

CLASSIFICATION OF FINITE COMMUTATIVE GROUP SCHEMES OVER COMPLETE DISCRETE VALUATION RINGS; THE TANGENT SPACE AND SEMISTABLE REDUCTION OF ABELIAN VARIETIES

M. V. BONDARKO

ABSTRACT. A complete classification is obtained for finite connected flat commutative group schemes over mixed characteristic complete discrete valuation rings. The group schemes are classified in terms of their Cartier modules. The equivalence of various definitions of the tangent space and the dimension for these group schemes is proved. This shows that the minimal dimension of a formal group law that contains a given connected group scheme S as a closed subgroup is equal to the minimal number of generators for the coordinate ring of S . The following reduction criteria for Abelian varieties are deduced.

Suppose K is a mixed characteristic local field with residue field of characteristic p , L is a finite extension of K , and $\mathfrak{D}_K \subset \mathfrak{D}_L$ are the rings of integers for K and L . Let e be the absolute ramification index of L , let $s = \lceil \log_p(pe/(p-1)) \rceil$, let e_0 be the ramification index of L/K , and let $l = 2s + v_p(e_0) + 1$.

For a finite flat commutative \mathfrak{D}_L -group scheme H , denote by TH the \mathfrak{D}_L -dual to J/J^2 . Here J is the augmentation ideal of the coordinate ring of H .

Let V be an m -dimensional Abelian variety over K . Suppose that V has semistable reduction over L .

Theorem (A). V has semistable reduction over K if and only if for some group scheme H over \mathfrak{D}_K there exist embeddings of H_K in $\text{Ker}[p^l]_{V,K}$ and of $(\mathfrak{D}_L/p^l\mathfrak{D}_L)^m$ in $TH_{\mathfrak{D}_K}$.

Theorem (B). V has ordinary reduction over K if and only if for some $H_K \subset \text{Ker}[p^l]_{V,K}$ and M unramified over K we have $H_M \cong (\mu_{p^l, M})^m$. Here μ denotes the group scheme of roots of unity.

INTRODUCTION

In the paper [3], the Cartier modules of finite, flat, connected, commutative group schemes over the rings of integers of complete discrete valuation fields (defined by Oort) were used to obtain results on the generic fiber of (finite, flat, commutative) group schemes. As an application, a certain finite (see below) p -adic good reduction criterion for Abelian varieties was proved.

In the present paper, we use the results of the previous work to obtain a complete classification of finite, connected, flat, commutative group schemes over mixed characteristic complete discrete valuation rings in terms of their Cartier modules. We give an explicit description of the image of the Oort functor.

We also prove the equivalence of various definitions of the tangent spaces and the dimension of these group schemes. As an application, certain finite p -adic criteria for semistable and ordinary reduction of Abelian varieties are proved. We call these criteria

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finite because, in contrast to Grothendieck's criteria (see [6]), it suffices to check certain conditions on some finite p -torsion subgroup of V (instead of the entire p -torsion).

We introduce some notation.

K is a complete discrete valuation field of characteristic 0 with residue field of characteristic p , L is a finite extension of K , and $\mathfrak{D}_K \subset \mathfrak{D}_L$ are the corresponding rings of integers. Let e be the absolute ramification index of L , and let $s = \lceil \log_p(pe/(p-1)) \rceil$. Put $e_0 = [L : K_0]$, where K_0 is the maximal unramified subextension of L/K (for the usual local fields, e_0 is equal to $e(L/K)$), and $l' = s + v_p(e_0) + 1$, $l = 2s + v_p(e_0) + 1$.

For a \mathfrak{D}_L -group scheme H , we denote by TH the \mathfrak{D}_L -dual of the module J/J^2 . Here J is the augmentation ideal of the coordinate ring of H .

Theorem (A). *Let V be an m -dimensional Abelian variety over K . Suppose that V has semistable reduction over L .*

Then V has semistable reduction over K if and only if for some finite flat group scheme H/\mathfrak{D}_K we have $TH_{\mathfrak{D}_L} \supset (\mathfrak{D}_L/p^l\mathfrak{D}_L)^m$ (i.e., there exists an embedding); moreover, there exists a monomorphism $g : H_K \rightarrow \text{Ker}[p^l]_{V,K}$.

Note that for $e < p - 1$ we have $l = 1$; hence, Theorem (A) is a generalization of Theorem 5.3 of [5] (where only the potentially good reduction case was considered).

Our technique also allows us to prove the following criteria for ordinary reduction (see Subsection 5.2).

Theorem (B). I. *Let V be an Abelian variety of dimension m over K that has good reduction over L .*

The following conditions are equivalent.

- (1) *V has good ordinary reduction over K .*
- (2) *For a certain multiplicative type (i.e., dual-étale) group scheme H/\mathfrak{D}_K , we have $TH_{\mathfrak{D}_L} \approx (\mathfrak{D}_L/p^{l'}\mathfrak{D}_L)^m$; moreover, there exists a monomorphism $g : H_K \rightarrow \text{Ker}[p^{l'}]_{V,K}$.*
- (3) *For some $H_K \subset \text{Ker}[p^{l'}]_{V,K}$ and some M unramified over K , we have $H_M \cong (\mu_{p^{l'},M})^m$. Here $\mu_{p^{l'},M}$ is the group scheme of $p^{l'}$ th roots of unity.*

II. *Let V be an Abelian variety of dimension m over K that has semistable reduction over L .*

The following conditions are equivalent.

- (1) *V has ordinary reduction over K .*
- (2) *For some multiplicative type group scheme H over \mathfrak{D}_K , there exist embeddings of H_K in $\text{Ker}[p^l]_{V,K}$ and of $(\mathfrak{D}_L/p^l\mathfrak{D}_L)^m$ in $TH_{\mathfrak{D}_L}$.*
- (3) *For some $H_K \subset \text{Ker}[p^l]_{V,K}$ and some M unramified over K , we have $H_M \cong (\mu_{p^l,M})^m$.*

In §1 we recall the definition of the Cartier modules for formal groups. As a rule, in this paper we shall use the so-called invariant Cartier modules (i.e., D_F) defined in [4] and [2]. We reproduce Oort's definition of the Cartier module $C(S)$ for a finite connected group scheme S (see [9]) and recall that the Oort functor is fully faithful. We recall the definitions of a closed submodule of a Cartier module and of a separated Cartier module. We also recall the relationship between closed submodules and closed subgroups (see [3]). We prove three technical lemmas on the Cartier modules related to the reduction of formal groups and finite group schemes.

§2 is devoted to the main classification theorem. We describe the image of the Oort functor; this gives us a full classification of finite, flat, connected, commutative group schemes over the rings of integers of complete discrete valuation fields. The proof of the fact that for any S the module $C(S)$ satisfies the conditions of Theorem 2.1.1 is easy. In the proof of the converse statement, we present a given Cartier module as a factor

of $C(F)$ for some formal group F ; next, we prove that F can be chosen to be of finite height. Our classification implies immediately that Ext^1 for connected group schemes coincides with Ext^1 for their Cartier modules.

In §3 we prove that the tangent space functor can easily be described in terms of Oort modules. This implies that the minimal dimension of a finite height formal group F that contains a given connected group scheme S as a closed subgroup is equal to the minimal number of generators for the coordinate ring of S . We also prove that a connected group scheme of exponent p^r is a truncated Barsotti–Tate group if and only if $TH \approx (\mathfrak{D}_L/p^r\mathfrak{D}_L)^m$.

At the beginning of §4, we recall the main generic fiber result and a certain descent result of [3]. Next, we present some new statements about descent for p -divisible groups. We prove that an Abelian variety has semistable reduction if and only if its “formal part” has “good reduction”. Using these statements and the results on tangent spaces, we prove a criterion for good reduction of an Abelian variety with potentially good reduction. This criterion provides a positive answer to the question on the existence of a finite p -adic criterion, attributed by B. Conrad (see [5]) to N. Katz. Two more versions of such a criterion were proved in [3].

In §5 we prove Theorems (A) and (B).

Theorem (A) seems to be less convenient than the corresponding finite l -adic criteria (see [12, 13]). On the other hand, the property of being ordinary is p -adic. Apparently, it is difficult to check it l -adically; in particular, no l -adic analog of Theorem (B) is known.

It is impossible to prove these criteria without using tangent spaces (even in the case where $e < p - 1$).

For the reader’s convenience, we mention that the proofs of reduction criteria involve neither the classification results of §2 nor the technical lemmas of Subsection 1.2.

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Notation and conventions. We keep the notation of the Introduction ($K, L, p, e, s, e_0, l, l', \mathfrak{D}_K, \mathfrak{D}_L, V, \mu_{p^{l'}}$).

Let \bar{L} denote the residue field of \mathfrak{D}_L , let \mathfrak{M} denote the maximal ideal of \mathfrak{D}_L , and let $\pi \in \mathfrak{M}$ be some uniformizing element of L .

We shall need the quantity $t = v_p(e) + 1$.

$X = (X_i) = X_1, \dots, X_m$ and x will be formal variables.

F will usually denote an m -dimensional formal group law over \mathfrak{D}_L ; $\lambda_F = (\lambda_i(X))$, $1 \leq i \leq m$, denotes the logarithm of F ; exp_F is its composition inverse.

In this paper, if it is not explicitly stated otherwise, a “group scheme” will mean a finite, flat, commutative p -group scheme (i.e., a scheme annihilated by a power of p), and S/\mathfrak{D}_L will mean a finite, flat, commutative p -group scheme over \mathfrak{D}_L .

$\text{Cart} = \text{Cart}_p(\mathfrak{D}_L)$ will denote the p -Cartier ring over \mathfrak{D}_L (see below); a Cartier module is a module over Cart .

For a ring \mathfrak{A} , we denote by $M_{m \times n}(\mathfrak{A})$ the module of $(m \times n)$ -matrices over \mathfrak{A} ; $M_m(\mathfrak{A}) = M_{m \times m}(\mathfrak{A})$.

For finite group schemes S and T , we write $S \triangleleft T$ if S is a closed subgroup scheme of T ; we write $S \subset T$ if there is a group scheme morphism $f : S \rightarrow T$ that is injective on the generic fiber.

The number of indecomposable summands of a finitely generated \mathfrak{D}_L -module M is called the dimension of M .

A formal group F is said to be of *finite height* if $[p]_F$ is an isogeny, i.e., the scheme-theoretic kernel $\text{Ker}[p]_F$ of $[p]_F : F \rightarrow F$ is a finite group scheme. This property is stable with respect to base change (see [15]). In particular, F is a finite height formal group

over \mathfrak{D}_L if and only if it is finite height over some $P \supset \mathfrak{D}_L$, where the residue field of P is perfect.

For a finite height formal group F and a group scheme H , we write $H \triangleleft F$ if H is a closed subgroup of $\text{Ker}[p^r]_F$ for some $r > 0$ (and hence, a subgroup of the p -divisible group of F).

A group scheme or a p -divisible group (in particular, a finite height formal group) is of *multiplicative type* if its Cartier dual is étale.

§1. CARTIER MODULES; REMINDER AND SOME LEMMAS

In this section we recall the relationship between invariant Cartier modules and ordinary ones; we also recall the result of Oort and some results of earlier papers. Next, we prove a few technical lemmas.

1.1. The Cartier ring, the Cartier theory. In this subsection we introduce some notation and recall briefly the relationship between invariant Cartier modules and the “classical” ones.

For a \mathfrak{D}_L -algebra Q , we denote by $\text{Cart} = \text{Cart}(Q)$ the ring obtained by factorizing $\mathbb{Z}\langle \mathbf{f}, \langle a \rangle \rangle \langle \langle \mathbf{V} \rangle \rangle$ (here $a \in Q$, we consider noncommutative series over noncommutative polynomials) modulo the following relations:

- (1) $\langle a \rangle \langle b \rangle = \langle ab \rangle$ for all $a, b \in Q$; $\mathbf{fV} = p$;
- (2) $\langle a \rangle \mathbf{V} = \mathbf{V} \langle a^p \rangle$, $\langle a^p \rangle \mathbf{f} = \mathbf{f} \langle a \rangle$ for any $a \in Q$;
- (3) $\langle a \rangle + \langle b \rangle = \sum_{n \geq 0} \mathbf{V}^n \langle r_{p^n}(a, b) \rangle \mathbf{f}^n$

for all $a, b \in Q$, where the r_{p^n} are the polynomials defined in [7, Subsection 16.2]. In [7], \mathbf{V} was denoted by V_p , and \mathbf{f} was denoted by F_p . We shall need the property $r_{p^n}(0, x) = r_{p^n}(x, 0) = 0$.

A natural analog of (3) is valid also for $\langle a \rangle - \langle b \rangle$.

Since Cart is \mathbf{V} -complete, any finitely generated Cart -module is also \mathbf{V} -complete.

If P is a Q -algebra, we can define the Cart -module structure on $P[[\Delta]]$ in the following way: for $f = \sum_{i \geq 0} c_i \Delta^i \in P[[\Delta]]$, $a \in Q$, we define

$$\mathbf{V}f = f\Delta; \quad \mathbf{f}f = \sum_{i > 0} p c_i \Delta^{i-1}; \quad \langle a \rangle f = \sum a^{p^i} c_i \Delta^i.$$

We also recall that Cart is p -complete if Q is; if M is a $\text{Cart}(Q)$ -module, then $M/\mathbf{V}M$ has a natural structure of a Q -module defined via

$$a \cdot (x \bmod \mathbf{V}M) = \langle a \rangle x \bmod \mathbf{V}M$$

for any $x \in M$.

The main ingredients of the classical Cartier theory were the modules of curves. For a formal group F/Q and any $i > 0$, the Cartier module structure of $F(x^i Q[[x]]) = \text{Ker}(F(Q[[x]]) \rightarrow F(Q[[x]]/x^i))$ can be defined. The p -typical (abstract) Cartier module $C(F)$ is a certain direct $\text{Cart}(Q)$ -summand of $F(xQ[[x]])$.

For $h \in (\mathfrak{A}[[\Delta]])^m$, the coefficients of h are equal to $h_i = \sum_{l \geq 0} c_{il} \Delta^l$, $c_{il} \in \mathfrak{A}$; we define

$$(4) \quad h(x) = (h_i(x)), \quad 1 \leq i \leq m, \quad \text{where } h_i(x) = \sum_{l > 0} c_{il} x^{p^l}.$$

Now, let $Q = \mathfrak{D}_L$. Define $D_F = \{f \in L[[\Delta]]^m : \exp_F(f(x)) \in \mathfrak{D}_L[[x]]^m\}$. In particular, for $m = 1$ we have $\sum a_i \Delta^i \in D_F \iff \exp_F\left(\sum a_i x^{p^i}\right) \in \mathfrak{D}_L[[x]]$.

In [2, 4] it was proved that $D_F \approx C(F)$ as a Cart-module. Moreover, for $\dim F_i = m_i$, $i = 1, 2$, we have

$$\text{Cart}(D_{F_1}, D_{F_2}) = \{A \in M_{m_2 \times m_1} \mathfrak{D}_L : AD_{F_1} \subset D_{F_2}\},$$

and if $f : F_1 \rightarrow F_2$, $f \equiv AX \pmod{\text{deg } 2}$, $A \in M_{m_2 \times m_1} \mathfrak{D}_L$, then the corresponding map $f_* : D_{F_1} \rightarrow D_{F_2}$ results from an application of A .

It is easily seen (cf. the argument in Subsection 2.3 below) that for a large class of Cart-modules, any $x_i \in M$, $1 \leq i \leq m$, such that $\mathfrak{D}_L(x_i \pmod{\mathbf{V}M}) = M \pmod{\mathbf{V}M}$, generate M over Cart . Moreover, any $x \in M$ can be presented as $\sum_{1 \leq i \leq m} \sum_{0 \leq j} \mathbf{V}^j \langle a_{ij} \rangle x_i$.

In particular, this is true for $M = D_F$. Hence, we can take $x_i \in D_F$ such that $x_i \equiv e_i \pmod{\Delta}$, $1 \leq i \leq m$ (here $e_i = (0, \dots, 1, \dots, 0)$ is the i th basis vector of \mathfrak{D}_L^m).

We shall also need the following properties of the Cartier modules of formal groups.

Proposition 1.1.1. 1. *If P is the ring of integers of a complete discrete valuation field containing L , then $D_{F,P} = \text{Cart}_P D_F \subset PL[[\Delta]]^m$ and $D_F = D_{F,P} \cap L[[\Delta]]^m$.*

2. *The map*

$$D_F \pmod{\Delta} \rightarrow \{f \in xL[[x]]^m : \exp_F(f) \in \mathfrak{D}_L[[x]]^m\} / x^2L[[x]]^m$$

induced by (4) is an isomorphism.

1.2. Some results on Cartier modules related to reduction. The definition of the Cartier ring is functorial. In particular, a Cart-module structure on M induces a $\text{Cart}(\overline{L})$ -module structure on $M / \text{Cart}\langle \pi \rangle M$.

If \overline{L} is perfect, the module $C(F) / \text{Cart}\langle \pi \rangle C(F)$ can be identified with the usual (co-variant) Dieudonné module of the reduction of F .

We denote by F_π the formal group law $F_\pi(X, Y) = \pi^{-1}(\pi X, \pi Y)$. Obviously, its coefficients belong to \mathfrak{D}_L .

In the proof of the main classification theorem, we shall need the following technical results.

We assume that $x_i \equiv e_i \pmod{\Delta L[[\Delta]]^m}$.

Lemma 1.2.1. 1. $\text{Cart}\langle \pi \rangle D_F = \pi D_{F_\pi}$.

2. $p^s \pi D_{F_\pi} \subset \mathfrak{D}_L[[\Delta]]^m$.

Proof. 1. We may assume that F is p -typical (see [7]), i.e., the logarithm $\lambda = (\lambda_i)$ of F satisfies $\lambda_i = \sum_{1 \leq j \leq m, l \geq 0} a_{ijl} X_j^{p^l}$. Then for $b_i = (b_{ij})$, $b_{ij} = \sum a_{ijl} \Delta^l$, we have $b_i \in D_F$, $1 \leq i \leq m$ (see [7]). Since $b_i \equiv e_i \pmod{\Delta}$, the b_i generate D_F as a Cart-module. Similarly, the $\pi^{-1}\langle \pi \rangle b_i$ generate D_{F_π} . Since $\langle \pi \rangle b_i \in \langle \pi \rangle D_F$, we obtain $\pi D_{F_\pi} \subset \text{Cart}\langle \pi \rangle D_F$.

On the other hand, πD_{F_π} is a Cart-module. We have

$$\langle \pi \rangle \left(\sum_{1 \leq i \leq m} \sum_{0 \leq j} \mathbf{V}^j \langle a_{ij} \rangle x_i \right) = \sum_{1 \leq i \leq m} \sum_{0 \leq j} \mathbf{V}^j \langle a_{ij} \pi^{p^j - 1} \rangle \langle \pi \rangle x_i \in \pi D_{F_\pi}.$$

Consequently, $\text{Cart}\langle \pi \rangle D_F \subset \pi D_{F_\pi}$.

2. Easy calculation; see the proof of part 2 of Theorem 6.2.2 in [2]. □

For the same x_i , the following statement is true.

Lemma 1.2.2. *Let $y, z \in D_F$, $z \equiv y \pmod{\pi D_{F_\pi}}$, $y = \sum_{1 \leq i \leq m} \sum_{0 \leq j} \mathbf{V}^j \langle y_{ij} \rangle x_i$. Then z can be presented in the form*

$$(5) \quad z = \sum_{1 \leq i \leq m} \sum_{0 \leq j} \mathbf{V}^j \langle c_{ij} \rangle x_i, \quad \text{where } c_{ij} \equiv y_{ij} \pmod{\pi}.$$

Proof. We use induction on l to construct a sequence of z_l, r_l such that $z = z_l + r_l$, the z_l satisfy condition (5) with some b_{ijl} in place of c_{ij} , and $r_l \in \mathbf{V}^l \pi D_{F_\pi}$.

For $l = 0$ we take $z_0 = y, r_0 = z - y$.

Now, suppose that we have z_l, r_l for some $l \geq 0$. We consider the expansion

$$r_l = \sum_{1 \leq i \leq m} \sum_{0 \leq j} \mathbf{V}^j \langle d_{ijl} \pi \rangle x_i.$$

Then, in accordance with (3), we have

$$\mathbf{V}^l \left(\sum_{1 \leq i \leq m} (\langle b_{iil} \rangle + \langle d_{iil} \pi \rangle) x_i \right) = \mathbf{V}^l \left(\sum_{1 \leq i \leq m} \langle b_{iil} + d_{iil} \pi \rangle x_i \right) + s_l$$

for some $s_l \in \mathbf{V}^{l+1} \pi D_{F_\pi}$. Thus, we may take

$$z_{l+1} = z_l + \sum_{1 \leq i \leq m} (\langle b_{iil} + d_{iil} \pi \rangle - \langle b_{iil} \rangle) x_i.$$

Since D_F is \mathbf{V} -complete, passing to the limit yields the claim. □

We denote by W the twisted power series ring $\overline{L}[[\mathbf{V}^l]] = \sum_{i \geq 0} \mathbf{V}^i x_i$; multiplication is defined by the rule $x \mathbf{V} = \mathbf{V} x^p, x \in \overline{L}$.

Let \tilde{M} be a module over $\text{Cart}' = \text{Cart}(\overline{L})$ (hence, also a Cart-module). Since the characteristic of \overline{L} is p , we have $\mathbf{V} \mathbf{f} = p = \mathbf{f} \mathbf{V}$ in Cart' (see [7]).

Suppose that any $x \in \tilde{M}$ can be presented in the form $\sum_{1 \leq i \leq m} \sum_{0 \leq j} \mathbf{V}^j \langle y_{ij} \rangle x_i$ for $y_{ij} \in \overline{L}$. For any $z = (z_i) \in W^m$, where $z_i = \sum \mathbf{V}^j z_{ij}$, we consider

$$(z)(x) = \sum_i \sum_{j \geq 0} \mathbf{V}^j \langle z_{ij} \rangle x_i.$$

Note that, in general, $(z_1 + z_2)(x) \neq (z_1)(x) + (z_2)(x)$.

In W , we introduce a topology whose basic neighborhoods of 0 are $V^i W$. Obviously, if $s_i \rightarrow 0$ in W^m , then $\sum (s_i)(x)$ converges in \tilde{M} .

Lemma 1.2.3. *Let X be a W -submodule of W^m ; we consider the set $Y = (X)(x) = \{(z)(x); z \in X\}$. Suppose that $\mathbf{f} x_i \in Y$ for $1 \leq i \leq m$.*

Then the following statements are true.

1. *For any $y, z \in X$ there exist $u, u' \in X$ such that $(u)(x) = (y)(x) + (z)(x), (u')(x) = (y)(x) - (z)(x)$.*

Moreover, if $z \in \mathbf{V}^r W^m$ for $r > 0$, then we can choose $u, u' \equiv y \pmod{\mathbf{V}^r W^m}$.

2. *Y is a Cart' -submodule of \tilde{M} .*

3. *For any $y \in X$ and $z \in W^m \setminus X$ there exists $u \in W^m \setminus Y$ such that $(u)(x) = (z)(x) - (y)(x)$.*

Proof. 1. The proof is similar to that of the preceding lemma.

First, we construct u . We denote $(y)(x) + (z)(x)$ by a . Arguing inductively (with respect to l), we construct a sequence of $b_l, r_{li} \in X, l, i \geq 0$, such that $\lim_i r_{li} = 0$ for any $i \geq 0, (b_l)(x) + \sum_i (r_{li})(x) = a$, and $r_{li} \in \mathbf{V}^l X$.

For $l = 0$ we take $b_0 = y, r_{0,0} = z, r_{0,i} = 0$ for $i > 0$.

Now, suppose we have b_l, r_{li} for some $l \geq 0$. For $a \in \overline{L}, c \in W^m$ we have $\langle a \rangle (c)(x) = (ac)(x)$. Hence,

$$(6) \quad \langle a \rangle Y \subset Y \quad \text{for any } a \in \overline{L}.$$

Now, as in the proof of Lemma 1.2.2, we see that we can take $b_{l+1} = b_l + \sum_{i \geq 0} r_{l,i}$. Indeed, by (3),

$$(b_{l+1})(x) - (b_l)(x) - \sum_{i \geq 0} (r_{l,i})(x)$$

can be written as a sum of $(w_{l+1,i})(x)$ for some $w_{l+1,i} \in \mathbf{V}^{j_{li}} \langle \overline{L} \rangle \mathbf{f}W^m \subset \mathbf{V}^{j_{li}} X$. Here each j_{li} is greater than l and $\lim_i j_{li} = \infty$. Thus, we can take $r_{l+1,i} = w_{l+1,i}$.

Finally, we take $u = \lim_l b_l$.

Obviously, for $z \in \mathbf{V}^r W^m$ this construction yields an element u that is congruent to y modulo $\mathbf{V}^r W^m$.

The construction of u' is obtained from that of u by replacing some $+$ signs by $-$.

2. We have $\mathbf{V}Y \subset Y$. Part 1 states that $Y \pm Y \subset Y$. We also have (6). It remains to check that $\mathbf{f}Y \subset Y$.

Let $d = \sum \mathbf{V}^l d_l \in W^m$, $d_l \in \overline{L}^m$. Then

$$\mathbf{f}(d)(x) = \sum_{l \geq 0} \mathbf{V}^l (\mathbf{f}d_l)(x),$$

and $\mathbf{f}(d_l)(x) \in Y$. In accordance with Part 1, there exists a sequence $u_i \in X$, $i \geq 0$, such that $(u_i)(x) = \sum_{0 \leq j \leq i} \mathbf{V}^j (\mathbf{f}d_j)(x)$, and $u_{i+1} \equiv u_i \pmod{\mathbf{V}^{i+1}}$. Hence, the limit $u = \lim u_i$ exists; we have $(u)(x) = \mathbf{f}((d)(x))$.

Thus, $\mathbf{f}Y \subset \mathbf{f}M \subset Y$.

3. The proof is very similar to that of Part 1. We argue inductively (with respect to l) to construct a sequence of $b_l, r_{li} \in X$, $l, i \geq 0$, such that $(b_l)(x) + \sum_i (r_{l,i})(x) = (z)(x) - (y)(x)$, $\lim_i r_{l,i} = 0$, and $r_{l,i} \in \mathbf{V}^l X$.

In order to pass from $b_l \notin X$ to $b_{l+1} \notin X$, we use the fact that $(W^m \setminus X) \pm X = W^m \setminus X$. □

1.3. Oort modules of group schemes. Let S be a connected group scheme over \mathfrak{D}_L , and let $0 \rightarrow S \rightarrow F \rightarrow G \rightarrow 0$ be its resolution by means of finite height formal groups. Following Oort, we define the Cartier module of S as $C(S) = \text{Coker } f_C : C(F) \rightarrow C(G)$.

We formulate the main result of [9].

Proposition 1.3.1. *The correspondence $S \mapsto C(S)$ is a well-defined functor on the category of connected (finite, flat, commutative) group schemes over \mathfrak{D}_L ; it is fully faithful, i.e., it determines an embedding of this category in the category of Cart-modules.*

We call $C(S)$ the Oort module of S .

1.4. Closed submodules and closed subschemes. The following definition played a crucial role in [3].

Definition 1.4.1. 1. For Cart-modules $M \subset N$, we write $M \triangleleft N$ if for any $x \in M$ the condition $\mathbf{V}x \in N$ implies that $x \in M$. We call M a *closed* submodule of N .

2. A Cart-module N is said to be *separated* if $\{0\} \triangleleft N$, i.e., N has no \mathbf{V} -torsion.

The following properties can be verified easily (see [3]).

Proposition 1.4.2. I.1. *If $M_i \triangleleft M$ for $i \in I$, then $\bigcap_{i \in I} M_i \triangleleft M$.*

2. *If $M_2 \triangleleft M_1$ and $M_1 \triangleleft M$, then $M_2 \triangleleft M$.*

3. *If $M_1 \triangleleft M$ and $M_2 \subset M_1$, then $M_1/M_2 \triangleleft M/M_2$.*

II. *If $f : N \rightarrow O$ is a Cart-morphism and $M \triangleleft O$, then $f^{-1}(M) \triangleleft N$.*

For a Cartier module N , $O \subset N$, we denote by $\text{Cl}_N(O)$ the smallest closed Cartier submodule of N that contains O . In particular, $\text{Cl}_N(O) \supset \text{Cart } O$.

Obviously, if $M \subset N$ are Cart-modules, then $\text{Cl}_N(M) = M \iff \text{Cl}_{N/M}\{0\} = N/M$.

Now we recall the relationship between closed modules and closed subgroup schemes.

Theorem 1.4.3. I. For a (finite, flat, commutative) connected group scheme S , the module $C(S)$ is separated.

II.1. The closed submodules of $C(S)$ are in one-to-one correspondence with the closed subgroup schemes of S .

2. If

$$M = C(S), \quad C(H) = N \triangleleft M,$$

where $H \triangleleft S$, then

$$M/N \approx C(S/H).$$

3. Conversely, the exact sequences of connected group schemes (viewed as fppf-sheaves, i.e., the inclusion is a closed embedding) induce exact sequences of Oort modules.

III. If $f : S \rightarrow T$ is a morphism of connected schemes, then $\text{Ker } f_C = C(\text{Ker } f)$, where f_C is the induced Oort module homomorphism; we consider the kernel in the category of flat group schemes.

IV. Let M be a closed finite index submodule of $C(F)$, where F is a finite height formal group. Then M is isomorphic to $C(G)$, where G is a formal group isogenous to F .

V. If

$$M \triangleleft L[[\Delta]]^m, \quad D_F \subset M,$$

and

$$M \bmod \Delta = \mathfrak{D}_L^m,$$

then

$$M = D_F.$$

Parts I–III were proved in [3, Theorem 3.4.1].

In part IV, the existence of G follows from [3, Proposition 3.1.5]; the fact that G is isogenous to F follows from the Cartier theory immediately (see Subsection 1).

Part V was proved in [2, §5] (in a somewhat different form, yet the same proof is valid).

§2. THE MAIN CLASSIFICATION THEOREM; EXTENSIONS OF GROUP SCHEMES

This section is devoted to the classification of group schemes in terms of their Oort modules. We describe the image of the Cartier–Oort functor, thus giving a complete classification.

2.1. Formulation.

Theorem 2.1.1. I. A Cart-module M is isomorphic to $C(S)$ for a finite flat commutative connected group scheme S over \mathfrak{D}_L if and only if M satisfies the following conditions:

1. $M/\mathbf{V}M$ is a finite length \mathfrak{D}_L -module;
2. M is separated;
3. $\bigcap_{i \geq 0} \mathbf{V}^i M = \{0\}$;
4. $M = \text{Cl}_M(\langle \pi \rangle M)$.

II. The minimal dimension of a finite height formal group F such that S can be embedded in F is equal to $\dim_{\mathfrak{D}_L}(C(S)/\mathbf{V}C(S))$ (i.e., to the number of indecomposable \mathfrak{D}_L -summands in the decomposition of $C(S)/\mathbf{V}C(S)$).

III. $M = C(\text{Ker}[p^r]_F)$ for an m -dimensional formal group F if and only if, in addition to the conditions of Part I, $p^r M = 0$ and $M/\mathbf{V}M \approx (\mathfrak{D}_L/p^r \mathfrak{D}_L)^m$.

We denote $\dim_{\mathfrak{D}_L}(M/\mathbf{V}M)$ by $\dim M$.

2.2. The proof of the “only if” part of Theorem 2.1.1. Let $M = C(S)$.

Suppose that $S = \text{Ker } f$, where $f : F \rightarrow G$ is an isogeny of finite height formal group laws of dimension m . Then for $M = \text{Coker } f_*$ we have $M/\mathbf{V}M = D_G/(\Delta D_G + AD_F)$, where $f \equiv AX \pmod{\text{deg } 2}$. Since $D_F \triangleleft L[[\Delta]]^m$, we have

$$(7) \quad \begin{aligned} M/\mathbf{V}M &= D_G/(\Delta D_G + AD_F) \\ &= (D_G \pmod{\Delta})/(AD_F \pmod{\Delta}) \approx \mathfrak{D}_L^m/A\mathfrak{D}_L^m. \end{aligned}$$

We see that M satisfies condition I.1. We also see that the minimal dimension of a finite height formal group F such that S can be embedded in F is at least $\dim(C(S)/\mathbf{V}C(S))$. Moreover, if $S = \text{Ker}[p^r]_F$, then $C(S)/\mathbf{V}C(S) \approx (\mathfrak{D}_L/p^r\mathfrak{D}_L)^m$.

Condition I.2 for M is Part I of Theorem 1.4.3.

Since $f_*D_G \triangleleft D_F$, we obtain

$$C(S)/\mathbf{V}^r C(S) \cong D_G/(\mathbf{V}^r D_G + f_*D_F) \cong (D_G/\mathbf{V}^r D_G)/(f_*D_F/\mathbf{V}^r f_*D_F).$$

Hence,

$$\varprojlim M/\mathbf{V}^r M \cong \varprojlim (D_G/\mathbf{V}^r D_G)/(f_*D_F/\mathbf{V}^r f_*D_F) = D_G/f_*D_F \cong M.$$

We immediately obtain condition I.3 for M .

Now we prove I.4. Suppose that $N = \text{Cl}_M(\langle \pi \rangle M) \neq M$. Then N corresponds to some $T \triangleleft S$. We see that $U = \{x \in D_G : x \pmod{f_*D_F} \in N\} \supset \pi D_{G_\pi}$. Since $p^r S = 0$ for some $r > 0$, we have $p^r M = 0$, whence D_G/U is a module of finite period. Therefore, U corresponds to a formal group isogenous to G (see Theorem 1.4.3, Part IV). Since these conditions are stated in terms of formal groups and their Cartier modules, they are not affected by complete discrete valuation field extensions (see Proposition 1.1.1). Hence, we may assume that \bar{L} is perfect. Therefore, $M/\langle \pi \rangle M$ is the (covariant) Dieudonné module of the reduction of S . Since S is connected, its reduction is also connected. It is well known that in this case $M/\langle \pi \rangle M$ is a \mathbf{V} -torsion module, which contradicts the existence of T .

2.3. The proof of the “if” part in Theorem 2.1.1: construction of a certain formal group. Suppose that M satisfies the conditions of Part I of Theorem 2.1.1 and $\dim M = m$. First, we prove that there exists a formal group F , $\dim F = m$, such that $M \approx D_F/C$, where $C \triangleleft D_F$. Next, we verify that F can be chosen to be a finite height formal group.

We choose representatives $l_1, \dots, l_m \in M$ for (some) \mathfrak{D}_L -generators of $M/\mathbf{V}M$. We check that any $x \in M$ can be expressed as

$$(8) \quad \sum_{1 \leq i \leq l} \sum_{0 \leq j} \mathbf{V}^j \langle a_{ij} \rangle l_i.$$

Indeed, for any $x \in M$ there exist $a_i \in \mathfrak{D}_L$ such that $x = \sum_{0 \leq i} \langle a_i \rangle l_i + Vx_1$ for some $x_1 \in M$. Repeating this procedure r times, $r > 0$, we obtain

$$x = \sum_{1 \leq i \leq l} \sum_{0 \leq j < r} \mathbf{V}^j \langle a_{ij} \rangle l_i + V^r x_r$$

for some $x_r \in M$. Since $\sum \mathbf{V}^j \langle a_{ij} \rangle$ converges in Cart for any $a_{ij} \in \mathfrak{D}_L$, there exists $y \in M$ that can be presented in the form (8) and $x - y \in V^r M$ for any $r > 0$. Now condition I.3 for M implies that $x = y$.

Next, we present $\mathbf{f}l_i$ in the form (8) for all $1 \leq i \leq m$. We obtain

$$(9) \quad \mathbf{f}l_i = \sum_{1 \leq k \leq l} \sum_{0 \leq j} \mathbf{V}^j \langle a_{ijk} \rangle l_k.$$

By [7, §27.7], there exists an m -dimensional formal group law F such that

$$C(F) \approx \left(\bigoplus \text{Cart } e_i \right) / \sum \text{Cart} \left(\mathbf{f}e_i - \sum_{1 \leq k \leq l} \sum_{0 \leq j} \mathbf{V}^j \langle a_{ijk} \rangle e_k \right)$$

(here the e_i are some Cart-generating elements). We see that there exists a Cart-homomorphism h of $C(F)$ to M that maps e_i to l_i . Since the l_i generate M over Cart, h is onto.

Since M is separated, $\text{Ker } h \triangleleft C(F)$.

2.4. Choosing a finite height formal group. Now we prove that F can be chosen to be a finite height group. Since the height is determined by the reduction of F , it suffices to consider the residues of a_{ijl} , which we denote by $\overline{a_{ijk}}$. Again we denote $M / \text{Cart}\langle \pi \rangle M$ by \tilde{M} . By Lemma 1.2.2, the calculation of $\overline{a_{ijk}}$ can be done in \tilde{M} . We denote by \tilde{l}_i the images of l_i in \tilde{M} .

We embed \tilde{L} into a perfect field U . It is well known (see, e.g., Proposition 4.6.1 in [2]) that F has finite height if and only if the matrix $B = b_{ki}$, where $b_{ki} = \sum_{j \geq 0} \overline{a_{ijk}} \mathbf{V}^j$, is nondegenerate over the skew-field $W' = U((V))' = \sum_{i \gg -\infty} \mathbf{V}^i c_i$. Here $W' \supset W$ is the skew field of the twisted Laurent series over U , with multiplication defined by the rule $u^p \mathbf{V} = \mathbf{V}u$, $u \in U$.

As in Subsection 1.2, for any $z = (z_i) \in W^m$ we define $(z)(\tilde{l}) = \sum z_i \tilde{l}_i \in \tilde{M}$.

Let $v \leq m$ be the maximal possible rank of B for all possible choices of the expansion (9). We fix F and B for which this rank is attained.

Suppose that $v < m$. We fix numbers i_1, \dots, i_v and the corresponding columns b_{i_l} of B such that the b_{i_l} are W' -independent. Then all other columns belong to the W' -span of b_{i_l} for any possible choice of their coefficients. We denote $(\sum W' b_{i_l}) \cap W^m$ by X and $(X)(\tilde{l}_i) \subset \tilde{M}$ by Y . Since $W'(W^m) \neq \sum W' b_{i_l}$, we have $X \neq W^m$. Also, we have $\mathbf{V}X \subset X$, $\mathbf{V}(W^m \setminus X) \subset W^m \setminus X$. Since (for some choice of F) all columns of B belong to X , we have $\mathbf{f}\tilde{l}_i \in Y$ for $1 \leq i \leq m$. Hence, we may apply Lemma 1.2.3 with \tilde{l}_i in the role of x_i .

We choose a column $b = b_h$ for $h \notin \{i_l\}$. We have $b \in X$. We prove that we can modify F so that the new b will not belong to X (without changing b_{i_l}).

We need to choose a certain $z \in W^m \setminus X$; the choice depends on whether $Y = \tilde{M}$ or not.

If $Y = \tilde{M}$, we choose any $z \in W^m \setminus X$.

Suppose $Y \neq \tilde{M}$. By Part 3 of Lemma 1.2.3, Y is a Cart' -module. Since the closure of $\{0\}$ in \tilde{M} is equal to \tilde{M} , we obtain $Y \not\triangleleft \tilde{M}$. Hence, there exists $y \in \tilde{M} \setminus Y$ such that $\mathbf{V}y \in \tilde{M}$. We expand $y = (z)(\tilde{l}_i)$, $z = (z_i)$, $z_i \in W$. We have $z \notin X$.

In both cases, we see that there exists $c \in X$ such that $(c)(\tilde{l}) = \mathbf{V}(z)(\tilde{l})$. By Part 3 of Lemma 1.2.3, there exists $g \in W^m \setminus X$ such that $(g)(\tilde{l}) = (c)(\tilde{l}) - \mathbf{V}(z)(\tilde{l}) = 0$. By Part 1 of Lemma 1.2.3, there exists $d \in X$ such that $(d)(\tilde{l}) = -(b)(\tilde{l})$. Applying Part 3 of Lemma 1.2.3 for $z = g$, $y = d$, we conclude that b can be replaced by $b' \notin X$.

Thus, the case where $v < m$ is impossible. Therefore, $v = m$, and we can choose F so that B is invertible over W' , whence F is a finite height formal group.

Remark 2.4.1. 1. In the perfect residue field case, the argument could be simplified by using the fact that $\mathbf{V}^u M \subset \text{Cart}\langle \pi \rangle M$ for some $u > 0$. This is no longer true in the imperfect residue field case.

2. It is possible to argue in a different way. Suppose that for a map $f : F \rightarrow G$ of (not necessarily finite height) formal groups we have $C(G)/f_*C(F) \approx M$ with M satisfying the conditions I.1–I.4. Then M can be obtained from $\text{Cart}\langle \pi \rangle M$ with the help of the successive adjoining of roots of equations of the type $\mathbf{V}y = x$. Hence, $C(\overline{G})$ can be

obtained from the Cartier module for the reduction of $f_*C(F)$ by successively adjoining the roots of the equation $\mathbf{V}(x_i) = (y_i)$, $1 \leq i \leq m$. Next, an Oort-type argument can be applied to prove that $\text{Ker } f$ is a finite flat group scheme and $M = C(\text{Ker } f)$.

2.5. Completing the proof of Parts I and II. We take any m -dimensional finite height group F such that M can be presented as a factor of D_F (as a Cart-module).

We prove that the exponent of M is finite, i.e., for some $u > 0$ we have $p^u M = 0$. This is equivalent to $p^u D_F \subset \text{Ker } h$, where $h : D_F \rightarrow M$ is the map constructed in Subsection 2.3.

First, we check that for some $r > 0$ we have $p^r \tilde{M} = 0$. Recall that

$$\tilde{M} = \sum_{1 \leq i \leq m} \text{Cart}' \tilde{l}_i = \text{Cl}_{\tilde{M}}\{0\}.$$

Hence, there exists a sequence $\{0\} = M_0 \subset M_1 \subset M_2 \subset \dots \subset M_r = \tilde{M}$ of Cart' -modules such that for any $0 \leq i < r$ we have $M_{i+1} = M_i + \text{Cart } x_i$ for some $x_i \in M_{i+1}$ satisfying $\mathbf{V}x_i \in M_i$. We have $px_i = \mathbf{fV}x_i \in M_i$. Therefore, $pM_{i+1} \subset M_i$. Thus, $p^r \tilde{M} = 0$.

Next we verify that $p^g \pi D_{F_\pi} \subset \text{Ker } h$ for some $g > 0$. Since $M/\mathbf{V}M$ is a finite length \mathfrak{D}_L -module, $\text{Ker } h \text{ mod } \Delta$ is a finite index submodule of $D_F \text{ mod } \Delta = \mathfrak{D}_L^m$. If $x = \sum_{i \geq 0} x_i \Delta^i$, $x_i \in L^m$, then $\mathbf{fV}x - \mathbf{Vf}x = px_0$. Consequently, for some $d > 0$ we have $p^d \mathfrak{D}_L^m \subset \text{Ker } h$, whence $p^d \mathfrak{D}_L[[\Delta]]^m \subset \text{Ker } h$. Using Part 2 of Lemma 1.2.1, we obtain $p^{d+s} \pi D_{F_\pi} \subset \text{Ker } h$.

We have an exact sequence $0 \rightarrow \pi D_{F_\pi} / (\pi D_{F_\pi} \cap \text{Ker } h) \rightarrow M \rightarrow \tilde{M} \rightarrow 0$. Thus, we can take $u = r + g$.

Since F is of finite height, the module $D_F/p^u D_F$ corresponds to a finite, flat, connected group scheme $\text{Ker}[p^u]_F$.

Since $\text{Ker } h \triangleleft D_F$, we obtain $\text{Ker } h/p^u D_F \triangleleft D_F/p^u D_F$. Hence, by Part II.2 of Theorem 1.4.3, M corresponds to some group scheme S that is a factor of $\text{Ker}[p^u]_F$ by a closed subscheme.

2.6. The proof of sufficiency in Theorem 2.1.1, part III. Again, we choose an arbitrary m -dimensional finite height group F such that M can be presented as a factor of D_F .

It suffices to prove that for F and h constructed above we have $p^r D_F = \text{Ker } h$. Since $p^r M = 0$, we have $p^r D_F \subset \text{Ker } h$.

Since $M/\mathbf{V}M \approx (\mathfrak{D}_L/p^r \mathfrak{D}_L)^m$, relation (7) implies that $\text{Ker } h \text{ mod } \Delta = (p^r \mathfrak{D}_L)^m$. Hence, we have $p^r D_F \subset \text{Ker } h$ and $p^r D_F \text{ mod } \Delta = \text{Ker } h \text{ mod } \Delta$. By Part V of Theorem 1.4.3, we have $p^{-r} \text{Ker } h = D_F$. Hence, $p^r D_F = \text{Ker } h$.

2.7. Extensions of group schemes. Theorem 2.1.1 implies easily that Ext^1 in the category of connected group schemes can be calculated in the category of Cartier modules.

Proposition 2.7.1. *If S and T are connected group schemes, then*

$$\text{Ext}^1(S, T) = \text{Ext}_{\text{Cart}}^1(C(S), C(T)).$$

Here we consider extensions in the category of finite flat group schemes (the definition of an exact sequence is the same as always in this paper).

Proof. Since S and T are connected, any extension of T by S is also connected.

We must prove that the conditions of Part I of Theorem 2.1.1 are preserved by extensions of Cart-modules.

Suppose we have an exact sequence $0 \rightarrow C(T) \rightarrow M \rightarrow C(S) \rightarrow 0$ of Cart-modules. We check that M satisfies conditions I.1–I.4.

The sequence $C(T)/\mathbf{V}C(T) \rightarrow M/\mathbf{V}M \rightarrow C(S)/\mathbf{V}C(S) \rightarrow 0$ is right exact, and we obtain I.1.

Since $C(S)$ is separated, we obtain $C(T) \triangleleft M$. Hence, $\{0\} \triangleleft C(T) \triangleleft M$, and we obtain I.2.

Since $\bigcap_{i \geq 0} \mathbf{V}^i C(S) = \{0\}$, for any $x \in \bigcap_{i \geq 0} \mathbf{V}^i M$ we have $x \in C(T)$. Since $C(T) \triangleleft M$, we also obtain $x \in \bigcap_{i \geq 0} \mathbf{V}^i C(T)$. Now condition I.3 for $C(T)$ implies $x = 0$. We obtain I.3 for M .

Since $C(T) \subset M$, we obtain $\langle \pi \rangle C(T) \subset \langle \pi \rangle M$. Hence, $\text{Cl}_M(\langle \pi \rangle M) \supset C(T) \triangleleft M$. Therefore,

$$\text{Cl}_M(\langle \pi \rangle M) = \{x \in M : x \bmod C(T) \in \text{Cl}_C(S)(\text{Cart} \langle \pi \rangle C(S))\} = M.$$

We obtain I.4 for M . □

§3. THE TANGENT SPACE OF A FINITE GROUP SCHEME

The tangent space of a group scheme (defined in a natural way) can easily be expressed in terms of its Oort module; this makes the tangent space an important invariant of group schemes.

3.1. Definition; expression in terms of the Oort module. We introduce a natural definition of the tangent space TS for a finite group scheme S .

Definition 3.1.1. For a finite flat group scheme S , we denote by TS the \mathfrak{D}_L -dual of J/J^2 (i.e., $\text{Hom}_{\mathfrak{D}_L}(J/J^2, L/\mathfrak{D}_L)$), where J is the augmentation ideal of the coordinate ring of S .

Remark 3.1.2. 1. Obviously, the tangent space is an additive functor on the category of finite group schemes.

2. The definition implies that the tangent space of $S_P = S \times_{\text{Spec } \mathfrak{D}_L} \text{Spec } P$ (defined as the projective limit of $\text{Hom}_P(J_P/J_P^2, P/\pi^i P)$) is equal to $TS \otimes_{\mathfrak{D}_L} P$. Here P is any \mathfrak{D}_L -algebra (commutative, with unit).

3. It is well known that the tangent space of a group scheme is equal (i.e., naturally isomorphic) to the tangent space of its connected component.

Proposition 3.1.3. 1. TS is naturally isomorphic to $C(S_0)/\mathbf{V}C(S_0)$, where S_0 is the connected component of S .

2. A connected group scheme morphism $f : S \rightarrow T$ is a closed embedding if and only if the induced map of the tangent spaces is an embedding.

3. If $0 \rightarrow H \rightarrow S \rightarrow T \rightarrow 0$ is an exact sequence of connected group schemes (in the category of fppf-sheaves, i.e., $H \triangleleft S$), then the corresponding sequence of tangent spaces is also exact.

4. If $H \subset S$, then the \mathfrak{D}_L -length of TH is not greater than the length of TS ; they are equal if and only if $H = S$.

Proof. 1. Since the connected component is functorial, it suffices to prove the statement for $S_0 = S$. We denote the isomorphism we want to construct by i_S .

For a formal group law (i.e., for a formal Lie group) F , let J_F denote the augmentation ideal of the coordinate ring of F . It is well known that J_F/J_F^2 is the cotangent space of F (at 0); i.e., we have a canonical functorial isomorphism

$$TF = \text{Hom}_{\mathfrak{D}_L}(J_F/J_F^2, \mathfrak{D}_L) \cong TF' = \text{Ker}(F(x\mathfrak{D}_L[[x]]) \rightarrow F(x\mathfrak{D}_L[[x]]/x^2)).$$

Via \log_F , the module TF' is canonically isomorphic to $\{f \in xL[[x]]^m : \exp_F(f) \in \mathfrak{D}_L[[x]]^m\}/x^2L[[x]]^m$. Applying Part 6 of Proposition 1.1.1, we obtain $T_F \cong D_F \bmod \Delta$. Hence, we have a functorial isomorphism $i_F : D_F/\mathbf{V}D_F \rightarrow TF$.

Let $0 \rightarrow S \rightarrow F \xrightarrow{h} G \rightarrow 0$ be a resolution of S via finite height formal groups of dimension $r > 0$. Since $h_*D_F \triangleleft L[[\Delta]]^r$, we have $h_*D_F \triangleleft D_G$. By (7),

$$C(S)/\mathbf{VC}(S) = D_G/(h_*D_F + \mathbf{V}D_G) = (D_G/\mathbf{V}D_G)/h_*(D_F/\mathbf{V}D_F).$$

Also, h_* is injective on $(D_F/\mathbf{V}D_F)$. The standard schematic construction of the kernel of a formal group scheme homomorphism yields $J \cong J_F/h^*J_G$. Moreover, the existence of logarithms of formal groups implies $h^*J_G^2 = h^*J_G \cap J_F^2$. Therefore, we can define i_S by means of i_G .

We note that the construction of i_S is functorial with respect to morphisms of isogenies of formal groups (i.e., with respect to commutative squares of morphisms of formal groups).

It remains to check that i_S does not depend on the choice of a resolution for S and is functorial with respect to group scheme morphisms.

Let $0 \rightarrow S \rightarrow F_1 \xrightarrow{h_1} G_1 \rightarrow 0$ and $0 \rightarrow S \rightarrow F_2 \xrightarrow{h_2} G_2 \rightarrow 0$ be two different resolutions of S via formal groups.

In [9] it was proved (see also Proposition 2.1.1 in [3]) that there exists a resolution $0 \rightarrow S \rightarrow F \xrightarrow{h} G \rightarrow 0$ such that, for $i = 1, 2$, the following commutative diagram with exact rows can be constructed:

$$(10) \quad \begin{array}{ccccc} S & \longrightarrow & F_i & \xrightarrow{h_i} & G_i \\ \downarrow \text{id}_S & & \downarrow & & \downarrow \\ S & \longrightarrow & F & \xrightarrow{h} & G. \end{array}$$

Hence, the i_S for the first two resolutions are the same as that for the third resolution.

Now, let $f : S \rightarrow T$ be a morphism of connected (finite, flat, commutative) group schemes. Then, in accordance with [9], we can construct a commutative diagram

$$(11) \quad \begin{array}{ccccc} S & \longrightarrow & F_1 & \longrightarrow & G_1 \\ \downarrow f & & \downarrow & & \downarrow \\ T & \longrightarrow & F_2 & \longrightarrow & G_2, \end{array}$$

where the rows are resolutions of S and T . Since i_S is functorial with respect to morphisms of isogenies of formal groups, we see that i_S is natural.

2. If $H \triangleleft S$, then $C(H) \triangleleft C(S)$, so that $C(H) \cap \mathbf{VC}(S) = \mathbf{VC}(H)$. Therefore, the kernel of the map $C(H)/\mathbf{VC}(H) \rightarrow C(S)/\mathbf{VC}(S)$ is zero.

We prove the converse implication.

We may decompose f as $g \circ i \circ h$, where g is a closed embedding, i is bijective on the generic fiber, and h is epimorphic. Here h is the morphism $H \rightarrow H/\text{Ker } f$, and g corresponds to the injection $\text{Im } f_L \rightarrow S_L$; i.e., g is defined on a scheme whose generic fiber is isomorphic to $\text{Im } f_L$ (see Proposition 4.1.1 below).

Since the induced tangent space map g_* is injective, we may assume that f_L is surjective (i.e., $g = \text{id}_S$).

If the composition $i_* \circ h_*$ of the maps of tangent spaces is injective, then h_* is injective.

If we have an exact sequence $0 \rightarrow B \rightarrow H \rightarrow S \rightarrow 0$, then $\{0\} \neq TB \subset TH$, and TB maps to 0 via the map $TH \rightarrow TS$. Hence, h is an isomorphism.

It remains to consider the case where $H \subset S$, $H_L = S_L$. The map $TH \rightarrow TS$ is injective if and only if $C(H) \cap \mathbf{VC}(S) = \mathbf{VC}(H)$. Then for $x \in C(H)$ with $\mathbf{V}x \in C(S)$ we obtain $\mathbf{V}x = \mathbf{V}y$, $y \in C(H)$. Since $C(S)$ is separated, we obtain $x = y \in C(H)$. Hence, $C(H) \triangleleft C(S)$. Applying Theorem 1.4.3, we obtain $H \triangleleft S$.

3. From Theorem 1.4.3 we obtain immediately that the sequence $TH \rightarrow TS \rightarrow TT \rightarrow 0$ is right exact. It remains to apply Part 2.

4. We may assume that H and S are connected.

Suppose we have exact sequences $0 \rightarrow H_0 \rightarrow H \rightarrow H_1 \rightarrow 0$ and $0 \rightarrow S_0 \rightarrow S \rightarrow S_1 \rightarrow 0$, where $H_i \subset S_i$ for $i = 0, 1$. Then, applying the assertion of Part 3, we see that it suffices to check the claim for the pairs $H_i \subset S_i$. The results of Raynaud (see Proposition 4.1.1 below) allow us to reduce the statement to the case where $H_L = S_L$ and H_L is L -irreducible. Since the claim in question is not affected by finite extensions of L (see Remark 3.1.2), we may also assume that the order of H is p . Then the coordinate ring of H can be presented as $\mathfrak{D}_L[x]/x^p - ax$ for some $a \in \mathfrak{D}_L$ (see the classification in [10]), and the augmentation ideal is equal to (x) . Then $TH \approx \mathfrak{D}_L/a\mathfrak{D}_L$. It remains to observe that $\text{Spec}(\mathfrak{D}_L[x]/x^p - ax) \subset \text{Spec} \mathfrak{D}_L[x]/x^p - bx$ implies $a \mid b$; if $b \sim a$, then this inclusion is an isomorphism. \square

Remark 3.1.4. It can also be proved that $d = l\#S$, where l is the length of TS and d is the valuation of the discriminant of the coordinate ring of S . This generalizes the Tate formula for $d(\text{Ker}[p^r]_F)$ (see [14]).

3.2. The dimension of a group scheme. Now we rewrite Parts II and III of Theorem 2.1.1 in terms of tangent spaces.

Theorem 3.2.1. I. For a connected group scheme S , the following numbers are equal:

1. the \mathfrak{D}_L -dimension of J/J^2 ;
2. the \mathfrak{D}_L -dimension of $C(S)/\mathbf{VC}(S)$;
3. the minimal dimension of a finite height formal group F such that $S \triangleleft F$.

II. S is equal to $\text{Ker}[p^r]_F$ for some m -dimensional finite height formal group F/\mathfrak{D}_L if and only if $p^r S = 0$ and $TS \approx (\mathfrak{D}_L/p^r \mathfrak{D}_L)^m$.

Proof. This follows immediately by combining Theorem 2.1.1 and Proposition 3.1.3. \square

It seems natural to call the number $\dim_{\mathfrak{D}_L}(J/J^2)$ the dimension of S .

Remark 3.2.2. 1. We have proved that the minimal number of generators for the coordinate ring of S is equal to the minimal possible dimension of F with $S \triangleleft F$. Obviously, the dimension of F cannot be smaller than the minimal number of generators; yet the reverse inequality is much more difficult and was not known earlier.

For $m = 1$, this result was proved in [1] (though there it was formulated in a different way).

2. It seems very plausible that the fact that the numbers I.1–I.3 are equal can be generalized to not necessarily connected p -group schemes (and p -divisible groups). For this, one should modify the Cartier module theory so that it would describe not necessarily connected group schemes. This would probably be done in one of the subsequent papers; see also the paper [16].

§4. DESCENT; A FINITE GOOD REDUCTION CRITERION

In the paper [3] it was proved that the question as to whether an Abelian variety of potentially good reduction has good reduction over K can be answered if we know $\text{Ker}[p']_{V,K}$. In this section we modify that criterion, using the results of the preceding section. We also prove that the study of semistable reduction for an Abelian variety can be reduced to the study of its “formal part” (see Subsection 4.4 below).

4.1. The generic fiber results: reminder. We often use the following fact (see [11]).

Proposition 4.1.1. *The closed subgroup schemes of S/\mathfrak{D}_L are in one-to-one correspondence with the closed subgroup schemes of S_L .*

In particular, for any set $S_i \triangleleft S$ there exists the smallest (in the sense of inclusion) group scheme H such that $S_i \triangleleft H$ (it is the subscheme “generated by S_i ”). H is also a closed subgroup scheme of S .

Now we recall the main result of [3]. It is a finite analog of the fullness of the generic fiber functor for p -divisible groups (proved by Tate). It also implies Tate’s result (see [14]) easily.

Theorem 4.1.2. *If S and T are \mathfrak{D}_L -group schemes, and $g : S_L \rightarrow T_L$ is a morphism of L -group schemes, then there exists a morphism $h : S \rightarrow T$ over \mathfrak{D}_L such that $h_L = p^s g$.*

Certainly, h is unique.

4.2. Descent: reminder. The following result was proved in [3].

Proposition 4.2.1. *Let F be a finite height formal group over \mathfrak{D}_L . Suppose that its generic fiber (as the fiber of a p -divisible group) $F_L = F \times_{\text{Spec } \mathfrak{D}_L} \text{Spec } L$ is defined over K ; i.e., there exists a p -divisible group Z_K over K such that*

$$(12) \quad Z_K \times_{\text{Spec } K} \text{Spec } L \cong F_L.$$

Suppose that $\text{Ker}[p^t]_Z \approx T \times_{\text{Spec } \mathfrak{D}_K} \text{Spec } K$ for some group scheme T/\mathfrak{D}_K ; this isomorphism combined with the isomorphism (12) is the generic fiber of a certain isomorphism $T \times_{\text{Spec } \mathfrak{D}_K} \text{Spec } \mathfrak{D}_L \cong \text{Ker}[p^t]_F$. Then $Z_K \approx F'_K$ for some formal group F'/\mathfrak{D}_K .

4.3. Descent for p -divisible groups in terms of tangent spaces. We recall that, by definition, the dimension of a p -divisible group over \mathfrak{D}_L is the dimension of its connected component (as a formal group law).

Proposition 4.3.1. *Let V be a p -divisible group over K , and let Y be a p -divisible group of dimension m over \mathfrak{D}_L . Suppose that $V \times_{\text{Spec } K} \text{Spec } L \cong Y \times_{\text{Spec } \mathfrak{D}_L} \text{Spec } L$. We denote by G the connected component of Y and by J the corresponding subgroup of V .*

Then the following conditions are equivalent:

- I. *There exists a p -divisible group Z over \mathfrak{D}_K such that $J \cong Z \times_{\text{Spec } \mathfrak{D}_K} \text{Spec } K$.*
- II. *For some (finite, flat, commutative) group scheme H/\mathfrak{D}_K we have*

$$TH_{\mathfrak{D}_L} \approx (\mathfrak{D}_L/p^l \mathfrak{D}_L)^m,$$

and there exists a monomorphism $g : H_K \rightarrow \text{Ker}[p^l]_{V,K}$.

- III. *We have $TH_{\mathfrak{D}_L} \supset (\mathfrak{D}_L/p^l \mathfrak{D}_L)^m$ (i.e., there exists an embedding), and there exists a monomorphism $g : H_K \rightarrow \text{Ker}[p^l]_{V,K}$.*

Proof. I \implies II. We can take $H = [\text{Ker } p^l]_{Z, \mathfrak{D}_K}$. Then, by Part II of Theorem 3.2.1, $TH_{\mathfrak{D}_L} \approx (\mathfrak{D}_L/p^l \mathfrak{D}_L)^m$.

II \implies III. This is obvious.

Now we prove the implication III \implies I. We may assume that H is connected.

By Theorem 4.1.2, there exists a morphism $h : H_{\mathfrak{D}_L} \rightarrow \text{Ker}[p^t]_Y$ whose generic fiber is equal to $p^s g$. Let S denote the kernel of h (i.e., $S \triangleleft H$; $S_L = \text{Ker } h_L$; see Proposition 4.1.1); we take $T = H/S$. Since h is defined over \mathfrak{D}_K , so are S and T .

We have an exact sequence $0 \rightarrow S_{\mathfrak{D}_L} \rightarrow H_{\mathfrak{D}_L} \rightarrow T_{\mathfrak{D}_L} \rightarrow 0$, and $T_{\mathfrak{D}_L} \subset \text{Ker}[p^t]_G$. Since $p^s S = 0$, we obtain $p^s C(S) = 0$, whence $p^s T S = 0$. Since exact sequences of group schemes correspond to exact sequences of tangent spaces (see Part 3 of Proposition 3.1.3), we see that $TT \supset (\mathfrak{D}_L/p^t \mathfrak{D}_L)^m$.

Now, Part 4 of Proposition 3.1.3 implies that $\text{Ker}[p^t]_G = T$.

Since T and the embedding are defined over \mathfrak{D}_K , Proposition 4.2.1 shows that G is also defined over \mathfrak{D}_K . □

4.4. Duality and semistable reduction of Abelian varieties. In order to reduce the descent question for Abelian varieties to the study of certain formal groups, we shall need the following statement.

Let V be an Abelian variety that has semistable reduction over L , and let V' denote the dual variety. We denote by V_p the p -torsion of V (as a p -divisible group over L), and by V_f the formal part of the p -torsion of V (i.e., the part corresponding to the formal group F of the Néron model of V over \mathfrak{D}_L). Obviously, V_f corresponds to the maximal p -divisible subgroup V_p that is a generic fiber of a connected scheme over \mathfrak{D}_L . Hence, the formal part is functorial with respect to isogenies of Abelian varieties (this fact can also be deduced from Part 1 of Theorem 4.4.1).

We note also that the formal part of an Abelian variety is Galois-stable over its field of definition. Hence, if V is defined over K , then V_f is equal to $V_{fK} \times_{\text{Spec } K} \text{Spec } L$ for a certain uniquely determined p -divisible subgroup $V_{fK} \subset V_p$ over K .

Theorem 4.4.1. *Let $V_m \subset V_f$ denote the multiplicative-type part of V_p , i.e., the part corresponding to the maximal multiplicative-type subgroup of F . Let V'_m denote the multiplicative-type part of V'_p .*

1. *The Weil pairing for V_p and V'_p induces the Cartier duality of V_p/V_f with V'_m .*
2. *Suppose also that V is defined over K . Then V has semistable reduction over K if and only if there exists a formal p -divisible group G/\mathfrak{D}_K such that the generic fiber of G (as a p -divisible group) is isomorphic to V_{fK} .*

Proof. 1. We denote by $V_{\text{fin}}, V'_{\text{fin}}$ the finite parts of V_p and V'_p , respectively, and by V_t and V'_t the toric parts of V_p and V'_p , respectively (see the definition of the toric part in [6]). Obviously, $V'_t \subset V'_m$.

By Theorem 5.2 of [6], V'_{fin} is the Cartier dual of V_p/V_t , and V'_t is the Cartier dual of V_p/V_{fin} . Denoting by D the Cartier dual of V_p/V_f , we have $V'_t \subset D \subset V'_{\text{fin}}$. Since V_{fin}/V_f is the maximal possible étale factor of V_{fin} (over \mathfrak{D}_L), it follows that D/V'_t is the maximal possible multiplicative-type subgroup of V'_{fin}/V'_t . Hence, $D = V'_m$.

2. We define V'_{fK} by analogy with V_{fK} . Let $\alpha : V \rightarrow V'$ be some polarization over K . Since the formal part is functorial, α induces an isogeny $\beta : V_{fK} \rightarrow V'_{fK}$. Then $\text{Ker } \beta$ is a finite K -group scheme; hence, it corresponds to some $S \triangleleft G$ (see Proposition 4.1.1). Therefore, $V'_{fK} \cong Z' \times_{\text{Spec } \mathfrak{D}_K} \text{Spec } K$, where $Z' = Z/S$ is a formal p -divisible group over \mathfrak{D}_K .

We see that V'_{fK} can be defined over \mathfrak{D}_K . Since the multiplicative-type part of an Abelian variety is Galois-stable, V'_{mK} (defined in the same way as V'_{fK}) is also defined over \mathfrak{D}_K . Using the assertion of Part 1, we conclude that V_{fK} and V_{pK}/V_{fK} are defined over \mathfrak{D}_K . Hence, V_{pK} is a Barsotti–Tate group of echelon 2 over \mathfrak{D}_K . By [6, Proposition 5.13c], V has semistable reduction over K . □

Remark 4.4.2. In [6], V_{fin} was called the fixed part of V_p , V_m was called the toroidal part of V_p , and $V_{K,\text{fin}}$ was called the effectively fixed part.

4.5. Criterion for good reduction of an Abelian variety with potentially good reduction.

Theorem 4.5.1. *Let V be an Abelian variety of dimension m over K that has good reduction over L . Then V has good reduction over K if and only if for some (finite, flat, commutative) group scheme H/\mathfrak{D}_K we have $TH_{\mathfrak{D}_L} \supset (\mathfrak{D}_L/p^l \mathfrak{D}_L)^m$ (i.e., there exists an embedding) and there exists a monomorphism $g : H_K \rightarrow \text{Ker}[p^l]_{V,K}$.*

Proof. If V has good reduction over K , then we can take $H = [\text{Ker } p^l]_{Y, \mathfrak{D}_K}$, where $Y_{\mathfrak{D}_K}$ is the Néron model of V over \mathfrak{D}_K .

Now we prove the reverse implication. By Proposition 4.3.1, the formal part of V_p is defined over \mathfrak{D}_K . By Part 2 of Theorem 4.4.1, V has semistable reduction over K . Hence, V has good reduction over K . \square

In [3], instead of the tangent space condition, it was required that $\text{Ker}[p^l]_V$ over K be the generic fiber of some truncated Barsotti–Tate group H/\mathfrak{D}_K .

§5. FINITE CRITERIA FOR SEMISTABLE AND ORDINARY REDUCTION

5.1. Proof of Theorem (A). We prove Theorem (A) (see the Introduction). We adopt the notation of Theorem 4.4.1.

If V has semistable reduction over K , then we can take $H = [\text{Ker } p^l]_G$.

Now we prove the reverse implication in Theorem (A). We may assume that H is connected.

Since the finite part of an Abelian variety is Galois-stable over its field of definition, V_{fin} is equal to $V_{\text{fin}K} \times_{\text{Spec } K} \text{Spec } L$ for some p -divisible group $V_{\text{fin}K}$ over K . By [6, Proposition 5.6], the factor V_p/V_{fin} is the generic fiber of some étale p -divisible group over \mathfrak{D}_L . Let H_0 denote the closed group scheme of H that corresponds to the preimage of V_{fin} in H_L (see Proposition 4.1.1). Then $(H/H_0)_L$ is isomorphic to a generic fiber of some \mathfrak{D}_L -group scheme. Since H is connected, Theorem 4.1.2 implies that $p^s(H/H_0) = 0$ (in fact, this statement is very easy to prove).

We denote by H_1 the maximal closed subgroup scheme of H that is killed by p^l . The same tangent space argument as in the proof of Proposition 4.3.1 shows that $TH_1 \supset (\mathfrak{D}_L/p^l \mathfrak{D}_L)^m$.

We have $H_1 \triangleleft H_0$, whence $H_{0K} \triangleleft V_{\text{fin}K}$. Therefore, $TH_0 \supset (\mathfrak{D}_L/p^l \mathfrak{D}_L)^m$. We use the same arguments as in the proof of Theorem 4.5.1. By Proposition 4.3.1, the formal part of V_p is defined over \mathfrak{D}_K . Using Part 2 of Theorem 4.4.1, we conclude that V has semistable reduction over K .

5.2. Ordinary reduction: reminder. We recall that an Abelian variety (over \mathfrak{D}_K or \mathfrak{D}_L) is said to have an ordinary reduction (or, simply, is ordinary) if the formal group of its Néron module is of finite height and of multiplicative type.

In particular, an ordinary reduction variety has semistable reduction.

For example, the reduction of an elliptic curve with semistable reduction is either ordinary or supersingular.

It is easily seen that the definition given above is equivalent to the usual ones. In [8], an Abelian variety was called ordinary over \mathfrak{D}_L if the connected component of 0 of the reduction of the Néron model of A over \mathfrak{D}_L is an extension of a torus by an ordinary Abelian variety (over \overline{L}).

If an Abelian variety with good reduction has an ordinary reduction, we say that it has a good ordinary reduction.

5.3. Proof of Theorem (B).

Proof. I. (1) \implies (2). We can take $H = \text{Ker}[p^l]_{F, \mathfrak{D}_K}$ where F is the formal group of the Néron model of V (over \mathfrak{D}_K).

(2) \implies (1). By Theorem 4.5.1, if such a scheme H exists, then V has good reduction over K . The same argument as in the proof of Theorem 4.3.1 shows that $\text{Ker}[p^t]_{F, \mathfrak{D}_L}$ is multiplicative. Then F is also multiplicative. In order to see this, one may pass to the duals and observe that a p -divisible group U is étale if and only if $\text{Ker}[p^t]_U$ is.

(2) \implies (3). Let H' denote the Cartier dual of H . Since H' is étale, H' is constant over some ring A unramified over \mathfrak{O}_K . For the role of M we take the fraction field of A .

Since the scheme H'_A is constant, H_A is isomorphic to $\sum_{1 \leq i \leq r} \mu_{p^{n_i}}$ for some $n_i, r > 0$. We have $\mu_{p^{n_i}, A} \cong \text{Spec } B$, where $B = A[x]/((x+1)^{p^{n_i}} - 1)$ and $J(B) = (x)$. Hence, $T\mu_{p^{n_i}} \approx A/p^{n_i}A$. Since $TH_{A\mathfrak{O}_L} \approx (A\mathfrak{O}_L/p^{l'}A\mathfrak{O}_L)^m$, using the additivity of the tangent space functor we obtain $r = m$ and $n_i = l'$ for $1 \leq i \leq r$. Thus, $H_M \cong (\mu_{p^{l'}, M})^m$.

(3) \implies (1). We denote by A the ring of integers of M . We take $H_A = (\mu_{p^{l'}, A})^m$. Since $TH \approx (A/p^{l'}A)^m$, H_A satisfies the conditions of (2) over M . Consequently, V has good reduction over M . Since M/K is unramified, V also has good reduction over K .

II. (1) \implies (2). If V has ordinary reduction, we can take $H = \text{Ker}[p^l]_{F, \mathfrak{O}_K}$.

(2) \implies (1). We apply Theorem (A). As in the proof of Part I, we show that $\text{Ker}[p^l]_F$ is multiplicative, whence F is multiplicative. We also note that (1) and (2) are equivalent to (2) with the condition $(\mathfrak{O}_L/p^l\mathfrak{O}_L)^m \subset TH$ replaced by $TH_{\mathfrak{O}_L} \approx (\mathfrak{O}_L/p^l\mathfrak{O}_L)^m$.

(2) \iff (3). The proof (for $TH_{\mathfrak{O}_L} \approx (\mathfrak{O}_L/p^l\mathfrak{O}_L)^m$) is the same as in Part I. \square

Remark 5.3.1. 1. By passing to the duals, a multiplicative subgroup scheme of $\text{Ker}[p^l]_V$ can be replaced with an étale factor scheme in Part II(3) of Theorem (B).

2. The equivalence (1) \iff (3) of Theorem (B) is quite similar to the finite l -adic semistable reduction criteria of [12, 13].

REFERENCES

- [1] M. V. Bondarko, *Local Leopoldt's problem for rings of integers in abelian p -extensions of complete discrete valuation fields*, Doc. Math. **5** (2000), no. 5, 657–693. MR1808921 (2002b:11166)
- [2] ———, *Explicit classification of formal groups over complete discrete valuation fields with imperfect residue field*, Trudy S.-Peterburg. Mat. Obshch. **11** (2005), 1–36; English transl., Proceedings of the St. Petersburg Mathematical Society. Vol. XI, Amer. Math. Soc. Transl. Ser. 2, vol. 218, Amer. Math. Soc., Providence, RI, 2006, pp. 1–29. MR2279302
- [3] ———, *The generic fiber of finite group schemes; a “finite wild” criterion for good reduction of abelian varieties*, Izv. Ross. Akad. Nauk Ser. Mat. **70** (2006), no. 4, 21–52; English transl., Izv. Math. **70** (2006), no. 4, 661–691. MR2261169
- [4] M. V. Bondarko and S. V. Vostokov, *Explicit classification of formal groups over local fields*, Trudy Mat. Inst. Steklov. **241** (2003), 43–67; English transl., Proc. Steklov Inst. Math. **2003**, no. 2 (241), 35–57. MR2024043 (2005b:11191)
- [5] B. Conrad, *Finite group schemes over bases with low ramification*, Compositio Math. **119** (1999), 239–320. MR1727133 (2001c:14071)
- [6] A. Grothendieck, *Modèles de Néron et monodromie*, Groupes de Monodromie en Géométrie Algébrique, Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969. I (SGA 7 I), Lecture Notes in Math., vol. 288, Springer-Verlag, Berlin–New York, 1972, Exp. IX, pp. 313–523. MR0354656 (50:7134)
- [7] M. Hazewinkel, *Formal groups and applications*, Pure Appl. Math., vol. 78, Acad. Press, New York–London, 1978. MR0506881 (82a:14020)
- [8] B. Mazur and J. Tate, *Canonical height pairings via biextensions*, Arithmetic and Geometry, Vol. I, Progr. Math., vol. 35, Birkhäuser Boston, Boston, MA, 1983, pp. 195–237. MR0717595 (85j:14081)
- [9] F. Oort, *Dieudonné modules of finite local group schemes*, Indag. Math. **36** (1974), 284–292. MR0354694 (50:7171)
- [10] F. Oort and J. Tate, *Group schemes of prime order*, Ann. Sci. École Norm. Sup. (4) **3** (1970), 1–21. MR0265368 (42:278)
- [11] M. Raynaud, *Schémas en groupes de type (p, \dots, p)* , Bull. Soc. Math. France **102** (1974), 241–280. MR0419467 (54:7488)
- [12] A. Silverberg and Yu. G. Zarhin, *Reduction of abelian varieties*, The Arithmetic and Geometry of Algebraic Cycles (Banff, AB, 1998), NATO Sci. Ser. C Math. Phys. Sci., vol. 548, Kluwer Acad. Publ., Dordrecht, 2000, pp. 495–513. MR1744959 (2001g:14073)
- [13] ———, *Semistable reduction of abelian varieties over extensions of small degree*, J. Pure Appl. Algebra **132** (1998), no. 2, 179–193. MR1640087 (99k:11085)
- [14] J. Tate, *p -divisible groups*, Proc. Conf. Local Fields (Driebergen, 1966), Springer, Berlin, 1967, pp. 158–183. MR0231827 (38:155)

- [15] Th. Zink, *Cartiertheorie kommutativer formaler Gruppen*, Teubner-Texte zur Math., Bd. 68, BSB B. G. Teubner Verlagsgesellschaft, Leipzig, 1984. MR0767090 (86j:14046)
- [16] ———, *The display of a formal p -divisible group*, *Cohomologies p -Adiques et Applications Arithmétiques*, I, Astérisque No. 278 (2002), 127–248. MR1922825 (2004b:14083)

DEPARTMENT OF MATHEMATICS AND MECHANICS, ST. PETERSBURG STATE UNIVERSITY, UNIVERSITETSKIĬ PROSPECT 28, STARYĬ PETERHOF, ST. PETERSBURG 198504, RUSSIA

E-mail address: `mbondarko@hotmail.com`

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