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MODULES OVER THE RING OF PSEUDORATIONAL NUMBERS AND QUOTIENT DIVISIBLE GROUPS

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Abstract. Structure theorems are obtained for some classes of modules over the ring of pseudorational numbers and some classes of quotient divisible mixed groups.

Introduction

The ring of pseudorational numbers and modules over this ring were introduced by Fomin in [2] and Krylov in [12, 13]. Later, in [3, 4] Fomin applied such modules to the study of quotient divisible mixed groups and torsion free groups of finite rank. In [12], Krylov used modules over the ring of pseudorational numbers for the study of the so-called *sp*-groups.

Many important Abelian groups are additive groups of modules over the ring of pseudorational numbers. The periodic, divisible, and algebraically compact groups and the groups with π-regular endomorphism ring are classical examples (all these classes were considered in [1]). The mixed groups contained in the direct product of their p-components (called the *sp*-groups) give another important example. The *sp*-groups were studied in the papers [5]–[12]; this list is far from being exhaustive.

Besides [2], modules over the ring of pseudorational numbers were treated in [14], where the injective modules were described, and in [13], where the ideals of the ring of pseudorational numbers were described and some questions about endomorphism rings and groups of homomorphisms for *sp*-groups were answered.

This paper can be divided into three parts. In the first part (§§1, 2) we introduce the principal notions and properties of modules over the ring of pseudorational numbers. §§3 and 4 constitute the second part. In these sections we study the structure of some classes of finitely generated modules. Theorems 3.1 and 4.1 and Corollaries 3.1 and 4.1 are the main results of that part. In the concluding part (§5) we discuss relationships between modules over the ring of pseudorational numbers and quotient divisible mixed groups and torsion free groups of finite rank.

Throughout the paper, Z, Q, and $\widehat{Z}_p$ denote (respectively) the rings of integers, rational numbers, and p-adic integers, as well as the additive groups of these rings. Next, $Z(m)$ stands for the residue class ring modulo m, P denotes the set of all prime integers, and $\mathbb{N}$ is the set of positive integers. By $r(G)$ we denote the torsion free rank of the group G, which will be called simply the rank of G, and $r^*(M)$ is the pseudorational rank of an R-module M. The periodic part of a group G will be denoted by $t(G)$. For a subset S of a group G, we denote by $\langle S \rangle_*$ the pure hull of S in G, i.e., the set of all $g \in G$ such that $mg \in \langle S \rangle$ for some nonzero integer m. The remaining notation and definitions are standard and correspond to those in [1].

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§1. The ring of pseudorational numbers

In the ring $\widehat{Z} = \prod_{p\in P} \widehat{Z}_p$, we consider the subring generated as a pure subgroup by the ideal $\bigoplus_{p\in P} \widehat{Z}_p$ and the unity of the ring.

Definition 1.1. The ring $R = \langle 1, \bigoplus_{p\in P} \widehat{Z}_p\rangle_*$ is called the ring of pseudorational numbers.

We shall also use some other rings introduced in the papers [2] and [13]. Let $\chi = (m_p)$ be an arbitrary characteristic, and let $K_p = Z/p^{m_p}Z$ if $m_p < \infty$ and $K_p = \widehat{Z}_p$ otherwise. If the characteristic χ contains infinitely many nonzero components, then we denote by $R_\chi \subset \prod_{p\in P} K_p$ the subring whose additive group is the pure hull $\langle 1, \bigoplus_{p\in P} K_p\rangle_*$. If the p-components of χ are nonzero only for $p = p_1, \dots, p_n$, then we put $K_\chi = K_{p_1} \oplus \cdots \oplus K_{p_n}$ and $R_\chi = Q \oplus K_\chi$. Observe that if $\chi = (\infty)$, then the ring R_χ is precisely the ring of pseudorational numbers.

The following properties of the rings R_χ were obtained in [2].

Properties:

1^0. *An element $r = (\alpha_p) \in \prod_{p\in P} \widehat{Z}_p$ is contained in the ring R if and only if there exists a rational number $|r| = m/n$ such that $n\alpha_p = m$ for almost all prime integers p.*

2^0. *For every $r \in R$ the rational number $|r|$ described in 1^0 is unique.*

3^0. *The elements of the form $\varepsilon_p = (0, \dots, 0, 1_p, 0, \dots)$ are idempotents of the ring R. Moreover, each idempotent of the ring of pseudorational numbers has the form $\varepsilon = \varepsilon_{p_1} + \cdots + \varepsilon_{p_n}$ or $1 - \varepsilon$.*

4^0. *$T = \bigoplus_{p\in P} \widehat{Z}_p$ is a maximal ideal of the ring of pseudorational numbers; it consists of all $r \in R$ such that $|r| = 0$, and $R/T \cong Q$.*

5^0. *Each epimorphic image of the ring R_χ has the form R_φ or K_φ, where $\chi \ge \varphi$.*

Next we consider some invariants of modules over the ring of pseudorational numbers and recall some properties of such modules.

Definition 1.2 ([2])**.** We say that an R-module M is *divisible* if its additive group is torsion free and divisible and $rm = |r|m$ for all $r \in R$, $m \in M$. We say that an R-module is *reduced* if it contains no divisible submodules.

Theorem 1.1 ([2])**.** *Let M be an arbitrary R-module. Then*:

1. *Either the module M is reduced, or it contains the greatest divisible submodule* $\operatorname{div} M$;
2. $\operatorname{div} M = \{m \in M \mid tm = 0$ *for every* $t \in T\}$;
3. $\operatorname{div} M$ *is a direct summand of M.*

Theorem 1.2 ([2])**.** *For an arbitrary R-module M, the set*

$$TM = \{tm \mid t \in T, m \in M\}$$

is a submodule of the module M, and $TM = \bigoplus_{p\in P} M_p$, where $M_p = \varepsilon_p M$.

Let M be a finitely generated R-module, and let $\{x_1, \dots, x_n\}$ be a generating set of M. Obviously, the $\widehat{Z}_p$-module $M_p = \varepsilon_p M$ is generated by the elements $\{\varepsilon_p x_1, \dots, \varepsilon_p x_n\}$. As any finitely generated p-adic module, M_p decomposes into the direct sum of cyclic $\widehat{Z}_p$-modules:

$$M_p = \langle a_1\rangle_{\widehat{Z}_p} \oplus \cdots \oplus \langle a_n\rangle_{\widehat{Z}_p};$$

some of these modules may be zero.

Any cyclic $\widehat{Z}_p$-module is isomorphic either to $\widehat{Z}_p$ or to $Z/p^{k_{ip}}Z$, where k_{ip} is a nonnegative integer. Therefore, the isomorphism

$$M_p \cong Z(p^{k_{1p}}) \oplus \cdots \oplus Z(p^{k_{tp}}) \oplus \bigoplus_s \widehat{Z}_p \quad (t+s=n)$$

gives rise to an ordered sequence of nonnegative integers and ∞ symbols,

$$0 \le k_{1p} \le \cdots \le k_{np} \le \infty, \tag{1.1}$$

the last s terms of which are the ∞ symbols ($0 \le s \le n$). The sequences (1.1) constructed for all prime integers p determine a sequence of characteristics. Several terms at the beginning of this sequence may be the zero characteristics; deleting them, we obtain a sequence

$$\chi_1 \le \cdots \le \chi_k. \tag{1.2}$$

The sequence of characteristics (1.2) will be called the *generalized characteristic* of the finitely generated R-module M.

Definition 1.3 ([2]). Let M be an R-module; by the *pseudorational rank* of M we mean the dimension $\dim_Q\big(M/TM\big)$ of the factor module M/TM over the field $Q \cong R/T$.

Properties:

6^0. *If $M = \langle x_1, \dots, x_n\rangle_R$, then $r^*(M) \le n$.*

7^0. *If N is a submodule of an R-module M, then $r^*(N) \le r^*(M)$.*

Since $N \subseteq M$ and $TN \subseteq TM$, there exists a monomorphism of vector spaces $N/TN \to M/TM$, which means that $\dim_Q(N/TM) \le \dim_Q(M/TM)$.

8^0. *If N is a submodule of an R-module M, then $r^*(M) = r^*(M/N) + r^*(N)$.*

Since $T(M/N) = TM/TN$ and $(M/N)/(TM/TN) \cong (M/TM)/(N/TN)$, we see that $r^*(M) = \dim_Q((M/N)/(TM/TN)) + \dim_Q(N/TN) = r^*(M/N) + r^*(N)$.

9^0. *If M is an R-module whose generalized characteristic is locally free* (*i.e., it does not contain any ∞ symbol*), *then $r^*(M) = r(M)$.*

If the generalized characteristic of the R-module M is locally free, then $TM = t(M)$; hence, $r^*(M) = r(M/t(M)) = r(M)$.

§2. The module of pseudorational relations

Let $X = \{x_1, \dots, x_n\}$ be any finite system of elements of an R-module M. Obviously, the set

$$\Delta M_X = \{(r_1, \dots, r_n) \in R^n \mid r_1x_1 + \cdots + r_nx_n = 0\}$$

is an R-module. If X is a generating system of M, then ΔM_X will be called the *module of pseudorational relations* of M.

Proposition 2.1. *Let M and L be arbitrary modules over the ring R, let $X=\{x_1, \dots, x_n\}$ be a generating system of M, and let $Y = \{y_1, \dots, y_n\}$ be any system of elements of L. A homomorphism $f : M \to L$ such that $f(x_i) = y_i$ $(1 \le i \le n)$ exists if and only if $\Delta M_X \subseteq \Delta L_Y$.*

Proof. We prove the "only if" part. Let $f : M \to L$ be a homomorphism such that $f(x_i) = y_i$ $(1 \le i \le n)$. If $(r_1, \dots, r_n) \in \Delta M_X$, then $r_1x_1 + \cdots + r_nx_n = 0$, whence

$$f(r_1x_1 + \cdots + r_nx_n) = r_1y_1 + \cdots + r_ny_n = 0.$$

Thus, $\Delta M_X \subseteq \Delta L_Y$.

Now we prove the "if" part. Let $\Delta M_X \subseteq \Delta L_Y$; we construct a homomorphism $f : M \to L$ such that $f(x_i) = y_i$ $(1 \le i \le n)$.

We define a correspondence $f : M \to L$ by the rule

$$f(r_1x_1 + \cdots + r_nx_n) = r_1y_1 + \cdots + r_ny_n.$$

Let $g = r_1x_1 + \cdots + r_nx_n = s_1x_1 + \cdots + s_nx_n$ be two arbitrary representations of an element $g \in M$. Then

$$(r_1 - s_1)x_1 + \cdots + (r_n - s_n)x_n = 0,$$

i.e., $((r_1 - s_1), \ldots, (r_n - s_n)) \in \Delta M_X$. The condition $\Delta M_X \subseteq \Delta L_Y$ implies

$$(r_1 - s_1)y_1 + \cdots + (r_n - s_n)y_n = 0,$$

which means that

$$f(r_1x_1 + \cdots + r_nx_n) = r_1y_1 + \cdots + r_ny_n = s_1y_1 + \cdots + s_ny_n = f(s_1x_1 + \cdots + s_nx_n).$$

Thus, the correspondence f is a mapping. Obviously, f is compatible with the operations; i.e., f is a homomorphism of R-modules. It remains to observe that $f(x_i) = y_i$ $(1 \le i \le n)$. $\square$

Corollary 2.1. *Let M and L be finitely generated R-modules. They are isomorphic if and only if they admit equal modules of pseudorational relations.*[1]

Proposition 2.2. *Let $X = \{x_1, \ldots, x_n\}$ be a generating system of an R-module M; then*

$$n = r^*(M) + r^*(\Delta M_X).$$

Proof. Let $\varphi : R^n \to M$ be the mapping defined by the rule

$$\varphi(r_1, \ldots, r_n) = r_1x_1 + \cdots + r_nx_n.$$

It is easily seen that φ is a homomorphism. But X is a generating system of the R-module M; hence, φ is an epimorphism. Observe that

$$(r_1, \ldots, r_n) \in \ker \varphi \iff r_1x_1 + \cdots + r_nx_n = 0,$$

i.e., $\ker \varphi = \Delta M_X$. Therefore,

$$M \cong R^n/\Delta M_X \quad \text{and} \quad n = r^*(M) + r^*(\Delta M_X). \qquad \square$$

§3. Modules of pseudorational rank 1

Lemma 3.1. *If M is a cyclic R-module, then its pseudorational rank is equal to 1 or 0. In the first case $M \cong R_\chi$, where χ is an arbitrary characteristic, and in the second case $M \cong K_\varphi$, where φ is an almost zero characteristic.*

Proof. Let $M = \langle x \rangle_R$; then, obviously, the mapping $\varphi : R \to M$ defined by the rule $\varphi(r) = rx$ is an epimorphism of modules, and $\ker \varphi$ is an ideal of the ring R. Thus, the module M is isomorphic to an epimorphic image of the ring R, which is isomorphic to one of the modules R_χ or K_φ.

Since $r^*(R_\chi) = 1$ and $r^*(K_\varphi) = 0$, we obtain the second statement of the lemma. $\square$

Lemma 3.2. *If $M = \langle m_1, \ldots, m_n \rangle_R$ is a finitely generated R-module of pseudorational rank 1, then we can choose a generating system $\{s_1, t_2, \ldots, t_n\}$ of the module M so that $s_1 \notin TM$ and $t_2, \ldots, t_n \in TM$.*

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[1] *Translator's Note*: Probably, here (as well as in the formulation of Proposition 5.1) the author has in mind the following: there exist generating systems of these modules for which the modules of their pseudorational relations coincide.

Proof. We introduce the notation

$$m_{i_1} = s_1 \notin TM, \ \ldots, \ m_{i_k} = s_k \notin TM;$$
$$m_{i_{k+1}} = t_{k+1} \in TM, \ \ldots, \ m_{i_n} = t_n \in TM.$$

Since

$$\begin{aligned} M/TM &= \langle s_1 + TM, \ldots, s_k + TM, t_{k+1} + TM, \ldots, t_n + TM \rangle_Q \\ &= \langle s_1 + TM, \ldots, s_k + TM \rangle_Q \quad \text{and} \quad \dim_Q M/TM = 1, \end{aligned}$$

we have $k \geq 1$. Since the module M/TM is a one-dimensional vector space over Q, any two elements of M are linearly dependent modulo TM: if $l, m \notin TM$, then $l = rm + t$ for some $r \in R$, $t \in TM$. Hence, we can represent the elements $s_1, \ldots, s_k$ in the following form:

$$\begin{aligned} &s_1 = s_1; \\ &s_2 = r_2 s_1 + t_2, \quad \text{where } r_2 \in R,\ t_2 \in TM; \\ &\quad \vdots \\ &s_k = r_k s_1 + t_k, \quad \text{where } r_k \in R,\ t_k \in TM. \end{aligned}$$

It follows that $M = \langle s_1, \ldots, s_k, t_{k+1}, \ldots, t_n \rangle_R = \langle s_1, t_2, \ldots, t_n \rangle_R$. □

Theorem 3.1. *If M is a finitely generated R-module of pseudorational rank* 1, *then*

$$M \cong R_{\chi_1} \oplus K_{\chi_2} \oplus \cdots \oplus K_{\chi_m},$$

where $\chi_2, \ldots, \chi_m$ are almost zero characteristics and χ_1 is an arbitrary characteristic.

Proof. Let M be an arbitrary R-module of pseudorational rank 1. By Lemma 3.2, we have $M = \langle s_1, t_2, \ldots, t_n \rangle_R$, where $s_1 \notin TM$ and $t_2, \ldots, t_n \in TM$. Then

$$M = \langle s_1, t_2, \ldots, t_n \rangle_R = \langle s_1 \rangle_R + \langle t_2, \ldots, t_n \rangle_R = Rs_1 + S,$$

where $S = \langle t_2, \ldots, t_n \rangle_R \subseteq TM$. Moreover, since $t_2, \ldots, t_n \in TM$, we can find an idempotent $\varepsilon \in R$ such that $S = \langle t_2, \ldots, t_n \rangle_R \subseteq \varepsilon M$. Then

$$M = ((1-\varepsilon)Rs_1 + \varepsilon Rs_1) + S = (1-\varepsilon)Rs_1 + (\varepsilon Rs_1 + S) = (1-\varepsilon)Rs_1 + S_1,$$

where $S_1 = \varepsilon Rs_1 + S$. Since εRs_1 and S are contained in εM, the module S_1 is also contained in εM, and $(1-\varepsilon)Rs_1 \cap S_1 = 0$ because $(1-\varepsilon)Rs_1 \cap \varepsilon M = 0$. Therefore,

$$M = (1-\varepsilon)Rs_1 \oplus S_1.$$

Being a cyclic R-module of pseudorational rank 1, the module Rs_1 is isomorphic to a module of the form R_χ. Then $(1-\varepsilon)Rs_1 \cong (1-\varepsilon)R_\chi = R_{\chi_1}$.

Let $\varepsilon = \varepsilon_{p_1} + \cdots + \varepsilon_{p_t}$. For all $i \in \{1, \ldots, t\}$, we define mappings $\pi_i : S_1 \to \varepsilon_{p_i} S_1$ by the rules $\pi_i(s) = \varepsilon_{p_i} s$. These mappings satisfy the following conditions:

a) $\pi_i \pi_j = \begin{cases} 0 & \text{if } i \neq j, \\ \pi_i & \text{if } i = j; \end{cases}$

b) for each $s \in S_1$, $s = \varepsilon s = \varepsilon_{p_1} s + \cdots + \varepsilon_{p_t} s = \pi_{p_1}(s) + \cdots + \pi_{p_t}(s)$.

It follows that $S_1 = \pi_{p_1}(S_1) \oplus \cdots \oplus \pi_{p_t}(S_1) = \varepsilon_{p_1} S_1 \oplus \cdots \oplus \varepsilon_{p_t} S_1$.

Since $\varepsilon_{p_i} R_{\chi_1} = 0$ for each $i \in \{1, \ldots, t\}$, we conclude that

$$\varepsilon_{p_i} S_1 = \varepsilon_{p_i} TM = M_{p_i} \cong K_{2p_i} \oplus \cdots \oplus K_{mp_i},$$

where each of the modules K_{jp_i} is isomorphic either to $Z(p_i^{k_j})$ or to $\widehat{Z}_{p_i}$. Therefore,

$$S_1 \cong \bigoplus_{i=1}^{t} K_{2p_i} \oplus \cdots \oplus \bigoplus_{i=1}^{t} K_{mp_i} \cong K_{\chi_2} \oplus \cdots \oplus K_{\chi_m}.$$

Thus, $M \cong R_{\chi_1} \oplus K_{\chi_2} \oplus \cdots \oplus K_{\chi_m}$. □

Observe that the R-module S_1 occurring in the proof of Theorem 3.1 has pseudorational rank 0, and there are no other restrictions imposed on that module. Thus, we may assume that S_1 is an arbitrary finitely generated R-module of pseudorational rank 0. Taking into account the decomposition of S_1 obtained in the above proof, we arrive at the following statement.

Corollary 3.1. *If M is a finitely generated R-module of pseudorational rank* 0, *then*

$$M \cong K_{\chi_1} \oplus \cdots \oplus K_{\chi_m},$$

where $\chi_1, \ldots, \chi_m$ is a sequence of almost zero characteristics.

Corollary 3.2. *The generalized characteristics constitute a complete and independent system of invariants for the class of finitely generated R-modules whose pseudorational rank is equal to* 0 *or* 1.

§4. Decomposability of certain R-modules

Let M be a finitely generated R-module. We denote by $\rho(M)$ the smallest number of generators of the module M. It is clear that $\rho(M) \geq r^*(M)$.

Theorem 4.1. *If the generalized characteristic of a finitely generated reduced R-module M consists of equal characteristics $\chi_1, \chi_2, \ldots, \chi_n$, and the number of these characteristics is equal to $\rho(M)$, then*

$$M \cong R_{\chi_1} \oplus R_{\chi_2} \oplus \cdots \oplus R_{\chi_n} \quad \textit{or} \quad M \cong K_{\chi_1} \oplus K_{\chi_2} \oplus \cdots \oplus K_{\chi_n}.$$

Proof. Let M be a finitely generated R-module, and let the sequence $(\chi_1, \chi_2, \ldots, \chi_n)$ be its generalized characteristic; assume that

$$\chi_1 = \chi_2 = \cdots = \chi_n = \chi \quad \text{and} \quad n = \rho(M).$$

We consider two cases.

Case 1. The characteristic χ contains infinitely many nonzero elements. Let $P_1 = \{p \in P \mid \chi_p \neq 0\}$. Since $\rho(M) = n$, there exist elements $x_1, \ldots, x_n \in M$ such that $\langle x_1, \ldots, x_n \rangle_R = M$. Consider an arbitrary combination

$$r_1 x_1 + \cdots + r_n x_n = 0, \quad \text{where } r_1, \ldots, r_n \in R. \tag{4.1}$$

For each prime integer p, (4.1) induces the combination

$$\alpha_{p1}\varepsilon_p x_1 + \cdots + \alpha_{pn}\varepsilon_p x_n = 0, \quad \text{where } \alpha_{p1} = \varepsilon_p r_1, \ \ldots, \ \alpha_{pn} = \varepsilon_p r_n. \tag{4.2}$$

Recalling the restrictions imposed on the characteristic, we see that

$$\langle \varepsilon_p x_1, \ldots, \varepsilon_p x_n \rangle_{\widehat{Z}_p} = \begin{cases} 0 & \text{if } \chi_p = 0, \\ \bigoplus_n Z(p^{m_p}) & \text{if } 0 < \chi_p = m_p < \infty, \\ \bigoplus_n \widehat{Z}_p & \text{if } \chi_p = \infty. \end{cases} \tag{4.3}$$

Formulas (4.2) and (4.3) imply that

$$\chi_p = \infty \implies \alpha_{p1} = \cdots = \alpha_{pn} = 0 \quad \text{and} \quad \chi_p = m_p \in \mathbb{N} \implies \alpha_{p1}, \ldots, \alpha_{pn} \in p^{m_p}\widehat{Z}_p.$$

If the characteristic χ contains infinitely many ∞ symbols, then infinitely many p-components of the coefficients $r_1, \ldots, r_n$ in (4.1) are equal to 0, which means, by properties 1^0 and 2^0 of the ring of pseudorational numbers, that these coefficients belong to the ideal T.

If the characteristic χ contains only a finite number of ∞ symbols, then it contains infinitely many p-components that are positive integers. Consider the set $P_2 = \{p \in P \mid \chi_p \in \mathbb{N}\}$. Since a prime integer p divides a pseudorational number r if and only if p divides

$\varepsilon_p r$, it follows that p divides $r_1, \ldots, r_n$ for all $p \in P_2$. But a pseudorational number has infinitely many prime divisors only if it belongs to T. Therefore, $r_1, \ldots, r_n \in T$.

Thus, we have proved that the elements $x_1, \ldots, x_n$ are independent modulo TM, which means that $r^*(M) = n$. Now, by Proposition 2.2, we have $r^*(\Delta M_X) = 0$ and

$$\Delta M_X = \bigoplus_{p\in P} \varepsilon_p \Delta M_X = \bigoplus_{p\in P} K_p, \tag{4.4}$$

where $K_p = \bigoplus_n p^{m_p} \widehat{Z}_p$ if $0 \le \chi_p = m_p < \infty$, and $K_p = 0$ if $\chi_p = \infty$.

In the R-module $L = R_{\chi_1} \oplus R_{\chi_2} \oplus \cdots \oplus R_{\chi_n}$, we take a system of elements $Y = \{y_1, \ldots, y_n\}$, where

$$y_1 = (1, 0, \ldots, 0),\ y_2 = (0, 1, \ldots, 0),\ \ldots,\ y_n = (0, 0, \ldots, 1).$$

Obviously, Y is a generating system of the module L, and $\Delta L_Y = \Delta M_X$. By Corollary 2.1, the modules M and L are isomorphic.

Case 2. χ is an almost zero characteristic. Then $r^*(M) = 0$, and consequently, M is determined by its generalized characteristic, i.e.,

$$M \cong K_{\chi_1} \oplus K_{\chi_2} \oplus \cdots \oplus K_{\chi_n}. \qquad \square$$

Corollary 4.1. *If the generalized characteristic of a finitely generated reduced R-module M consists of characteristics $\chi_1, \chi_2, \ldots, \chi_n$ that differ only at a finite number of positions, and if $n = \rho(M)$, then*

$$M \cong R_{\chi_1} \oplus R_{\chi_2} \oplus \cdots \oplus R_{\chi_n} \quad \text{or} \quad M \cong K_{\chi_1} \oplus K_{\chi_2} \oplus \cdots \oplus K_{\chi_n}.$$

Proof. If an R-module M satisfies the assumptions of the corollary, then in the ring R there exists an idempotent $(1-\varepsilon)$ such that the module $(1-\varepsilon)M$ satisfies the conditions of Theorem 4.1. If $r^*(M) = 0$, then

$$M \cong K_{\chi_1} \oplus K_{\chi_2} \oplus \cdots \oplus K_{\chi_n},$$

and if $r^*(M) \neq 0$, then

$$M = (1-\varepsilon)M \oplus \varepsilon M \cong R_{\chi_1} \oplus R_{\chi_2} \oplus \cdots \oplus R_{\chi_n}. \qquad \square$$

§5. Quotient divisible mixed groups

In this section we consider the categories $\mathcal{QTF}$ and $\mathcal{QD}$, the objects of which are, respectively, the torsion free Abelian groups of finite rank and the quotient divisible mixed groups, and the morphisms of which are quasihomomorphisms.

In [3] it was proved that the categories $\mathcal{QTF}$ and $\mathcal{QD}$ are mutually dual. On the other hand, in [4] Fomin constructed a category $\mathcal{F}$ whose objects are finitely generated R-modules with a chosen free generating system, and whose morphisms are pairs of quasihomomorphisms; he proved that the categories $\mathcal{F}$ and $\mathcal{QD}$ are equivalent. In what follows, we shall try to use the "nearness" of the groups in $\mathcal{QTF}$ and $\mathcal{QD}$ to R-modules, in order to extend the above results to these categories.

Definition 5.1 ([3])**.** A group G is said to be quotient divisible if it contains no periodic divisible subgroup but contains a free subgroup F of finite rank such that G/F is a periodic divisible group.

The group F occurring in Definition 5.1 is called the *fundamental subgroup*, and a fixed linearly independent generating system $X = \{x_1, \ldots, x_n\}$ of the group F is the *fundamental system* of the quotient divisible group G.

Let G be any reduced quotient divisible mixed group; we denote by $\widehat{G}$ its Z-adic completion. The canonical homomorphism $\alpha\colon G \to \widehat{G}$ is a monomorphism, because

$\ker\alpha = \bigcap_{n\in N} nG = G^1 = 0$. The group $\widehat{G}$ is a $\widehat{Z}$-module, and consequently a module over the ring of pseudorational numbers.

Definition 5.2 ([4]). The R-module $\mathcal{R}(G) = \operatorname{div} G \oplus \langle \alpha(G) \rangle_R$ is called the *pseudorational type* of the quotient divisible group G.

The *pseudorational rank* $r^*(G)$ of a quotient divisible group G is defined as the pseudorational rank of its pseudorational type:

$$r^*(G) = r^*(\mathcal{R}(G)).$$

Obviously, there exists an embedding $\varphi : G \to \mathcal{R}(G)$; below we shall always identify the group G with its image $\varphi(G)$.

By definition, the *generalized characteristic* of a quotient divisible group is the generalized characteristic of its pseudorational type.

Theorem 5.1. *A quotient divisible group is an R-module if and only if its generalized characteristic is locally free.*

Proof. If G is the additive group of an R-module, then, for every prime integer p,

$$G = (1-\varepsilon_p)G \oplus \varepsilon_p G = (1-\varepsilon_p)G \oplus \widehat{G}_p.$$

In [4] it was proved that $\mathcal{R}(G)$ is a finitely generated R-module; therefore, $\widehat{G}_p$ is a finitely generated $\widehat{Z}_p$-module. Consequently, it decomposes into a direct sum of finitely many cyclic $\widehat{Z}_p$-modules. If $\widehat{G}_p$ is not finite, then G contains a direct summand of the form $\widehat{Z}_p$, which is impossible, because Definition 5.1 implies that G is a group of finite rank. Thus, for every prime integer p the group $\widehat{G}_p$ is finite, which means that the generalized characteristic of G is locally free.

Let G be a quotient divisible group whose generalized characteristic is locally free. To prove the "if" part, it suffices to show that $G = \mathcal{R}(G)$, i.e., that $rg \in G$ for all $r \in R$, $g \in G$.

Property 1^0 of the ring of pseudorational numbers shows that each number $r \in R$ can be represented in the form

$$r = (1-\varepsilon)|r| + \varepsilon r,$$

where $|r| = \frac{m}{n} \in Q$, $\varepsilon = \varepsilon_{p_1} + \cdots + \varepsilon_{p_k}$ is an idempotent of the ring R, and all prime divisors of n are contained in the set $\{p_1, \dots, p_k\}$. Since the generalized characteristic of G is locally free, we obtain

$$G = (1-\varepsilon)G \oplus \varepsilon G, \tag{5.1}$$

where the group $(1-\varepsilon)G$ is p-divisible for each $p \in \{p_1, \dots, p_k\}$. If $g \in G$, then

$$rg = (1-\varepsilon)\frac{m}{n}g + \varepsilon rg. \tag{5.2}$$

But we have proved above that the element $(1-\varepsilon)\frac{m}{n}g$ is contained in $(1-\varepsilon)G$, and the element εrg is contained in εG. Thus, by (5.1) and (5.2), we have $rg \in G$, whence $G = \mathcal{R}(G)$. $\square$

In [2] it was proved that the class of quotient divisible groups with locally free generalized characteristic coincides with the well-known class $\mathcal{G}$ of self-small groups of finite torsion and free rank, which was introduced by Glaz and Wickless in [6].

The following facts are immediate consequences of Theorem 3.1, Property 9^0, and Theorem 4.1.

Corollary 5.1. *If G is a quotient divisible group with locally free generalized characteristic, then $r^*(G) = r(G)$.*

Corollary 5.2. *Let G be a quotient divisible group with locally free generalized characteristic. G is a group of pseudorational rank 1 if and only if $G \cong R_\chi$, where χ is a characteristic containing no infinite p-component.*

Corollary 5.3. *If G is a quotient divisible group with locally free generalized characteristic, and if the reduced part of G is a group of pseudorational rank 1, then*

$$G \cong R_{\chi_1} \oplus R_{\chi_2} \oplus \cdots \oplus R_{\chi_n},$$

where the characteristics $\chi_1, \chi_2, \ldots, \chi_n$ do not contain the ∞ symbols, the characteristic χ_1 contains infinitely many nonzero elements, and $\chi_2, \ldots, \chi_n$ are almost zero characteristics.

Let G be a quotient divisible group, and let $F = \bigoplus_{i=1}^{n} Zx_i$ be its fundamental subgroup. We define two sets depending on G:

$$\nabla G_X = \{(r_1, \ldots, r_n) \in R^n \mid r_1x_1 + \cdots + r_nx_n \in G\},$$
$$\Delta G_X = \{(r_1, \ldots, r_n) \in R^n \mid r_1x_1 + \cdots + r_nx_n = 0\}.$$

Obviously, ∇G_X is a group, ΔG_X is a module over the ring of pseudorational numbers, and $\Delta G_X \subset \nabla G_X$. We call ΔG_X the *module of pseudorational relations* of the quotient divisible group G.

Lemma 5.1. *If G is a quotient divisible mixed group, then*

$$G \cong \nabla G_X / \Delta G_X.$$

Proof. Let $\varphi : \nabla G_X \to G$ be the mapping defined by the rule

$$\varphi(r_1, \ldots, r_n) = r_1x_1 + \cdots + r_nx_n.$$

This is an epimorphism of groups, and its kernel coincides with ΔG_X; the homomorphism theorem yields $G \cong \nabla G_X / \Delta G_X$. □

Theorem 5.2 ([4])**.** *If H is a reduced R-module, or G is a divisible R-module, then* $\mathrm{Hom}_Z(G, H) = \mathrm{Hom}_R(G, H)$.

Lemma 5.2. *Let G, H be quotient divisible groups; assume that either H is a reduced group, or G is a divisible group. Next, let $\bigoplus_{i=1}^{n} Zx_i$ be a fundamental subgroup of G, and let $\varphi : G \to H$ be a group homomorphism. If*

$$g = r_1x_1 + \cdots + r_nx_n \in G, \quad r_1, \ldots, r_n \in R,$$

then $\varphi(g) = r_1\varphi(x_1) + \cdots + r_n\varphi(x_n)$.

Proof. We analyze several cases.

Case 1. The groups G and H are reduced. Let $\widehat{G}$ and $\widehat{H}$ denote the Z-adic completions of the groups G and H; there exists a unique homomorphism φ^* such that the diagram

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & H \\ \downarrow \mu & & \downarrow \nu \\ \widehat{G} & \xrightarrow{\varphi^*} & \widehat{H} \end{array}$$

is commutative. Since the mappings μ and ν are monomorphisms, we may assume that $G \subset \widehat{G}$ and $H \subset \widehat{H}$. Since $\widehat{G}$ and $\widehat{H}$ are reduced R-modules, Theorem 5.2 applies, showing that

$$\varphi(g) = \varphi(r_1x_1 + \cdots + r_nx_n) = \varphi^*(r_1x_1 + \cdots + r_nx_n)$$
$$= r_1\varphi^*(x_1) + \cdots + r_n\varphi^*(x_n) = r_1\varphi(x_1) + \cdots + r_n\varphi(x_n).$$

Case 2. The groups G and H are torsion free and divisible. Then they are divisible R-modules, and by Theorem 5.2 we have

$$\varphi(g) = \varphi(r_1 x_1 + \cdots + r_n x_n) = r_1\varphi(x_1) + \cdots + r_n\varphi(x_n).$$

Case 3. The group G is divisible and $H = D \oplus H_1$, where D is a divisible group and H_1 is a reduced group. Since $\operatorname{Hom}(G, H) = \operatorname{Hom}(G, D)$, this case reduces to Case 2.

Case 4. The group H is reduced and $G = D \oplus G_1$, where D is a divisible group and G_1 is a reduced group. Since $\operatorname{Hom}(G, H) = \operatorname{Hom}(G_1, H)$, this case reduces to Case 1. □

Proposition 5.1. *Two quotient divisible mixed groups are isomorphic if and only if they admit equal modules of pseudorational relations.*

Proof. Let G and H be quotient divisible mixed groups, and let ΔG_X and ΔH_Y be their modules of pseudorational relations. Suppose

$$\Delta G_X = \Delta H_Y. \tag{5.3}$$

If $(r_1, \dots, r_n)$ is an element of ∇G_X, then

$$r_1 x_1 + \cdots + r_n x_n = g \in G. \tag{5.4}$$

Since X is a fundamental system of G, the group $G/\langle X\rangle$ is periodic. Therefore, there exists $m \in \mathbb{N}$ such that

$$mg = m_1 x_1 + \cdots + m_n x_n, \tag{5.5}$$

where $m_1, \dots, m_n$ are integers. By (5.4) and (5.5), we have

$$(mr_1 - m_1)x_1 + \cdots + (mr_n - m_n)x_n = 0,$$

i.e., $((mr_1 - m_1), \dots, (mr_n - m_n)) \in \Delta G_X$. Recalling (5.3), we see that

$$(mr_1 - m_1)y_1 + \cdots + (mr_n - m_n)y_n = 0$$

or

$$m(r_1 y_1 + \cdots + r_n y_n) = m_1 y_1 + \cdots + m_n y_n \in \langle Y\rangle \subset H. \tag{5.6}$$

Since H is a pure subgroup of the group $\mathcal{R}(H)$, relation (5.6) implies the existence of an element $h \in H$ such that

$$mh = m_1 y_1 + \cdots + m_n y_n. \tag{5.7}$$

By (5.6) and (5.7), we have $m(r_1 y_1 + \cdots + r_n y_n - h) = 0$, i.e.,

$$r_1 y_1 + \cdots + r_n y_n - h = t \in t(\mathcal{R}(H)).$$

But $t(\mathcal{R}(H)) = t(H)$, whence $r_1 y_1 + \cdots + r_n y_n = h - t \in H$, and

$$(r_1, \dots, r_n) \in \nabla H_Y. \tag{5.8}$$

This shows that $\nabla G_X \subseteq \nabla H_Y$. Similarly we can prove that $\nabla H_Y \subseteq \nabla G_X$, i.e., $\nabla G_X = \nabla H_Y$. Thus,

$$G \cong \nabla G_X/\Delta G_X = \nabla H_Y/\Delta H_Y \cong H.$$

The "only if" part is proved.

Conversely, let $\varphi : G \to H$ be an isomorphism, and let $X = \{x_1, \dots, x_n\}$ be a fundamental system in G. Then $Y = \{\varphi(x_1), \dots, \varphi(x_n)\}$ is a fundamental system in H; moreover,

$$r_1 x_1 + \cdots + r_n x_n = 0 \iff r_1\varphi(x_1) + \cdots + r_1\varphi(x_n) = 0,$$

which means that $\Delta G_X = \Delta H_Y$. □

Theorem 5.3. *If the generalized characteristic of a quotient divisible group G consists of several copies of one and the same characteristic χ, and their number is equal to $r(G)$, then $G \cong \bigoplus_{r(G)} Q_\chi$, where Q_χ is a quotient divisible group of rank 1 and characteristic χ.*

Proof. Let G be a group satisfying the assumptions of the theorem, and let $X = \{x_1, \dots, x_n\}$ be a fundamental system of this group. We consider two cases.

Case 1. G is a reduced group.

In [4] it was shown that X is a generating system of the pseudorational type of the group G, i.e., $\mathcal{R}(G) = \langle x_1, \dots, x_n\rangle_R$. Precisely as in the proof of Theorem 4.1, it can be shown that the system X is independent modulo $T\mathcal{R}(G)$. Therefore, $r(G) = r^*(\mathcal{R}(G)) = n$. It follows that X is a minimal generating system of the module $\mathcal{R}(G)$, which means that $\mathcal{R}(G)$ satisfies the assumptions of Theorem 4.1. Hence,

$$\mathcal{R}(G) \cong \bigoplus_n R_\chi. \tag{5.9}$$

This implies that $r^*(\mathcal{R}(G)) = n$; hence, by Proposition 2.2, $r^*(\Delta\mathcal{R}(G)_X) = r^*(\Delta G_X) = 0$. Therefore,

$$\Delta G_X = \bigoplus_{p\in P} \varepsilon_p \Delta G_X = \bigoplus_{p\in P} K_p, \tag{5.10}$$

where $K_p = \bigoplus_n p^{m_p}\widehat{Z}_p$ if $0 \le \chi_p = m_p < \infty$, and $K_p = 0$ if $\chi_p = \infty$.

Let $H = \bigoplus_n Q_\chi$. Since the decomposition (5.10) of the module ΔG_X is completely determined by the generalized characteristic of the group G, and the group H has the same generalized characteristic, we obtain $\Delta G = \Delta H$. Thus, the groups G and H are isomorphic by Proposition 5.1.

Case 2. $G = \operatorname{div} G \oplus G/\operatorname{div} G$ and $\operatorname{div} G \ne 0$.

Observe that the pseudorational types of the groups G and $G/\operatorname{div} G$ differ only in the direct summand $\operatorname{div} G$; consequently, their generalized characteristics coincide. Let

$$z_1 = x_1 + \operatorname{div} G,\ \ z_2 = x_2 + \operatorname{div} G,\ \ \dots,\ \ z_n = x_n + \operatorname{div} G.$$

Then $\mathcal{R}(G/\operatorname{div} G) = \langle z_1, \dots, z_n\rangle_R$. The conditions of the theorem imply that

$$\langle z_1\rangle_{\widehat{Z}_p} \cong \langle z_2\rangle_{\widehat{Z}_p} \cong \dots \cong \langle z_n\rangle_{\widehat{Z}_p}$$

for every prime integer p. Thus,[2] $o(\varepsilon_p z_1) = \dots = o(\varepsilon_p z_n)$ for each prime p, which means that $o(z_1) = \dots = o(z_n)$. Since the system $\{x_1, \dots, x_n\}$ is independent, we see that either $r(G/\operatorname{div} G) = r(G)$ or $r(G/\operatorname{div} G) = 0$. Since $\operatorname{div} G \ne 0$, we conclude that $r(G/\operatorname{div} G) = 0$, i.e., $r(\operatorname{div} G) = r(G)$. Consequently, the group $G/\operatorname{div} G$ is periodic.

Thus, $G/\operatorname{div} G$ is a quotient divisible group, the generalized characteristic of which is locally free and has pseudorational rank 0. Now, by Theorem 5.1 and Corollary 3.1, we have $G \cong \bigoplus_n Q_\chi$, where $Q_\chi = Q \oplus K_\chi$ is a quotient divisible group of rank 1. □

Observe that a quotient divisible group Q_χ of rank 1 is determined (up to isomorphism) by its characteristic χ and can be described as the pure hull of 1 in the group R_χ. In particular, if χ does not contain the ∞ symbols, then $Q_\chi = R_\chi$.

Corollary 5.4. *If the generalized characteristic of a quotient divisible mixed group consists of characteristics of one and the same type, and the number of these characteristics is equal to the rank of the group, then this group decomposes into a direct sum of quotient divisible groups of rank* 1.

[2] Recall that $o(x)$ is the standard notation for the order of the element x; see [1].

Proof. Let G be a group satisfying the assumptions of the corollary, and let $(\chi_1, \dots, \chi_n)$ be its generalized characteristic:

$$\chi_1 = (m_{1p})_{p\in P}, \chi_2 = (m_{2p})_{p\in P}, \dots, \chi_n = (m_{np})_{p\in P}.$$

Consider the set $P_1 = \{p_1, \dots, p_k\}$ of all prime integers such that at least one of the equalities

$$m_{1p} = m_{2p} = \dots = m_{np}$$

fails. Since all m_{ip_j} $(1 \le i \le n,\ 1 \le j \le k)$ are nonnegative integers, the groups $\varepsilon_p \mathcal{R}(G) = \widehat{G}_p = G_p$ are finite for all $p \in P_1$. Let $\varepsilon = \varepsilon_{p_1} + \dots + \varepsilon_{p_k}$; we have the direct decomposition $G = (1-\varepsilon)G \oplus \varepsilon G$. Obviously, $(1-\varepsilon)G$ is a quotient divisible group satisfying the requirements of Theorem 5.3, and εG is an R-module of pseudorational rank 0. Hence,

$$G = (1-\varepsilon)G \oplus \varepsilon G \cong \bigoplus_n Q_\chi \oplus K_{\varphi_1} \oplus \dots \oplus K_{\varphi_n} = Q_{\chi_1} \oplus \dots \oplus Q_{\chi_n}. \qquad \square$$

For torsion free groups of finite rank, there is a result similar to Corollary 5.4. Namely, in [15] it was proved that if the Richman type of a torsion free group of finite rank consists of several copies of one and the same type and the number of these copies is equal to the rank of the group, then the group is completely decomposable; i.e., it decomposes into a direct sum of groups of rank 1.

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