Алгебра и анализ
Том. 17 (2005), вып. 6

St. Petersburg Math. J.
Vol. 17 (2006), No. 6, Pages 1015–1030
S 1061-0022(06)00938-1
Article electronically published on September 20, 2006

ANTIMONOTONE QUADRATIC FORMS AND PARTIALLY ORDERED SETS

L. A. NAZAROVA, A. V. ROĬTER, AND M. N. SMIRNOVA

Abstract. Representations of partially ordered sets (posets) and quivers are an important part of the theory of matrix problems and algebra representations. Along with chains (linearly ordered sets), a special role is played by certain special posets; in this paper it is shown that they are in one-to-one correspondence with the rational numbers that are greater than or equal to 1.

A wattle $\langle n_1, \dots, n_t\rangle$ is a union of nonintersecting chains Z_i ($|Z_i| = n_i$) such that the minimal element of Z_i is smaller than the maximal element of Z_{i+1} ($i = 1, \dots, t-1$) (and these are the only possible comparisons). The known lists of critical (i.e., minimal) infinitely representable and wild posets consist of cardinal chains, with the exception of one poset in the first list (namely, $\langle 2,2\rangle + Z_4$) and one in the second (namely, $\langle 2,2\rangle + Z_5$). At the same time, the authors have assigned a rational number $P(S)$ to each poset S in such a way that $P(S) < 4$ if and only if S is finitely representable and $P(S) = 4$ if and only if S is tame. A poset S is said to be *P-faithful* if $P(S') < P(S)$ whenever $S' \subset S$.

From the work of Zel′dich, Sapelkin, and the authors it follows that the P-faithful posets are cardinal sums of r-sets, i.e., they are wattles of a special type (chains can be regarded as a partial case of r-sets).

In the present paper, the notion of an antimonotone poset is introduced, and a criterion for a poset to be antimonotone is presented under the assumption that the quadratic form $\sum_{s_i\le s_j} x_i x_j$ ($S = \{s_1, \dots, s_n\}$) is positive semidefinite. At the same time, we manage to substantially simplify the proof of the criterion for a poset to be P-faithful, avoiding an item-by-item examination of several dozens of various cases. Also, simple explicit formulas for calculation of $P(S)$ are obtained, which lead in an elementary way to the lists of critical posets (originally, they arose as a result of a cumbersome and complex argument).

Let P be a bounded set in the n-dimensional space $\mathbb{R}_n$, and let $f(x_1, \dots, x_n) = f(x)$ ($x \in \mathbb{R}_n$) be a continuous function. By the well-known second Weierstrass theorem, $\inf\{f(\overline{P})\}$ ($= \inf_{\overline{P}} f(x)$) is attained. We say that a function f is *P-faithful* if $\inf\{f(\overline{P})\}$ is not attained on $\overline{P} \setminus P$ and $\inf\{f(\overline{P})\} > 0$ (i.e., f is positive on $\overline{P}$). Observe that if $n = 1$ and $P = (a, b)$, then any P-faithful function is not monotone.

In what follows we assume that $P = P_n = \{(x_1, \dots, x_n) \mid 0 < x_i \le 1,\ i = 1, \dots, n,\ x_1 + \dots + x_n = 1\}$. If $n > 1$, then $x_i < 1$, $i = 1, \dots, n$. Then $\overline{P} = \{(x_1, \dots, x_n) \mid 0 \le x_i \le 1,\ i = 1, \dots, n,\ x_1 + \dots + x_n = 1\}$. In this case the P-faithfulness of f depends substantially on the behavior of f on the hyperplane $H_n = \{(x_1, \dots, x_n) \mid x_1 + \dots + x_n = 0\}$.

For a differentiable function f we put $C^-(f) = \{h \in H_n \setminus \{0\} \mid \frac{\partial f}{\partial x_i}(h) \le 0,\ i = 1, \dots, n\}$, $C^+(f) = \{h \in H_n \setminus \{0\} \mid \frac{\partial f}{\partial x_i}(h) \ge 0,\ i = 1, \dots, n\}$, $C(f) = C^+(f) \cup C^-(f)$. A function f is said to be *antimonotone* if $C(f) = \varnothing$. If $n = 1$, then $P_1 = (1)$, $H_1 \setminus \{0\} = \varnothing$, and any function is antimonotone.

2000 *Mathematics Subject Classification.* Primary 06-99.

Key words and phrases. Quiver, graph, wattle, faithful poset, antimonotone poset.

In Subsection 3 (see Proposition 1) we prove that any P-faithful quadratic form is antimonotone; therefore, in this case antimonotonicity is a generalization of P-faithfulness.

Example 1. A linear function $f=\sum_{i=1}^n a_i x_i$, $n>1$, is antimonotone if and only if $a_i>0$, $a_j<0$ for some i,j. The quadratic forms $x_1^2+x_2^2$ and $x_1^2+x_2^2+x_1x_2$ are antimonotone, but the forms $x_1^2-x_2^2$ and $x_1^2+x_2^2+x_3^2+x_1x_2+x_1x_3$ are not.

Apparently, the problem of obtaining an efficient criterion for antimonotonicity is hard even for quadratic forms.

In this paper we solve this problem for the quadratic forms f_S corresponding to (finite) partially ordered sets (*posets*) $S=\{s_1,\dots,s_n\}$: $f_S(x_1,\dots,x_n)=\sum_{s_i\le s_j} x_i x_j$ (see [2]) under the additional requirement that f_S be positive semidefinite (i.e., $f_S(x)\ge 0$). The posets with antimonotone form generalize the P-faithful posets, defined in [3] and studied in [3]–[7], and (as is shown below) coincide with them not only for positive definite forms, but also for positive semidefinite ones.

An explicit construction of a vector belonging to $C(f_S)$ allows us to simplify the proof of the P-faithfulness criterion (see [3] and [5]–[7]), avoiding consideration of many different cases.

We also deduce an explicit formula for the calculation of $\inf\{f_S(\overline{P})\}$ for P-faithful S; on the basis of this formula, we give simple proofs of the criteria for finite representativity (see [8] and also [9]) and tameness (see [10] and also [11]) of partially ordered sets.

1. In this subsection, f is a differentiable function defined on $\mathbb{R}_n$. The elements of $\mathbb{R}_n$ will be called vectors.

We put $\mathbb{R}_n^+=\{x\in\mathbb{R}_n \mid x_i>0,\ i=1,\dots,n\}$, $\overline{\mathbb{R}}_n^+=\{x=(x_1,\dots,x_n)\in\mathbb{R}_n \mid 0\le x_i, i=1,\dots,n; x\ne 0\}$; then $P_n=\overline{P}_n\cap\mathbb{R}_n^+$ ($\mathbb{R}^+=\mathbb{R}_1^+$).

If f_1 and f_2 are defined on $\mathbb{R}_m$ and on $\mathbb{R}_n$ (respectively), we put $(f_1\oplus f_2)(x_1,\dots,x_m, x_{m+1},\dots,x_{n+m})=f_1(x_1,\dots,x_m)+f_2(x_{m+1},\dots,x_{m+n})$.

We say that a twice differentiable function f is *concave* if

a) $\frac{\partial f}{\partial x_i}(0)=0$, $i=1,\dots,n$, and

b) $\frac{\partial^2 f}{\partial x_i\partial x_j}\ge 0$, $i,j=1,\dots,n$, and f is *q-concave*, $q\in\mathbb{R}^+$, if, in addition,

c) $\frac{\partial^2 f}{\partial x_i^2}\ge q$, $i=1,\dots,n$.

In particular, the quadratic form f_S corresponding to S is 2-concave.

Remark 1. By the Lagrange theorem, for $d\ge 0$, b) implies I), and c) implies II$_q$):

I) $\frac{\partial f}{\partial x_i}(x_1,\dots,x_{j-1},x_j+d,x_{j+1},\dots,x_n)\ge\frac{\partial f}{\partial x_i}(x_1,\dots,x_n)$, $i,j\in\{1,\dots,n\}$.

II$_q$) $\frac{\partial f}{\partial x_i}(x_1,\dots,x_{i-1},x_i+d,x_{i+1},\dots,x_n)\ge\frac{\partial f}{\partial x_i}(x_1,\dots,x_n)+qd$, $i=1,\dots,n$.

We put $\widehat{C}^-(f)=\left\{x\in\mathbb{R}_n\setminus\{0\} \mid \sum_{i=1}^n x_i\ge 0, \frac{\partial f}{\partial x_i}(x)\le 0, i=1,\dots,n\right\}$ and $\widehat{C}^+(f)=\left\{x\in\mathbb{R}_n\setminus\{0\} \mid \sum_{i=1}^n x_i\le 0,\ \frac{\partial f}{\partial x_i}(x)\ge 0,\ i=1,\dots,n\right\}$.

Lemma 1. *If f is concave, then f is antimonotone if and only if $\widehat{C}^+(f)\cup\widehat{C}^-(f)=\varnothing$.*

Proof. Let $x\in\widehat{C}^-(f)$ (the case of $x\in\widehat{C}^+(f)$ is similar), and let $\sum_{i=1}^d x_i=d\in\mathbb{R}^+$. Then $\{x_1-d,x_2,\dots,x_n\}\in C(f)$ (by I)) unless $x=(d,0,\dots,0)$. But in the latter case we have $(d,-d,0,\dots,0)\in C(f)$. If $y\in C(f)$, then, clearly, $y\in\widehat{C}^+(f)\cup\widehat{C}^-(f)$. □

Lemma 2. *If f_1 and f_2 are concave, then the function $f_1\oplus f_2$ is antimonotone if and only if f_1 and f_2 are.*

Proof. We must prove that $C(f_1\oplus f_2)\ne\varnothing$ if and only if either $C(f_1)\ne\varnothing$ or $C(f_2)\ne\varnothing$. If $(x_1,\dots,x_{n_1},y_1,\dots,y_{n_2})\in C(f_1\oplus f_2)$, then either $(x_1,\dots,x_{n_1})\in\widehat{C}^+(f_1)\cup\widehat{C}^-(f_1)$

or $(y_1,\dots,y_{n_2}) \in \widehat{C}^+(f_2) \cup \widehat{C}^-(f_2)$, and by Lemma 1, $C(f_1) \neq \varnothing$ in the first case and $C(f_2) \neq \varnothing$ in the second case.

If $x = (x_1,\dots,x_{n_1}) \in C(f_1)$, then $(x_1,\dots,x_{n_1},\underbrace{0,\dots,0}_{n_2}) \in C(f_1 \oplus f_2)$ by a); if $(y_1,\dots,y_{n_2}) \in C(f_2)$, then $(\underbrace{0,\dots,0}_{n_1},y_1,\dots,y_{n_2}) \in C(f_1 \oplus f_2)$. □

We say that a nonzero vector $d \in \mathbb{Z}_n$ is *m-Dynkin* ($1 \le m \le n$) for a q-concave function f if 1) $0 \le \frac{\partial f}{\partial x_m}(d) \le q$, and 2) $\frac{\partial f}{\partial x_j}(d) = 0$ for $j \neq m$, $j = 1,\dots,n$.

We say that a function f is *m-isolated* if $\frac{\partial f}{\partial x_k}(s_m) = 0$ for $1 \le k \le n$, $k \neq m$, $s_m = (\underbrace{0,\dots,0}_{m-1},1,0,\dots,0)$.

Lemma 3. *Let f be q-concave and not m-isolated. If it admits an m-Dynkin vector, then $C(f) \neq \varnothing$.*

Proof. Let $\sum_{i=1}^n d_i = \overline{d}$. If $\overline{d} \le 0$, then $d \in \widehat{C}^+(f)$ and $C(f) \neq \varnothing$ by Lemma 1. Let $\overline{d} > 0$. Putting $u_j = d_j$ if $j \neq m$ and $u_m = d_m - \overline{d}$, we prove that $u = (u_1,\dots,u_n) \in C(f)$. Clearly, $u \in H_n$. We have $\frac{\partial f}{\partial x_j}(u) \le 0$ for $j \neq m$ by I) and 2), and $\frac{\partial f}{\partial x_m}(u) \le 0$ by II_q) and 1).

It remains to show that $u \neq 0$. If $u = 0$, then $d = \lambda s_m$, $\lambda \neq 0$ (because $d \neq 0$), which implies that $\frac{\partial f}{\partial x_k}(d) \neq 0$, $k \neq m$, because f is not m-isolated. □

Example 2. Let $S = \{s_1,s_2,s_3,s_4,s_5 \mid s_1 < s_i, i = 2,\dots,5\}$; then $f_S = \sum_{i=1}^5 x_i^2 + x_1\sum_{j=2}^5 x_j$, and $d = (-2,1,1,1,1)$ is an i-Dynkin vector for f_S ($i = 1,\dots,5$). The vectors $(-2,1,1,1,-1)$, $(-2,1,1,-1,1)$, $(-2,1,-1,1,1)$, $(-2,-1,1,1,1)$, and $(-4,1,1,1,1)$ belong to $C(f_S)$.

We turn to P-faithfulness. We denote $\mathrm{St}(f) = \{a \in \mathbb{R}_n^+ \mid \frac{\partial f}{\partial x_i}(a) = \frac{\partial f}{\partial x_j}(a),\ i,j = 1,\dots,n\}$; $\mathrm{St}^+(f) = \{a \in \mathrm{St} \mid \frac{\partial f}{\partial x_i}(a) > 0\}$.

We say that a vector $u \in P_n$ is *P-faithful* for f if $f(u) > 0$ and $w \in \overline{P}_n$ implies $f(u) \le f(w)$; moreover, if $w \notin P_n$, then $f(u) < f(w)$.

Let $\widetilde{\mathrm{St}}(f)$ denote the set of P-faithful vectors for f. The P-faithfulness of f is equivalent to the fact that $\widetilde{\mathrm{St}}(f) \neq 0$.

Lemma 4. $\widetilde{\mathrm{St}}(f) \subseteq \mathrm{St}(f)$ *for any f.*

Proof. Let $n > 1$. We write $x_n = 1 - \sum_{i=1}^{n-1} x_i$ and consider the function $\widehat{f}(x_1,\dots,x_{n-1}) = f(x_1,\dots,x_{n-1},1-\sum_{i=1}^{n-1} x_i)$. If $u = (u_1,\dots,u_n)$ is a P-faithful vector for f, then $\widehat{f}(\widehat{u})$ attains its minimum at the point $\widehat{u} = (u_1,\dots,u_{n-1})$. We have $\frac{\partial \hat{f}}{\partial x_i} = \frac{\partial f}{\partial x_i} + \frac{\partial f}{\partial x_n} \cdot \frac{\partial x_n}{\partial x_i}$, $x_n = 1 - \sum_{i=1}^{n-1} x_i$, $\frac{\partial x_n}{\partial x_i} = -1$ ($i = 1,\dots,n-1$). Therefore, $\frac{\partial \hat{f}}{\partial x_i} = \frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_n} = 0$. □

Let f be a homogeneous function of degree k (i.e., $f(\lambda x_1,\dots,\lambda x_n) = \lambda^k f(x_1,\dots,x_n)$). For such f, if $k \neq 1$ and $\inf\{f(P)\} > 0$, we put $P(f) = \inf\{f(P)\}^{\frac{1}{1-k}}$. In particular, $P(f) = \inf\{f(\overline{P})\}^{-1}$ for $k = 2$.

Lemma 5. *Suppose $f_1(x_1,\dots,x_{n_1})$ and $f_2(x_{n_1+1},\dots,x_{n_2})$ are two homogeneous functions of degree k, $n_1 + n_2 = n$, $\inf\{f_j(P_{n_j})\} > 0$, $j = 1,2$. Then $P(f_1 \oplus f_2) = P(f_1) + P(f_2)$.*

Proof. The values of a homogeneous function f on $\overline{\mathbb{R}}_n^+$ are determined by its values on $\overline{P}_n$, namely, for $y \in \overline{\mathbb{R}}_n^+$ we have $f(y) = \lambda^k f(u)$, where $u \in \overline{P}_n$, $\lambda = \sum_{i=1}^n y_i$, $u = \lambda^{-1} y$. Therefore,

$$\inf\{(f_1 \oplus f_2)(\overline{P}_n)\} = \inf_{0 \le \lambda \le 1} \Big[\lambda^k \inf\{f_1(\overline{P}_{n_1})\} + (1-\lambda)^k \inf\{f_2(\overline{P}_{n_2})\}\Big].$$

We put $\inf\{f_1(\overline{P}_{n_1})\} = a$, $\inf\{f_2(\overline{P}_{n_2})\} = b$.

Consider the function $\Phi_{ab}(\lambda) = a\lambda^k + b(1-\lambda)^k$, $a > 0$, $b > 0$; we find $\inf_{0\le\lambda\le1} \Phi_{ab}(\lambda)$. The derivative of $\Phi_{ab}(\lambda)$ with respect to λ (u and v are viewed as constants) is $(\Phi_{ab}(\lambda))'_\lambda = ka\lambda^{k-1} - kb(1-\lambda)^{k-1}$. Let $\overline{\lambda}$ be a positive root of the equation $(\Phi_{ab}(\lambda))'_\lambda = 0$. Then $a\overline{\lambda}^{k-1} = b(1-\overline{\lambda})^{k-1}$, whence $a\frac{1}{k-1}\overline{\lambda} = b^{\frac{1}{k-1}}(1-\overline{\lambda})$ and $\overline{\lambda} = \frac{b^{\frac{1}{k-1}}}{a^{\frac{1}{k-1}} + b^{\frac{1}{k-1}}}$. Thus, $\inf_{0\le\lambda\le1} \Phi_{ab}(\lambda) = \min\{\Phi_{ab}(0), \Phi_{ab}(1), \Phi_{ab}(\overline{\lambda})\} = \min\{a, b, \Phi_{ab}(\overline{\lambda})\}$.

We show that $\Phi_{ab}(\overline{\lambda}) < \Phi_{ab}(0) = b$. Indeed, $a\overline{\lambda}^{k-1} = b(1-\overline{\lambda})^{k-1}$ and $\Phi_{ab}(\overline{\lambda}) = a\overline{\lambda}^k + b(1-\overline{\lambda})^k = b(1-\overline{\lambda})^{k-1}\overline{\lambda} + b(1-\overline{\lambda})^k = b(1-\overline{\lambda})^{k-1} < b$, because $a > 0$, $b > 0$, and $0 < \overline{\lambda} < 1$; similarly, $\Phi_{ab}(\overline{\lambda}) = a\overline{\lambda}^{k-1} < \Phi_{ab}(1) = a$. Therefore, $\inf\{(f_1 \oplus f_2)(\overline{P}_n)\} = \Phi_{ab}(\overline{\lambda}) = \frac{ab}{\left(a^{\frac{1}{k-1}} + b^{\frac{1}{k-1}}\right)^{k-1}}$.

Returning to $P(f_1 \oplus f_2)$, $P(f_1)$, and $P(f_2)$, we have $P(f_1) = a^{\frac{1}{1-k}}$, $P(f_2) = b^{\frac{1}{1-k}}$, $P(f_1 \oplus f_2) = \Big(\frac{ab}{\left(a^{\frac{1}{k-1}} + b^{\frac{1}{k-1}}\right)^{k-1}}\Big)^{\frac{1}{1-k}} = b^{\frac{1}{1-k}} + a^{\frac{1}{1-k}}$. □

Corollary 1. *Under the conditions of Lemma 5, $f_1 \oplus f_2$ is P-faithful if and only if f_1 and f_2 are.*

2. In what follows, $f = \sum_{i,j=1}^n a_{ij} x_i x_j$ $(a_{ij} = a_{ji})$ is a quadratic form over the field $\mathbb{R}$; $A = (a_{ij})$ is the symmetric matrix of f. We have $\frac{\partial^2 f}{\partial x_i \partial x_j} = 2a_{ij}$, and f is 2-concave if $a_{ii} \in \mathbb{N}$, $a_{ij} + a_{ji} \in \mathbb{N}_0$ $(i, j = 1, \dots, n)$. Fixing f, for $x = (x_1, \dots, x_n) \in \mathbb{R}_n$ we put

$$x'_i = \frac{\partial f}{\partial x_i}(x_1, \dots, x_n) = 2\sum_{j=1}^n a_{ji} x_j, \quad x' = (x'_1, \dots, x'_n) = 2xA.$$

We need the following identity, which can easily be checked:

$$f(u+v) = f(u) + f(v) + \sum_{i=1}^n u'_i v_i, \quad u, v \in \mathbb{R}_n, \tag{1}$$

whence

$$\sum_{i=1}^n u'_i v_i = \sum_{i=1}^n v'_i u_i, \quad f(u + \varepsilon v) = f(u) + \varepsilon^2 f(v) + \varepsilon \sum_{i=1}^n u_i v'_i, \ \ \varepsilon \in \mathbb{R}. \tag{2}$$

Putting $u = v$ in (1), we obtain (cf. [1, Subsection 178])

$$f(u) = \frac{1}{2}\sum_{i=1}^n u_i u'_i. \tag{3}$$

Using (3), we can reformulate Lemma 4 as follows.

Lemma 4′. *For any quadratic form f, we have $\widetilde{\mathrm{St}}(f) \subset \mathrm{St}^+(f)$.*

We denote $\widetilde{C}(f) = \{(v_1, \dots, v_n) \in C(f) \mid (v'_1, \dots, v'_n) \ne 0\}$. Since $\frac{\partial f}{\partial x_i}(-x) = -\frac{\partial f}{\partial x_i}(x)$, the relation $C(f) \ne \varnothing$ implies that $C^-(f) \ne \varnothing$ and $C^+(f) \ne \varnothing$. Therefore, choosing a vector $v \in C(f) \ne \varnothing$, in the sequel we assume that $v \in C^-(f)$.

Proposition 1. *For any quadratic form f, 1) at least one of the sets* $\mathrm{St}(f)$ *and* $\widetilde{C}(f)$ *is empty, and* 2) *at least one of the sets* $C(f)$ *and* $\widetilde{\mathrm{St}}(f)$ *is empty.*

Proof. 1) Suppose $u \in \mathrm{St}(f)$, $v \in \widetilde{C}(f)$. Then $\sum_{i=1}^n u_i' v_i = u_1' \sum_{i=1}^n v_i = 0$. On the other hand, if $v_j' < 0$, then $\sum_{i=1}^n u_i v_i' < 0$ (because $v_i' \le 0$, $u_i > 0$, $i = 1, \dots, n$), which contradicts (2).

2) Suppose $u \in \widetilde{\mathrm{St}}(f)$, $v \in C(f)$. If $v \in \widetilde{C}(f)$, then 1) implies the claim. Let $v \in C(f) \setminus \widetilde{C}(f)$, i.e., $v_i' = 0$ for $i = 1, \dots, n$. Then $f(v) = 0$ (e.g., by (3)), whence $f(u + \varepsilon v) = f(u)$ for any ε. We put $|\varepsilon| = \min_i \frac{u_i}{|v_i|}$ and take the sign of ε to be opposite to the sign of one of the v_i at which the minimum is attained. Then $u + \varepsilon v \in \overline{P}_n \setminus P_n$, which contradicts the P-faithfulness of u. □

Corollary 2. *Any P-faithful quadratic form is antimonotone.*

Example 3. Let $f = \sum_{i=1}^4 x_i^2 + (x_1 + x_2)(x_3 + x_4)$. Then

$$A = \begin{pmatrix} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 1 \end{pmatrix}, \quad \mathrm{St}(f) \ni (1,1,1,1), \;\; C(f) \ni (1,1,-1,-1).$$

Proposition 1 implies that $\widetilde{\mathrm{St}}(f) = \varnothing$ and $\widetilde{C}(f) = \varnothing$.

In this example, $|A| = 0$.

Proposition 2. *If $|A| \ne 0$, then one of the sets $C(f)$ and $\mathrm{St}(f)$ is not empty, but the other is empty.*

Proof. First, suppose that $\varnothing \ne C(f) \ni v$ and $\varnothing \ne \mathrm{St}(f) \ni u$. If $v \in \widetilde{C}(f)$, then $\mathrm{St}(f) = \varnothing$ by Proposition 1. If $v \in C(f) \setminus \widetilde{C}(f)$, then $v' = 0$ and $vA = \frac{1}{2}v' = 0$. Therefore, $v = 0$, which contradicts the fact that $v \in H_n \setminus \{0\}$ (see the definition of $C(f)$).

Now we prove that either $\mathrm{St}(f) \ne \varnothing$ or $C(f) \ne \varnothing$. Let $e_n = (1, \dots, 1) \in \mathbb{R}_n$, and let $y = e_n A^{-1}$, $yA = e_n$. If $y \in \mathbb{R}_n^+$ or $-y \in \mathbb{R}_n^+$, then $y \in \mathrm{St}(f)$. If $\{y, -y\} \cap \mathbb{R}_n^+ = \varnothing$, then either $y_k = 0$ for some k or $y_s < 0$ and $y_t > 0$ for some s and t. It is easily seen that in both cases there exists $w \in \overline{\mathbb{R}}_n^+$ such that $wy^T \left(= \sum_{i=1}^n w_i y_i\right) = 0$ (in the first case we can put $w_k > 0$, $w_i = 0$ for $i \ne k$, and in the second case $w_s = y_t$, $w_t = -y_s$, $w_i = 0$ for $i \notin \{s, t\}$, $i = 1, \dots, n$). We prove that $v = -wA^{-1} \in C(f)$. We have $-v' = wA^{-1}A = w \in \overline{\mathbb{R}}_n^+$, whence $v_i' \le 0$. Next, $v \ne 0$ because $w \ne 0$ and $|A| \ne 0$. It remains to check that $v \in H_n$, which is equivalent to $ve_n^T = 0$. We have $ve_n^T = -wA^{-1}e_n^T$ and $y^T = (A^{-1})^T e_n^T = A^{-1}e_n^T$ (because $A^T = A$). Therefore, $-wA^{-1}e_n^T = -wy^T = 0$. □

Proposition 3 (see [7, Part II, Remark to Theorem 1])**.** 1) *If* $\widetilde{\mathrm{St}}(f) \ne \varnothing$, *then f is positive definite.* 2) *If f is positive definite, then* $\widetilde{\mathrm{St}}(f) = \mathrm{St}(f) \cap P_n$ (*thus,* $\widetilde{\mathrm{St}}(f) = \varnothing$ *if and only if* $\mathrm{St}(f) = \varnothing$).

Proof. 1) Suppose the contrary: $f(v) \le 0$ ($v \ne 0$), $u \in \widetilde{\mathrm{St}}(f)$. a) First, we assume that $v \in H_n$, i.e., $\sum_{i=1}^n v_i = 0$ and $f(v) < 0$. Then $f(u + \varepsilon v) = f(u) + \varepsilon^2 f(v) + \varepsilon \sum_{i=1}^n u_i' v_i$. We have $u \in \mathrm{St}(f)$ by Lemma 4′, whence $\varepsilon \sum_{i=1}^n u_i' v_i = 0$, i.e., $f(u + \varepsilon v) = f(u) + \varepsilon^2 f(v)$. Since $f(v) < 0$, it follows that $f(u + \varepsilon v) < f(u)$, which contradicts the P-faithfulness of u.

b) Now, let $v \in H_n$, $f(v) = 0$. Then $f(u + \varepsilon v) = f(u)$ for any ε. Put $\varepsilon = \min_i \frac{u_i}{|v_i|}$. The sign of ε is opposite to the sign of one of the v_i for which this minimum is attained. Then $u + \varepsilon v \in \overline{P}_n \setminus P_n$, again contradicting the P-faithfulness of u.

c) Finally, let $\sum_{i=1}^n v_i \neq 0$. We may assume that $\sum_{i=1}^n v_i = 1$. Put $w = u - v$. Formula (2) with $\varepsilon = -1$ and Lemma 4 imply $f(w) = f(u) + f(v) - u'$, $u' = u'_i$, $i = 1, \dots, n$. Formula (3) yields $f(u) = \frac{u'_1}{2}$, whence $f(w) = f(v) - \frac{u'_1}{2}$. Since $u' > 0$ by Lemma 4$'$, we have $f(w) < 0$; $w \in H_n$. Also, $w \neq 0$, because if $w = 0$, then $u = v$ by (3), but $f(v) \leq 0$. Thus, c) reduces to a).

2) Suppose $u \in \mathrm{St}(f) \cap P_n$, $v \in \overline{P}_n$, $v \neq u$. Then $u \neq 0$ because $u \in \mathrm{St}(f)$, so that $f(u) > 0$. We show that $f(u) < f(v)$. We have $0 < f(u-v) \overset{(2)}{=} f(u) + f(v) - \sum_{i=1}^n u'_i v_i \overset{(3)}{=} \frac{u'}{2} + f(v) - u' = f(v) - \frac{u'}{2} = f(v) - f(u)$, i.e., $f(v) > f(u)$. □

3. In the sequel we shall consider the 2-concave form f_S for a poset $S = \{s_1, \dots, s_n\}$, $f_S = \sum_{s_i \leq s_j} x_i x_j$. Put $C(S) = C(f_S)$, $\mathrm{St}(S) = \mathrm{St}(f_S)$, and $\widetilde{\mathrm{St}}(S) = \widetilde{\mathrm{St}}(f_S)$. The poset S is *antimonotone* if f_S is antimonotone.

A poset S is *P-faithful* if $\widetilde{\mathrm{St}}(f_S) \neq \varnothing$. (This is equivalent to the definition of P-faithfulness given in [3].) In this case, $C(S) = \varnothing$ by Proposition 1. Observe that $\inf\{f_S(\overline{P})\} > 0$ because $a_{ij} \geq 0$, $i, j = 1, \dots, n$, $A \neq (0)$.

When talking of graphs, we always mean nonoriented graphs. Oriented graphs will be called quivers. All graphs and quivers are assumed to be finite and not involving loops and multiple edges or arrows (i.e., two edges or arrows between two given points). Every quiver Q gives rise to the graph $\Gamma(Q)$ in which all arrows are replaced by edges.

The Hasse quiver (orgraph) $Q(S)$ of a poset S is a quiver whose vertices are elements of S and two vertices are connected by an arrow $s_i \to s_j$ if $s_i < s_j$ and no $s_k \in S$ satisfies $s_i < s_k < s_j$. Drawing lines (edges) instead of arrows, we obtain the (nonoriented) Hasse graph $\Gamma(S)$ of the partially ordered set S. Usually, a finite poset S is depicted by a diagram, i.e., by the graph $\Gamma(S)$, assuming that lesser elements are drawn below the greater ones.

The elements of the poset S and the corresponding elements of $Q(S)$ and $\Gamma(S)$ will be denoted by the same symbols.

A *path of length* k ($k \geq 1$) from s_1 to s_{k+1} in a graph (quiver) is a sequence $s_1, \dots, s_{k+1}$ of vertices such that s_i and s_{i+1} are joined by an edge (by an arrow starting at s_i and terminating at s_{i+1}), $i = 1, \dots, k$. A path in a quiver Q is a path in the graph $\Gamma(Q)$, but the converse may fail; s_1 is the origin and s_{k+1} is the end of a path.

A path $s_1, \dots, s_{k+1}$ in a graph Γ is called a *cycle* if the s_i are different for $i = 1, \dots, k$, $k > 2$, and $s_1 = s_{k+1}$. A cycle is said to be *simple* (and is denoted by $\tilde{A}_k$) if there are no other edges joining $s_k, \dots, s_{k+1}$. A graph Γ and a poset S with $\Gamma(s) = \Gamma$ are said to by *cyclic* if Γ involves a cycle, and *acyclic* otherwise. It is easily seen that a cyclic graph involves a simple cycle; accordingly, a cyclic poset S includes a subset S' with $\Gamma(S') = \tilde{A}_m$.

To a quiver Q with vertices $s_1, \dots, s_n$, we assign an $(n \times n)$-matrix $\tilde{Q}$ such that Q_{ij} is the number of arrows (0 or 1) from s_i to s_j. Then $(\tilde{Q}^t)_{ij}$ is the number of paths of length t from s_i to s_j. A path $s_1, \dots, s_{k+1}$ in a quiver is called an *oriented cycle* if $s_1 = s_{k+1}$. Two paths $s_1, \dots, s_{k+1}$ and $t_1, \dots, t_{k+1}$ in a quiver Q are said to be *parallel* [12] if $s_1 = t_1$ and $s_{k+1} = t_{k+1}$. If a quiver involves an oriented cycle, it also involves parallel paths. The quiver $Q(S)$ has no oriented cycles (because the relation $\leq$ is antisymmetric), but it may have parallel paths. It is easily seen that if the graph $\Gamma(Q)$ is acyclic, then Q has no parallel paths. If Q has no oriented cycles, then the length of any path does not exceed n and $\tilde{Q}^n = 0$. If, moreover, there are no parallel paths, then the entries of $\tilde{Q}^t$ are equal to 0 or 1. Moreover, if $Q = Q(s)$, then the matrix A of the quadratic from f_s is given by

$$A = E + \frac{1}{2}\Big(\sum_{i=1}^{n=1} \tilde{Q}^i + \sum_{i=1}^{n-1} (\tilde{Q}^T)^i\Big).$$

Let $\mathcal{T}_S(x_1,\dots,x_n)=\sum_{i=1}^n x_i^2-\sum_{s_i-s_j}x_ix_j$ be the Tits quadratic form of the graph $\Gamma(S)$ (the second sum is taken over all edges of the graph $\Gamma(S)$). The matrix of the form $\mathcal{T}_S$ is denoted either by $\mathcal{A}$ or by $\mathcal{A}(S)$.

It is well known that the Tits form of Γ is positive definite (respectively, positive semidefinite) if Γ is a Dynkin graph (respectively, extended graph), i.e. A_n, D_n, E_6, E_7, E_8 (respectively, $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$; see Subsection 4).

We put $E-\tilde{Q}(s)=\hat{Q}$, $|\hat{Q}|=1$. It is easily seen that $\hat{Q}^{-1}=(E+\sum_{i=1}^{n-1}\hat{Q}^i)$, $A=\frac{1}{2}(\hat{Q}^{-1}+(\hat{Q}^{-1})^T)$, $\mathcal{A}=\frac{1}{2}(\hat{Q}+\hat{Q}^T)$.

Proposition 4 (See [4])[1]**.** *If there are no parallel paths in $Q(S)$, then the forms $\mathcal{T}_S$ and f_S are equivalent over $\mathbb{Z}$.*

Proof. Indeed, $\hat{Q}^{-1}\mathcal{A}_T(\hat{Q}^{-1})^T=\frac{1}{2}\hat{Q}^{-1}(\hat{Q}+\hat{Q}^T)(\hat{Q}^{-1})^T=\frac{1}{2}[(\hat{Q}^{-1})^T+\hat{Q}^{-1}]=A$. □

Propositions 1–4 imply the following statement.

Corollary 3. *Suppose $\Gamma(S)$ is an acyclic graph and at least one of the forms f_S and $\mathcal{T}_S$ is positive definite* (*this is true if $\Gamma(S)$ is a Dynkin graph, see Subsection* 4)*. Then the other form is also positive definite, and the following statements are equivalent*:

a) *S is antimonotone*;

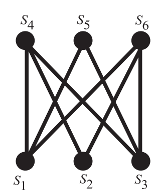
b) *S is P-faithful*;

c) $\mathrm{St}(S)\neq\varnothing$.

For $s_i\in S$, we denote by $I(s_i)$ the number of edges of the graph $\Gamma(S)$ that end at s_i.

We call s_i a *terminal point* if $I(s_i)\le 1$; s_i is a *branch point* if $I(s_i)\ge 3$; s_i is *a junction point* if it is either the end of at least two arrows or the origin of at least two arrows of the quiver $Q(S)$. We denote by $S^\times$ the set of junction points.

Example 4.

$S=$ 

Here $S^\times=S$ and $C(S)\cup\widetilde{\mathrm{St}}(S)=\varnothing$. Indeed, since the values of f_S can be negative ($f_S(1,1,1,-1,\ -1,-1)=-2$), we have $\widetilde{\mathrm{St}}(S)=\varnothing$ by Proposition 3, but $\mathrm{St}(S)\ni(1,2,1,1,2,1)$, $|A|=-48$. Proposition 2 implies $C(S)=\varnothing$.

Throughout in what follows (except in the Appendix) we assume that the graph $\Gamma(S)$ is connected (any two points are joined by a path).

If $\Gamma(S)=\Gamma(\vec{S})$, then $Q(\vec{S})$ can be obtained from $Q(S)$ by "reorientation" (i.e., by changing the direction) of several arrows. If $\Gamma(S)$ is acyclic and a quiver $\vec{Q}$ is obtained from $Q(S)$ by reorientation of arrows, then there exists $\vec{S}$ such that $\vec{Q}=Q(\vec{S})$.

A poset S and the quiver $Q=Q(S)$ are said to be *standard* if $I(s_i)=2$ implies that s_i is the origin of one arrow and the end of one arrow, and $I(s_i)\neq 2$ implies that s_i is either the origin of $I(s_i)$ arrows or the end of $I(s_i)$ arrows ($i=1,\dots,n$). It is easy to check that exactly one standard poset corresponds to each acyclic graph (up to antiisomorphism). If S^* is antiisomorphic to S, then $\Gamma(S)=\Gamma(S^*)$, and $Q(S^*)$ is obtained from $Q(S)$ by reorientation of all arrows.

[1]In [6], Sapelkin called this statement the Zel′dich lemma. Zel′dich in [7] said that it is "important and surprising", with which the authors agree, in spite of the brevity of the proof.

If φ is an arrow of $Q(S)$, then we denote by $S(\varphi)$ the poset obtained from S after reorientation of the arrow φ, and by A_φ the matrix of $f_{S(\varphi)}$. Obviously, $\mathcal{A}(S(\varphi)) = \mathcal{A}(S)$.

We say that a point $s_m \in S$ is a *Dynkin point* if there exists an m-Dynkin vector for the form f_S.

Remark 2. The function f_S is m-isolated in the sense of Subsection 1 if s_m is comparable with no other point of S. Therefore, for connected S, the requirement that f_S be not m-isolated (in Lemma 3) is fulfilled automatically.

Lemma 6. *Let $\Gamma(S)$ be acyclic, and let $s_i \xrightarrow{\varphi} s_j \in Q(S)$. Suppose $d \neq 0$ is a vector such that $d'_i = 2(dA)_i = 0$, $d'_j = 2(dA)_j = 0$. Then there exists a vector $\widehat{d} \neq 0$ such that $dA = \widehat{d}A_\varphi$.*

Proof. The proof of Proposition 4 shows that $\widetilde{Q}^{-1}\mathcal{A}(\widetilde{Q}^{-1})^T = A$, $\widetilde{Q}_\varphi^{-1}\mathcal{A}_\varphi(\widetilde{Q}_\varphi^{-1})^T = A_\varphi$, $\widetilde{Q}_\varphi^{-1}\widetilde{Q}A\widetilde{Q}^T(\widetilde{Q}_\varphi^T)^{-1} = A_\varphi$ ($\mathcal{A}_\varphi = \mathcal{A}$). Let $\widehat{d} = d\widetilde{Q}^{-1}\widetilde{Q}_\varphi$ ($\widehat{d} \neq 0$); then $\widehat{d}A_\varphi = d\widetilde{Q}^{-1}\widetilde{Q}_\varphi\widetilde{Q}_\varphi^{-1}\widetilde{Q}A\widetilde{Q}^T(Q_\varphi^T)^{-1} = dA\widetilde{Q}^T(\widetilde{Q}_\varphi^T)^{-1}$.

Recalling that $d'_i = d'_j = 0$, we put $dA = \sum_{k\notin\{i,j\}} \alpha_k s_k = b$. We need to show that $b\widetilde{Q}^T(\widetilde{Q}_\varphi^T)^{-1} = b$. This is equivalent to $s_k\widetilde{Q}^T(\widetilde{Q}_\varphi^T)^{-1} = s_k$, i.e., to $s_k\widetilde{Q}^T = s_k\widetilde{Q}_\varphi^T$, which follows from the definition of $\widetilde{Q}$ and from the fact that $k \notin \{i, j\}$. □

Lemma 7. *If $\Gamma(S)$ is acyclic and s_t is a Dynkin terminal point of S, then it is a Dynkin point for the poset $\vec{S}$ provided that $\Gamma(\vec{S}) = \Gamma(S)$.*

Proof. If $\vec{S} = S(\varphi)$, then the claim follows from Lemma 6. Turning to the general case ($\vec{S}$ is not $S(\varphi)$), first we note that if S^* and S are antiisomorphic, then $f_S = f_{S^*}$ and s_t is a Dynkin point also for S^*.

Let ψ (respectively, $\hat{\psi}$) denote a unique arrow of $Q(S)$ (respectively, of $Q(\vec{S})$) for which s_t is either the end or the origin. Then the condition $s_t \notin \{s_i, s_j\}$ is equivalent to $\varphi \neq \psi$.

Without loss of generality we assume that ψ (in $Q(S)$) and $\hat{\psi}$ (in $Q(\vec{S})$) have the same orientation (otherwise we pass to $\vec{S}^*$). Then we can pass from S to $\vec{S}$ by reversing several arrows different from ψ; therefore, the partial cases where $\vec{S} = S(\varphi)$ (Lemma 6) and $\vec{S} = S^*$ considered above imply the statement of the lemma. □

4. Let Γ be a connected acyclic graph with one branch point and three terminal points. Γ is the union of three chains A_{n_1}, A_{n_2}, A_{n_3} intersecting at a branch point s_1, $\Gamma = A_{n_1} \cup A_{n_2} \cup A_{n_3}$, $A_{n_1} \cap A_{n_2} = A_{n_1} \cap A_{n_3} = A_{n_2} \cap A_{n_3} = \{s_1\}$, and $|A_{n_j}| = n_j$, $j = 1, 2, 3$, $|\Gamma| = n_1 + n_2 + n_3 - 2$. We shall denote such Γ by $\Gamma(n_1, n_2, n_3)$ (the graph will not change if we permute the n_j).

All Dynkin graphs except for A_n (i.e., D_n, E_6, E_7, E_8) and the extended Dynkin graphs $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$ are of the form $\Gamma(n_1, n_2, n_3)$. It is well known that $\Gamma(n_1, n_2, n_3)$ is a Dynkin graph if and only if $n_1^{-1} + n_2^{-1} + n_3^{-1} > 1$, and $\Gamma(n_1, n_2, n_3)$ is an extended Dynkin graph if $n_1^{-1} + n_2^{-1} + n_3^{-1} = 1$.

Namely, $\Gamma(n_1, n_2, n_3)$ is E_6, E_7, E_8, or D_n if $(n_1, n_2, n_3) = (3, 3, 2)$, $(2, 4, 3)$, $(2, 3, 5)$, or $(1, 1, n-2)$, respectively. $\Gamma(m_1, m_2, m_3)$ is $\widetilde{E}_6$, $\widetilde{E}_7$, or $\widetilde{E}_8$ if $(m_1, m_2, m_3) = (3, 3, 3)$, $(2, 4, 4)$, or $(2, 3, 6)$, respectively.

Here the numeration of m_j and n_j is fixed so that $m_1 \le m_2 \le m_3$, and for E_n with $n = 6, 7, 8$ we have $n_1 = m_1$, $n_2 = m_2$, $n_3 = m_3 - 1$.

Observe that in all cases m_1 and m_2 divide m_3.

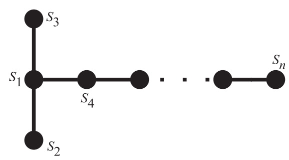
Proposition 5. *If $\Gamma(S) = \Gamma(n_1, n_2, n_3)$ is a Dynkin graph or an extended Dynkin graph (i.e., D_n, E_6, E_7, E_8, $\widetilde{E}_6$, $\widetilde{E}_7$, or $\widetilde{E}_8$), then S contains a terminal Dynkin point.*

Proof. Lemma 7 shows that there is no loss of generality in assuming that S is standard.

For any $\Gamma(m_1, m_2, m_3)$ ($I(s_1) = 3$) we construct a vector $\widetilde{d}$ by putting $\widetilde{d}_1 = -m_3$, $\widetilde{d}_i = \frac{m_3}{m_j}$ for $s_i \in A_{n_j}$, $i \neq 1$. It is easily seen that $\widetilde{d}'_i = 0$ for $i \neq 1$, and $\widetilde{d}'_1 = m_3(1 - m_1^{-1} - m_2^{-1} - m_3^{-1})$. If, moreover, $\Gamma(S)$ is an extended Dynkin graph, then $\widetilde{d} \in \mathbb{Z}_n$ and $\widetilde{d}'_1 = 0$, i.e., $\widetilde{d}$ is an i-Dynkin vector for any i.

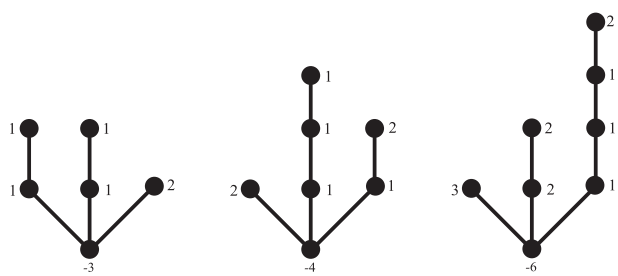
Let $\Gamma(S)$ be E_n, $|S| = n$, and let $\widetilde{S}$ be a standard poset such that $\Gamma(\widetilde{S}) = \widetilde{E}_n$, $|\widetilde{S}| = n + 1$, $S \subset \widetilde{S}$, $\widetilde{S} \setminus S = \{s_{n+1}\} \subset A_{n_3}$. We construct a Dynkin vector d for S, modifying the Dynkin vector $\widetilde{d}$ for $\Gamma(\widetilde{S})$. We put $d_i = \widetilde{d}_i$ for $i < n$ and $d_n = 2$ $(= \widetilde{d}_n + \widetilde{d}_{n+1})$, $d'_n = 1$ ($d'_i = 0$ for $i = 1, \ldots, n-1$).

Let

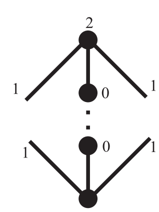
$\Gamma(S) = D_n =$ 

Then $w = (w_1, \ldots, w_n)$, where $w_1 = -2$, $w_2 = w_3 = 1$, $w_n = 2$, $w_i = 0$ for $i \notin \{1, 2, 3, n\}$, is an n-Dynkin vector s_n ($w'_n = 2$). $\square$

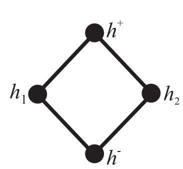
Dynkin vectors for the standard posets S such that $\Gamma(S) = E_6$, E_7, or E_8 can be written out explicitly:



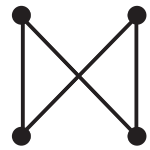
Example 5. A Dynkin vector for the standard poset S such that $\Gamma(S) = \widetilde{D}_n$, $n > 4$ (for $n = 4$ see Example 2) has the following form (all points are Dynkin points):



5. Consider the posets

$V =$ 

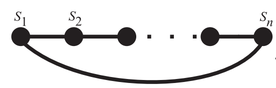
and $W^{2k} = \{s_1^-, \dots, s_k^-, s_1^+, \dots, s_k^+ \mid s_i^- < s_i^+, s_i^- < s_{i+1}^-, s_k^- < s_1^+, i = 1, \dots, k\}$, $k > 1$; in particular,

$W^4 =$ 

(see Example 3).

Lemma 8. *If S is a cyclic poset and each $S' \subset S$ is acyclic, then S is either V or W^{2k} $(k \geq 2)$.*

Proof. Without loss of generality we may assume that $\Gamma(S)$ is a simple cycle

$\widetilde{A}_n =$ 

.

Let $S \setminus S^\times \ni s$. Then $s^- < s < s^+$ and in $\Gamma(S)$ the vertices s^- and s^+ are connected with s by edges. Let $S' = S \setminus \{s\}$. If there is an edge s^-—s^+ in the graph $\Gamma(S')$, then $\Gamma(S')$ is a cycle, which contradicts the condition of the lemma.

If s^- and s^+ are not connected by any edge in $\Gamma(S')$, then in S there is a point $\bar{s} \neq s$ such that $s^- < \bar{s} < s^+$, $\bar{s} \not\lessgtr s$ (i.e., $\bar{s}$ and s are not comparable), because otherwise s^- and s^+ would not be connected with s in $\Gamma(S)$. Thus, $\{s^-, s^+, s, \bar{s}\} = V$ and $S = V$.

So, if $S \neq V$, then $S^\times = S$, and then, since $\Gamma(S) = \widetilde{A}_n$, it is easy to check that $S = W^{2k}$ for some $k > 1$. □

Lemma 9. *If $S \supseteq V$ and $S \not\supset W^4$, then $C(S) \neq \varnothing$.*

Proof. An arbitrary vector $v \in \mathbb{R}_n$ can be viewed as a function on S with values in $\mathbb{R}$.

Let $v : S \to \mathbb{R}$ be such that $v(h^-) = v(h^+) = -1$, $v(h_1) = v(h_2) = 1$, $v(t) = 0$ for $t \in S \backslash V$; then $v \in H_n$. We prove that $v'(s) \leq 0$ for $s \in S$. We have $v'(h^-) = v'(h^+) = -1$ and $v'(h_1) = v'(h_2) = 0$. If t is comparable neither with h_1 nor with h_2, then, clearly, $v'(t) \leq 0$. If t is only comparable with one of h_1, h_2, then it is comparable either with h^- or with h^+, and also $v'(t) \leq 0$. Suppose t is comparable with h_1 and h_2. Let $t < h_1$ (the case where $t > h_1$ is similar); then $t < h_2$ ($h_1 < t < h_2$ is impossible, so that $t < h^+$). Then if t is comparable with h^- also, then $v'(t) = 0$, and otherwise we have $S \supset W^4 = \{t, h_2, h^-, h_1\}$. □

Lemma 10. *If $S \supseteq W^{2k}$ $(k \geq 2)$ and the form f_S is positive semidefinite, then $C(S) \neq \varnothing$.*

Proof. Let $t \in T = S \setminus W^{2k}$. Putting $S^-(t) = |\{s_i^- \mid t < s_i^-\} \cup \{s_i^- \mid t > s_i^-\}|$ and $S^+(t) = |\{s_i^+ \mid t < s_i^+\} \cup \{s_i^+ \mid t > s_i^+\}|$, we prove that if f_S is positive semidefinite, then $S^-(t) = S^+(t)$.

Indeed, let $S^-(t_0) > S^+(t_0)$ for a fixed $t_0 \in T$ (the case where $S^-(t_0) < S^+(t_0)$ is similar). We consider $x : S \to \mathbb{R}_n$ with $x(s_i^-) = -1$, $x(s_i^+) = 1$ $(i = 1, \dots, k)$, $x(t_0) = \varepsilon$, $0 < \varepsilon < 1$, and $x(t) = 0$ for $t \in T \setminus \{t_0\}$. It is easily seen that $f_S(x) < 0$.

Now, let $v : S \to \mathbb{R}_n$ be a vector such that $v(s_i^-) = -1$, $v(s_i^+) = 1$, $v(t) = 0$ for $t \in T$. The relation $S^{-1}(t) = S^+(t)$, $t \in T$, implies that $v'(s) = 0$ for any $s \in S$. Clearly, $v \in H_n$, whence $v \in C(S)$. □

Proposition 6. *If S is an antimonotone poset and the form f_S is positive semidefinite, then $\Gamma(S) = A_n$.*

Proof. If S is cyclic, then Lemmas 8, 9, and 10 imply the statement. If S is acyclic, then the Tits form $\mathcal{T}_S$ is positive semidefinite by Proposition 4, so that $\Gamma(S)$ is one of A_n, D_n, E_6, E_7, E_8, $\widetilde{D}_n$, $\widetilde{E}_6$, $\widetilde{E}_7$, or $\widetilde{E}_8$ ($\Gamma(\widetilde{A}_n)$ is cyclic). If $\Gamma(S) \neq A_n$, then Proposition 5, Examples 2 and 5, and Lemma 7 imply the existence of a Dynkin point and, by Lemma 3 and Remark 2, $C(S) \neq 0$. □

6. Now, let $\Gamma(S) = A_n$. In this case, up to antiisomorphism, the poset S is determined by its order and by the subset $S^\times$ of junction points (see Subsection 3). Clearly, $S^\times = \varnothing$ if and only if S is a chain.

Consider the posets $W^{k,k+1} = \{s_1^-, \dots, s_k^-, s_1^+, \dots, s_{k+1}^+ \mid s_i^- < s_i^+, s_i^- < s_{i+1}^+, i = 1, \dots, k\}$ and $W^{k+1,k} = \{s_1^-, \dots, s_{k+1}^-, s_1^+, \dots, s_k^+ \mid s_i^+ > s_i^-, s_i^+ > s_{i+1}^-, i = 1, \dots, k\}$.

Lemma 11. *If* $\Gamma(S) = A_n$ *and* $\Gamma(S)$ *contains* W *of the form* $W^{k,k+1}$ (*respectively*, $W^{k+1,k}$), *and moreover*, $s_1^+, s_{k+1}^+ \notin S^\times$ (*respectively*, $s_1^-, s_{k+1}^- \notin S^\times$), *then* $C(S) \neq \varnothing$.

Proof. For definiteness, let $S \supset W^{k,k+1}$. Consider a vector v such that $v(s_i^-) = -2$ for $i = 1, \dots, k$, $v(s_i^+) = +2$ for $i = 2, \dots, k$, $v(s_1^+) = v(s_{k+1}^+) = 1$, and $v(t) = 0$ for $t \in S \setminus W^{k,k+1}$. We prove that $v \in C(f)$.

Indeed, $v \in H_n$ and $v'(s_1^-) = v'(s_k^-) = -1$, $v'(s_i^-) = 0$ for $i = 2, \dots, k-1$, and $v'(s_i^+) = 0$ for $i = 1, \dots, k+1$. The absence of branch points implies that if $t \notin W$ is comparable with $w \in W$, then $w \in \{s_1^+, s_{k+1}^+\}$. If t is comparable with both s_1^+ and s_{k+1}^+, then S is cyclic. If s_1^+, $s_{k+1}^+ \notin S^\times$, then $t > w$. Therefore, each t either is comparable with exactly one s_i^- and one s_i^+ or is not comparable with any $w \in W$, whence $v'(t) \leq 0$. □

A poset ζ will be called a *wattle* [3] if it is a union of mutually disjoint chains Z_i, $|Z_i| \geq 2$, $i = 1, \dots, t$, $t > 1$, such that the minimal element of Z_i is less than the maximal element of Z_{i+1} and there are no other comparisons between elements of different Z_i. We have $\Gamma(\zeta) = A_n$. In accordance with [3], we denote $\zeta = \langle n_1, \dots, n_t \rangle$, where $n_i = |Z_i|$.

For a poset S, $\Gamma(S^\times)$ can be viewed as a disconnected subgraph of $\Gamma(S)$. Let $S_i^\times$ denote its connected components.

Lemma 12. *A poset* S *with* $\Gamma(S) = A_n$ *is either a chain or a wattle if* (*and only if*) *the orders of all* $S_i^\times$ *are even.*

Proof. If S is a wattle, then the claim is evident (and we shall not use it). The converse statement will be proved by induction on $|S|$. The induction base is evident. Let $|S| = n+1$. We write $\Gamma(S) = \cdots s_{n-1}$—$s_n$—$s_{n+1}$, where s_{n+1} is a terminal point (therefore, $s_{n+1} \notin S^\times$). For definiteness, we assume that $s_n > s_{n+1}$, so that s_{n+1} is minimal. Put $S' = S \setminus \{s_{n+1}\}$ and $S'' = S \setminus \{s_{n+1}, s_n\}$. We have two possibilities: 1) $s_{n-1} > s_n$ and 2) $s_{n-1} < s_n$.

1) $s_n \notin S^\times$, $(S')^\times = S^\times$. By the inductive hypothesis, S' is a wattle in which s_n is a minimal terminal point. Clearly, S is either a wattle or a chain. (If $S' = \langle n_1, \dots, n_t \rangle$, then $S' = \langle n_1, \dots, n_t + 1 \rangle$).

2) $s_n \in S^\times$ (S' does not satisfy the inductive hypothesis), and $s_n \in S_p^\times$, $|S_p^\times| \equiv 0 \pmod 2$. Then $s_{n-1} \in S_p^\times \subset S^\times$ and s_{n-1} is a terminal point of S'', whence $s_{n-1} \notin (S'')^\times$. If $S^\times = \bigcup_{i=1}^p S_i^\times$, then $(S'')^\times = \bigcup_{i=1}^{p-1} S_i^\times \cup (S_p^\times \setminus \{s_n, s_{n+1}\})$.

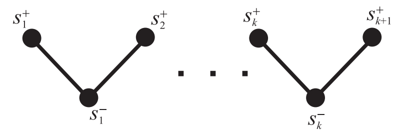
Consequently, S'' satisfies the inductive hypothesis, and hence, is either a chain or a wattle in which s_{n-1} is the minimal point. If $S'' = \langle n_1, \dots, n_t \rangle$, then $S = \langle n_1, \dots, n_t, 2 \rangle$. □

Proposition 7. *If the form* f_S *is positive semidefinite and* $C(S) = \varnothing$, *then* S *is either a chain or a wattle.*

Proof. By Proposition 6, we have $\Gamma(S) = A_n$.

If S is neither a chain nor a wattle, then, by Lemma 12, there exists $S_p^\times$ such that $|S_p^\times| \equiv 1 \pmod 2$. It is easily seen that $S_p^\times$ is $W^{k,k+1}$ or $W^{k+1,k}$. For definiteness, let

$$S_p^\times = W^{k,k+1} =$$



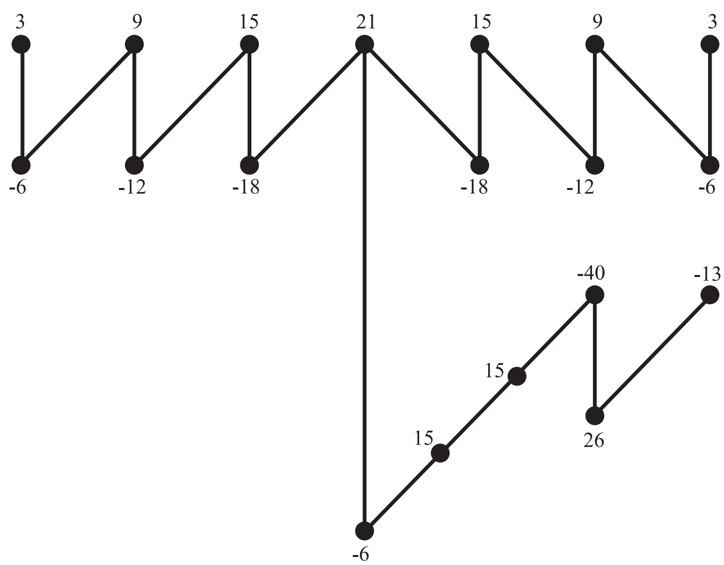
Since $s_1^+ \in S^\times$ and $S_p^\times$ is a connected component in $S^\times$, we see that there exists $s_0^- \in S \setminus S^\times$ such that $s_0^- \to s_1^+$. Similarly, there exists $s_{k+1}^- \in S \setminus S^\times$ such that $s_{k+1}^- \to s_{k+1}^+$. Then $S_p^\times \cup \{s_0^-, s_{k+1}^-\} = W^{k+2,k+1}$ and $C(S) \neq \varnothing$ by Lemma 11. $\square$

Example 4 in Subsection 3 shows that, generally speaking, the requirement that f_S be positive semidefinite cannot be lifted.

Conjecture. *If S is acyclic and $\Gamma(S) \neq A_n$, then $C(S) \neq \varnothing$.*

In some cases the existence of $v \in C(f)$ for acyclic S is obvious. However, we present a computer-made example of an acyclic poset S and $v \in C(S)$.

Example 6.



7. Let $\zeta = \langle n_1, \dots, n_t \rangle$ $(t > 1)$ be a wattle, where $n_i = |Z_i|$, $\sum_{i=1}^t n_i = n$, $n_i > 1$, $i = 1, \dots, t$.

In [3], the minimal points of the chains Z_i, $i = 1, \dots, t-1$, were denoted by z_i^-, and the maximal points of the chains Z_i, $i = 2, \dots, t$, were denoted by z_i^+, $z_i^- < z_{i+1}^+$. The remaining (i.e., not junction) points were called *common* points (including the maximal point of Z_1 and the minimal point of Z_t). They are only comparable to points within their chains.

The *width* $\omega(S)$ of a partially ordered set S is the maximal number of its pairwise incomparable elements. With each poset S, we associate the rational number $r(S) = \frac{n+1}{t} - 1$, where $n = |S|$ and $t = \omega(S)$. If S is a chain, then $w(s) = 1$ and $r(S) = n$.

Clearly, there exist many wattles with the same r. However, below we prove that any noninteger $r > 1$ corresponds to exactly one (uniform in the sense of [3]) P-faithful (= antimonotone; see Corollary 3) wattle $\zeta(r)$, which will be called the *r-wattle*.

For a positive rational a we put $\{a\} = a - [a]$. Let r be a positive nonintegral rational number exceeding 1, and let q/t be the representation of $\{r\}$ in the form of an irreducible fraction. We indicate a sequence of integers $n_1, \dots, n_t$ that will be the orders of the sets Z_i in $\zeta(r)$. Put $n_1 = n_t = [r]+1$ and $n_i = [ri]-[r(i-1)]+1$ for $i = 2, \dots, t-1$. Clearly, $[\{r\}i] - [\{r\}(i-1)]$ is either 1 or 0. Therefore, n_i is either $1+[r]$ or $2+[r]$. The number of i for which $n_i = 2+[r]$ is $q-1$, and $n = t([r]+1)+q-1$, $r(\zeta(r)) = r$.

Observe that the r-wattles are uniform in the sense of [3].

Thus, to each nonintegral rational number $r > 1$ we assigned a wattle $\zeta(r)$. The integers numbers can also be considered if we agree that, for any integer r, $\zeta(r)$ is a chain of length r. All posets of the form $\zeta(r)$, $r \geq 1$ (i.e., the uniform wattles and chains), will be called the r-sets.

Theorem. *Suppose that the form f_S is positive semidefinite* ($\Gamma(S)$ *is connected*). *Then $C(S) = \varnothing$ if and only if S is an r-set.*

Proof. If r is an integer, then the statement is obvious (see [3]). Therefore, by Proposition 7, we only need to prove that $C(\zeta) = \varnothing$ if and only if ζ is an r-wattle. □

For any r-wattle $\zeta(r)$, we consider a vector $x : \zeta \to \mathbb{R}^+$ such that $x(s) = 1$ for $s \in \zeta \setminus \zeta^\times$, $x(z_i^-) = \{ir\}$, and $x(z_i^+) = 1 - x(z_{i-1}^-)$. Since $(q,t) = 1$, we have $x(s) > 0$ for any $s \in \zeta(r)$.

The fact that $x \in \mathrm{St}(\zeta)$ can be checked either directly, by using the definition of $\mathrm{St}(\zeta)$, or with the help of the following lemma.

Lemma 13 (See [3, Lemma 5]). *The vector $x : \zeta \to \mathbb{R}$ belongs to* $\mathrm{St}(\zeta)$ *if and only if there exist positive numbers α and β such that*

1) $x(s) = \alpha$ *for* $s \in \zeta \setminus \zeta^\times$ (*multiplying x by $\lambda \in \mathbb{R}^+$, we can assume that $\alpha = 1$*);
2) $x(z_i^-) + x(z_{i+1}^+) = \alpha$ *for* $i = 1, \dots, t-1$;
3) $\sum_{s \in Z_i} x(s) = \beta$, $i = 1, \dots, t$.

This lemma is almost evident. We only mention that first we prove 2), using the relations $\frac{\partial f_S}{\partial z_i^-}(x) = \frac{\partial f_S}{\partial z_i^+}(x)$, $i = 2, \dots, t-1$, and then 1).

The vector x constructed above satisfies conditions 1) and 2) of Lemma 13. It is easy to check (for $\alpha = 1$) that

$$\sum_{s \in Z_i} x(s) = r \quad (i = 1, \dots, t), \tag{4}$$

$$x'(s) = 1 + r, \tag{5}$$

$$\sum_{s \in \zeta(r)} x(s) = tr. \tag{6}$$

Thus, $x \in \mathrm{St}(\zeta)$, whence $\widetilde{\mathrm{St}}(\zeta) \neq \varnothing$, and $C(S) = \varnothing$ by Corollary 3.

It remains to show that any P-faithful wattle ζ is an r-wattle (where $[z] = |Z_1| - 1$, $\{r\} = x(z_1^-)$). This is a consequence of the next statement.

Lemma 14. *Let $\zeta = \langle z_1, \dots, z_t \rangle$ and $\hat{\zeta} = \langle \hat{z}_1, \dots, \hat{z}_t \rangle$ be two P-faithful wattles, and let $x \in \mathrm{St}(\zeta)$, $\hat{x} \in \mathrm{St}(\hat{\zeta})$ ($\hat{\alpha} = \alpha = 1$). If $Z_1 = \hat{Z}_1$ and $x(s) = \hat{x}(s)$ for $s \in Z_1 = \hat{Z}_1$, then $\zeta = \hat{\zeta}$ and $x(s) = \hat{x}(s)$ for $s \in \zeta$.*

Proof. It suffices to check that if $m \leq \max\{t, \hat{t}\}$, then $z_i = \hat{z}_i$ for $i \leq m$ and $x(s) = \hat{x}(s)$ for $s \in \bigcup_{i=1}^m Z_i$, and this follows from Lemma 13 by induction on m (see [3]). □

Now, we calculate $P(\zeta(r))$. In [3], the numerical function $\rho(r) = 1 + \frac{r-1}{r+1}$, where $r \in \mathbb{N}$, was introduced. We extend this definition to the case of an arbitrary rational $r \geq 1$. Put

$\rho(r_1, \dots, r_t) = \sum_{i=1}^{t} \rho(r_i)$. If Z_n is a chain of order n, then $P(Z_n) = \rho(n)$ (see [3]). Let $\zeta(r)$ be a wattle. By (6), the vector $\overline{x} = (tr)^{-1}x$ belongs to $P_n \cap \mathrm{St}(\zeta(r))$ (here x is the vector constructed in the proof of the theorem). We have

$$P(\zeta(r)) = f^{-1}_{\zeta(r)}(\overline{x}) = (tr)^2 f^{-1}_{\zeta(r)}(x) \overset{(3)}{=} 2(tr)^2 \Bigg(\sum_{s\in\zeta(r)} x'(s)x(s) \Bigg)^{-1} \overset{(5),\,(6)}{=} \frac{2t^2r^2}{(1+r)tr}$$
$$= \frac{2tr}{1+r} = t\rho(r).$$

The same formula is valid if $t = 1$ (i.e., in the case of a chain). For any positive rational $r = \frac{l}{t}$ ($(l,t) = 1$) we have $t\rho(r) = \frac{2lt}{l+t}$. We introduce the function $P(r) = \frac{2lt}{l+t}$ ($P(n) = \rho(n)$ for $n \in \mathbb{N}$). Thus, for any $r \geq 1$ we have

$$P(\zeta(r)) = t\rho(r) = P(r). \tag{7}$$

Appendix

We say that a poset S is *connected* if the graph $\Gamma(S)$ is connected. The theorem and Corollary 3 imply that a connected poset S is P-faithful if and only if it is an r-set. On the other hand, the results of [3]–[7] imply our theorem only if f_S is positive definite (not merely positive semidefinite). Characterization of disconnected antimonotone posets with positive semidefinite f_S and of P-faithful posets reduces to connected posets by Lemmas 2 and 5.

We say a few words about the role played by P-faithful posets in representation theory. We write $S = S_1 \sqcup S_2$ if $S = S_1 \cup S_2$, $S_1 \cap S_2 = \varnothing$, and the elements of S_1 are not comparable to the elements of S_2. A poset $S = Z_1 \sqcup \cdots \sqcup Z_p$ is *primitive* if the Z_i are chains, $i = 1, \dots, p$. We denote such S by $(n_1, \dots, n_p)$ if $n_i = |Z_i|$.

Any poset can be represented as $S = \bigsqcup_{i=1}^{p} S_i$, where the S_i are connected components. By Lemma 5, we have $P(S) = \sum_{i=1}^{p} P(S_i)$, and if S is primitive, then $P(S) = \sum_{i=1}^{p} \rho(n_i) = \rho(n_1, \dots, n_t)$.

The role of quadratic forms in the theory of representations of quivers and posets is well known (see [12]).

The norm of a relation, $\|S, \leq\| = \inf_{u\in\overline{P}_n} f_S(u)$, was introduced in [2] in terms of the form f_S. Lemma 5 shows that, instead of $\|S, \leq\|$, it is natural to consider the function $P(S) = \|S, \leq\|^{-1}$. The following statement was proved in [2].

Proposition 8. *S has finite* (*respectively, tame*) *type if and only if $P(S) < 4$* (*respectively, $P(S) = 4$*).

With this viewpoint, Kleiner's list of critical posets (see [8]) is the list of *P-faithful* posets S_i for which $P(S) = 4$.

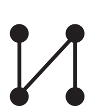
Four posets of Kleiner's list are primitive:

$$\text{(I)} \quad (1,1,1,1), \quad (2,2,2), \quad (1,3,3), \quad (1,2,5),$$

and the fifth is

$$(4) \sqcup K,$$

where

$K =$  $= \langle 2; 2\rangle = \zeta\Big(1\frac{1}{2}\Big).$

It is easily seen that any chain is P-faithful, and Subsection 7 implies that K is also P-faithful ($P(K) = 2,4$). By Lemma 5, a disconnected poset is P-faithful if and only if all its components are.

The list of critical sets presented in [10],

$$\text{(II)}\quad (1,1,1,1,1),\quad (1,1,1,2),\quad (2,2,3),\quad (1,3,4),\quad (1,2,6),\quad (6)\sqcup K,$$

can be characterized as the list of all S with the following properties:

1) $P(S) > 4$;

2) if $S' \subset S$, then $P(S') \leq 4$.

The following statement plays a key role in the theory of representations of posets (see [8, 10]).

A poset S is finitely represented (*respectively, tame*) *if and only if S contains no subsets of the form* I (*respectively*, II).

It seemed natural to conjecture that all P-faithful posets are either chains or belong to a collection for which K is the least representative. This was a reason for introducing the notion of a P-faithful poset [3].

Now we show how the lists (I), (II) can be obtained from the characterization of the (connected) P-faithful sets and formula (7). It is easy to check that $P(S) = 4$ for $S \in \mathrm{I}$ and $P(S) > 4$ for $S \in \mathrm{II}$ (we recall Lemma 5 and formula (7)).

We say that a P-faithful poset S is *utmost* if $P(S) \geq 4$ and $P(S') \leq 4$ for any $S' \subset S$ (here S' can be assumed to be P-faithful).

Lemma 15. *Any nonprimitive utmost S is of the form $K \sqcup Z_m$ with m equal to* 4 *or* 5.

Proof. Suppose S contains a connected component $\zeta(r)$, where $\{r\} = \frac{q}{t}$, $t > 1$, $q < t$, $(q,t) = 1$. The characterization of the P-faithful posets implies that $\omega(S) < 4$, because otherwise $S \supset S' = (2,1,1,1)$, $\rho(2,1,1,1) = 4\frac{1}{3} > 4$. Consequently, $t \leq 3$, and moreover, if $t = 3$, then $\zeta(r) = S$.

Let $t = 3$, and let $1 \leq q \leq 2$. If $[r] \geq 2$, then $S \supset S' = S \setminus \{z_1^-, z_3^+\}$. S' is a primitive poset containing $(2,3,2)$, $\rho(2,2,3) = 4\frac{1}{6}$, $\rho(S') > 4$. If $[r] = 1$, then either $r = 1\frac{1}{3}$ or $r = 1\frac{2}{3}$. We see that $\rho(r) \leq 1\frac{1}{4}$, and then $P(S) = 3\rho(r) < 4$ (see (7)).

Let $t = 2$, and let $S \neq \zeta(1\frac{1}{2}) \sqcup \hat{S}$. If $S = \zeta(r)$, then $P(S) = 2\rho(r) < 4$ because $\rho(r) < 2$ for any r. So, $S = \zeta(r) \sqcup \hat{S}$, $r > 1\frac{1}{2}$, i.e., $r \geq 2\frac{1}{2}$, $\zeta(r) \supset \{\zeta(r) \setminus z_1^-\} \supseteq (2,3)$. If $|\hat{S}| > 1$, then we can find S' contained in S, containing $(2,2,3)$ or $(1,1,2,3)$, and such that $P(S') > 4$. Hence, $|\hat{S}| = 1$. Then $[r] < 3$, because otherwise $\zeta(r) \supset S' = \zeta(r) \setminus z_1^- \supseteq (3,4)$, and $P(S' \sqcup (1)) > 4$ because $\rho(3,4,1) > 4$. For $[r] = 2$ we obtain $P(S) < 4$ because $P(\zeta(2\frac{1}{2})) = 2 \cdot 1\frac{3}{7}$, $P(\zeta(2\frac{1}{2}) \sqcup (1)) = 2\frac{6}{7} + 1$ (Lemma 5).

Finally, let $S = \zeta(1\frac{1}{2}) + \widehat{S}$ ($S \neq \zeta(1\frac{1}{2})$ because $P(\zeta(1\frac{1}{2})) = 2,4$). If $w(\widehat{S}) > 1$, then $S \supset S' = \{\zeta(1,\frac{1}{2}) \setminus z_1^-\} \sqcup (1,1) = (2,1,1,1)$, $\rho(S') > 4$. If $\widehat{S} = Z_m$, then for $m < 4$ we have $\rho(m) < 1,6$, and $P(S) < 4$, and for $m > 5$ we have $S \supset S' = (\zeta(1\frac{1}{2}) \sqcup Z_5)$, $\rho(S') > 4$, $P(K \sqcup Z_4) = 4$, $P(K \sqcup Z_5) = 4\frac{1}{15}$. $\square$

Proposition 9. *A P-faithful S is utmost if and only if $S \in \mathrm{I} \cup \mathrm{II}$.*

Proof. If S is not primitive, then the claim follows from Lemma 15. Let S be primitive. Then $w(S) > 2$ and $S \notin \{(1,1,n),(1,2,2),(1,2,3),(1,2,4)\}$ (otherwise $P(n_1,\dots,n_t) = \rho(n_1,\dots,n_t) < 4$). In the remaining cases direct inspection shows that if $S \notin \mathrm{I} \cup \mathrm{II}$, then $S \supset S' \in \mathrm{II}$, and if $S \in \mathrm{I} \cup \mathrm{II}$, then $S \not\supset S' \in \mathrm{II}$. $\square$

Since $P(S) = 4$ for $S \in \mathrm{I}$ and $P(S) > 4$ for $S \in \mathrm{II}$, Propositions 8 and 9 imply the main theorems of [8] and [10].

The P-faithful posets for which $P = 4$ play an important role in representation theory. We do not know whether the same can be said about P-faithful posets with $P = n > 4$. The primitive posets with $P = 5$ were listed in [14]. As a (probably unique) example of a nonprimitive poset S with $P(S) = 5$ we mention $\zeta(3\frac{1}{2}) \sqcup (17)$ (see (7) and Lemma 3).

Example 4 in Subsection 3 presents a poset S such that $C(S) = \varnothing$ but S is not P-faithful. We hope that the study of $C(S)$ can be of interest for representation theory. We note that in [2] the norm $\|P\|$ of an arbitrary binary relation P (on a finite set) and the corresponding notion of a P-faithful set were introduced (these notions can be used for locally scalar representations (see [9]) in Hilbert spaces). However, in this case the structure of the collection of all P-faithful sets is more complicated, and obtaining a full description of such sets seems a difficult and interesting problem.

References

[1] G. M. Fikhtengol′ts, *A course of differential and integral calculus*. Vol. 1, Gostekhizdat, Moscow–Leningrad, 1947. (Russian)

[2] A. V. Roĭter, *The norm of a relation*, Representation Theory. I. Finite Dimensional Algebras (Ottawa, 1984), Lecture Notes in Math., vol. 1177, Springer-Verlag, Berlin, 1986, pp. 269–271. MR0842470 (87e:16076)

[3] L. A. Nazarova and A. V. Roĭter, *The norm of a relation, separation functions, and representations of marked quivers*, Ukrain. Mat. Zh. **54** (2002), no. 6, 808–840; English transl., Ukrainian Math. J. **54** (2002), no. 6, 990–1018. MR1956639 (2003k:16027)

[4] M. V. Zel′dich, *On characteristic forms of partially ordered sets with simple connected Hasse graph*, Visn. Kiiv. Univ. Ser. Fiz.-Mat. Nauki **2001**, no. 4, 36–44. (Ukrainian) MR1935927 (2003i:16023)

[5] ———, *On ρ-faithful partially ordered sets*, Visn. Kiiv. Univ. Ser. Fiz.-Mat. Nauki **2001**, no. 4, 45–51. (Ukrainian) MR1935928 (2003h:16018)

[6] A. I. Sapelkin, *P-faithful partially ordered sets*, Ukrain. Mat. Zh. **54** (2002), no. 10, 1381–1395; English transl., Ukrainian Math. J. **54** (2002), no. 10, 1669–1688. MR2015489 (2004h:06004)

[7] M. V. Zel′dich, *On characteristic and multiply transitive forms of partially ordered sets. On P-exact partially ordered sets*, Preprint, Kiev. Nat. Univ. T. Shevchenka, Kiev, 2002, 64 pp.

[8] M. M. Kleĭner, *Partially ordered sets of finite type*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **28** (1972), 32–41; English transl., J. Soviet Math. **3** (1975), no. 5, 607–615. MR0332585 (48:10911)

[9] C. M. Ringel, *Tame algebras and integral quadratic forms*, Lecture Notes in Math., vol. 1099, Springer-Verlag, Berlin, 1984, 376 pp. MR0774589 (87f:16027)

[10] L. A. Nazarova, *Partially ordered sets of infinite type*, Izv. Akad. Nauk SSSR Ser. Mat. **39** (1975), no. 5, 963–991; English transl. in Math. USSR–Izv. **9** (1975), no. 5. MR0406878 (53:10664)

[11] A. G. Zavadskiĭ and L. A. Nazarova, *Partially ordered sets of tame type*, Matrix Problems, Akad. Nauk Ukrain. SSR Inst. Mat., Kiev, 1977, pp. 122–143. (Russian) MR0505911 (58:21874b)

[12] P. Gabriel and A. V. Roĭter, *Representations of finite-dimensional algebras*, Algebra, VIII, Encyclopaedia Math. Sci., vol. 73, Springer-Verlag, Berlin, 1992, pp. 1–177. MR1239447 (94h:16001b)

[13] S. A. Kruglyak and A. V. Roĭter, *Locally scalar representations of graphs in the category of Hilbert spaces*, Funktsional. Anal. i Prilozhen. **39** (2005), no. 2, 13–30; English transl., Funct. Anal. Appl. **39** (2005), no. 2, 91–105. MR2161513 (2006g:16030)

[14] I. K. Redchuk and A. V. Roĭter, *Singular locally scalar representations of quivers in Hilbert spaces, and separating functions*, Ukrain. Mat. Zh. **56** (2004), no. 6, 796–809; English transl., Ukrainian Math. J. **56** (2004), no. 6, 947–963. MR2106639 (2006a:16025)

Institute of Mathematics, National Academy of Sciences of Ukraine, Tereshchenkovska 3, Kiev 01601, Ukraine

Institute of Mathematics, National Academy of Sciences of Ukraine, Tereshchenkovska 3, Kiev 01601, Ukraine
E-mail address: roiter@imath.kiev.ua

Institute of Mathematics, National Academy of Sciences of Ukraine, Tereshchenkovska 3, Kiev 01601, Ukraine

Received 14/FEB/2005

Translated by THE AUTHORS