\\ &\le c\|\rho(1-X)D(\partial_x)^\top D(\partial_x)v;L_2(T_X^h)\|\,\|\rho^{-1}\mathcal{R}^h;L_2(\Omega(h))\|.\end{aligned}$$

Observe that $\rho(x)\le ch$ for $x\in T_X^h$. Applying the Hardy inequality (4.10) to the vector-valued function $\mathcal{R}^h$, we get

$$\begin{aligned}|\mathbf{I}_4|&\le ch^1\|v;H^2(\Omega(h))\|\,\|D(\nabla_x)\mathcal{R}^h;L_2(\Omega(h))\|\\ &\le ch^1\mathbf{N}\|D(\nabla_x)\mathcal{R}^h;L_2(\Omega(h))\|.\end{aligned}$$

The term $h^{-1}\mathbf{I}_7$ is estimated similarly. Namely, from relations (4.10) and (4.11) for $\mathcal{R}^h$ and $D(\partial_x)v$ (respectively), we deduce the inequalities

$$\begin{aligned}h^{-1}|\mathbf{I}_7|&\le c\|\rho(D(e_2\partial_\zeta)X^h)^\top A^{[0]}D(\partial_y)W^{[0]}D(\partial_x)v;L_2(T_X^h)\|\,\|\rho^{-1}\mathcal{R}^h;L_2(\Omega(h))\|\\ &\le ch^0\|D(\partial_x)v;L_2(T_X^h)\|\,\|D(\nabla_x)\mathcal{R}^h;L_2(\Omega(h))\|\\ &\le ch^{1/2}\mathbf{N}\|D(\nabla_x)\mathcal{R}^h;L_2(\Omega(h))\|.\end{aligned}$$

Now, we consider $\mathbf{I}_6$. Combining (4.27) and the inequality

$$(4.31)\qquad \partial_\zeta^k(\partial_\zeta W^{[1]}(x,y^{(j)},\zeta)+\partial_yU^{[1]}(x,y^{(j)},\zeta))\le ch^0q^{-j},\quad x\in\Pi_j(h),$$

with $k = 0$, we get

$$\sup_{x\in\Omega(h)} \big\{\rho(D(e_2\partial_\zeta)W^{[1]} + D(\partial_y)U^{[1]})\big\} \\ \leq c \sup_{x\in\Omega(h)} \big\{|x_2|(D(e_2\partial_\zeta)W^{[1]} + D(\partial_y)U^{[1]})\big\} \leq ch^1.$$

Recalling (4.10) and (4.2), we see that

$$|\mathbf{I}_6| \leq c\|\rho D(\partial_x)^\top X^h(D(e_2\partial_\zeta)W^{[1]} + D(\partial_y)U^{[1]})D(\partial_x)v; L_2(\Omega(h))\|\|\rho^{-1}\mathcal{R}^h; L_2(\Omega(h))\| \\ \leq ch\|v; H^2(\Omega(h))\|\,\|\rho^{-1}\mathcal{R}^h; L_2(\Omega(h))\| \leq ch\mathbf{N}\|D(\nabla_x)\mathcal{R}^h; L_2(\Omega(h))\|.$$

To the term $\mathbf{I}_9$, we apply Lemma 4.2. This yields

$$|\mathbf{I}_9| \leq ch^\varkappa\|F; H^\varkappa(Q \to L_\infty(S))\|\,\|\mathcal{R}^h; H^1(\Omega(h))\|.$$

It remains to handle $\mathbf{I}_5$ and $\mathbf{I}_8$. By formulas (4.28), (4.2), and (4.31) (with $k = 0, 1$ in the last of them), for any $\varepsilon > 0$ we have

$$|\mathbf{I}_5|^2 \leq \sum_{j=1}^{N_2}\bigg\{\varepsilon^{-1}\int_{\Pi_j} x^{-2}\mathcal{R}^2\,dx + \varepsilon c\mathbf{N}^2 q^{-2j}\int_{\Pi_j} x_2^2\,dx\bigg\}. \tag{4.32}$$

Clearly, $\int_{\Pi_j} x_2^2\,dx \leq ch^3q^{-2j}j^2 \leq Ch^3$. Thus, the algebraic fact mentioned at the end of the proof of Lemma 4.2 reshapes (4.9) to the relation

$$h^{-1}|\mathbf{I}_5| \leq ch^{1/2}\mathbf{N}\|D(\nabla_x)\mathcal{R}^h; L_2(\Omega(h))\|.$$

To estimate the integral $|\mathbf{I}_8|$, we use (4.11). Fixing $\varepsilon > 0$, we obtain

$$|\mathbf{I}_8| \leq c\sum_{j=1}^{N_2-1}\|v; H^2(\tau_j^h)\|\|\mathcal{R}; L_2(\tau_j^h)\| \\ \leq c\bigg\{\varepsilon\sum_{j=1}^{N_2-1}\|v; H^2(\tau_j^h)\|^2 + \varepsilon^{-1}\sum_{j=1}^{N_2-1}\|\mathcal{R}; L_2(\tau_j^h)\|^2\bigg\} \\ \leq c\bigg\{\varepsilon\|v; H^2(Q)\| + \varepsilon^{-1}\|\mathcal{R}^h; H^1(\Omega(h))\|\sum_{j=1}^{N_2-1}\big(\mathrm{meas}_2(\tau_j^h)\big)\bigg\}.$$

Here $\tau_j^h = \tau_{j,+}^h \cup \tau_{j,-}^h$, and $\tau_{j,\pm}^h$ is the support of $\chi_{j,\pm}^h$. Since $\mathrm{meas}_2(\tau_j^h) \leq chq^{-j}$, the same algebraic rule (4.9) as before shows that

$$|\mathbf{I}_8| \leq ch^{1/2}\mathbf{N}\|\mathcal{R}; L_2(\Omega)\|.$$

4. Estimation of the discrepancy: the second step. To the third summand in (4.18), we join the discrepancies generated by the expressions $(4.14)_1$ occurring in the sum over j in the definition (4.13). We have

(4.33)

$$\sum_{j=1}^{N_2-1}\Big\{\big(\mathcal{L}\mathfrak{Z}_j^h,\mathcal{R}^h\big)_{\Pi_j(h)}+\big(\mathcal{N}\mathfrak{Z}_j^h,\mathcal{R}^h\big)_{\partial\Pi_j(h)\setminus\partial Q}\Big\}+\big([\mathcal{L},\chi^h]\mathfrak{W}^h,\mathcal{R}^h\big)_{\Omega(h)}$$
$$+\big([\mathcal{N},\chi^h]\mathfrak{W}^h,\mathcal{R}^h\big)_{\partial\Omega(h)\setminus\partial Q}$$
$$=h\Big([\mathcal{L},\chi^h]\mathcal{V}^{[0]}D(\partial_x)v-[\mathcal{L}_0,\chi^h]\mathcal{V}_0^{[0]}D(\partial_x)v,\mathcal{R}^h\Big)_{\Omega(h)}$$
$$+h\sum_{j=1}^{N_2-1}\Big\{\Big(\mathcal{X}_j^h\,(\mathcal{L}w_0)\,D(\partial_x)v,\mathcal{R}^h\Big)_{\Pi_j(h)}+\Big(\mathcal{X}_j^h\,(\mathcal{L}w_0)\,D(\partial_x)v,\mathcal{R}^h\Big)_{\Pi_j(h)}$$
$$+\Big(\big([\mathcal{L},\mathcal{X}_j^h]-[\mathcal{L}_0,\mathcal{X}_j^h]\big)\,w_0D(\partial_x)v,\mathcal{R}^h\Big)_{\Pi_j(h)}$$
$$+\Big(\Big\{\mathfrak{X}_j^h(\mathcal{L}-\mathcal{L}_0)\{\widetilde{\boldsymbol{\mathcal{Z}}}^{(j)}+\chi_{j,+}\boldsymbol{\mathcal{C}}^{(j)}\}$$
$$+[\mathcal{L},\mathfrak{X}_j^h]\{\widetilde{\boldsymbol{\mathcal{Z}}}^{(j)}+\chi_{j,+}\boldsymbol{\mathcal{C}}^{(j)}\}\Big\}D(\partial_x)v,\mathcal{R}^h\Big)_{\Pi_j(h)}$$
$$+\Big(\Big\{\mathfrak{X}_j^h(\mathcal{N}-\mathcal{N}_0)\{\widetilde{\boldsymbol{\mathcal{Z}}}^{(j)}+\chi_{j,+}\boldsymbol{\mathcal{C}}^{(j)}\}$$
$$+[\mathcal{N},\mathfrak{X}_j^h]\{\widetilde{\boldsymbol{\mathcal{Z}}}^{(j)}+\chi_{j,+}\boldsymbol{\mathcal{C}}^{(j)}\}\Big\}D(\partial_x)v,\mathcal{R}^h\Big)_{\partial\Pi_j(h)\setminus\{\partial Q\cup\Gamma_{j-1}\cup\Gamma_j\}}\Big\}$$
$$=:h\boldsymbol{\mathcal{J}}+h\sum_{j=1}^{N_2-1}\mathbf{K}^{(j)}.$$

Here $A_0^{[0]}(x,y_1)=A^{[0]}(x,y_1,0)$,

$$\mathcal{L}_0(\nabla_x,x,y_1)=D(-\nabla_x)^\top A_0^{[0]}(x,y_1)D(\nabla_x),$$
$$\mathcal{N}_0(\nabla_x,x,y_1)=D(\nu(x))^\top A_0^{[0]}(x,y_1)D(\nabla_x),$$

and $\mathcal{V}_0$ is the matrix function defined in the strip $\Pi_j(h)$ by the formula

$$\mathcal{V}_0(x,y^{(j)},\zeta)=w(x,0)+\zeta\partial_\zeta w(x,0)+q^{-j}W(x,y^{(j)},0).$$

Integrating by parts, we rewrite the integral $\boldsymbol{\mathcal{J}}$ as follows:

$$\Big(\Big\{A^{[0]}\big(D(\nabla_x)\chi^h\big)\mathcal{V}^{[0]}-A_0^{[0]}\big(D(\nabla_x)\chi^h\big)\mathcal{V}_0^{[0]}\Big\}D(\partial_x)v,D(\nabla_x)\mathcal{R}^h\Big)_{\Omega(h)}$$
$$-\sum_{j=1}^{N_2-1}\Big\{\Big(\mathcal{X}_j\Big\{A^{[0]}\big(D(\nabla_x)\mathcal{V}^{[0]}\big)$$
$$-A_0^{[0]}\big(D(\nabla_x)\mathcal{V}_0^{[0]}\big)\Big\}D(\partial_x)v,D(\nabla_x)\mathcal{R}^h\Big)_{\Omega(h)}$$
$$+\Big(\mathcal{X}_j\Big\{A^{[0]}\big(D(\nabla_x)\mathcal{V}^{[0]}\big)$$
$$-A_0^{[0]}\big(D(\nabla_x)\mathcal{V}_0^{[0]}\big)\Big\}D(\nabla_x)^\top D(\partial_x)v,\mathcal{R}^h\Big)_{\Omega(h)}$$
$$+\Big(\mathcal{X}_j\{\mathcal{L}\mathcal{V}^{[0]}-\mathcal{L}_0\mathcal{V}_0^{[0]}\}D(\partial_x)v,\mathcal{R}^h\Big)_{\Omega(h)}\Big\}$$
$$=:\mathbf{J}_0-\sum_{j=1}^{N_2-1}\Big\{\mathbf{J}_1^{(j)}+\mathbf{J}_2^{(j)}+\mathbf{J}_3^{(j)}\Big\}.$$

We have used the fact that the supports of $\nabla_x X$ and $\mathcal{X}_j$ are disjoint.

Inequalities (4.29), (4.2), and (4.28) imply the estimate

$$\begin{aligned} h|\mathbf{J}_1^{(j)}| &\le c\|v; H^{\varkappa+2}(Q)\|\,\|D(\nabla_x)\mathcal{R}^h; L_2(\Omega(h))\|\big(\operatorname{meas}_2(\sigma_j^h)\big)^{1/2} \\ &\le ch^{1/2}\mathbf{N}\|D(\nabla_x)\mathcal{R}; L_2(\Omega)\|. \end{aligned}$$

Combining the same inequalities and the relations

$$\begin{gathered} |D(\nabla_x)\chi^h(x)| \le ch^{-1}q^j, \quad x \in \overline{\sigma_j^h}, \\ \sum_{j=1}^{N_2-1} \operatorname{meas}_2(\sigma_j^h) \le ch \sum_{j=1}^{N_2-1} \operatorname{meas}_2 q^{-j} \le ch, \end{gathered} \tag{4.34}$$

which follow from $(4.12)_3$ and (4.29), we see that

$$h|\mathbf{J}_0| \le ch^{1/2}\mathbf{N}\|D(\nabla_x)\mathcal{R}; L_2(\Omega)\|.$$

Next, using (4.27) and also (4.10), (1.9) and (4.2), (4.28), we obtain

$$\begin{aligned} h|\mathbf{J}_2^{(j)}| &\le c\|v; H^2(\Omega(h))\|\,\|x_2^{-1}\mathcal{R}^h; L_2(\Omega(h))\|\bigg(\sum_{j=1}^{N_2-1}\operatorname{meas}_2(\sigma_j^h)\bigg)^{1/2} \\ &\le ch^{1/2}\mathbf{N}\|D(\nabla_x)\mathcal{R}; L_2(\Omega)\|. \end{aligned}$$

Since the function $\mathcal{V}$ constructed in §2 satisfies equation (1.5) up to $O(q^{-j})$, the above estimate is valid for the sum of the expressions $\mathbf{J}_3^{(j)}$.

Now we consider the discrepancies related to the boundary layers. Integrating by parts, we transform the scalar products $\mathbf{K}^{(j)}$ occurring in (4.33) as follows:

$$\begin{aligned} \mathbf{K}^{(j)} = {} & q^{-j}\Big(\mathfrak{X}_j^h(A^{[0]} - A_0^{[0]})\Big(D(\nabla_x)\widetilde{\boldsymbol{\mathcal{Z}}}^{(j)}\Big)D(\partial_x)v, D(\nabla_x)\mathcal{R}^h\Big)_{\Pi_j(h)} \\ & + q^{-j}\Big(\widetilde{\boldsymbol{\mathcal{Z}}}^{(j)}(A^{[0]} - A_0^{[0]})\left(D(\nabla_x)\mathfrak{X}_j^h\right)D(\partial_x)v, D(\nabla_x)\mathcal{R}^h\Big)_{\Pi_j(h)} \\ & - q^{-j}\Big([\mathcal{L}, \mathfrak{X}_j^h]\widetilde{\boldsymbol{\mathcal{Z}}}^{(j)}D(\partial_x)v, \mathcal{R}^h\Big)_{\Pi_j(h)} \\ & - \Big(\mathcal{X}_j^h A^{[0]}\left(D(\nabla_x)w_0\right)D(\partial_x)v, D(\partial_{x_1}, 0)\mathcal{R}^h\Big)_{\Pi_j(h)} \\ & + \Big(\mathcal{X}_j^h D(0, \partial_{x_2})^\top \Big(A^{[0]}D(\nabla_x)w_0\Big)D(\partial_x)v, \mathcal{R}^h\Big)_{\Pi_j(h)} \\ & + \Big((A^{[0]} - A_0^{[0]})w_0\left(D(\nabla_x)\mathcal{X}_j^h\right)D(\partial_x)v, D(\nabla_x)\mathcal{R}^h\Big)_{\Pi_j(h)} \\ & - \Big(\left(D(\nabla_x)^\top\mathcal{X}_j^h\right)(A^{[0]} - A_0^{[0]})\left(D(\nabla_x)w_0\right)D(\partial_x)v, \mathcal{R}^h\Big)_{\Pi_j(h)} \\ & + q^{-j}\Big((A^{[0]} - A_0^{[0]})\left(D(\nabla_x)\chi_{j,+}\right)\boldsymbol{\mathcal{C}}_j^h D(\partial_x)v, \mathcal{R}^h\Big)_{\Pi_j(h)} \\ & + q^{-j}\Big(A^{[0]}D(\nabla_x)\mathfrak{X}_j^h\boldsymbol{\mathcal{C}}^{(j)}D(\partial_x)v, \mathcal{R}^h\Big)_{\Pi_j(h)}. \end{aligned}$$

The first five terms on the right will be denoted by $\mathbf{K}_1^{(j)}, \dots, \mathbf{K}_5^{(j)}$, and the sum of the remaining four terms by $\mathbf{K}_6^{(j)}$.

By Proposition 3.1, the quantity $\widetilde{\boldsymbol{\mathcal{Z}}}^{(j)}(\mathfrak{y}^{(j)})$ behaves like $O(\frac{1}{\exp(\frac{\delta_0}{h})})$ on the support of $D(\nabla_x)\mathfrak{X}_j^h(x)$, i.e.,

$$h\sum_{j=1}^{N_2-1}\Big\{\big|\mathbf{K}_2^{(j)}\big|+\big|\mathbf{K}_3^{(j)}\big|\Big\}\le c\exp\Big(-\frac{\delta_0}{h}\Big)\mathbf{N}\|D(\nabla_x)\mathcal{R};L_2(\Omega)\|,\quad \delta_0>0.$$

For any $\varepsilon>0$, for the scalar product $\mathbf{K}_1^{(j)}$ we have

$$h\big|\mathbf{K}_1^{(j)}\big|\le ch^{1/2}\|v;C(\overline{Q})\|\Big(q^{-3j/2}\varepsilon\|\widetilde{\boldsymbol{\mathcal{Z}}}^{(j)};W^1_{\beta,\mathrm{reg}}(G)\|+\varepsilon^{-1}\|D(\nabla_x)\mathcal{R};L_2(\Pi_j(h))\|\Big).$$

Consequently, by (4.9), (4.28), and (4.2),

$$h\sum_{j=1}^{N_2-1}\big|\mathbf{K}_1^{(j)}\big|\le ch^{1/2}\mathbf{N}\|D(\nabla_x)\mathcal{R}^h;L_2(\Omega(h))\|.$$

Since

$$|D(\nabla_x)w_0(\zeta)|\le c,\quad \Big|D(0,\partial_{x_2})^\top\Big(A^{[0]}(x,y_1,\zeta)D(\nabla_x)w_0(\zeta)\Big)\Big|\le c \tag{4.35}$$

for any $x\in\Pi_j(h)$, the same arguments as in the preceding case lead to the inequality

$$h\sum_{j=1}^{N_2-1}\Big\{\big|\mathbf{K}_4^{(j)}\big|+\big|\mathbf{K}_5^{(j)}\big|\Big\}\le ch^1\mathbf{N}\|D(\nabla_x)\mathcal{R}^h;L_2(\Omega(h))\|.$$

Finally, recalling that $\operatorname{supp}(\nabla_x\mathfrak{X}_j^h)\subset\sigma_j^h$, and applying (4.34) and (4.35), we finish the treatment of expression (4.33) with the estimate

$$h\sum_{j=1}^{N_2-1}\big|\mathbf{K}_6^{(j)}\big|\le ch^{1/2}\mathbf{N}\|D(\nabla_x)\mathcal{R}^h;L_2(\Omega(h))\|.$$

5. Statement of the main theorems on asymptotics. In the above subsections we estimated all the expressions that form the left-hand side of (4.17). As a result, (4.17) takes the form

$$\|\mathcal{R}^h;H^1(\Omega(h))\|^2\le c\|F;H^\varkappa(Q\to L_\infty(S\times(0,b_2)))\|\,\|\mathcal{R}^h;H^1(\Omega(h))\|.$$

Thus, the following statement is true.

Theorem 4.1. *Suppose that the matrix D, the matrix function A, and the vector-valued function f^h satisfy the conditions indicated above; in particular, for the right-hand side f^h of the system* (1.5)–(1.7) *we have a representation of the form* (2.23) *in which $F\in H^\varkappa(Q\to L_\infty(S\times(0,b_2)))$ and the smoothness exponent $\varkappa$ belongs to the interval* $(0,1/2]$. *Then*

$$\begin{aligned}\|u^h-\mathcal{U}^h;L_2(\Omega(h))\|&+\|\nabla_x\{u^h-\mathcal{U}^h\};L_2(\Omega(h))\|\\&\le c_\varkappa h^\varkappa\|F;H^\varkappa(Q\to L_\infty(S\times(0,b_2)))\|,\end{aligned} \tag{4.36}$$

where $c_\varkappa$ is a constant independent of F and of the parameter $h\in(0,1]$.

The global asymptotic approximation (4.13) involves terms introduced for technical purposes only. The cut-off functions and boundary layer type solutions are used for ensuring a sufficient smoothness of our asymptotic approximation and for the substitution of it in the integral identity that serves problem (1.5)–(1.7). However, in the final asymptotic formulas for the solution $u^h\in\mathring{H}^1_{\mathrm{reg}}(\Omega)^n$ and for its gradient, the "extra" terms can be lifted. Calculating the norm of each summand in (4.13)–(4.14) directly, and including the irrelevant terms in the remainder, we arrive at considerably simpler asymptotic formulas.

Theorem 4.2. *Under the same conditions and in the same notation as above, we have*

$$\begin{aligned}\|u^h - v; L_2(\Omega(h))\| + \|\nabla_x u^h - \nabla_x v - \mathfrak{w}^{[0]} D(\nabla_x) v; L_2(\Omega(h))\| \\ \le c_\varkappa h^\varkappa \|F; H^\varkappa(Q \to L_\infty(S \times (0, b_2)))\|,\end{aligned} \tag{4.37}$$

where $\mathfrak{w} = (\partial_y, 0)W + (0, \partial_\zeta)w$, *and* $c_\varkappa$ *is a constant independent of* F *and of the parameter* h.

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