

## GEOMETRY OF 1-TORI IN $GL_n$

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ABSTRACT. We describe the orbits of the general linear group  $GL(n, T)$  over a skew field  $T$  acting by simultaneous conjugation on pairs of 1-tori, i.e., subgroups conjugate to  $\text{diag}(T^*, 1, \dots, 1)$ , and identify the corresponding spans. We also provide some applications of these results to the description of intermediate subgroups and generation. These results were partly superseded by A. Cohen, H. Cuypers, and H. Sterk, but our proofs use only elementary matrix techniques. As another application of our methods, we enumerate the orbits of  $GL(n, T)$  on pairs of a 1-torus and a root subgroup, and identify the corresponding spans. This paper constitutes an elementary invitation to a series of much more technical works by the author and V. Nesterov, where similar results are established for microweight tori in Chevalley groups over a field.

### INTRODUCTION

The geometry of unipotent *long* root subgroups in Lie type groups and related geometries have been extensively studied and are well understood. This subject has been explored by outstanding masters such as J. McLaughlin, A. Wagner, M. Aschbacher, G. M. Seitz, W. M. Kantor, B. N. Cooperstein, A. E. Zalesskii, and many others; see, in particular, [17, 21, 24, 40, 60, 67, 68], [73]–[76], [83, 92, 100, 127] and further references in [92, 47], [36]–[38].

In the mid-1990s, M. W. Liebeck and G. M. Seitz proposed remarkably transparent proofs and broad generalizations of the main results on generation by long root unipotents in the context of algebraic groups [99].

On the other hand, F. G. Timmesfeld developed an extraordinarily deep geometric theory of *abstract root subgroups* [115]–[121]. In the work of Timmesfeld himself, A. Steinbach, H. Cuipers, and others, this theory has brought explicit answers in many cases; see [72]–[82], [109]–[113]. Among other things, Timmesfeld’s theory is a very broad generalization of Thompson’s theory of quadratic pairs [114, 108, 65].

However, the remarkable papers [1]–[7] and [62]–[64] by E. L. Bashkirov, devoted to subgroups of the classical groups over skew fields generated by the root subgroups parametrised by a subfield, show that, in general, getting explicit answers is blocked by the arithmetics of skew fields, and is an immensely complicated problem. Among other things, E. L. Bashkirov succeeded in getting *final* answers for the case of quadratic unipotent elements of residue 2, which is closely related to the arithmetics of quaternion skew fields.

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Another remarkable advance of the last decade was the results by V. V. Nesterov on the geometry of *short* root subgroups [52]–[55], which finalized the research by B. Stark [105]–[107] and Li Shangzhi pertaining to the classical groups, and obtained similar results for exceptional groups.

At the same time, very little is known about the geometry of tori. Anisotropic groups do not have unipotents, and in this case the study of tori is a strict necessity. Here one could mention, for example, extremely interesting results by B. N. Weisfeiler and D. G. James [131]–[133], [91].

However, even for Chevalley groups, it might be of interest to describe orbits on pairs, triples, etc., of tori of various classes, to identify the corresponding spans, to classify the irreducible subgroups generated by tori from a particular class, etc. This may lead to geometries similar to, but different from, buildings and their residues; compare [133, p. 365]. Results in this spirit might shed a new light on the known results concerning the description of subgroups; see the surveys [25, 26], [37]–[39], [48, 123]. They would also serve as a first step towards the generalization of results on subgroups generated by pseudoreflections, two-dimensional elements, or the like (see [41, 42], [49]–[51], [70, 71, 77], [87]–[90], [104], [128]–[130]) to other types of groups and to other types of semisimple elements.

In the present paper we study in some detail the model case of 1-tori in the general linear group  $G = \text{GL}(n, T)$  over a skew field  $T$ . The 1-tori are one-parameter groups of pseudoreflections = *dilations* with the same axis and center. First we study the orbits of  $G$  acting by conjugation on pairs  $(X, Y)$  of 1-tori. In the sequel we call such orbits *orbitals*. It turns out that when the skew field  $T$  contains at least 7 elements, there is a close analogy between the description of such orbitals and the description of orbits on pairs of root subgroups.

Namely, as for the root subgroups, one can introduce the *angle*  $\phi$  between two 1-tori  $X$  and  $Y$ , taking values  $0, \pi/3, \pi/2, 2\pi/3$ , and  $\pi$ . When  $\phi < \pi$ , the above orbitals are almost completely determined by the angle between  $X$  and  $Y$ . Just as for root subgroups, the angle  $\pi/3$  leads to two possible configurations, fused by the external automorphism of  $G$ . Unlike the case of root subgroups, the angle  $\pi$  leads to  $|T| - 1$  distinct orbitals, falling into two essentially different cases, in one of which the span  $\langle X, Y \rangle$  is isomorphic to  $\text{GL}(2, T)$ , whereas in the other case it is triangularizable.

There is another important aspect in which this result differs from the case of root subgroups. Namely, whereas for pairs  $(X, Y)$  of root subgroups the orbitals and the corresponding spans  $\langle X, Y \rangle$  do not depend on the skew field, in our case the fields  $K$  with  $|K| \leq 7$  are true exceptions. In these cases the number of orbitals and/or the shape of the span  $\langle X, Y \rangle$  for a given orbital differ from the generic case.

Furthermore, we show that our results, in particular, immediately imply the following two results.

- Description of the irreducible subgroups in  $G$  generated by 1-tori, an analog of McLaughlin's theorem for pseudoreflections; see [23, 98, 72].
- Description of the overgroups of the diagonal subgroup  $D = \text{D}(n, T)$ ; see [8, 9, 13] for the first proof of this result; [103] for a proof for finite fields; [14, 16] for a proof based on Bruhat decomposition; [30, 101] for an exceedingly simple modern proof based on small Bruhat cells; [10] and [44]–[47] for the description of intermediate subgroups in the exceptional cases  $|K| \leq 5$ ; [122, 19, 26] for various proofs of the conjugacy theorem; [84, 97] for proofs of a weaker statement about the maximality of the monomial subgroup; [15, 26, 94, 95] for the case of the special linear group; and the surveys [25, 26], [37]–[39], [123], as well as the papers [29, 86] for the general background and further references.

Theorem 3 of the present paper was announced in [23], but its complete proof has never been published. In the 1990s, I noticed that the same methods afford a description of subgroups generated by pairs of 1-tori, which is Theorem 1 of the present paper. The proofs of this result circulated as a preprint [126], but for various reasons have not yet been published. Several years later, an important paper [72] by A. M. Cohen, H. Cuypers, and H. Sterk appeared, which contained geometric generalizations of these results, also to the infinite-dimensional case. Nevertheless, I decided to publish my original computational proofs, since they are easier, and, what is more important, more *elementary* than the proofs of those more general results. Moreover, in the present paper I prove some further results in this direction, in particular, describe the orbits of  $GL(n, T)$  on pairs consisting of a 1-torus and a root subgroup.

Of course, for a *commutative* field  $T = K$ , the results of the present paper are a very special case of the forthcoming papers by V.V. Nesterov and the author, where similar results are obtained for the microweight tori (see [66], [18]–[20], [24, 26, 27, 124] for the definition of weight elements in Chevalley groups) and for long root tori in arbitrary Chevalley groups. However, there are at least three reasons to publish this elementary case separately.

- First, this is the only case among the groups of normal type that makes sense over skew fields, so that the results of this paper are slightly more general in this respect.
- Second, here it is easy to completely describe the orbitals also in the exceptional cases of very small fields. Nothing like that is available for other groups and tori, where we could only treat the *generic* case of *sufficiently large* fields.
- Third, as opposed to Chevalley groups, in the present paper we can afford to use only very rudimentary matrix calculations.

Thus, the present paper is addressed to a much larger audience, and may serve as an elementary introduction and an invitation to the forthcoming much more sophisticated and technical papers by V. V. Nesterov and the author.

The paper is organized as follows. In §1 we recall the necessary notation and some background. In §2 we introduce some obvious invariants of the orbitals, and reduce the description of the orbits of  $GL(n, T)$  on the pairs of 1-tori to the cases of  $B(3, T)$  and  $GL(2, T)$ , which are treated in §3 and §4, respectively. In §5 we summarize our results and discuss the exceptional cases. In §6 we show how to deduce the description of the subgroups of  $GL(n, T)$  containing  $D(n, T)$  from the results of the present paper. In §7 we prove an analog of McLaughlin’s theorem on irreducible linear groups generated by 1-tori. Finally, in §8 we describe the orbits of  $GL(n, T)$  on pairs consisting of a 1-torus and a root subgroup, while in §9 we state some unsolved problems in this field.

### §1. NOTATION

The following standard notation will be used throughout the paper. Let  $T$  be a skew field, let  $T^* = T \setminus \{0\}$  be its multiplicative group, and let  $G = GL(n, T)$  be the general linear group of degree  $n$  over  $T$ . We consider the following subgroups of  $G$ : the group  $B = B(n, T)$  of upper triangular matrices, the group  $B^- = B^-(n, T)$  of lower triangular matrices, the group  $D = D(n, T)$  of diagonal matrices, and the monomial subgroup  $N = N(n, T)$ . By  $U = U(n, T)$  and  $U^- = U^-(n, T)$  we denote the unipotent radicals of  $B$  and  $B^-$ , respectively, i.e., the groups of upper and lower unitriangular matrices.

The quotient group  $N/D$  is isomorphic to  $S_n$ , the symmetric group on  $n$  letters. We denote by  $W = W_n$  the group of permutation matrices in  $G$ ; in the sequel it is sometimes mentioned as the Weyl group. We identify  $S_n$  and  $W_n$  via the isomorphism  $\pi \mapsto w_\pi$ ,

where  $w_\pi$  is the matrix whose entry at the position  $(i, j)$  is  $\delta_{i, \pi j}$ . As usual, we denote by  $w_{ij}$  the permutation matrix corresponding to the transposition  $(i, j)$ .

Denote by  $V = T^n$  the *right* vector space of columns of height  $n$  over  $T$ . Sometimes, we identify a matrix  $g \in G$  with the corresponding linear transformation of the space  $T^n$ . This transformation multiplies a column by  $g$  from the *left*. When we want to emphasize this geometric viewpoint, we refer to the elements of  $G$  as *transformations*. By  $v_1, \dots, v_n$  we denote the standard base of  $T^n$ ; i.e.,  $v_i$  is the column that has 1 as its  $i$ th coordinate, while all other coordinates are zeros. The dual space  $V^* = {}^nT$  is the *left* vector space of rows of length  $n$ . By  $u_1, \dots, u_n$  we denote the standard basis of  ${}^nT$ , dual to  $v_1, \dots, v_n$  with respect to the standard pairing.

For a matrix  $g \in G$  we denote by  $g_{ij}$  its entry at the position  $(i, j)$ , so that  $g = (g_{ij})$ ,  $1 \leq i, j \leq n$ . As usual,  $g^{-1} = (g'_{ij})$  denotes the inverse of  $g$ ,  $e$  denotes the identity matrix, and  $e_{ij}$  is a standard matrix unit, i.e., the matrix whose entry at the position  $(i, j)$  is 1 and all the remaining entries are zeros. Thus,  $g = \sum g_{ij}e_{ij}$ . By  $g^t$  we denote the *formal* transpose of  $g$ , whose entry at the position  $(i, j)$  equals  $g_{ji}$  viewed as an element of  $T$ . (In the proper definition of a transpose,  $g_{ji}$  should be viewed as an element of the *opposite* skew field  $T^0$ ).

For  $\xi \in T$  and  $1 \leq i \neq j \leq n$ , we denote by  $t_{ij}(\xi) = e + \xi e_{ij}$  an *elementary transvection*. For given  $i \neq j$ , we consider the corresponding unipotent *root subgroup*  $X_{ij} = \{t_{ij}(\xi), \xi \in T\}$ . The subgroup  $E(n, T)$  of  $G$  generated by all  $X_{ij}$ ,  $1 \leq i \neq j \leq n$ , is called the *elementary* subgroup of  $G$ . When  $T = K$  is commutative, it coincides with the special linear group  $SL(n, K)$ . Similarly, by  $d_i(\varepsilon) = e + (\varepsilon - 1)e_{ii}$  we denote an *elementary pseudoreflexion*. For a given  $i$ , we consider the corresponding *1-torus*  $Q_i = \{d_i(\varepsilon), \varepsilon \in T^*\}$ . We write  $Q = Q_1$ . Clearly,  $GL(n, T)$  is generated by  $E(n, T)$  and  $Q$ .

Recall that the *residual space* of a transformation  $g \in G$  is the space

$$S(g) = \{ga - a \mid a \in T^n\},$$

and the *residue* of  $g$  is the dimension of  $S(g)$  or, in other words, the rank of the transformation  $g - e$ . We denote the residue of  $g$  by  $\text{res}(g)$ . It follows immediately from the definition that  $\text{res}(gh) \leq \text{res}(g) + \text{res}(h)$  for all  $g, h \in G$ , and in [85] one can find many further properties of  $S(g)$  and  $\text{res}(g)$ . Sometimes, an element  $g$  such that  $\text{res}(g) = m$  is called an *m-dimensional transformation*.

The most important individual elements of  $GL(n, T)$  are, of course, the *1-dimensional transformations*, which play a crucial role in the study of linear groups. The general form of a 1-dimensional transformation is  $x_{ab}(\xi) = e + a\xi b$ , where  $a = (\alpha_1, \dots, \alpha_n)^t$  is a column of height  $n$ ,  $\xi \in T$ , and  $b = (\beta_1, \dots, \beta_n)$  is a row of length  $n$ .

The subspace generated by the column  $a$  is called the *center* of the transformation  $x_{ab}(\xi)$ . The hyperplane orthogonal to the row  $b$  is called the *axis* of  $x_{ab}(\xi)$ . Sometimes, we loosely apply the terms *center* and *axis* to  $a$  and  $b$  themselves. The center and the axis of a 1-dimensional transformation  $x$  will be denoted by  $a(x)$  and  $b(x)$ , respectively. Note that they are defined only projectively, i.e., only up to an invertible scalar factor. For any matrix  $g \in GL(n, T)$  we have  $gx_{ab}(\xi)g^{-1} = x_{ga, bg^{-1}}(\xi)$ . In particular,  $a(gxg^{-1}) = ga(x)$  and  $b(gxg^{-1}) = b(x)g^{-1}$ .

Let  $ba = \beta_1\alpha_1 + \dots + \beta_n\alpha_n = \delta$ . If  $\delta = 0$ , the transformation  $x_{ab}(\xi)$  is a *transvection* for every  $\xi \in T$ . If  $\delta \neq 0$ , then, changing  $\xi$  if necessary, we can always assume that  $ba = \delta = 1$ . In that case, to guarantee the invertibility of  $x_{ab}(\xi)$ , we must require that  $\xi \neq -1$ . Set  $T^\# = T \setminus \{-1\}$ . In this case the transformations  $x_{ab}(\xi)$ ,  $\xi \in T^\#$ , are called *pseudoreflexions*. Usually, we write a pseudoreflexion in the form  $d_{ab}(\varepsilon) = x_{ab}(\varepsilon - 1)$ , where  $\varepsilon = \xi + 1 \in T^*$ . Clearly,  $d_{ab}(\varepsilon)d_{ab}(\eta) = d_{ab}(\varepsilon\eta)$  for two  $\varepsilon, \eta \in T^*$ .

Fix some  $r$  and consider the conjugate  $y(\varepsilon) = xd_r(\varepsilon)x^{-1}$  of  $d_r(\varepsilon)$  by  $x$ :

$$y(\varepsilon) = \begin{pmatrix} 1 + \alpha_1(\varepsilon - 1)\beta_1 & \alpha_1(\varepsilon - 1)\beta_2 & \dots & \alpha_1(\varepsilon - 1)\beta_n \\ \alpha_2(\varepsilon - 1)\beta_1 & 1 + \alpha_2(\varepsilon - 1)\beta_2 & \dots & \alpha_2(\varepsilon - 1)\beta_n \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_n(\varepsilon - 1)\beta_1 & \alpha_n(\varepsilon - 1)\beta_2 & \dots & 1 + \alpha_n(\varepsilon - 1)\beta_n \end{pmatrix},$$

where the  $\alpha_i = x_{ir}$  are the entries of the  $r$ th column of  $x$ , while the  $\beta_j = x'_{rj}$  are the entries of the  $r$ th row of the inverse matrix  $x^{-1} = (x'_{ij})$ . Thus,  $xd_r(\varepsilon)x^{-1} = d_{ab}(\varepsilon)$ .

In this paper we consider *one-parameter groups of pseudoreflections*

$$Q_{ab} = \{d_{ab}(\varepsilon), \varepsilon \in T^*\}.$$

For brevity, such one-parameter subgroups are called *1-tori*. Over a skew field, every row and column whose product equals 1 may be interpreted as the first row of an invertible matrix and the first column of the inverse of that matrix, respectively. It follows that over a skew field every 1-torus is conjugate to the torus  $Q = Q_1$ .

### §2. INITIAL REDUCTION

Our objective is to describe the orbits of  $G = GL(n, T)$  acting by simultaneous conjugation on pairs of 1-tori,

$$(X, Y) \mapsto (gXg^{-1}, gYg^{-1}), \quad g \in G,$$

and to identify the corresponding spans. In this section we discuss the obvious invariants of these orbits and reduce their description to two special cases, of degrees 2 and 3, respectively.

We start with some obvious general observations. Let  $X$  and  $Y$  be two 1-tori with the corresponding centers  $a_1$  and  $a_2$  and axes  $b_1$  and  $b_2$ . Let  $l = l(X, Y) = \dim\langle a_1, a_2 \rangle$  and  $m = m(X, Y) = \dim\langle b_1, b_2 \rangle$ . Then, clearly,  $l$  and  $m$  can take values 1 or 2 and are not changed by simultaneous conjugation. The products  $b_1a_2$  and  $b_2a_1$  also remain invariant in the projective sense. This means that if  $b_1a_2$  is equal to zero (respectively, distinct from zero), for a given pair  $(X, Y)$ , then the same is true for any conjugate pair  $(gXg^{-1}, gYg^{-1})$ . Denote by  $p = p(X, Y)$  and  $q = q(X, Y)$  the invariants of a pair  $(X, Y)$  that take the value 0 if  $b_1a_2 = 0$  or  $b_2a_1 = 0$ , respectively, or 1 otherwise. It turns out that, generically, these obvious discrete invariants almost suffice to distinguish all *small* orbitals.

However, to distinguish orbitals *in the general position*, we need to introduce one more continuous parameter. Namely, assume that both products  $b_1a_2$  and  $b_2a_1$  are distinct from zero (which is, of course, the *general position* case) and consider the matrix

$$\begin{pmatrix} b_1a_1 & b_1a_2 \\ b_2a_1 & b_2a_2 \end{pmatrix}.$$

This matrix is defined only up to multiplication of its columns and rows by invertible factors from  $T$ , but, clearly, the *conjugacy class* of the element

$$c = c(X, Y) = (b_1a_1)^{-1}(b_1a_2)(b_2a_2)^{-1}(b_2a_1) \in T^*$$

is well defined and is an actual invariant of the orbital specified by the pair  $(X, Y)$ . In particular, in the *commutative* case, the product

$$c = c(X, Y) = \frac{(b_1a_2)(b_2a_1)}{(b_1a_1)(b_2a_2)} \in K^*$$

is itself invariant.

It turns out — and this is one of our main results — that an orbital is completely determined by the invariants  $l, m, p, q$ , and  $c$  and in each particular case the corresponding span can easily be identified. Generically (when  $l = m = 2, p = q = 1$ , and  $c \neq 1$ ), it is  $\mathrm{GL}(2, T)$ .

As a first step, we reduce the problem to degree  $n \leq 3$ .

**Lemma 1.** *Let  $X$  and  $Y$  be two 1-tori in  $\mathrm{GL}(n, T)$ ,  $n \geq 4$ . Then there exists  $g \in \mathrm{GL}(n, T)$  such that  $gXg^{-1}, gYg^{-1} \leq H$ , where  $H$  is the subgroup of degree 3 generated by  $Q, X_{12}, X_{21}, X_{13}$ , and  $X_{23}$ .*

*Proof.* Let  $(X, Y) = (xQx^{-1}, yQy^{-1})$  be two 1-tori. Without loss of generality, we may assume from the outset that  $x = 1$  and, thus,  $X = Q$ . Let  $Y = Q_{ab}$  as above, where  $a = (\alpha_1, \dots, \alpha_n)^t$  is the first column of  $y$  and  $b = (\beta_1, \dots, \beta_n)$  is the first row of  $y^{-1}$ .

Either  $\alpha_2 = \dots = \alpha_n = 0$ , or at least one of the  $\alpha_i, 2 \leq i \leq n$ , is distinct from zero. If not all  $\alpha_i, 2 \leq i \leq n$ , are zeros, then, possibly after conjugation by a permutation matrix, we may assume that  $\alpha_2 \neq 0$ . Conjugating by  $d_2(\alpha_2^{-1})$ , we may even assume that  $\alpha_2 = 1$ . Now, conjugating by a transvection  $t_{32}(-\alpha_3) \dots t_{n2}(-\alpha_n)$ , we may assume that  $\alpha_3 = \dots = \alpha_n = 0$ . All these conjugations preserve  $X = Q$ . In any case, after conjugation we may assume that the centers of  $X$  and  $Y$  are both contained in the subspace  $v_1T \oplus v_2T \leq T^n$ .

It remains to repeat the same argument for the  $\beta_i$ 's. Either  $\beta_3 = \dots = \beta_n = 0$  and we are done, or at least one of the  $\beta_i, 3 \leq i \leq n$ , is distinct from zero. In the second case, possibly after conjugation by a permutation matrix, we may assume that  $\beta_3 \neq 0$ . Conjugating by  $d_3(\beta_3)$ , we may even assume that  $\beta_3 = 1$ . Now, conjugating by a transvection  $t_{34}(\beta_4) \dots t_{3n}(\beta_n)$ , we may assume that  $\beta_4 = \dots = \beta_n = 0$ . These conjugations preserve  $X$  and the subspace  $v_1T + v_2T \leq T^n$ . Now the axes of  $X$  and  $Y$  are both contained in the subspace  $Tu_1 \oplus Tu_2 \oplus Tu_3 \leq {}^nT$ , which, together with the above claim about the centers, proves the lemma.  $\square$

From the above lemma, it follows that (possibly after a simultaneous conjugation) we may assume that

$$X, Y \leq \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 1 \end{pmatrix}.$$

Repeating the same argument as above, but with columns and rows interchanged, we see that (again up to simultaneous conjugation) we have

$$X, Y \leq \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & 1 \end{pmatrix}.$$

Obviously, any pair of parabolic subgroups is simultaneously conjugate to the pair  $P_1, wP_2w^{-1}$ , where  $P_1$  and  $P_2$  are standard parabolic subgroups and  $w$  is an element of the Weyl group. In particular, only one of the following two possibilities may occur for the intersection of two nonconjugate maximal parabolic subgroups in  $\mathrm{GL}(3, T)$ : either their intersection is  $\mathrm{GL}(2, T)$  (if they are opposite), or it is  $B(3, T)$  (otherwise). Thus, the preceding lemma immediately implies the following result.

**Lemma 2.** *Let  $X$  and  $Y$  be two 1-tori in  $\mathrm{GL}(n, T)$ ,  $n \geq 4$ . Then there exists  $g \in \mathrm{GL}(n, T)$  such that  $gXg^{-1}, gYg^{-1} \leq H$ , where  $H$  is either the subgroup of degree 2 generated by  $Q, X_{12}$ , and  $X_{21}$ , and isomorphic to  $\mathrm{GL}(2, T)$ , or the subgroup of degree 3 generated by  $Q_2, X_{12}, X_{13}$ , and  $X_{23}$ .*

Thus, from now on we may assume that  $X$  and  $Y$  are both contained in one of the following subgroups:

$$\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & * & * \\ 0 & * & * \\ 0 & 0 & 1 \end{pmatrix}.$$

In §§3 and 4, we treat these cases under the assumption  $|T| \geq 7$  and then in §5 summarize the results. In the sequel we refer to the case where  $|T| \geq 7$  as *generic*, whereas the cases where  $|T| \leq 5$  constitute genuine exceptions for our results.

§3. THE CASE OF  $B(3, T)$

In this section we describe the orbits of  $GL(3, T)$  (or, in fact, of  $B(3, T)$ ) acting by conjugation on pairs of 1-tori contained in the following subgroup:

$$P = \begin{pmatrix} 1 & * & * \\ 0 & * & * \\ 0 & 0 & 1 \end{pmatrix}.$$

It turns out that this case furnishes *exactly one* new orbital, as compared with the orbitals occurring already in  $GL(2, T)$ . The span of  $X$  and  $Y$ , where  $(X, Y)$  is a representative of this orbital, coincides with  $P$ .

The following lemma is a variation of Lemma 1.

**Lemma 3.** *Any pair  $(X, Y)$  of 1-tori in  $GL(n, T)$  such that  $l(X, Y) = 1$  or  $m(X, Y) = 1$  is conjugate to a pair of 1-tori in  $B(2, T)$ .*

*Proof.* We keep the setting from the proof of Lemma 1. Let, moreover,  $a_1, a_2$  be the centres and  $b_1, b_2$  the axes of the tori  $X$  and  $Y$ , respectively. As in Lemma 1, we may assume that  $X = Q$ .

If  $a_1 = a_2$ , then  $a_1 = a_2 = v_1$ . Either  $\beta_2 = \dots = \beta_n = 0$ , or at least one of the  $\beta_i$ ,  $2 \leq i \leq n$ , is distinct from zero. In the first case we are done; in the second case we may assume that  $\beta_2 \neq 0$  and  $\beta_i = 0$  for all  $3 \leq i \leq n$ . Thus,  $X, Y \leq QX_{12} \leq B(2, T)$ .

If  $b_1 = b_2$ , a similar argument shows that  $X, Y \leq QX_{21} \leq B^-(2, T)$ . However,  $B^-(2, T)$  is conjugate to  $B(2, T)$  by a permutation matrix.  $\square$

**Lemma 4.** *There is a unique  $B(3, T)$ -orbit of pairs of 1-tori in  $P$  such that  $l(X, Y) = m(X, Y) = 2$ .*

*Proof.* Denote by  $a_1$  and  $a_2$  (respectively, by  $b_1$  and  $b_2$ ) the centers (respectively, the axes) of  $X$  and  $Y$ . Then, clearly,  $a_1, a_2 \in v_1T \oplus v_2T$  and  $b_1, b_2 \in Tu_2 \oplus Tu_3$ . This means that conjugation by  $X_{12}$  preserves the axes, whereas conjugation by  $X_{23}$  preserves the centers. Conjugating by a transvection in  $X_{12}$ , we may assume that  $a_1 = v_2$ , and conjugating by a transvection in  $X_{23}$  (which does not change  $a_1$ ), we may assume that  $b_2 = u_2$ . Now,  $l = 2$  implies that  $a_2 \notin v_2T$ , and  $m = 2$  implies that  $b_1 \notin Tu_2$ . This means that conjugating by an element in  $Q_3$  and by an element in  $Q_1$ , we may assume that  $a_2 = v_1 + v_2$  and  $b_1 = u_2 + u_3$ . This shows that *any* pair of 1-tori in  $P$  such that  $l = m = 2$  is conjugate to the pair

$$X = \left\{ x(\varepsilon) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \varepsilon & \varepsilon - 1 \\ 0 & 0 & 1 \end{pmatrix}, \varepsilon \in T^* \right\},$$

$$Y = \left\{ y(\varepsilon) = \begin{pmatrix} 1 & \varepsilon - 1 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & 1 \end{pmatrix}, \varepsilon \in T^* \right\},$$

as claimed.  $\square$

Clearly, for this pair we have  $p = q = 1$  and  $c = 1$ . Now, we identify the span of this pair in the generic case.

**Lemma 5.** *Let  $T$  be a skew field containing at least 5 elements. If  $X$  and  $Y$  are two 1-tori in  $P$  such that  $l(X, Y) = m(X, Y) = 2$ , then  $\langle X, Y \rangle = P$ .*

*Proof.* By the preceding lemma, we may assume from the outset that  $X$  and  $Y$  have the above form. Denote the subgroup  $\langle X, Y \rangle$  by  $Z$ . We want to prove that  $Z = P$ .

Take any three commuting elements  $\varepsilon, \eta, \theta \neq 0, 1$ , and form the product

$$z = x(\varepsilon)y(\eta)x(\theta) = \begin{pmatrix} 1 & (\eta - 1)\theta & (\eta - 1)(\theta - 1) \\ 0 & \varepsilon\eta\theta & \varepsilon\eta(\theta - 1) + (\varepsilon - 1) \\ 0 & 0 & 1 \end{pmatrix} \in Z.$$

Now, we set  $\theta = \eta^{-1}\varepsilon^{-1}(\varepsilon\eta - \varepsilon + 1)$ , so that  $\varepsilon\eta(\theta - 1) + (\varepsilon - 1) = 0$ . Then  $\theta \neq 0$  imposes one additional restriction on  $\eta$ , namely  $\eta \neq 1 - \varepsilon^{-1}$ , whereas  $\theta \neq 1$  automatically, because  $\eta \neq 0$  and  $\varepsilon \neq 1$ .

Consider the element  $u = zy(\varepsilon\eta\theta)^{-1} \in Z$ . Clearly,  $u \in X_{12}X_{13}$ , and we want to prove that we may choose  $\varepsilon$  and  $\eta$  so that  $u \notin X_{13}$ . Indeed,  $u = t_{12}(\lambda)t_{13}(*),$  where

$$\lambda = (\eta - 1)\eta^{-1}\varepsilon^{-1}(\varepsilon\eta - \varepsilon + 1) - (\varepsilon\eta - \varepsilon + 1) + 1,$$

and an easy calculation shows that under the above restrictions on  $\varepsilon$  and  $\eta$  we have  $\lambda \neq 0$ , provided that  $\varepsilon\eta + 1 \neq 0$ . For a given  $\varepsilon \neq 0, 1$ , this imposes on  $\eta$  four linear restrictions altogether, so that the desired  $\varepsilon$  and  $\eta$  may be found whenever  $T$  contains at least 5 elements (it is well known [43] that every noncentral element of an infinite skew field is contained in an infinite subfield).

Now, the proof can be finished as follows. Since  $X_{13}$  coincides with the center of  $P$ , we have  $[g, u] = [g, t_{12}(\lambda)]$  for any  $g \in P$ . Taking  $g = y(\xi^{-1})$ , where  $\xi \in T^*$ , we get  $t_{12}(\lambda(\xi - 1)) \in Z$ . Since  $\xi$  is arbitrary, this shows that  $X_{12} \leq Z$ . Since, moreover,  $Y \leq Z$ , we have  $Q_2 \leq Z$ , and now, since  $X \leq Z$ , also  $X_{23} \leq Z$ . But then  $X_{13} = [X_{12}, X_{23}] \leq Z$ , and thus, finally,  $Z = P$ .  $\square$

The cases of fields  $K$  containing less than 5 elements, excluded here, are true exceptions. For the field  $\mathbb{F}_2$  there is nothing to talk about. On the other hand, for the fields  $\mathbb{F}_3$  and  $\mathbb{F}_4$  a straightforward calculation shows that  $X$  and  $Y$  generate a proper subgroup in  $P$  of order 6 or 24, respectively (in these cases the group  $P$  itself has order  $2 \cdot 3^3$  or  $2^6 \cdot 3$ ).

#### §4. THE CASE OF $\text{GL}(2, T)$

Now, we start to describe the orbits of  $\text{GL}(2, T)$  acting by conjugation on pairs of 1-tori in the generic case. In this section we identify the corresponding spans. Under a slightly different disguise, this problem was treated in [8, 9, 13], and in terms even closer to our present terms in [14, 16]. These papers gave (among other things) a description of the subgroups of  $\text{GL}(n, T)$  over a skew field  $T$ ,  $|T| \geq 7$ , that contain the group  $D(n, T)$  of diagonal matrices and, more generally, are normalized by it. In the special case where  $n = 2$ , one of the main results of those papers may be summarized as follows.

**Lemma 6.** *Let  $T$  be a skew field containing at least 7 elements. Then there are only three proper intermediate subgroups between  $D(2, T)$  and  $\text{GL}(2, T)$ , namely,  $N(2, T)$ ,  $B(2, T)$  and  $B^-(2, T)$ .*

It is an elementary exercise in matrix calculus to give a direct proof of this result independent of the general case; see, for example, [14]. To the contrary, an analog of this result for  $\text{SL}(2, K)$  turned out to be *extremely* deep [94]; see also [26, 123].

This result, and its higher-dimensional analogs, are in general no longer valid when  $|K| \leq 5$ . A very easy argument (see [9] or [13]) shows that this result implies the following slightly more general one.

**Lemma 7.** *Under the assumptions of Lemma 3, let  $H$  be a subgroup of  $GL(2, T)$  normalized by  $D(2, T)$ . Then one of the following is true:*

- (1)  $E(2, T) \leq H \leq GL(2, T)$ ;
- (2)  $U^\pm(2, T) \leq H \leq B^\pm(2, T)$ ;
- (3)  $H \leq N(2, T)$ .

This result already *almost* answers our question. Indeed, in the case of  $n = 2$ , the group  $D(2, T)$  is the product of  $Q$  and the center of  $GL(2, T)$ . It follows that any subgroup containing  $Q$  is normalized by  $D(2, T)$  and thus satisfies the conclusion of Lemma 4. In particular, if  $H$  is generated by  $Q$  and another 1-torus  $Q_{ab}$ , then it must either coincide with  $GL(2, T)$ , or be contained in  $B(2, T)$  or  $N(2, T)$ , up to conjugacy.

The following *connectedness theorem* shows that the case of  $N(2, T)$  may be discarded right away. This is essentially a special case of [122, 7<sup>o</sup>] or of [16, Proposition 1]. In these papers we assumed that  $x Dx^{-1} \leq N$ , but we may repeat essentially the same argument. Below we follow [122] since the argument there is more elementary; the proof in [16] (see also [19, 26]) is much more natural, but it involves some facts about Bruhat decomposition.

**Lemma 8.** *Let  $T$  be a skew field containing at least 4 elements. Then*

$$xQx^{-1} \leq N(n, T) \implies xQx^{-1} \leq D(n, T).$$

*Proof.* Suppose  $x$  has at least two nonzero entries  $x_{p1}$  and  $x_{q1}$ ,  $p \neq q$ , in the first column. The inverse  $x^{-1} = (x'_{ij})$  has at least one nonzero entry in the first row, say  $x'_{1s} \neq 0$ . For all  $\varepsilon \in T^*$  we have

$$y(\varepsilon) = x d_1(\varepsilon) x^{-1} = e + \sum_{i,j} x_{i1}(\varepsilon - 1) x'_{1j} e_{ij}.$$

At least one of the indices  $p, q$  is distinct from  $s$ . If  $p \neq s$ , then clearly  $y(\varepsilon)_{ps} = x_{p1}(\varepsilon - 1)x'_{1s} \neq 0$  for all  $\varepsilon \neq 0, 1$ . If  $q$  is also distinct from  $s$ , we are done, since in this case  $y(\varepsilon)_{qs}$  is also distinct from 0 for all  $\varepsilon \neq 0, 1$ , and thus  $y(\varepsilon)$  cannot be monomial. Thus, we are left with the case where  $q = s$ . In this case  $y(\varepsilon)_{ss} = 1 + x_{s1}(\varepsilon - 1)x'_{1s}$  equals zero for at most one value of  $\varepsilon \neq 0$ . Altogether, this prohibits three values of  $\varepsilon$ . Since  $|T| \geq 4$ , we can find an  $\varepsilon$  such that  $y(\varepsilon)$  is not monomial.

The above contradiction shows that  $x$  has exactly one nonzero entry in the first column, say  $x_{p1} \neq 0$ . The same argument, with the roles of  $x$  and  $x^{-1}$  interchanged, shows that  $x^{-1}$  has exactly one nonzero entry in the first row, say  $x'_{1s} \neq 0$ . Since  $x^{-1}$  and  $x$  are mutually inverse, it follows that  $p = s$ . Thus, finally,  $xQx^{-1} = Q_p$ , which proves the lemma.  $\square$

The above two lemmas immediately imply the following result.

**Lemma 9.** *Let  $X$  and  $Y$  be two 1-tori in  $GL(2, T)$ ,  $|T| \geq 7$ . Then either  $X, Y \leq B$ , up to simultaneous conjugacy, or  $\langle X, Y \rangle = GL(2, T)$ .*

### §5. PAIRS OF 1-TORI

In this section we prove the first main result of the present paper. In analogy with the long root subgroups (see, in particular, [60, 92] [73]–[76], [83]), we introduce the *angle* between two 1-tori, which describes almost completely, if not the orbitals themselves, then, at least, their spans. Unlike root subgroups, this angle is purely conventional, and

expresses what the angle between two root subgroups with the same values of  $l, m, p, q$  would have been. For example, from a different viewpoint one could say that in case 7 below the angle is  $2\pi/3$ .

**Theorem 1.** *Assume that  $|T| \geq 7$ . Then for any  $n \geq 3$  the orbits of  $GL(n, T)$  acting by simultaneous conjugation on pairs  $(X, Y)$  of 1-tori admit the following description. These orbits can be distinguished by the values of  $l, m, p, q$ , and  $c$ . The values of these invariants on orbits and the corresponding spans are identified in the following table.*

$NN$	$\phi$	$l$	$m$	$p$	$q$	$c$	$\langle X, Y \rangle$
1	0	1	1	1	1	1	$Q_1$
2	$\pi/3$	1	2	1	1	1	$Q_1 X_{12}$
3	$\pi/3$	2	1	1	1	1	$Q_1 X_{21}$
4	$\pi/2$	2	2	0	0	–	$Q_1 Q_2$
5	$2\pi/3$	2	2	0	1	–	$Q_1 Q_2 X_{12}$
6	$2\pi/3$	2	2	1	0	–	$Q_1 Q_2 X_{12}$
7	$\pi$	2	2	1	1	1	$Q_2 X_{12} X_{13} X_{23}$
8*	$\pi$	2	2	1	1	$\neq 1$	$GL(2, T)$

In particular, in the commutative case we get exactly  $|K| + 5$  orbitals.

*Proof.* Lemmas 3, 4, and 9 imply that, with exactly one exception, for all orbitals  $(X, Y)$  we have the following alternative: either  $X, Y \leq B(2, T)$ , or  $\langle X, Y \rangle = GL(2, T)$ . The exceptional orbital is represented by

$$(Q_{v_1, u_1+u_2}, Q_{v_1+v_3, u_1}) \sim (Q, Q_{v_1+v_3, u_1+u_2}).$$

This is exactly the case where  $\phi = \pi$  and  $c = 1$ . Now, Lemma 5 implies that under the assumption  $|T| \geq 5$  the span of this pair has the required shape. On the other hand, the following lemma settles the case where  $X, Y \in B(2, T)$ .

**Lemma 10.** *Assume that  $|T| \geq 3$ . Then all pairs  $X, Y \leq B(2, T)$  belong to one of the orbits 1 – 6 listed above, and their spans have the required shape.*

*Proof.* Without loss of generality we may assume that  $X = Q_i, i = 1, 2$ . In its turn, for  $Y = Q_{ab}$  we have an alternative: either  $a = v_1$ , or  $b = u_2$ .

If  $X = Q_1$  and  $a = v_1$ , then either  $b = u_1$ , and then  $X = Y$ , or  $b = u_1 + \lambda u_2, \lambda \neq 0$ . Conjugating by an element of  $Q_2$ , we may assume that  $\lambda = 1$ , which leads to one orbit represented by  $(X, Y) = (Q_1, Q_{v_1, u_1+u_2})$ . Clearly,  $\langle X, Y \rangle \geq X_{12}$ . This gives us the orbits corresponding to  $\phi = 0$  and the first case for  $\phi = \pi/3$ .

The case of  $X = Q_2$  and  $b = u_2$  is similar, but it leads to the second case for  $\phi = \pi/3$ , represented by  $(Q_2, Q_{v_1+v_2, u_2})$ , or, what is the same, by  $(Q_1, Q_{v_1+v_2, u_1})$ . Clearly, this case is fused with the first one by the contragredient map.

On the other hand, if  $X = Q_1$  and  $b = u_2$ , then either  $a = v_2$ , and then  $Y = Q_2$ , or  $a = \lambda u_1 + u_2, \lambda \neq 0$ . Conjugating by an element of  $Q_1$ , we may assume that  $\lambda = 1$ , which leads to one orbital represented by  $(X, Y) = (Q_1, Q_{v_1+v_2, u_2})$ . Since  $|T| \geq 3$ , we have  $[X, Y] = X_{12}$ . This gives us the orbit corresponding to  $\phi = \pi/2$  and the first case for  $\phi = 2\pi/3$ .

The case of  $X = Q_2$  and  $a = v_1$  is similar, but it leads to the second case for  $\phi = 2\pi/3$ , represented by  $(Q_2, Q_{v_1, u_1+u_2})$ . Clearly, this case is fused with the first one if we are allowed to permute  $X$  and  $Y$ . This finishes the proof of the lemma.  $\square$

Finally, if  $\langle Q, Q_{ab} \rangle = GL(2, T)$ , then  $a = v_1 + \lambda v_2$ ,  $\lambda \neq 0$ , and  $b = u_1 + \mu u_2$ ,  $\mu \neq 0$ . Conjugating by an element of  $Q_2$ , we may even assume that  $a = v_1 + v_2$ . Now for this orbital we have  $\phi = \pi$  and  $c = 1 + \mu \neq 1$ . Since  $c$  is left invariant by simultaneous conjugation, for every  $c \neq 1$  we get one orbital represented by  $(Q, Q_{v_1+v_2, u_1+(c-1)u_2})$ .  $\square$

It only remains to describe the orbitals and the corresponding spans in the exceptional cases of fields containing less than 7 elements. Only the case of  $\phi = \pi$  is problematic. The case of  $K = \mathbb{F}_2$  is vacuous. In the cases of  $\mathbb{F}_3$ ,  $\mathbb{F}_4$ , and  $\mathbb{F}_5$  the 1-torus is generated by a pseudoreflection of order 2 (i.e., a reflection), 3, or 4, respectively. Thus, our problem amounts to a description of the orbits of  $GL(2, K)$  on pairs of such conjugate pseudoreflections.

In the case of  $\mathbb{F}_3$ , Lemma 8 breaks down. Out of 12 reflections, 10 lie in  $B(2, 3)$  or  $B^-(2, 3)$ , but each of the remaining two  $\pm(e_{12} + e_{21})$  generates  $N(2, 3)$  together with  $d_1(-1)$ . Thus, in this case the spans in 7 and 8(-1) differ from the generic case.

As was observed in §3, for the case of  $\mathbb{F}_4$  the span for the orbit 7 differs from the generic case. Otherwise, no alterations are necessary. The results of [45] show that Lemma 6 remains valid for  $\mathbb{F}_4$ . Indeed, let  $\omega$  be a generator of  $\mathbb{F}_4$ ,  $\omega^3 = 1$ . Then 14 out of 20 pseudoreflections conjugate to  $d_1(\omega)$  lie in  $B(2, 4)$  or  $B^-(2, 4)$ . Up to conjugation by a monomial matrix, the remaining 6 have the form

$$\begin{pmatrix} 0 & \omega^2 \\ \omega^2 & \omega^2 \end{pmatrix}.$$

It is immediate that each of these matrices together with  $d_1(\omega)$  generates  $GL(2, 4)$ .

Finally, let  $K = \mathbb{F}_5$ . In this case, 18 out of 30 pseudoreflections conjugate to  $d_1(2)$  lie in  $B(2, 5)$  or  $B^-(2, 5)$ . Up to conjugation by a monomial matrix, 8 of the remaining 12 have the form

$$\begin{pmatrix} 0 & 1 \\ 3 & 3 \end{pmatrix}$$

and thus generate the entire group  $GL(2, 5)$  together with  $d_1(2)$ . The remaining 4 are conjugate by a monomial matrix to

$$\begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix},$$

and a straightforward computation (see, for example, [10]) shows that each of these matrices together with  $d_1(2)$  generates a subgroup of order 96.

### §6. OVERGROUPS OF THE DIAGONAL SUBGROUP

In this section we show that Theorem 1 immediately implies results on overgroups of the diagonal subgroup  $D = D(n, T)$  in the general linear group  $G = GL(n, T)$  over a skew field  $T$ ; see, for example, [8, 9, 13]. These overgroups are described in terms of closed sets of roots in the root system  $\Phi$  of type  $A_{n-1}$ .

Recall that  $\Phi$  may be realized as follows:

$$\Phi = \{(i, j) \mid 1 \leq i \neq j \leq n\}.$$

The Weyl group  $W(A_{n-1}) = S_n$  acts in an obvious way, and it remains to define the sum of two roots. We put  $(i, j) + (j, h) = (i, h)$  and agree that  $(i, j) + (h, k)$  is not a root if  $j \neq h$  and  $i \neq k$ .

A subset  $S \subseteq \Phi$  is said to be *closed* if for any two roots  $\alpha, \beta \in S$  such that their sum is a root, we have  $\alpha + \beta \in S$ . Several subgroups in  $G = GL(n, T)$  may be associated with  $S$ . Let  $E(S)$  be the subgroup of  $G$  spanned by all  $X_{ij}$  such that  $(i, j) \in S$ . Then, clearly,  $E(S)$  is normalized by  $D$ , and the subgroup  $G(S) = DE(S)$  contains  $D$ . The normalizer of  $G(S)$  in  $G$  is denoted by  $N(S)$ . If  $T$  contains at least 3 elements, then

$N(S)$  is spanned over  $G(S)$  by all permutation matrices  $w \in S_n$  such that  $wS = S$  (see [8] or [9] for a direct proof, but in fact this is a very special case of the Tits conjugacy theorem).

Then the standard description of subgroups in  $G$  containing  $D$  asserts that for any such subgroup  $H$ ,  $D \leq H \leq G$ , there exists a unique closed set of roots  $S \subseteq \Phi$  such that  $G(S) \leq H \leq N(S)$ . In [8, 9, 13] it was proved that the standard description is valid whenever  $T$  contains at least 7 elements, i.e., that the following result holds.

**Theorem 2.** *Let  $T$  be a skew field containing at least 7 elements. Then for any subgroup  $H$  of  $\mathrm{GL}(n, T)$  containing  $D(n, T)$ , there exists a unique closed set  $S$  of roots such that*

$$G(S) \leq H \leq N(S).$$

When  $T = K$  is a (commutative) field, this theorem asserts precisely that all intermediate subgroups are *algebraic*, i.e., are groups of  $K$ -rational points of the overgroups of the corresponding algebraic torus  $\overline{D} = D(n, \overline{K})$  in the general linear group  $\overline{G} = \mathrm{GL}(n, \overline{K})$  over the algebraic closure  $\overline{K}$  of the field  $K$  (see [12]). Nothing like that holds for nonsplit tori, even for the field  $\mathbb{R}$  of real numbers; see references in [26, 123].

As a matter of fact, in the above papers a slightly different language was used. Closed sets of roots were identified with *D-nets of ideals* in  $T$ , i.e., square tables  $\sigma = (\sigma_{ij})$ ,  $1 \leq i, j \leq n$ , of ideals  $\sigma_{ij} \triangleleft T$  such that

- (1)  $\sigma_{ij}\sigma_{jh} \subseteq \sigma_{ih}$  for all  $i, j, h$ ;
- (2)  $\sigma_{ii} = T$  for all  $i$ .

Namely, for any closed set  $S$  of roots we may define the corresponding D-net  $\sigma$  of ideals as follows:  $\sigma_{ij} = T$  if either  $i = j$  or  $(i, j) \in S$ , and  $\sigma_{ij} = 0$  otherwise. Now, the group  $G(S)$  may be defined as follows:

$$G(S) = \{a = (a_{ij}) \in G \mid a_{ij} \in \sigma_{ij}, 1 \leq i, j \leq n\}.$$

This language is more suitable for generalizations to rings; see [9, 13, 25, 123].

Standard arguments of the theory of linear groups (see [39]) reduce the proof of Theorem 1 to the nonexistence of proper primitive irreducible subgroups of  $\mathrm{GL}(n, T)$  containing  $D(n, T)$ . Compare [15], where this was done for the case of the special linear group; the case of  $\mathrm{GL}_n$  is even simpler and was explicitly written up (and/or down) in [101]. Using McLaughlin's theorem [100] and its noncommutative version [98, 23] (Lemmas 11 and 12 below), one readily sees (compare Proposition 2 of [15, II]) that this is equivalent to the following statement.

**Lemma 11.** *Assume that  $T$  is a skew field containing at least 7 elements and  $H$  is a nonmonomial subgroup of the general linear group  $G = \mathrm{GL}(n, T)$ . If  $H$  contains the group  $D = D(n, T)$  of diagonal matrices, then  $H$  contains a unipotent root subgroup.*

Clearly, this lemma (and thus also Theorem 2) immediately follows from our Theorem 1. Indeed, if  $x \in H$  is a nonmonomial matrix, then, by Lemma 8, for some  $i$ , the 1-torus  $X = xQ_i x \leq H$  is nonmonomial (otherwise  $xDx^{-1} \leq D$ , which contradicts the assumption that  $x$  is nonmonomial). Being nonmonomial,  $X$  cannot coincide with  $Q_i$ . Since the centers  $v_1, \dots, v_n$  of the tori  $Q_1, \dots, Q_n$  span the entire space  $T^n$ , and the pairing  ${}^nT \times T^n \rightarrow T$  is nondegenerate, it follows that  $X$  cannot be orthogonal to all  $Q_i$ ,  $1 \leq i \leq n$ . Thus, there exists  $i$  such that  $X$  is distinct from  $Q_i$  and not orthogonal to it. Then by Theorem 1 already the span  $\langle X, Q_i \rangle \leq H$  contains a unipotent root subgroup.

Of course, for the case of  $\mathrm{GL}_n$  this proof is not much simpler than the direct proof in the papers [8, 9, 13]. But already for other classical groups, let alone for exceptional ones, it is usually an enormous advantage to work with two one-dimensional tori lying in something very small, like  $D_4$ , instead of calculating in the group itself.

§7. IRREDUCIBLE GROUPS GENERATED BY 1-TORI

In this section we prove an analog of McLaughlin’s theorem for irreducible subgroups generated by 1-tori. We recall the theorem first.

**Lemma 12.** *Let  $K$  be a field containing at least 3 elements, and let  $H$  be an irreducible subgroup of  $G = GL(n, K)$  generated by unipotent root subgroups. Then either  $H = SL(n, K)$ , or  $n = 2l$  is even and  $H \sim Sp(n, K)$ .*

Now, let  $T$  be a skew field. Recall that  $E(n, T)$  is the elementary subgroup of degree  $n$  over  $T$  generated by all elementary transvections of degree  $n$ . The following analog of McLaughlin’s theorem was established in [98] and [23]. It can be interpreted as the nonexistence of a *natural* analog of the symplectic group in the noncommutative case.

**Lemma 13.** *Let  $T$  be a noncommutative skew field, and let  $H$  be an irreducible subgroup of  $G = GL(n, T)$  generated by unipotent root subgroups. Then  $H = E(n, T)$ .*

We are going to prove an analog of these results for pseudoreflections. Like Theorem 2, the following result can be deduced from Theorem 1. However, we prefer to give a direct proof in the style of [8, 9, 14]. The following result was first published in [23]. Observe that [72] produces an infinite-dimensional generalization of this result.

**Theorem 3.** *Let  $T$  be a skew field containing at least 7 elements. Then any irreducible subgroup  $H$  of  $G = GL(n, T)$  generated by 1-tori coincides with  $G$ .*

*Proof.* To reduce our result to the preceding lemmas, we need to prove that  $H$  contains a unipotent root subgroup. Passing from  $H$  to its conjugate, we may assume that  $Q_1 \leq H$ . Let

$$Q_{ab} = \{x_{ab}(\xi), \xi \in T^\#\} \leq H$$

be another 1-torus contained in  $H$ . Here we keep the notation of §1; in particular,  $a = (\alpha_1, \dots, \alpha_n)^t$  is a column of height  $n$  and  $b = (\beta_1, \dots, \beta_n)$  is a row of length  $n$ , subject to the condition  $\beta_1\alpha_1 + \dots + \beta_n\alpha_n = 1$ .

Since  $H$  is irreducible, there exists a 1-torus  $Q_{ab} \leq H$  such that  $\beta_1 \neq 0$  and not all  $\alpha_2, \dots, \alpha_n$  are equal to 0. We fix such  $a$  and  $b$  and denote  $x_{ab}(\xi)$ ,  $\xi \in T^\#$ , simply by  $x(\xi)$ . We have  $x(\xi)^{-1} = x(\bar{\xi})$ , where  $\bar{\xi} = -\xi(1 + \xi)^{-1}$ .

If  $\alpha_1 \neq 0$ , we pick up commuting  $\xi, \eta \neq 0, -1$  such that  $\xi \neq \eta$ ,  $\xi \neq \bar{\eta}$ ,  $\alpha_1\xi\beta_1 \neq -1$ , and  $\alpha_1\eta\beta_1 \neq -1$ . Such  $\xi$  and  $\eta$  exist, because  $T$  contains at least 7 elements. When  $T$  is noncommutative,  $\beta_1\alpha_1$  is contained in an infinite subfield, so that we may always assume that  $\xi$  and  $\eta$  both commute with  $\beta_1\alpha_1$ . Now, consider the matrix

$$y = x(\xi)d_1(\varepsilon)x(\bar{\eta}) \in H,$$

where

$$\varepsilon = (1 + \alpha_1\xi\beta_1)^{-1}\alpha_1\xi\eta^{-1}\alpha_1^{-1}(1 + \alpha_1\eta\beta_1).$$

A straightforward calculation shows that  $y_{1j} = 0$  for all  $2 \leq j \leq n$ , and that  $y_{j1}$  differs from  $\alpha_j$  by an invertible factor. In particular, not all  $b_{j1}$ ,  $2 \leq j \leq n$ , are equal to 0. Take  $\theta \neq 0, 1$ , and set  $z = [y, d_1(\theta)] \in H$ . Thus,  $z \neq e$  is a nontrivial transvection in  $H$ . But then  $H$  contains the entire unipotent root subgroup

$$X = \{d_1(\varepsilon)zd_1(\varepsilon)^{-1}, \varepsilon \in T^*\} \cup \{e\}.$$

If  $\alpha_1 = 0$ , then we may set  $y = x(\xi)$ ,  $\xi \neq 0, -1$ , in the preceding argument.

Now, let  $L$  be the normal subgroup of  $H$  generated by  $X$ . Should  $H$  be imprimitive, it would be monomial ([40, Lemma 1.8]), and then it cannot contain a root subgroup. Thus, we may assume that  $H$  is primitive, and by Clifford’s theorem (which remains valid for skew fields, see [39]),  $L$  is irreducible (see [40, Lemma 1.10]). Then the preceding two

lemmas imply that either  $H = E(n, T)$ , or  $T = K$  is commutative,  $n = 2l$  is even, and  $L \sim \text{Sp}(n, K)$ . But the second possibility is excluded, because the normalizer of  $\text{Sp}(n, K)$  in  $\text{GL}(n, K)$  coincides with  $\text{GSp}(n, K)$  (see [96, 83]) and contains pseudoreflections. Since, moreover,  $Q_1 \leq H$ , we have  $H = \text{GL}(n, T)$ .  $\square$

§8. MUTUAL ARRANGEMENT OF A 1-TORUS AND A ROOT SUBGROUP

In the present section,  $Q$  denotes an arbitrary 1-torus. Our purpose is to describe the orbits of  $G = \text{GL}(n, T)$  acting by simultaneous conjugation on pairs that consist of a 1-torus and a root subgroup:

$$(Q, X) \mapsto (gQg^{-1}, gXg^{-1}), \quad g \in G,$$

and to identify the corresponding spans.

We start with listing the obvious invariants of such a pair. Again, there are four obvious discrete invariants. Let  $a_1$  and  $a_2$  be the centers of  $Q$  and  $X$ , respectively, and let  $b_1$  and  $b_2$  be the corresponding axes. Set  $l = l(X, Y) = \dim\langle a_1, a_2 \rangle$  and  $m = m(X, Y) = \dim\langle b_1, b_2 \rangle$ . Then, clearly,  $l$  and  $m$  can take values 1 or 2 and are not changed by simultaneous conjugation. The products  $b_1a_2$  and  $b_2a_1$  also remain invariant in the projective sense. This means that if  $b_1a_2$  is equal to zero (respectively, distinct from zero) for a given pair  $(Q, X)$ , then the same is true for any conjugate pair  $(gQg^{-1}, gXg^{-1})$ . We denote by  $p = p(X, Y)$  and  $q = q(X, Y)$  the invariants of a pair  $(Q, X)$  that take the value 0 if  $b_1a_2 = 0$  and  $b_2a_1 = 0$ , respectively, and 1 otherwise. It turns out that these obvious discrete invariants suffice to distinguish the orbits. Namely, the following result holds.

**Theorem 4.** *Assume that  $|T| \geq 7$ . Then for any  $n \geq 3$  there are 6 orbits of the group  $\text{GL}(n, T)$  acting by simultaneous conjugation on pairs  $(Q, X)$  that consist of a 1-torus and a root subgroup. These orbits can be distinguished by values of  $l, m, p, q$ . The values of these invariants on the orbits and the corresponding spans are identified in the following table.*

$NN$	$l$	$m$	$p$	$q$	$\langle Q, X \rangle$
1	1	2	1	0	$Q_1X_{21}$
2	2	1	0	1	$Q_1X_{12}$
3	2	2	0	0	$Q_2X_{13}$
4	2	2	0	1	$Q_2X_{12}X_{13}$
5	2	2	1	0	$Q_2X_{13}X_{23}$
6	2	2	1	1	$\text{GL}(2, T)$

First, observe that, since the invariants  $l, m, p$ , and  $q$  are preserved by simultaneous conjugation, all these cases actually lie in distinct orbits. What remains to prove is that no further possibilities for these invariants can arise, and that all pairs  $(Q, X)$  with a given 4-tuple of invariants actually form a single orbit.

As a first step, we reduce the problem to degree  $n \leq 3$ . The following reduction is proved by exactly the same argument as Lemma 2, because that proof is only based on the fact that  $Q$  is a 1-torus and  $X$  consists of 1-dimensional transformations.

**Lemma 14.** *Let  $Q$  be a 1-torus and  $X$  a root subgroup in  $\text{GL}(n, T)$ ,  $n \geq 4$ . Then there exists  $g \in \text{GL}(n, T)$  such that*

$$gXg^{-1}, gYg^{-1} \leq H,$$

where  $H$  is either the subgroup of degree 2 generated by  $Q$ ,  $X_{12}$ , and  $X_{21}$ , isomorphic to  $GL(2, T)$ , or the subgroup of degree 3 generated by  $Q_2$ ,  $X_{12}$ ,  $X_{13}$ , and  $X_{23}$ .

Thus, from now on we may assume that  $Q$  and  $X$  are both contained in one of the following subgroups:

$$\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & * & * \\ 0 & * & * \\ 0 & 0 & 1 \end{pmatrix}.$$

In the sequel we denote the second of these subgroups by  $P$ . By definition, we have  $P = Q_2 X_{12} X_{13} X_{23}$ .

Now, repeating the proof of Lemma 3 word for word, we see that the cases described in lines 1 and 2 of the above table are the only cases arising when either  $l = 1$ , or  $m = 1$ . Observe that since  $b_1 a_1 \neq 0$ , whereas  $b_1 a_2 = 0$  and  $b_2 a_1 = 0$ , unlike the case of two 1-tori the situation  $l = m = 1$  does not occur. As representatives of these orbits we may take  $(Q_1, X_{12})$  and  $(Q_1, X_{21})$ .

Thus, in the sequel we may assume that  $l = m = 2$ .

**Lemma 15.** *Let  $|T| \geq 3$ . There are three  $B(3, T)$ -orbits on the pairs  $(Q, X)$  in  $P$  such that  $l(Q, X) = m(Q, X) = 2$ , namely, the orbits listed in lines 3, 4, and 5 of the above table.*

*Proof.* Denote by  $a_1$  and  $a_2$  (respectively, by  $b_1$  and  $b_2$ ) the centers (respectively, the axes) of  $Q$  and  $X$ . Then, clearly,  $a_1, a_2 \in v_1 T \oplus v_2 T$  and  $b_1, b_2 \in Tu_2 \oplus Tu_3$ . Without loss of generality, we may assume that  $a_1 = v_2$  and  $b_1 = u_2$ . Now,  $l = 2$  implies that  $a_2 \notin v_2 T$ , and  $m = 2$  implies that  $b_2 \notin Tu_2$ . Since  $b_2$  and  $a_2$  are determined only up to an invertible factor, we may assume that  $a_2 = \alpha v_1 + p v_2$  and  $b_2 = q u_2 + \beta u_3$ , where  $p, q = 0, 1$ , whereas  $\alpha, \beta \neq 0$ . This means that, conjugating by an element of  $Q_1$  and by an element of  $Q_3$ , respectively, we may assume that  $a_2 = v_1 + p v_2$  and  $b_2 = q u_2 + u_3$ . Since  $X$  is a root subgroup, we have  $b_2 a_2 = 0$ , and thus,  $p$  and  $q$  cannot be both 1. This leaves us with the three possibilities represented in lines 4, 5, and 6 of the above table. For  $p = q = 0$  we can take  $(Q_2, X_{13})$  as a representative of the orbit. For the other two cases we can take  $(Q_2, X_{v_1+v_2, u_3})$  and  $(Q_2, X_{v_1, u_2+u_3})$ , respectively. It only remains to identify the resulting spans, but this is obvious in view of Theorem 5 of [8].  $\square$

The possibility presented in line 6 of the above table can only happen when, up to conjugacy,  $\langle Q, X \rangle \leq GL(2, T)$ . Let  $Q = Q_1$ . By the same argument as in the proof of Lemma 2, we may assume that  $a_2 = v_1 + \alpha v_2$  and  $b_2 = u_1 + \beta u_2$ . Since  $b_2 a_2 = 0$ , we have  $\beta = -\alpha^{-1}$ . Conjugating by an element of  $Q_2$ , we may even assume that  $a_2 = v_1 + v_2$  and  $b_2 = u_1 - u_2$ . Thus, we may take the pair  $(Q_1, X_{v_1+v_2, u_1-u_2})$  as a representative of the remaining orbit. Arguing exactly as in §4, we see that from the special case of [8, 9, 14] pertaining to  $GL(2, T)$  it follows that if  $|T| \geq 7$ , then the span of  $Q_1$  and  $X_{v_1+v_2, u_1-u_2}$  equals  $GL(2, T)$ . This finishes the proof of Theorem 4.

§9. WHERE NEXT?

In conclusion, I mention some unsolved problems, which we address in a current extensive series of joint papers by myself and Vladimir Nesterov. We assume that the reader is familiar with Chevalley groups and their representations [11, 58, 69]. Let  $\Phi$  be a reduced irreducible root system,  $K$  a field,  $G(\Phi, K)$  a Chevalley group of type  $\Phi$  over  $K$ , and  $\bar{G}(\Phi, K)$  an extended Chevalley group. The group  $\bar{G}(\Phi, K)$  is generated over  $G(\Phi, K)$  by some semisimple elements  $h_\omega(\varepsilon)$ ,  $\omega \in P(\Phi^\vee)$ ,  $\varepsilon \in K^*$ , which we call *weight semisimple elements*. For the adjoint case, a construction of such elements was contained already in the Chevalley original paper [59]; see also [69]. For the simply connected case,

it is considerably harder to construct weight elements. This task was only concluded in [66]; see also [18]–[20], [24, 26, 124], where all possible details can be found.

**Problem 1.** Describe the orbits of a Chevalley group  $G(\Phi, K)$  acting by simultaneous conjugation on pairs of microweight tori

$$X, Y \sim \{h_\omega(\varepsilon) \mid \varepsilon \in K^*\}, \quad \omega \in P(\Phi^\vee),$$

and the corresponding spans.

In the present paper we have used exclusively elementary matrix techniques. As opposed to that, the main technical tool in the work by Nesterov and the author is Bruhat decomposition, and first of all, the results related to Bruhat decomposition of microweight elements, obtained in my papers [18, 124].

Actually, the most interesting cases here, from the viewpoint of applications, are the tori of the type  $\{h_{\varpi_1}(\varepsilon) \mid \varepsilon \in K^*\}$  in the Chevalley group  $G(E_6, K)$  and the tori of the type  $\{h_{\varpi_7}(\varepsilon) \mid \varepsilon \in K^*\}$  in the Chevalley group  $G(E_7, K)$ . For the majority of real life applications it would suffice to have the following qualitative corollary to our main results, which reduces many problems of the geometry of microweight tori to Timmesfeld's theory of abstract root subgroups. If  $X, Y$  are two noncommuting microweight tori, then the subgroup  $\langle X, Y \rangle$  generated by them contains a small unipotent element. The most complicated element that can arise in this context for the group of type  $E_6$  has the form  $x_{\beta_1}(\xi_1)x_{\beta_2}(\xi_2)$ , while for the group of type  $E_7$  it has the form  $x_{\beta_1}(\xi_1)x_{\beta_2}(\xi_2)x_{\beta_3}(\xi_3)$ , where  $\beta_1, \beta_2, \beta_3$  are pairwise orthogonal roots.

However, if one is interested in *explicit* answers, already the case of  $GL_n$  is highly nontrivial. In this case microweight tori consist of quadratic semisimple matrices. By analogy with the 1-tori treated in the present paper, one can consider the case of  $m$ -tori

$$Q = \{\text{diag}(\varepsilon, \dots, \varepsilon, 1, \dots, 1) \mid \varepsilon \in K^*\},$$

where the number of  $\varepsilon$ 's equals  $m$ .

**Problem 2.** Describe the orbits of the group  $GL(n, K)$  acting by simultaneous conjugation on pairs of  $m$ -tori, and describe the corresponding spans.

The main additional complication, as compared with the present paper and [72], is that for  $m \geq 2$  the answer is not purely geometric any longer. As it happens, it depends on the structure of extensions of the ground field. Even worse, these extensions may fail to be commutative; they can be *skew fields*. In a series of remarkable papers, Bashkirov has *completely* treated the case of quadratic unipotent elements of residue 2. I believe that a combination of the methods of the present paper with the results by Bashkirov makes it possible to get an *explicit* answer for the case of 2-tori, approximately in the same style as in the present paper, naturally, with some nontrivial arithmetic parameters.

The following problem is even more ambitious and tempting, as far as applications to the structure theory of Chevalley groups are concerned.

**Problem 3.** Describe the orbits of the Chevalley group  $G(\Phi, K)$  acting by simultaneous conjugation on pairs of long root tori

$$X, Y \sim \{h_\alpha(\varepsilon) \mid \varepsilon \in K^*\}, \quad \alpha \in \Phi_l,$$

and describe the corresponding spans.

There is little doubt that a combination of our methods with the results by Franz Timmesfeld and Eugenio Bashkirov may serve as a bench mark towards a complete solution of this problem. Here again, our main tool would be the results on the Bruhat decomposition of long root semisimple elements, obtained by the author and Andrei Semenov [20, 22, 24, 31, 32, 57]. Moreover, the existing reductions completely reduce the

study of reductive parts of the spans in all Chevalley groups to the key case of type  $D_4$ , and the study of the spans themselves, together with the unipotent parts, to the type  $D_5$ . Getting exhaustive answers inside  $\text{Spin}(8, K)$  and even  $\text{Spin}(10, K)$  seems to be a fatiguing, but still perfectly realistic enterprise.

Again, for many real life applications, the following corollary to our main results would suffice. If  $X, Y$  are two noncommuting long root tori, then the subgroup  $\langle X, Y \rangle$  generated by them contains a small unipotent element. The most complicated element that can occur here is the product

$$x_{\beta_1}(\xi_1)x_{\beta_2}(\xi_2)x_{\beta_3}(\xi_3)x_{\beta_4}(\xi_4),$$

where  $\beta_1, \beta_2, \beta_3, \beta_4$  are four pairwise orthogonal roots.

As it happens, here, as in Problem 1, already the linear case is *absolutely nontrivial*. A solution of the following problem should give, in particular, a unified approach towards the description of overgroups of the diagonal subgroup *in natural generality*, not involving a separate study of small characteristics, finite and infinite fields, etc. Observe that recently Oliver King [95] obtained a definitive description of overgroups in this case, which simultaneously generalizes the results by the author [15] (for fields) and by Bui Xuan Hai (for skew fields with certain restrictions on the orders of their center). However, as of today, there are still no unified proofs which cover all cases simultaneously.

**Problem 4.** Describe the orbits of the group  $GL(n, T)$  acting by simultaneous conjugation on pairs of long root tori

$$X, Y \sim \{\text{diag}(\varepsilon, \varepsilon^{-1}, 1, \dots, 1) \mid \varepsilon \in T^*\},$$

and describe the corresponding spans. Using this result, give new proofs of all existing results on overgroups of split maximal tori.

In the following problem we propose to do the same for the *much* more general case of Bak's unitary group  $SU(n, T, \Lambda)$  over a skew field  $T$  with involution  $\lambda \rightarrow \bar{\lambda}$  and form parameter  $\Lambda$ , see [85, 61, 56, 102], where one can find references to the original work.

**Problem 5.** Describe the orbits of the group  $SU(n, T, \Lambda)$  acting by simultaneous conjugation on pairs of long root tori

$$X, Y \sim \{\text{diag}(\varepsilon, 1, \dots, 1, \bar{\varepsilon}^{-1}) \mid \varepsilon \in K^*\},$$

and describe the corresponding spans. Using this result, give new proofs of results on overgroups of quasisplit maximal tori.

There is no need to say that this problem is *terribly much* harder than the preceding one, since both noncommutativity and nontrivial involution do not hide deep inside the proofs, but rather leap out from the outset, at the level of definitions. Not to inflate the number of references, I merely quote some recent papers by Elizaveta Dybkova [33]–[36], where overgroups of maximal tori were discussed in this general setting. In [25, 26, 123], one can find references to our preceding papers pertaining to the *split* classical groups.

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