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QUASIANALYTIC CARLEMAN CLASSES ON BOUNDED DOMAINS

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Abstract. Several criteria for the quasianaliticity of Carleman classes at a boundary point of a Jordan domain with rectifiable boundary are found.

Introduction

We start with recalling the necessary definitions from the paper [1].

Let E be a perfect compact set in the plane $\mathbb{C}$. A complex-valued function f is said to be infinitely differentiable on E if there exist functions $f_0, f_1, \dots$ continuous on E with $f_0(z) \equiv f(z)$, $z \in E$, and such that, for any $n = 0, 1, 2, \dots, k = 0, 1, \dots, n$, the functions

$$R_{n,k}(\zeta, z) := f_k(\zeta) - \sum_{p=0}^{n-k} f_{k+p}(z)\frac{(\zeta - z)^p}{p!}$$

satisfy the estimate

$$|R_{n,k}(\zeta, z)| = o(|\zeta - z|^{n-k})$$

uniformly in $\zeta, z \in E$. Note that for any infinitely differentiable function f the functions f_k are determined uniquely by f via the recurrence relations

$$f_0(z) = f(z), \qquad f_{k+1}(z) = \lim_{\zeta \longrightarrow z} \frac{f_k(\zeta) - f(z)}{\zeta - z}, \quad k = 0, 1, \dots .$$

In particular, the function f turns out to be holomorphic at the interior points of E and, moreover, $f_k(z) = f^{(k)}(z)$, and the derivatives of f extend continuously up to the boundary of the set E. Having this in mind, in what follows we write $f^{(k)}$ in place of f_k for the functions infinitely differentiable on E.

For an increasing sequence of positive numbers $\mathcal{M} = (M_n)_{n=0}^{\infty}$ and for a positive integer q, we denote by $A_q(E, \mathcal{M})$ the class of functions f infinitely differentiable on E and satisfying the condition

$$|R_{n,k}(\zeta, z)| \le C_f q^{n+1} M_{n+1} \frac{|\zeta - z|^{n-k+1}}{(n-k+1)!}, \quad \zeta, z \in E,$$

where the constant C_f depends neither on n, k nor on $\zeta, z \in E$. The Carleman class $A(E, \mathcal{M})$ is defined as the union of all classes $A_q(E, \mathcal{M})$, $q \in \mathbb{N}$.

If E is a closed interval I of the real axis, then, by the Taylor formula, the Carleman classes can be described in a classical way as the classes of those infinitely differentiable functions f on the corresponding open interval I^0 for which there exist $q_f \in \mathbb{N}$ and $C_f > 0$ such that

$$|f^{(k)}(x)| \le C_f q_f^k M_k, \quad k = 0, 1, \dots, \ x \in I^0.$$

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In the present paper the set E will be the closure of a simply connected bounded domain D in $\mathbb{C}$ with rectifiable Jordan boundary. In this case, the functions of class $A(\overline{D},\mathcal{M})$ are holomorphic in D and admit continuous extensions to the boundary of D together with their derivatives. In what follows, we denote by $f^{(k)}(z)$ the kth order derivative of f extended continuously to the boundary of D. Thus, the class $A(\overline{D},\mathcal{M})$ consists of functions f holomorphic in D and satisfying the condition

$$\sup_{n\geq 0, k\leq n}\sup_{z,\zeta\in D}\frac{(n-k+1)!}{q^{n+1}M_{n+1}|\zeta-z|^{n-k+1}}\left|f^{(k)}(\zeta)-\sum_{p-0}^{n-k}f^{(k+p)}(z)\frac{(\zeta-z)^p}{p!}\right|<\infty$$

for some $q\in\mathbb{N}$. If the domain D is a quasidisk, i.e., there exists $\delta>0$ such that any two points $\zeta, z\in D$ can be connected by a curve of length at most $\delta|z-\zeta|$, then the Carleman class $A(\overline{D},\mathcal{M})$ coincides with the class of functions f that are holomorphic in D and, for some $q_f\in\mathbb{N}$ and $C_f>0$, satisfy $|f^{(k)}(z)|\leq C_f q_f^k M_k$, $k=0,1,2,\dots$, $z\in D$. Obviously, the convex domains enjoy this condition. The problem is as follows: if a point z_0 is on the boundary (in the plane sense) of E, then what conditions on E and the sequence $\mathcal{M}$ ensure that the uniqueness theorem at the point z_0 will hold true for the class $A(E,\mathcal{M})$? The classes where there is no nonzero function vanishing at the point z_0 together with all its derivatives are said to be *quasianalytic* at z_0.

The problem of finding conditions on the sequence $\mathcal{M}$ that are necessary and sufficient for quasianalyticity dates back to Hadamard who posed it in 1912 (see [2]).

Let I be an open interval in $\mathbb{R}$, and let

$$A(I,\mathcal{M})=\Big\{f\in C^\infty(I):\sup_{x\in I}|f^{(n)}(x)|\leq C_f q_f^n M_n \text{ for all } n\geq 0\Big\}.$$

The class $A(I,\mathcal{M})$ is said to be *quasianalytic at a point* $x_0\in I$ if

$$f\in A(I,\mathcal{M}),\ f^{(n)}(x_0)=0 \text{ for all } n\geq 0 \Longrightarrow f(x)\equiv 0.$$

A criterion for quasianalyticity is given by the Denjoy–Carleman–Ostrowski theorem [3]–[5].

Let $T(r)=\sup_{n\geq 0}\frac{r^n}{M_n}$ be the trace function for the sequence $\mathcal{M}$. The class $A(I,\mathcal{M})$ is quasianalytic at a point $x_0\in I$ if and only if

$$\int_1^\infty \frac{\ln T(r)}{r^2}\,dr=\infty.$$

As we see, this criterion does not depend on the point $x_0\in I$.

A criterion for quasianalyticity at the point $z=0$ for the class $A(\overline{\Delta}_\gamma,\mathcal{M})$, where

$$\Delta_\gamma=\Big\{z:|\arg z|<\frac{\pi}{2}\gamma,\ 0<|z|<\infty\Big\}$$

is the angle of opening $\gamma\pi$ with vertex at zero, was obtained by Salinas in [6]: for the quasianalyticity of $A(\overline{\Delta}_\gamma,\mathcal{M})$ at zero it is necessary and sufficient that

$$\int_1^\infty \frac{\ln T(r)}{r^{\frac{\gamma+2}{\gamma+1}}}\,dr=\infty.$$

For a boundary point of the disk, a quasianalyticity condition is given by a theorem due to Korenblum [7]: in this case the condition also does not depend on the point, and the class in question is quasianalytic if and only if

$$\int_1^\infty \frac{\ln T(r)}{r^{\frac{3}{2}}}\,dr=\infty.$$

The quasianalyticity problem for a boundary point z_0 of a convex bounded domain D was treated in [8] (see also [9]).

Consider the support lines to the convex domain D through the points at distance from z_0 equal to the length of an arc s, and let $\gamma(s)\pi$ be the size of the angle between these lines that contains the domain D. We put

$$R(x) = \exp^{\int_x^{x_0} \frac{1+\gamma(s)}{\gamma(s)}\, d\ln s}, \quad x \in (0; x_0),$$

where x_0 is any positive number less than the length of the boundary of D. Then the quasianalyticity property is equivalent to the condition

$$\int_1^{\infty} \frac{\ln T(r)}{r^2 R^{-1}(r)}\, dr = \infty,$$

where $R^{-1}(r)$ is the function inverse to $R(x)$.

In the present paper, we deal with the quasianalyticity problem at a boundary point z_0 of a nonconvex domain D. We shall pass to the dual problem, which is the problem of density for the system $\{(\zeta - z_0)^{(-n)}\}$, $n = 1, 2, \ldots$, in a certain weighted space of functions holomorphic in the complement of the domain $\overline{D}$. The corresponding spaces can be defined as follows.

In the space $A_q(\overline{D}, \mathcal{M})$, we introduce the norm

$$\|f\|_q := \max\left(\sup_{n\geq 0, k\leq n} \frac{(n-k+1)!}{q^{n+1} M_{n+1}} \sup_{z,\zeta\in D} \frac{|R_{n,k}(\zeta, z)|}{|\zeta - z|^{n-k+1}}, \frac{1}{M_0} \sup_{z\in D} |f(z)| \right).$$

The spaces $A_q(\overline{D}, \mathcal{M})$ are Banach, and obviously the space $A_q(\overline{D}, \mathcal{M})$ is continuously embedded into $A_{q+1}(\overline{D}, \mathcal{M})$. We consider the space $A(\overline{D}, \mathcal{M})$ with the inductive limit topology induced by the spaces $A_q(\overline{D}, \mathcal{M})$:

$$A(\overline{D}, \mathcal{M}) = \operatorname{ind}\lim_q A_q(\overline{D}, \mathcal{M}).$$

The sequence

$$m_n = \frac{M_n}{n!}, \quad n = 0, 1, \ldots,$$

is called the adjoint sequence. In what follows, we always assume that the sequence (m_n) is regular [1], i.e., it satisfies the following three conditions:

1) logarithmic convexity:

$$m_n^2 \leq m_{n-1} m_{n+1}, \quad n = 1, 2, \ldots; \tag{1.1}$$

2) there is an integer $Q > 0$ such that

$$m_{n+1} \leq Q^n m_n, \quad n = 0, 1, \ldots; \tag{1.2}$$

3) the following relation holds true:

$$\lim_{n\to\infty} m_n^{\frac{1}{n}} = \infty. \tag{1.3}$$

We define a function on the positive semiaxis by the formula $M(x) = \sup_{k\geq 0} \frac{1}{m_k x^k}$, $x > 0$. Clearly, $M(x)$ is a monotone decreasing function and

$$\lim_{x\to 0} M(x) = \infty, \quad M(x) \geq \frac{1}{m_0}. \tag{1.4}$$

The logarithmic convexity of the sequence (m_n) shows that we have an inverse representation:

$$m_k = \sup_{x>0} \frac{1}{x^k M(x)}, \quad k = 0, 1, \ldots. \tag{1.5}$$

Let G denote the complement of $\overline{D}$ in the extended complex plane, i.e., $G = \overline{\mathbb{C}} \setminus \overline{D}$, and let

$$d(\zeta) = \inf_{z\in D} |\zeta - z|, \quad \zeta \in G,$$

be the distance function to the boundary of G. For $q \in \mathbb{N}$, we introduce the Banach space

$$X_q = \Big\{\gamma \in H(G),\ \gamma(\infty) = 0,\ \|\gamma\|_{X_q} = \sup_{\zeta\in G} \frac{|\gamma(\zeta)|}{M(qd(\zeta))} < \infty\Big\}.$$

Since the function $M(x)$ is monotone decreasing, the space X_{q+1} is continuously embedded into X_q. We denote by $\widetilde{A}(G,\mathcal{M})$ the projective limit of the spaces X_q:

$$\widetilde{A}(G,\mathcal{M}) = \operatorname*{proj\,lim}_q X_q.$$

To simplify the notation, we assume that the point $z = 0$ lies on the boundary of D and consider the problem of quasianalyticity at the point $z = 0$.

§1. Isomorphism between the spaces $A(\overline{D},\mathcal{M})$ and $\widetilde{A}^*(G,\mathcal{M})$

Let $\widetilde{A}^*(G,\mathcal{M})$ denote the space of continuous linear functionals on $\widetilde{A}(G,\mathcal{M})$ equipped with the strong topology. It is known (see [10]) that

$$\widetilde{A}^*(G,\mathcal{M}) = \operatorname*{ind\,lim}_q X_q^*.$$

Since the function $M(x)$ is bounded from below, the function $(\zeta - z)^{-1}$ belongs to $\widetilde{A}(G,\mathcal{M})$ for any $z \in \overline{D}$. Hence, for every continuous linear functional S on $\widetilde{A}(G,\mathcal{M})$ we can define its Cauchy transform:

$$\widetilde{S}(z) := S_\zeta\Big(\frac{1}{\zeta - z}\Big), \quad z \in \overline{D}.$$

Lemma 1. *For any z_0, $z \in \overline{D}$ and any $k \geq 1$ and $q > 0$, we have*

$$\Big\|\frac{1}{(\zeta - z)^k} - \frac{1}{(\zeta - z_0)^k}\Big\|_{X_q} \leq q^{k+1} m_{k+1} k |z - z_0|$$

and

$$\Big|\frac{1}{(z - z_0)}\Big(\frac{1}{\zeta - z} - \frac{1}{\zeta - z_0}\Big) - \frac{1}{(\zeta - z_0)^2}\Big|_{X_q} \leq q^2 m_2 |z - z_0|.$$

Proof. For $\zeta \in G$, we have

$$\Big|\frac{1}{(\zeta - z)^k} - \frac{1}{(\zeta - z_0)^k}\Big| \leq |z - z_0| \sum_{j=1}^{k} \frac{1}{|\zeta - z|^j |\zeta - z_0|^{k-j+1}} \leq \frac{k|z - z_0|}{d(\zeta)^{k+1}}.$$

Therefore,

$$\begin{aligned}\Big|\frac{1}{(\zeta - z)^k} - \frac{1}{(\zeta - z_0)^k}\Big| &\leq \frac{k|z - z_0|}{d(\zeta)^{k+1}} = \frac{q^{k+1} m_{k+1} k |z - z_0|}{m_{k+1}(qd(\zeta))^{k+1}} \\ &\leq q^{k+1} m_{k+1} k |z - z_0| M(qd(\zeta)).\end{aligned}$$

This yields the first inequality. The proof of the second is similar. □

The second statement of the lemma shows that the function $\widetilde{S}(z)$ is holomorphic in D, and moreover,

$$\widetilde{S}'(z) = S_\zeta\Big(\frac{1}{(\zeta - z)^2}\Big).$$

In the same way we obtain a general formula for an arbitrary $k \geq 1$:

$$\widetilde{S}^{(k)}(z) = S_\zeta\Big(\frac{k!}{(\zeta - z)^{k+1}}\Big). \tag{1.6}$$

Also, the first inequality in Lemma 1 implies that the limit

$$\widetilde{S}^{(k)}(z_0) := \lim_{z \in D,\ z \to z_0} \widetilde{S}^{(k)}(z) = S_z\Big(\frac{k!}{(z - z_0)^{k+1}}\Big)$$

exists for any $z_0 \in \partial D$, i.e., the function $\widetilde{S}^{(k)}(z)$ has a continuous extension to $\overline{D}$.

Theorem 1. *Let (m_n) be a regular sequence, and let D be a Jordan domain. Then the mapping $C : S \longmapsto \widetilde{S}$ is a topological isomorphism between the spaces $\widetilde{A}^*(G, \mathcal{M})$ and $A(\overline{D}, \mathcal{M})$.*

Proof. First, we verify that C is a continuous mapping from $\widetilde{A}^*(G, \mathcal{M})$ to $A(\overline{D}, \mathcal{M})$.

Lemma 2. *For any continuous linear functional S on $\widetilde{A}(G, \mathcal{M})$, its Cauchy transform $\widetilde{S}(z)$ is in the space $A(\overline{D}, \mathcal{M})$, and moreover, for any $S \in X_q^*$, $q \in \mathbb{N}$, we have*

$$\|\widetilde{S}\|_{A_{qQ}(\overline{D}, \mathcal{M})} \leq q\|S\|_{X_q^*},$$

where Q is the number occurring in (1.2).

Proof. For any ζ, z, $w \in \mathbb{C}$, $w \neq \zeta$, z, and any $j = 0, 1, 2, \ldots$, the identity

$$\frac{1}{w - \zeta} - \sum_{p=0}^{j} \frac{(\zeta - z)^p}{(w - z)^{p+1}} \equiv \frac{(\zeta - z)^{j+1}}{(w - \zeta)(w - z)^{j+1}}$$

follows by a straightforward computation of the sum of the geometric progression on the left-hand side.

Differentiating this identity with respect to w, we obtain identities valid for $k = 0, 1, 2, \ldots$:

$$\frac{k!}{(w - \zeta)^{k+1}} - \sum_{p=0}^{j} \frac{(p + k)!(\zeta - z)^p}{p!(w - z)^{p+k+1}} \equiv (\zeta - z)^{j+1} \sum_{s=0}^{k} \frac{\binom{k}{s}(j + k - s)!s!}{j!(w - \zeta)^{s+1}(w - z)^{j+k-s+1}}.$$

We take a number $n \geq k$ and put $j = n - k$ in the last identity:

$$\begin{aligned} &\frac{k!}{(w - \zeta)^{k+1}} - \sum_{p=0}^{n-k} \frac{(p + k)!(\zeta - z)^p}{p!(w - z)^{p+k+1}} \\ &\qquad \equiv (\zeta - z)^{n-k+1} \sum_{s=0}^{k} \frac{\binom{k}{s}(n - s)!s!}{(n - k)!(w - \zeta)^{s+1}(w - z)^{n-s+1}}. \end{aligned} \tag{1.7}$$

Now, let S be a continuous linear functional on the space $\widetilde{A}(G, \mathcal{M})$. As has already been mentioned, the Cauchy transform $\widetilde{S}$ extends continuously to the closure of D together with all its derivatives, and formula (1.6) holds true. Therefore, for any $n = 0, 1, 2, \ldots, k \leq n$, by the linearity of the functional we have

$$\begin{aligned} &\widetilde{S}^{(k)}(\zeta) - \sum_{p=0}^{n-k} \widetilde{S}^{k+p}(z) \frac{(\zeta - z)^p}{p!} \\ &\qquad = \widetilde{S}_w\Big(\frac{k!}{(w - \zeta)^{k+1}} - \sum_{p=0}^{n-k} \frac{(p + k)!}{(w - z)^{p+k+1}} \frac{(\zeta - z)^p}{p!}\Big), \quad \zeta, z \in \overline{D}. \end{aligned} \tag{1.8}$$

Identity (1.7) implies that

$$\begin{aligned}\widetilde{S}^{(k)}(\zeta) - \sum_{p=0}^{n-k} \widetilde{S}^{k+p}(z)\frac{(\zeta-z)^p}{p!}\\ = (\zeta-z)^{n-k+1}\widetilde{S}_w\left(\sum_{s=0}^{k}\frac{\binom{k}{s}(n-s)!s!}{(n-k)!(w-\zeta)^{s+1}(w-z)^{n-s+1}}\right).\end{aligned} \tag{1.9}$$

We show that the argument of the function $\widetilde{S}$ on the right-hand side belongs to the space X_q for any $q \in \mathbb{N}$.

If $\zeta, z \in \overline{D}$ and $w \notin \overline{D}$, then $|w-\zeta|, |w-z| \geq d(w)$; therefore,

$$\begin{aligned}\left|\sum_{s=0}^{k}\frac{\binom{k}{s}(n-s)!s!}{(n-k)!(w-\zeta)^{s+1}(w-z)^{n-s+1}}\right|\\ \leq \frac{1}{d(w)^{n+2}}\sum_{s=0}^{k}\frac{\binom{k}{s}(n-s)!s!}{(n-k)!} = \frac{k!}{d(w)^{n+2}}\sum_{s=0}^{k}\binom{n-s}{k-s}.\end{aligned}$$

The sum of the binomial coefficients on the right-hand side can be evaluated by using the well-known recurrence relation $\binom{n+1}{m} = \binom{n}{m} + \binom{n}{m-1}$, $m = 1, 2, \dots, n$, and $\binom{n}{0} = \binom{n}{n} = 1$:

$$\begin{aligned}\sum_{s=0}^{k}\binom{n-s}{k-s} &= \sum_{p=0}^{k}\binom{n-k+p}{p}\\ &= \binom{n-k}{0} + \binom{n-k+1}{1} + \sum_{p=2}^{k}\binom{n-k+p}{p}\\ &= \binom{n-k+1}{0} + \binom{n-k+1}{1} + \sum_{p=2}^{k}\binom{n-k+p}{p}\\ &= \binom{n-k+2}{1} + \binom{n-k+2}{2} + \sum_{p=3}^{k}\binom{n-k+p}{p}\\ &= \dots = \binom{n+1}{k}.\end{aligned} \tag{1.10}$$

Thus, we obtain the estimate

$$\begin{aligned}\left|\sum_{s=0}^{k}\frac{\binom{k}{s}(n-s)!s!}{(n-k)!(w-\zeta)^{s+1}(w-z)^{n-s+1}}\right|\\ \leq \frac{k!}{d(w)^{n+2}}\binom{n+1}{k} = \frac{1}{d(w)^{n+2}}\frac{(n+1)!}{(n-k+1)!}.\end{aligned} \tag{1.11}$$

Next, for an arbitrary $q \in \mathbb{N}$ and for the number Q occurring in the regularity condition (1.2), by the definition of $M(x)$ we obtain

$$\begin{aligned}\left|\sum_{s=0}^{k}\frac{\binom{k}{s}(n-s)!s!}{(n-k)!(w-\zeta)^{s+1}(w-z)^{n-s+1}}\right| \leq \frac{1}{m_{n+2}(qd(w))^{n+2}}\frac{q^{n+2}m_{n+2}(n+1)!}{(n-k+1)!}\\ \leq qM(qd(w))(qQ)^{n+1}\frac{m_{n+1}(n+1)!}{(n-k+1)!} = qM(qd(w))(qQ)^{n+1}\frac{M_{n+1}}{(n-k+1)!}.\end{aligned}$$

Obviously, this implies that, for any $\zeta, z \in \overline{D}$,

$$\sup_{w\in G} \frac{1}{M(qd(w))} \left| \sum_{s=0}^{k} \frac{\binom{k}{s}(n-s)!s!}{(n-k)!(w-\zeta)^{s+1}(w-z)^{n-s+1}} \right| \le q\frac{(qQ)^{n+1}M_{n+1}}{(n-k+1)!}.$$

Since the expression on the left-hand side coincides with the norm in the space X_q, we have the following estimate for any $\zeta, z \in \overline{D}$ and $q \in \mathbb{N}$:

$$\left\| \sum_{s=0}^{k} \frac{\binom{k}{s}(n-s)!s!}{(n-k)!(w-\zeta)^{s+1}(w-z)^{n-s+1}} \right\|_{X_q} \le q\frac{(qQ)^{n+1}M_{n+1}}{(n-k+1)!}. \tag{1.12}$$

We return to identity (1.8). Since we consider the space $\widetilde{A}(G, \mathcal{M})$ with the topology of the canonical projective limit, the functional S on this space can be extended to a continuous linear functional on X_q for some q. For this number q, relation (1.9) implies

$$\begin{aligned} &\left| \widetilde{S}^{(k)}(\zeta) - \sum_{p=0}^{n-k} \widetilde{S}^{k+p}(z)\frac{(\zeta-z)^p}{p!} \right| \\ &\qquad \le |\zeta - z|^{n-k+1} \|S_w\|_{X_q^*} \left\| \sum_{s=0}^{k} \frac{\binom{k}{s}(n-s)!s!}{(n-k)!(w-\zeta)^{s+1}(w-z)^{n-s+1}} \right\|_{X_q}. \end{aligned}$$

Using this and (1.12), we see that the following estimate is true for all $\zeta, z \in \overline{D}$ and $n \ge 0$, $k \le n$:

$$\left| \widetilde{S}^{(k)}(\zeta) - \sum_{p=0}^{n-k} \widetilde{S}^{k+p}(z)\frac{(\zeta-z)^p}{p!} \right| \le q|\zeta - z|^{n-k+1} \|S_w\|_{X_q^*} \frac{(qQ)^{n+1}M_{n+1}}{(n-k+1)!},$$

or

$$\sup_{n\ge 0, k\le n} \frac{(n-k+1)!}{(qQ)^{n+1}M_{n+1}} \sup_{\zeta, z\in\overline{D}} \frac{\left| \widetilde{S}^{(k)}(\zeta) - \sum_{p=0}^{n-k} \widetilde{S}^{k+p}(z)\frac{(\zeta-z)^p}{p!} \right|}{|\zeta - z|^{n-k+1}} \le q\|S_w\|_{X_q^*}. \tag{1.13}$$

Finally, we estimate the supremum of the function $\widetilde{S}(z)$, where S is a functional that admits extension to the space X_q:

$$|\widetilde{S}(z)| = \left| S\Big(\frac{1}{w-z}\Big) \right| \le \|S\|_{X_q^*} \left\| \frac{1}{w-z} \right\|_{X_q}. \tag{1.14}$$

Since $|w - z| \ge d(w)$ for $w \notin \overline{D}$, $z \in \overline{D}$, it follows that

$$\left| \frac{1}{w-z} \right| \le \frac{1}{d(w)} = \frac{qm_1}{qd(w)m_1} \le qm_1 M(qd(w)).$$

Hence,

$$\left\| \frac{1}{w-z} \right\|_{X_q} \le qm_1 \le qQm_0 = qM_0.$$

Substituting this estimate in (1.14), we obtain the inequality

$$\frac{1}{M_0} \sup_{z\in\overline{D}} |\widetilde{S}(z)| \le q\|S\|_{X_q^*}.$$

This and estimate (1.13) imply that if a functional S extends to a bounded linear functional on X_q, then, by the definition of the space $A_{Qq}(\overline{D}, \mathcal{M})$, its Cauchy transform belongs to $A_{Qq}(\overline{D}, \mathcal{M})$, and moreover,

$$\|\widetilde{S}(z)\|_{A_{qQ}} \le q\|S\|_{X_q^*}.$$

Lemma 2 is proved. □

Our next step is to show that the mapping C is injective. By the Banach theorem, its injectivity will follow from the completeness of the system $\{(\zeta - z)^{-1},\ z \in D\}$ in the space $\widetilde{A}(G, \mathcal{M})$. Since, by Lemma 1, for any $z \in \partial D$ the function $\frac{1}{\zeta - z}$ can be approximated in $\widetilde{A}(G, \mathcal{M})$ by functions of the form $\{(\zeta - z)^{-1},\ z \in D\}$, it suffices to prove the completeness of the system $\{(\zeta - z)^{-1},\ z \in \overline{D}\}$.

If a function $\gamma(\zeta)$ is holomorphic in $\overline{G}$, we can take a contour Γ contained in the domain of analyticity of γ and in the domain D. We represent the function γ in G as the Cauchy integral over Γ. The integral sums converge to γ uniformly in $\overline{G}$. Since the function $M(x)$ is bounded from below, these sums, which are linear combinations of functions belonging to the system under consideration, will approximate γ also in the topology of the space $\widetilde{A}(G, \mathcal{M})$. Thus, it remains to prove that the space $H(\overline{G})$ of functions holomorphic in $\overline{G}$ is dense in $\widetilde{A}(G, \mathcal{M})$.

Lemma 3. *The space $H(\overline{G})$ is dense in $\widetilde{A}(G, \mathcal{M})$.*

Proof. The proof is based on the following theorem by N. Sibony [11].

Theorem A. *Let Φ be a positive function on a domain of holomorphy Ω. Assume that*

$$\Phi(z) - 2\ln \delta_\Omega(z) = \left(\sup_{i\in I} \varphi_i\right)^*(z), \quad z \in \Omega,$$

where each φ_i is a plurisubharmonic function in a domain of holomorphy $\Omega_i \supset \Omega$. Assume also that the family of restrictions to Ω of the functions φ_i, $i \in I$, is right-directed (i.e., for any $i, j \in I$ there exists $k \in I$ such that $\varphi_i(z), \varphi_j(z) \le \varphi_k(z)$ for all $z \in \Omega$). Then for any function $f \in H^2(\Omega, \exp(-\Phi))$ there exists a sequence of functions in

$$\bigcup_{i\in I} H^2(\Omega_i, \exp(-\varphi_i)\delta_0^4)$$

that converges to f in the norm of the space $H^2(\Omega, \exp(-\Phi)\delta_\Omega^2\delta_0^4)$.

Here Ω is a domain in $\mathbb{C}^n$, $d(z)$ stands for the usual distance to the boundary of Ω, $\delta_0(z) = (1+|z|^2)^{-1/2}$, and $\delta_\Omega = \min(d, \delta_0)$. We denote by $H^2(\Omega, w)$ the space of functions holomorphic in Ω and such that $\int_\Omega |f(z)|^2 w(z)\, dv(z) < \infty$, where dv is the area Lebesgue measure. The symbol $u^*(z)$ denotes the upper regularization of the function u:

$$u^*(z) = \overline{\lim_{w \longrightarrow z}} u(w).$$

Note that in the plane, any domain is a domain of holomorphy.

We choose an exhausting sequence of compact sets K_i for the domain D:

$$K_i \subset K_{i+1}, \qquad \bigcup_{i=1}^{\infty} K_i = D.$$

For K_i we can take $K_i = \{z \in D : \inf_{w\in\partial D} |w - z| \ge \frac{1}{i}\}$.

Let Ω_i denote the set $\mathbb{C} \setminus K_i$; the distance to the boundary of Ω_i will be denoted by $d_i(\zeta)$, for brevity. Put $\Omega = \mathbb{C} \setminus \overline{D} = G \setminus \{\infty\}$. We fix $q \in \mathbb{N}$ and put $q_1 = 2Qq$, where Q is as in (1.2). Let

$$\varphi_i(\zeta) = 2\ln M(q_1 d_i(\zeta)), \quad \zeta \in \Omega_i,$$
$$\Phi(\zeta) = 2\ln M(q_1 d(\zeta)) + 2\ln \delta_\Omega(\zeta), \quad \zeta \in \Omega.$$

By definition, $\ln M(qx)$ is a convex and monotone increasing function of $-\ln x$, and the function $-\ln d(\zeta)$ is subharmonic in Ω_i. Therefore, the functions φ_i are subharmonic in

Ω_i. Since, for any $\zeta \in \Omega$, the sequence $d_i(\zeta)$ is monotone nonincreasing and tends to $d(\zeta)$, and the function $M(x)$ is monotone nonincreasing, we have

$$\Phi(\zeta) - 2\ln\delta_\Omega(\zeta) = \sup_i \varphi_i(\zeta), \quad \zeta \in \Omega.$$

In what follows we shall need a property of the weight function $M(x)$.

Lemma 4. *For any $x > 0$, we have*

$$M(Qx) \leq xM(x),$$

where Q is the number occurring (1.2).

Proof. This follows immediately from the definition of $M(x)$ and condition (1.2):

$$M(Qx) = \sup_{k\geq 0} \frac{1}{m_k x^k Q^k} \leq \sup_{k\geq 0} \frac{1}{m_{k+1} x^k} \leq xM(x). \qquad \square$$

Theorem A deals with integral norms, whereas the norms in the spaces under consideration are uniform. Therefore, we need yet another lemma to translate the results from integral to uniform norms.

Lemma 5. *If $f \in H^2(\Omega, \exp(-\Phi)\delta_\Omega^2\delta_0^4)$, then*

$$|f(\zeta)| \leq \frac{2q\sqrt{6}}{\sqrt{\pi}} M(qd(\zeta))(1+|\zeta|^2)\|f\|, \ \zeta \in \Omega,$$

where $\|f\|$ denotes the norm in the space $H^2(\Omega, \exp(-\Phi)\delta_\Omega^2\delta_0^4)$.

Proof. By the subharmonicity of $|f|^2$, we have

$$\begin{aligned} |f(\zeta)|^2 &\leq \frac{4}{\pi d(\zeta)^2} \int_{|\lambda-\zeta|\leq d(\zeta)/2} |f(\lambda)|^2\, dv(\lambda) \\ &\leq \frac{4}{\pi d(\zeta)^2} \max_{|\lambda-\zeta|\leq d(\zeta)/2} \left(e^{\Phi(\lambda)}\delta_\Omega^{-2}(\lambda)\delta_0^{-4}(\lambda)\right) \|f\|. \end{aligned} \tag{1.15}$$

Obviously, in the disk of integration we have $d(\lambda) \geq d(\zeta)/2$; therefore,

$$M(q_1 d(\lambda)) \leq M\left(\frac{q_1}{2} d(\zeta)\right).$$

The estimate $d(z) \leq |z|$ shows that $|\lambda| \leq \frac{3}{2}|\zeta|$ for the points of the disk of integration. Hence, in that disk, $(1+|\lambda|^2)^2 \leq \frac{81}{16}(1+|\zeta|^2)^2$. Also, since $q_1 = 2Qq$, we have (Lemma 4)

$$M(q_1 d(\lambda)) \leq qd(\zeta)M(qd(\zeta)).$$

Thus,

$$\max_{|\lambda-\zeta|\leq d(\zeta)/2} \left(e^{\Phi(\lambda)}\delta_\Omega^{-2}(\lambda)\delta_0^{-4}(\lambda)\right) \leq 6q^2 d(\zeta)^2 M^2(qd(\zeta))(1+|\zeta|^2)^2.$$

Substituting this in (1.15), we obtain the required estimate.

Lemma 5 is proved. $\square$

Now, we take an arbitrary function $\gamma \in \widetilde{A}(G, \mathcal{M})$. By Lemma 1, the function ζ^{-k} can be approximated by functions in $H(\overline{G})$, and so we can omit any finite number of terms in the Laurent series for γ at ∞. We omit the first two terms and assume that

$$|\gamma(\zeta)|^2 = O\left(\frac{1}{|\zeta|^6}\right), \quad |\zeta| \to \infty. \tag{1.16}$$

The definition of the space $\widetilde{A}(G, \mathcal{M})$ implies the inequality

$$|\gamma(\zeta)| \leq CM(q_1 Qd(\zeta)), \quad \zeta \in \Omega.$$

Hence, by Lemma 4, we obtain $|\gamma(\zeta)| \le Cq_1 d(\zeta)M(q_1 d(\zeta))$, $\zeta \in \Omega$; together with (1.16), this yields

$$|\gamma(\zeta)|^2 e^{-\Phi(\zeta)} \le \frac{\text{const}}{(1+|\zeta|)^4}, \quad \zeta \in \Omega,$$

that is, $\gamma \in H^2(\Omega, \exp(-\Phi))$. We apply Theorem A to this function: the function γ can be approximated by functions belonging to

$$\bigcup_i H^2(\Omega_i, e^{-\varphi_i}\delta_0^4)$$

in the norm of the space $H^2(\Omega \exp(-\Phi)\delta_\Omega^2\delta_0^4)$. Unfortunately, the approximating functions may fail to be holomorphic at the point $z = \infty$, and so we need to correct them slightly. Let f_n be the approximating sequence and let $g_n = f_n - \gamma$; then, by Lemma 5, we have

$$|g_n(\zeta)| \le \epsilon_n M(qd(\zeta))(1+|\zeta|^2), \quad \zeta \in \Omega, \tag{1.17}$$

where $\epsilon_n \to 0$. The function $M(x)$ is bounded from above for $x \ge 1$. Hence, the regular part of the Laurent expansion for g_n in a neighborhood of $z = \infty$ contains at most three terms:

$$g_n(\zeta) = P_n(\zeta) + \gamma_n(\zeta),$$

where P_n is a polynomial of degree at most 2, the functions γ_n are holomorphic at the point $z = \infty$ and $\gamma_n(\infty) = 0$. For $k \in \mathbb{N}$, let Γ_k be the contour $\{\zeta \in \Omega : \ d(\zeta) = k\}$, and let $R_k = \max_{\zeta\in\Gamma_k} |\zeta|$. It is clear that $\min\{|\zeta - \lambda|, \ \zeta \in \Gamma_1, \ \lambda \in \Gamma_2\} \ge 1$. Hence, by (1.17), inside the contour Γ_1 we have

$$|P_n(\zeta)| = \left| \int_{\Gamma_2} \frac{g_n(\lambda)}{\lambda - \zeta}\, d\lambda \right| \le \epsilon_n l_2 (1 + R_2^2) M(2q),$$

where l_2 is the length of Γ_2. Since the function $M(x)$ is monotone decreasing and $d(\zeta) < 1$ inside the contour Γ_1, it follows that

$$|P_n(\zeta)| \le \epsilon_n l_2 (1 + R_2^2) M(qd(\zeta)).$$

Combining this and (1.17), we conclude that in Ω inside the contour Γ_1 we have

$$|\gamma_n(\zeta)| \le \epsilon_n (l_2 + 1)(1 + R_2^2) M(qd(\zeta)). \tag{1.18}$$

On the contour Γ_1 this estimate takes the form

$$|\gamma_n(\zeta)| \le \epsilon_n (l_2 + 1)(1 + R_2^2) M(q).$$

Using the maximum principle for the points outside Γ_1 and the monotonicity of $M(x)$, we obtain

$$\begin{aligned} |\gamma_n(\zeta)| &\le \epsilon_n (l_2 + 1)(1 + R_2^2) M(q) = \epsilon_n (l_2 + 1)(1 + R_2^2) M(q) m_0 \frac{1}{m_0} \\ &\le \epsilon_n (l_2 + 1)(1 + R_2^2) M(q) m_0 M(qd(\zeta)). \end{aligned}$$

Combined with (1.18), the latter estimate shows that

$$|\gamma_n(\zeta)| \le \epsilon_n' M(qd(\zeta)), \quad \zeta \in \Omega,$$

where $\epsilon_n' \to 0$. It remains to observe that $\gamma_n = g_n - P_n = (f_n - P_n) - \gamma$. Thus, the sequence $f_n - P_n$ approximates the function γ in the space X_q, and, by construction, these functions are holomorphic in $\overline{\Omega}$, including the point $z = \infty$, i.e., $f_n - P_n \in H(\overline{G})$.

Lemma 3 is proved. □

To complete the proof of Theorem 1, it remains to prove that the mapping C is surjective.

Let $f \in A(\overline{D}, \mathcal{M})$; we construct a continuous linear functional on $\widetilde{A}(G, \mathcal{M})$ such that $\widetilde{S} = f$. By Lemma 3, the space $H(\overline{G})$ is dense in $\widetilde{A}(G, \mathcal{M})$. Hence, it suffices to define a continuous linear functional on $H(\overline{G})$ and then extend it by continuity to $\widetilde{A}(G, \mathcal{M})$.

For any function $\gamma \in \widetilde{A}(G, \mathcal{M})$ holomorphic on $\overline{G}$, put

$$S(\gamma) = \frac{1}{2\pi i} \int_{\partial D} \gamma(z) f(z)\, dz.$$

Note that we can choose a smooth contour Γ_γ, contained in the intersection of the domain of holomorphy of the function γ with D, in such a way that

$$S(\gamma) = \frac{1}{2\pi i} \int_{\Gamma_\gamma} \gamma(z) f(z)\, dz.$$

We will need this remark when we apply the Green formula.

We make use of a theorem on pseudoanalytic continuation proved in [1].

Theorem B. *Let D be a domain in $\mathbb{C}$, and let $m_n = \frac{M_n}{n!}$ be a regular sequence. Then any function f in the class $A(\overline{D}, \mathcal{M})$ can be extended to a continuously differentiable function F with compact support in $\mathbb{C}$ such that*

$$\left| \frac{\partial F}{\partial \overline{\zeta}} \right| \le \frac{C}{M(Bd(\zeta))}, \quad \zeta \in \mathbb{C},$$

where C and B are positive constants.

With the help of this theorem, we extend the function $f \in A(\overline{D}, \mathcal{M})$ to a function F and apply the Green formula

$$\begin{aligned} S(\gamma) &= \frac{1}{2\pi i} \int_{\Gamma_\gamma} \gamma(z) f(z)\, dz = \frac{1}{2\pi i} \int_{\Gamma_\gamma} \gamma(z) F(z)\, dz \\ &= -\frac{1}{\pi} \int_G \gamma(\zeta) \frac{\partial F(\zeta)}{\partial \overline{\zeta}}\, dv(\zeta) = -\frac{1}{\pi} \int_{K \cap G} \gamma(\zeta) \frac{\partial F(\zeta)}{\partial \overline{\zeta}}\, dv(\zeta), \end{aligned}$$

where K is the support of F and $dv(\zeta)$ is the area Lebesgue measure. This representation implies the estimate

$$|S(\gamma)| \le \frac{|K| C}{\pi} \sup_{\zeta \in G} \frac{|\gamma(\zeta)|}{M(Bd(\zeta))} = 2|K|C \|\gamma\|_{X_B},$$

where $|K|$ is the area of the compact set K and $\|\gamma\|_{X_B}$ is the norm of γ in the space X_B. Thus, S is a linear functional on $H(\overline{G})$ continuous with respect to the norm of X_B, and hence, also in the topology of the space $\widetilde{A}(G, \mathcal{M})$. By the density of $H(\overline{G})$ in $\widetilde{A}(G, \mathcal{M})$, the functional S can be extended to a continuous linear functional on $\widetilde{A}(G, \mathcal{M})$. By the definition and the Cauchy formula, we conclude that $\widetilde{S}(z) = f(z)$, $z \in D$.

Theorem 1 is proved. □

§2. Quasianalyticity and an extremal problem for subharmonic functions

Theorem 1, Lemma 1, and the Banach theorem imply the following criterion for quasianalyticity.

Theorem 2. *Let the sequence (m_n) be regular, and let the point $z = 0$ be on the boundary of a bounded Jordan domain D. The class $A(\overline{D}, \mathcal{M})$ is quasianalytic at the point $z = 0$ if and only if the system ζ^{-n}, $n = 1, 2, \dots$, is dense in the space $\widetilde{A}(G, \mathcal{M})$.*

Proof. If S is a continuous linear functional on $\widetilde{A}(G,\mathcal{M})$ that is equal to zero on the elements of the system ζ^{-n}, $n = 1, 2, \dots$, then, by Lemma 1, the function $\widetilde{S}(z)$ in $A(\overline{D},\mathcal{M})$ satisfies the condition $\widetilde{S}^{(n)}(0) = 0$ for all $n = 1, 2, \dots$.

Theorem 2 is proved. □

Let $q \in \mathbb{N}$ be arbitrary. For any $w \in G$, there exists a number $p \in \mathbb{N}$ and a point $t \in \partial D$ such that

$$M(qd(w)) = \sup_{k\geq 0}\sup_{z\in D}\left|\frac{1}{q^k m_k(z-w)^k}\right| = \left|\frac{1}{q^p m_p(t-w)^p}\right|.$$

In what follows we denote the function $\frac{1}{q^p m_p(t-\zeta)^p}$ by $f_w(\zeta)$. Thus, the function $f_w(\zeta)$ has the following properties:

$$f_w(\zeta) \in \widetilde{A}(G,\mathcal{M}); \quad |f_w(\zeta)| \leq M(qd(\zeta)),\ \ \zeta \in G; \quad |f_w(w)| = M(qd(w)).$$

Assume that the system $\{\zeta^{-n},\ n = 1, 2, \dots\}$ is complete in $\widetilde{A}(G,\mathcal{M})$. This means that for any function γ in $\widetilde{A}(G,\mathcal{M})$ there exists a sequence of polynomials $P_n(z)$, $P_n(0) = 0$, $n = 0, 1, \dots$, such that

$$P_n\Big(\frac{1}{\zeta}\Big) \to \gamma(\zeta)$$

in the space $\widetilde{A}(G,\mathcal{M})$ as $n \to \infty$. In particular, for any $\epsilon > 0$ there exists a polynomial $P(z)$, $P(0) = 0$, such that

$$\Big|P\Big(\frac{1}{\zeta}\Big) - f_w(\zeta)\Big| \leq \epsilon M(qd(\zeta)), \quad \zeta \in G.$$

Then

$$\Big|P\Big(\frac{1}{\zeta}\Big)\Big| \leq (1+\epsilon)M(qd(\zeta)),\ \ \zeta \in G; \quad \Big|P\Big(\frac{1}{w}\Big)\Big| \geq (1-\epsilon)M(qd(w)).$$

For the polynomial $P_1(z) = P(z)/(1+\epsilon)$ we have the inequalities

$$\Big|P_1\Big(\frac{1}{\zeta}\Big)\Big| \leq M(qd(\zeta)),\ \ \zeta \in G; \quad \Big|P_1\Big(\frac{1}{w}\Big)\Big| \geq \frac{1-\epsilon}{1+\epsilon}M(qd(w)). \tag{2.1}$$

In view of these considerations, we introduce the class K_q of functions v satisfying the following conditions:

1) every function v is continuous and subharmonic in $\overline{\mathbb{C}} \setminus \{0\}$;
2) $v(\zeta) = O(\ln \frac{1}{|\zeta|})$ as $\zeta \to 0$;
3) $v(\zeta) \leq \ln M(qd(\zeta)), \quad \zeta \in G$.

For example, the functions $\max(\ln |P_1(\frac{1}{\zeta})|, -\ln m_0)$, where the polynomials P_1 satisfy (2.1), are in K_q.

Obviously, instead of the sequence M_n we may consider the sequence M_n/eM_0 and assume that $m_0 = 1/e$. Thus, $M(x) \geq 1/m_0 = e$ and $\ln M(x) \geq 1$. Therefore, to the definition of the class K_q we may add the following item:

4) $v(z) \geq 0$.

Theorem 3. *Let the sequence (m_n) be regular, and let the point $z = 0$ be on the boundary of a bounded Jordan domain D. The class $A(\overline{D},\mathcal{M})$ is quasianalytic at the point $z = 0$ if and only if the condition*

$$\sup\{v(\zeta),\ v \in K_q\} = \ln M(qd(\zeta)), \quad \zeta \in G, \tag{2.2}$$

is fulfilled for each $q \in \mathbb{N}$.

Proof. By Theorem 2, the quasianalyticity of the class $A(\overline{D}, \mathcal{M})$ implies that the system ζ^{-n}, $n = 1, 2, \dots$, is complete in the space $\widetilde{A}(G, \mathcal{M})$. Above, we have shown that the completeness of this system implies (2.2), because the functions $\max(\ln|(P_1(\frac{1}{\zeta})|, -\ln m_0)$ belong to the class K_q. To prove the converse statement, we use the following lemma.

Lemma 6. *Assume that the condition*

$$\sup\{v(\zeta),\ v \in K_q\} = \ln M(qd(\zeta)), \quad \zeta \in G,$$

is fulfilled for some q. Then any function in the space $\widetilde{A}(G, \mathcal{M})$ can be approximated by the system ζ^{-n}, $n = 1, 2, \dots$, in the norm of the space $X_{q/2Q}$, where Q is as in (1.2).

Proof. For the role of φ_i in Theorem A, we take functions $2v(\zeta)$, where $v \in K_q$. Let $\Omega_i = \mathbb{C} \setminus \{0\}$, $\Omega = \mathbb{C} \setminus \overline{D}$. We put

$$\Phi(\zeta) = 2\ln M(qd(\zeta)) + 2\ln \delta_G(\zeta), \quad \zeta \in \Omega.$$

Then the assumptions of Theorem A are satisfied in view of the assumptions of the lemma. Therefore, any function of the class $H^2(\Omega, \exp(-\Phi))$ can be approximated by functions belonging to the union

$$\bigcup_{v \in K_q} H^2(\Omega_i, \exp(-2v(\zeta))\delta_0^4)$$

in the norm of the space $H^2(\Omega, \exp(-\Phi)\delta_G^2\delta_0^4)$. Let $\gamma \in \widetilde{A}(G, \mathcal{M})$. Since we are interested in approximating the function γ by linear combinations of functions ζ^{-n}, we can omit several terms in the Laurent series expansion of γ near ∞. Thus, we may assume that

$$|\gamma(\zeta)|^2 = O\Big(\ln \frac{1}{|\zeta|^6}\Big), \quad |\zeta| \to \infty.$$

Also, by Lemma 4 we have

$$|\gamma(\zeta)| \le \|\gamma\|_{X_{qQ}} M(qQd(\zeta)) \le \|\gamma\|_{X_{qQ}} qd(\zeta) M(qd(\zeta)), \quad \zeta \in G.$$

These two relations show that $\gamma \in H^2(\Omega, \exp(-\Phi))$. Let

$$f_n \in \bigcup_{v \in K_q} H^2(\mathbb{C} \setminus \{0\}, \exp(-2v(\zeta))\delta_0^4)$$

be an approximating sequence and let $g_n = f_n - \gamma$. By Lemma 5, we have

$$|g_n(\zeta)| \le \epsilon_n M(q_1 d(\zeta))(1 + |\zeta|^2), \quad \zeta \in \Omega,$$

where $\epsilon_n \to 0$ and $q_1 = q/2Q$. The function $M(x)$ is bounded from above for $x \ge 1$, and so the last estimate yields

$$g_n(\zeta) = Q_n(\zeta) + \gamma_n(\zeta),$$

where Q_n is a polynomial of degree at most 2, and $\gamma_n(\infty) = 0$. Since γ is holomorphic at the point $\zeta = \infty$ and $\gamma(\infty) = 0$, we conclude that $Q_n(\zeta)$ is the regular part of the function f_n at ∞, and the function $f_n - Q_n$ is holomorphic at ∞ and vanishes there. As in the proof of Theorem 1, we can show that

$$|\gamma_n(\zeta)| \le \epsilon_n' M(q_1 d(\zeta)), \quad \zeta \in G,$$

where $\epsilon_n' \to 0$. This means that the functions $f_n(\zeta) - Q_n(\zeta)$ approximate γ in the norm of the space X_{q_1}. The function f_n belongs to one of the spaces

$$H^2(\mathbb{C} \setminus \{0\}, \exp(-2v(\zeta))\delta_0^4).$$

Hence,

$$\int_{\mathbb{C} \setminus \{0\}} |f_n(\zeta)|^2 \frac{e^{-2v(\zeta)}}{(1 + |\zeta|^2)^2}\, dv(\zeta) < \infty.$$

Now the definition of the classes K_q and the subharmonicity of $|f_n|$ show that the function f_n has a pole of order N at the point $\zeta = 0$. Since $f_n - Q_n$ is holomorphic at infinity, this function is a linear combination of the functions ζ^{-n} , $n = 1, 2, \dots, N$. Thus, we have shown that the function γ can be approximated by the system ζ^{-n} in the norm of the space $X_{q/2Q}$.

Lemma 6 is proved. □

Now we complete the proof of Theorem 3. If condition (2.2) is satisfied for all $q \in \mathbb{N}$, then, by Lemma 6, the functions $\widetilde{A}(G, \mathcal{M})$ can be approximated by the system (ζ^{-n}) in the norm of each of the spaces $X_{q/2Q}$, that is, in the topology of the space $\widetilde{A}(G, \mathcal{M})$.

Theorem 3 is proved. □

§3. Quasianalyticity and a Dirichlet problem

We introduce the function

$$U_q(\zeta) = \sup\{v(\zeta),\ v \in K_q\}, \quad \zeta \in \overline{\mathbb{C}}.$$

Lemma 7. *For any $q \in \mathbb{N}$, either $U_q(\zeta) \equiv \infty$ in D or $U_q(\zeta)$ is a harmonic function in D.*

Proof. Let D_1 denote the set of points z in D such that $U_q(z) = \infty$, and let $D_2 = \{z \in D :\ U_q(z) < \infty\}$. We fix an arbitrary point $z_0 \in D$ and a monotone increasing sequence of functions $v_n \in K_q$ such that $\lim_{n\to\infty} v_n(z_0) = U_q(z_0)$. Let $2d = \inf_{\zeta\in G} |\zeta - z_0|$. We extend each function v_n harmonically to the disk $B(z_0, d)$. Obviously, the resulting functions $\widetilde{v}_n$ are also in K_q, and moreover, since $\widetilde{v}_n \ge v_n$, we have $\lim_{n\to\infty} \widetilde{v}_n(z_0) = U_q(z_0)$. Applying the Harnack inequality to each of these functions, we see that, in the disk $B(z_0, d/2)$, we have

$$\frac{1}{3}\widetilde{v}_n(z_0) \le \widetilde{v}_n(z) \le 3\widetilde{v}_n(z_0). \tag{3.1}$$

The left-hand side inequalities show that if $z_0 \in D_1$, then $B(z_0, d/2) \subset D_1$, whereas the right-hand side inequalities show that for $z_0 \in D_2$ we have $B(z_0, d/2) \subset D_2$. Thus, the two sets D_1 and D_2 are open in D. Since D is connected, this means that one of them must be empty.

It remains to show that if $D_1 = \varnothing$, then the function U_q is harmonic in D.

We take an arbitrary point $w \in B(z_0, d/2)$ and, as for the point z_0, construct a monotone increasing sequence of functions $h_n(z) \in K_q$ such that $\lim_{n\to\infty} h_n(w) = U_q(w)$. Then, by harmonic extension to the disk $B(z_0, d)$, we obtain an increasing sequence of functions $\widetilde{h}_n$ with the same property: $\lim_{n\to\infty} \widetilde{h}_n(w) = U_q(w)$.

Now we put $s_n(z) = \max(v_n(z), h_n(z))$. Clearly,

$$\lim_{n\to\infty} s_n(w) = U_q(w), \quad \lim_{n\to\infty} s_n(z_0) = U_q(z_0).$$

Extending the s_n harmonically to the disk $B(z_0, d)$, we construct functions $\widetilde{s}_n$ with the same properties. Also, it is clear that $\widetilde{s}_n$ is greater than both $\widetilde{v}_n$ and $\widetilde{h}_n$. Put

$$\lim_{n\to\infty} \widetilde{v}_n(z) = V(z), \quad \lim_{n\to\infty} \widetilde{h}_n(z) = H(z), \quad \lim_{n\to\infty} \widetilde{s}_n(z) = S(z).$$

Then from our constructions it follows that

$$V(z_0) = U_q(z_0), \quad H(w) = U_q(w), \quad S(z_0) = U_q(z_0), \quad S(w) = U_q(w),$$
$$S(z) \ge V(z), \quad S(z) \ge H(z).$$

A nonnegative function $S - V$ harmonic in $B(z_0, d)$ vanishes at the interior point z_0. By the maximum principle, $S \equiv V$. Similarly, $S \equiv H$. Therefore, $V(w) = S(w) = H(w) = U_q(w)$, but the construction of V depends only on z_0 and not on w. Hence,

$V(w) = U_q(w)$ for all points in the disk $B(z_0, d/2)$, and $U_q(z)$ is harmonic in this disk. Since z_0 is an arbitrary point in D, we conclude that U_q is harmonic in D.

Lemma 7 is proved. □

Lemma 8. *If for $q \in \mathbb{N}$ we have $U_q(\zeta) \equiv \infty$ in D, then $U_q(\zeta) \equiv \ln M(qd(\zeta))$ in G.*

Proof. For a fixed $q \in \mathbb{N}$ and any $w \in G$, in §2 we introduced the function $f_w(\zeta)$ of the form $(m_p q^p (\zeta - z)^p)^{-1}$, where $p \in \mathbb{N}$ and z is a point on the boundary of D. These functions have the properties

$$|f_w(\zeta)| \le M(qd(\zeta)), \quad \zeta \in G, \quad |f_w(w)| = M(qd(w)).$$

Fixing a point $w \in G$ and a number $\epsilon > 0$, we replace the boundary point z in the definition of $f_w(\zeta)$ with a sufficiently close point $z' \in D$ so that the resulting function $\widetilde{f}_w(\zeta)$ satisfy

$$|\widetilde{f}_w(\zeta)| \le M(qd(\zeta)), \;\; \zeta \in G; \quad |\widetilde{f}_w(w)| \ge (1-\epsilon) M(qd(w)).$$

By the assumptions of the lemma, we have $U_q(z') = \infty$, and so there exists a sequence of functions $v_n \in K_q$ such that $v_n(z') \to \infty$. As in the proof of the preceding lemma, we may assume that the functions v_n are extended harmonically to the disk $B(z', d)$, where $2d$ is the distance from z' to the boundary D. Then, by (3.1), $v_n(z) \to \infty$ uniformly in the disk $B(z', d/2)$. Outside the disk $B(z', d/3)$, the function $|\widetilde{f}_w(\zeta)|$ is bounded:

$$|\widetilde{f}_w(\zeta)| \le \frac{3^p}{m_p q^p d^p} = M.$$

We choose n so large that $v_n(\zeta) > \ln M$ in the disk $B(z', d/2)$ and introduce the function

$$u(\zeta) = \begin{cases} \max(v_n(\zeta), \ln|\widetilde{f}_n(\zeta)|) & \text{if } \; \zeta \notin B(z', d/3), \\ v_n(\zeta) & \text{if } \; \zeta \in B(z', d/3). \end{cases}$$

By construction, we have $u \in K_q$ and

$$u(w) \ge \ln|\widetilde{f}_w(w)| \ge \ln(1-\epsilon) + \ln M(qd(w)).$$

Since $\epsilon > 0$ is arbitrary, $U_q(w) = \ln M(qd(w))$.

Lemma 8 is proved. □

Lemma 9. *If, for a given q, the function $U_q(z)$ is finite at some point $z_1 \in D$, then in a neighborhood of any point $z \in \partial D$ there are points $\zeta \in G$ such that $U_q(\zeta) < \ln M(qd(\zeta))$.*

Proof. Assume the contrary. Let $z_0 \in \partial D$ and suppose that $U_q(\zeta) \equiv \ln M(qd(\zeta))$ in the intersection of the disk $B(z_0, r)$ with G. Put $r_0 = \min(r/2, |z_0|/2)$ and let G' be the connected component of the intersection of G with $B(z_0, r_0)$ such that z_0 is on its boundary. We denote by G_0 the difference $G \setminus G'$ and let $\widetilde{K}_q$ be the class of functions that are subharmonic, nonnegative, and continuous in $\overline{\mathbb{C}} \setminus \{0\}$ and satisfy the conditions

$$v(z) = O\Big(\ln \frac{1}{|z|}\Big), \;\; |z| \to 0, \quad v(\zeta) \le \ln M(qd(\zeta)), \;\; \zeta \in G_0.$$

Obviously, we have $K_q \subset \widetilde{K}_q$, whence

$$\widetilde{U}_q(\zeta) = \sup\{v(\zeta), \; v \in \widetilde{K}_q\} \ge U_q(\zeta).$$

Hence, the function $\widetilde{U}_q(\zeta)$ is unbounded near the point $z_0 \in \mathbb{C} \setminus \overline{G}_0$. By Lemma 7, we see that $\widetilde{U}_q(\zeta) \equiv \infty$ in $D_0 = \mathbb{C} \setminus \overline{G}_0 \supset D$. Let $\widetilde{v}$ be a function in $\widetilde{K}_q$ such that

$$\widetilde{v}(z_1) \ge U_q(z_1) + 4.$$

Then the function $v(z) = \widetilde{v}(z) - 2$ satisfies the inequalities

$$v(z_1) \geq U_q(z_1) + 2, \quad v(\zeta) \leq \ln M(qd(\zeta)) - 2, \;\; \zeta \in G_0.$$

The subharmonic function $v(z)$ is bounded from above in the disk $B(z_0, r_0)$:

$$v(z) \leq M, \quad z \in B(z_0, r_0),$$

for some M. Since $M(x)$ is monotone, on the set $G'' = \{\zeta \in G' : \; d(\zeta) \leq \epsilon\}$ we have

$$\ln M(qd(\zeta)) \geq \ln M(q\epsilon).$$

Let $\epsilon > 0$ be so small that $\ln M(q\epsilon) > M + 1$. Then in G'' we have the inequality

$$\ln M(qd(\zeta)) \geq M + 1.$$

By assumption, the identity $U_q(z) \equiv \ln M(qd(z))$ is true on the set $G' \setminus G''$. For any point $w \in \partial G' \cap \partial G''$ there exists a function $u_w \in K_q$ such that $u_w(w) > M$ and, by the continuity of u_w, this inequality extends to some neighborhood V_w of w. The complementary part of the boundary $G' \setminus G''$ lies on the boundary of G_0 in the disk $B(z', r_0)$. Hence, for a point ζ in this part of the boundary, we have

$$U_q(\zeta) = \ln M(qd(\zeta)) \geq v(\zeta) + 2.$$

Hence, for each point w in this part of the boundary there exists a function $u_w \in K_q$ satisfying $u_w(w) > v(w) + 1$. Again, by the continuity of u_w and v, this inequality extends to some neighborhood V_w of w. Since $\partial(G' \setminus G'')$ is a compact set, we can choose a finite subcovering $V_{w_1}, \dots, V_{w_m}$ of the covering $\{V_w, \; w \in \partial(G' \setminus G'')\}$, where $w_1, \dots, w_m \in \partial(G' \setminus G'')$. Put $u(z) = \max_{k=1,\dots,m} u_{w_k}(z)$. Obviously, $u(z) \in K_q$ and, by construction, we have $u(z) > v(z)$ on the set $V = \bigcup V_{w_k}$. We introduce the function

$$u_0(\zeta) = \begin{cases} \max(u(\zeta), v(\zeta)) & \text{if } \; z \notin G' \setminus G'', \\ u(\zeta) & \text{if } \; \zeta \in G' \setminus G''. \end{cases}$$

Since in the neighborhood V of the boundary $\partial(G' \setminus G'')$ we have $u_0(\zeta) = u(\zeta)$, the function u_0 is subharmonic and continuous. Moreover, $u_0 \in K_q$. The necessary inequalities on the set G_0 follow from the fact that both functions u and v satisfy these inequalities there. On the set $G' \setminus G''$, the necessary inequalities follow because $u \in K_q$. Finally, in G'' we have

$$v(\zeta) \leq M < M + 1 \leq \ln M(qd(\zeta)).$$

Thus, $u_0 \in K_q$, and so $u_0(z_1) \leq U_q(z_1)$.

On the other hand, $u_0(z_1) \geq v(z_1) \geq U_q(z_1) + 2$, a contradiction.

Lemma 9 is proved. □

Lemmas 7, 8, and 9 make it possible to state new quasianalyticity criteria.

Theorem 4. *Let the sequence (m_n) be regular, and let $z = 0$ be on the boundary of a bounded Jordan domain D. Then the class $A(\overline{D}, \mathcal{M})$ is quasianalytic at the point $z = 0$ if and only if*

$$\sup\{v(z), \; v \in K_q\} = \infty, \quad \zeta \in D,$$

for any $q \in \mathbb{N}$.

Proof. If the assumptions of the theorem are fulfilled, then, by Lemma 8, for any $q \in \mathbb{N}$ we have

$$U_q(\zeta) \equiv \ln M(qd(\zeta)), \quad \zeta \in G. \tag{3.2}$$

By Theorem 3, in this case the class $A(\overline{D}, \mathcal{M})$ is quasianalytic. Conversely, if the class $A(\overline{D}, \mathcal{M})$ is quasianalytic, then we have (3.2) by Theorem 3. By Lemma 9, the function U_q cannot be finite in D, and therefore, the conclusion of Theorem 4 follows from Lemma 7.

Theorem 4 is proved. □

Theorem 5. *Let the sequence (m_n) be regular, and let $z = 0$ be on the boundary of a bounded Jordan domain D. Then the class $A(\overline{D}, \mathcal{M})$ is nonquasianalytic at the point $z = 0$ if and only if, for any $q \in \mathbb{N}$ greater than some q_0, there is a domain D_q containing $\overline{D} \setminus \{0\}$ and a function $h(\zeta)$ that is harmonic in D_q, equals $\ln M(qd(\zeta))$ on the boundary of D_q, and satisfies*

$$\lim_{|z| \longrightarrow 0} \frac{h(z)}{-\ln |z|} = +\infty. \tag{3.3}$$

Proof. Obviously, under the assumptions of the theorem, the class $A(\overline{D}, \mathcal{M})$ cannot be quasianalytic.

Assume that the class $A(\overline{D}, \mathcal{M})$ is nonquasianalytic. Then, for a fixed $q \in \mathbb{N}$, we extend the function $\ln M(qd(\zeta))$ to the entire plane assuming that it is equal $+\infty$ on $\overline{D}$. Let

$$D' = \{\zeta \in \overline{C} : U_q(\zeta) < \ln M(qd(\zeta))\}.$$

Under our assumptions, by Theorem 4, there exists $q_0 \in \mathbb{N}$ such that $D' \supset D$, and the intersection $D' \cap G$ is nonempty by Lemma 9. It is clear that the same is true for $q > q_0$. □

Lemma 10. *The sets*

$$G' = D' \cap G = \{\zeta \in G : U_q(\zeta) < \ln M(qd(\zeta))\},$$
$$G'' = \{\zeta \in G : U_q^*(\zeta) < \ln M(qd(\zeta))\}$$

coincide, are open in G, and the function $U_q(\zeta)$ is harmonic in G'.

Proof. Clearly, $G'' \subset G'$. Assume that there is a point $\zeta_0 \in G$ such that $\zeta_0 \in G' \setminus G''$. To simplify the notation, we put $T = \ln M(qd(\zeta_0))$ and assume that

$$U_q(\zeta_0) = (1 - 2a)T \tag{3.4}$$

for some $a \in (0, 1/2)$. Take the largest $r > 0$ such that the harmonic majorant $V(\zeta)$ of the function $\ln M(qd(\zeta))$ in the disk $B(\zeta_0, r)$ satisfies the condition

$$V(\zeta) \le \ln M(qd(\zeta)) + aT.$$

Now we fix an arbitrary $\epsilon > 0$ and choose a point w in $B(\zeta_0, r)$ with the property

$$U_q(w) > (1 - \epsilon)T.$$

This is possible because $\zeta_0 \notin G''$, and so $U_q^*(\zeta_0) = \ln M(qd(\zeta_0))$. Also, let $|w - \zeta_0| = \delta r$; we may assume that $\delta < \frac{1}{3}$. There exists $v \in K_q$ such that

$$v(w) > U_q(w) - \epsilon T.$$

Let $\widetilde{v}$ denote the harmonic extension of v to the disk $B(\zeta_0, r)$. Since $\widetilde{v} \le V$ in that disk, we have, by the choice of the number r,

$$\widetilde{v}(\zeta) - aT \le \ln M(qd(\zeta)), \quad \zeta \in G,$$

that is, $\widetilde{v}(\zeta) - aT \in K_q$, and moreover,

$$\widetilde{v}(w) - aT \ge v(w) - aT \ge U_q(w) - \epsilon T - aT \ge (1 - a - 2\epsilon)T.$$

Therefore, $\widetilde{v}(w) - aT \ge (1 - a - 2\epsilon)T$.

Applying the Harnack inequality to this function in the disk $B(w,(1-\delta)r)$, we obtain

$$\widetilde{v}(\zeta_0) - aT \geq \frac{(1-\delta)r - \delta r}{(1-\delta)r + \delta r}(\widetilde{v}(w) - aT).$$

The last two inequalities imply that

$$\widetilde{v}(\zeta_0) - aT \geq (1-2\delta)(1-a-2\epsilon)T.$$

Consequently,

$$U_q(\zeta_0) \geq (1-a)T - (2\epsilon + 2\delta - 2\delta a - 4\delta\epsilon)T.$$

Recalling (3.4), we see that $a \leq 2\epsilon + 2\delta - 2\delta a - 4\delta\epsilon$.

Letting δ and ϵ tend to zero, we obtain $a = 0$. This means that $\zeta_0 \notin G'$, a contradiction. Thus, $G' = G''$. Since U_q^* is upper semicontinuous and $\ln M(qd(\zeta))$ is continuous, the set G'' is open. Therefore, G' is also open.

We prove that the function U_q is harmonic in G''. Let $\zeta_0 \in G''$. Since

$$\lim_{r\to 0} \frac{1}{2\pi} \int_0^{2\pi} U_q^*(\zeta_0 + re^{i\varphi})\, d\varphi = U_q^*(\zeta_0),$$

for any $\epsilon > 0$ there exists $r' > 0$ such that for $r \leq r'$ we have

$$\frac{1}{2\pi} \int_0^{2\pi} U_q^*(\zeta_0 + re^{i\varphi})\, d\varphi \leq U_q^*(\zeta_0) + \epsilon.$$

We choose $\epsilon > 0$ so small that $U_q^*(\zeta_0) + 3\epsilon < \ln M(qd(\zeta_0))$, extend the function U_q^* harmonically to the disk $B(\zeta_0, r')$, and denote the resulting function on $\mathbb{C} \setminus \{0\}$ by $\widetilde{U}_q$. Observe that

$$\widetilde{U}_q(\zeta_0) = \frac{1}{2\pi} \int_0^{2\pi} U_q^*(\zeta_0 + r'e^{i\varphi})\, d\varphi \leq U_q^*(\zeta_0) + \epsilon.$$

For $\delta \in (0;1)$, by the Harnack inequality in the disk $B(\zeta_0, \delta r')$, we have

$$\widetilde{U}_q(\zeta) \leq \frac{1+\delta}{1-\delta}\widetilde{U}_q(\zeta_0) \leq \frac{1+\delta}{1-\delta}(U_q^*(\zeta_0) + \epsilon).$$

Thus, for sufficiently small $\delta > 0$ in the disk $B(\zeta_0, \delta r')$ we have

$$\widetilde{U}_q(\zeta) \leq U_q^*(\zeta_0) + 2\epsilon.$$

Let $r > 0$ be so small that in the disk $B(\zeta_0, r)$ we have the inequality

$$\ln M(qd(\zeta)) \geq \ln M(qd(\zeta_0)) - \epsilon.$$

(Such a choice is possible by the continuity of $M(x)$.) Put $r_0 = \min(\delta r', r)$. Then in the disk $B(\zeta_0, r_0)$ we have

$$\widetilde{U}_q(\zeta) \leq (U_q^*(\zeta_0) + 2\epsilon) \leq \ln M(qd(\zeta_0)) - \epsilon \leq \ln M(qd(\zeta)). \tag{3.5}$$

Each v in K_q is less than or equal to U_q^*, so v does not exceed $\widetilde{U}_q(\zeta)$ on the boundary of the disk $B(\zeta_0, r_0)$. If we extend v harmonically to $B(\zeta_0, r_0)$, then the resulting function $\widetilde{v}$ will not exceed $\widetilde{U}_q(\zeta)$ in this disk by the maximum principle. By (3.5), all functions $\widetilde{v}$ obtained in this way are in the class K_q and are harmonic in the disk under consideration. Then the function U_q will also be harmonic in $B(\zeta_0, r_0)$ as an upper envelope of a bounded family of harmonic functions; see [12].

It is obvious that relation (3.3) holds true for the function $U_q(z)$.

Lemma 10 is proved. □

To clarify what happens to the points of the boundary of D, we need the following lemma concerning the properties of the function $M(x)$.

Lemma 11. *Let $p(x)$ denote the smallest natural number p such that*

$$M(x) = \frac{1}{m_p x^p};$$

such numbers exist by the definition of the function $M(x)$. Then

$$\lim_{x\to 0} p(x) = \infty.$$

Also,

$$\lim_{x\to 0} \frac{\ln M(x)}{-\ln x} = \infty.$$

Proof. To prove the first statement, assume the contrary. Suppose that for a sequence $x_n \to 0$ we have $p(x_n) < p$. Then

$$\frac{1}{m_{p(x_n)} x_n^{p(x_n)}} \geq \frac{1}{m_p x_n^p}$$

and, for $x_n < 1$,

$$x_n > x_n^{p-p(x_n)} \geq \frac{\min\{m_k,\ k = 0, 1, \ldots, p-1\}}{m_p}.$$

Let n tend to infinity. Then either $\min\{m_k,\ k = 0, 1, \ldots, p-1\} = 0$ or $m_p = \infty$, which is impossible.

Now we take $j \in \mathbb{N}$ and let $1 > \delta > 0$ be so small that $p(x) > j$ for all $x < \delta$. Then

$$\ln M(x) = \ln \frac{1}{m_{p(x)} x^{p(x)}} \geq -\ln m_j - j \ln x, \quad x < \delta.$$

Hence,

$$\lim_{x\to 0} \frac{\ln M(x)}{-\ln x} \geq j.$$

Lemma 11 is proved. □

Now we study the points on the boundary of D.

Lemma 12. *The set $\partial D \setminus \{0\}$ is contained in D'; the function U_q is harmonic at the points of this set and satisfies* (3.3).

Proof. First, we prove that on ∂D there are points where U_q is locally bounded. It is clear that such points belong to D'. We fix an arbitrary point $z \in \partial D$ and some number $\rho > 0$. By property (1.3) of regular sequences, there exists a number $p_0 = p(\rho)$ such that for all $p \geq p_0$ we have

$$m_p \rho^p > 1,$$

whereas, by Lemma 11, there exists $\delta > 0$ such that $p(x) > p_0$ for all $x \in (0; \delta)$. By Lemma 9, all boundary points are limit points for G'. Take a point $w \in G'$ in the disk $B(z, \min(\rho, \delta))$. Let $\epsilon > 0$ be such that

$$U_q(w) \leq \ln M(qd(w)) - 2\epsilon, \tag{3.6}$$

and let $f_w(\zeta)$ be a function of the form $1/m_p(\zeta - t)^p$, where $t \in D$, $p \in \mathbb{N}$; this function satisfies the estimates

$$\ln |f_w(\zeta)| \leq \ln M(qd(\zeta)),\ \ \zeta \in G; \quad \ln |f_w(w)| \geq \ln M(qd(w)) - \epsilon.$$

Since $d(w) < \delta$, the number p in the formula for f_w is at least p_0, and, by the choice of p_0,

$$\ln |f_w(\zeta)| < 1, \quad \zeta \notin B(t, \rho). \tag{3.7}$$

Let $v \in K_q$ be an arbitrary function, and denote by V the connected component of the set

$$\{\zeta : v(\zeta) < \ln |f_w(\zeta)|\}$$

that contains the point w. Should the singularity point t be situated outside V, the function

$$\widetilde{v}(\zeta) = \begin{cases} \max(v(\zeta), \ln |f_w(\zeta)|) & \text{if } \zeta \in V, \\ v(\zeta) & \text{if } \zeta \notin V, \end{cases}$$

would belong to the class K_q, which leads to the estimate

$$U_q(w) \geq \widetilde{v}(w) = \ln |f_w(w)| \geq \ln M(qd(w)) - \epsilon,$$

and this contradicts (3.6). Thus, the component V contains the singularity point t of f_w. Note that, since $v \geq 0$, the component V is in $B(t, \rho)$ by (3.7). Thus, the point w can be connected with the point t by a path in $B(t, \rho)$ on which we have

$$v(\zeta) < \ln |f_w(\zeta)|.$$

Let $2\sigma = \inf\{|t - \zeta|,\ \zeta \in \partial D\}$, and let t_0 be the first point of the circle $|t - \zeta| = \sigma$ lying on this path (if we start from w). On the part of the path from w to t_0 we have

$$v(\zeta) < \ln |f_w(\zeta)| \leq \frac{1}{m_p \sigma^p}.$$

Note that the number $1/m_p\sigma^p$ does not depend on the function v.

Now, we take two points $z_1, z_2 \in \partial D$ and put $\rho = |z_1 - z_2|/3$. Repeating the above constructions for each of these points, we find points $w_i \in G'$, $t_i \in D$ for the points z_i, $i = 1, 2$, such that for any function $v \in K_q$ there is a path $\gamma_i \subset B(t_i, \rho)$ connecting w_i and t_i, and on each of these two paths we have

$$v(\zeta) \leq M_i,$$

where M_i does not depend on v. Put

$$s = \min\left(d(w_1), d(w_2), \inf_{\zeta \in \partial D} |t_1 - \zeta|, \inf_{\zeta \in \partial D} |t_2 - \zeta| \right).$$

We connect the points w_1, w_2 by a path l_1 in the domain $\{\zeta : d(\zeta) > s/2\}$, and the points t_1, t_2, by a path l_2 in D at the distance of at least $s/2$ from ∂D. Let $M' = \max\{U_q(\zeta),\ \zeta \in l_2\}$. Then on the contour composed of γ_i, l_i, $i = 1, 2$, we have the estimate

$$v(\zeta) \leq \max\left(M_1, M_2, M', \ln M\left(\frac{qs}{2}\right)\right) = M.$$

By the maximum principle, this estimate extends to the interior V_0 of the contour. In particular, this estimate holds true on the set $V_0 \setminus (B(t_1, \rho) \cup B(t_2, \rho))$, which does not depend on the function v. Since $|t_1 - t_2| \geq |z_1 - z_2| - |z_1 - t_1| - |z_2 - t_2| > 3\rho$, this set is nonempty. Hence, $U_q(\zeta) \leq M$ on this set.

Thus, we have found points on ∂D where U_q is locally bounded.

We prove that the set of points on ∂D where U_q is locally bounded is connected. Let $z, w \in \partial D$, and let the function U_q be bounded in $B(z, r_1), B(w, r_2)$. Consider the diameter d_1 of the disk $B(z, r_1)$ with endpoints $z_1 \in D$, $z_2 \in G$ and the diameter d_2 of the disk $B(w, r_2)$ with endpoints $w_1 \in D$, $w_2 \in G$. We connect the points z_1, w_1 by a continuous path l_1 in D and the points z_2, w_2, by a continuous path l_2 in G. Put

$$C_1 = \max\{U_q(\zeta),\ \zeta \in d_1 \cup l_1 \cup d_2\},$$
$$\epsilon = \min\{d(\zeta),\ \zeta \in l_2\},$$
$$C = \max(C_1, \ln M(q\epsilon)).$$

Obviously, on the contour composed of the diameters d_1, d_2 and the paths l_1, l_2, we have $U_q(\zeta) \le C$. By the maximum principle, this estimate extends to the domain bounded by the contour. Hence, the function U_q is locally bounded at the points of the arc of the boundary of D between z and w.

Now we show that U_q is locally bounded at all points of the set $\partial D \setminus \{0\}$. Let Γ_0 be the maximal connected arc of the boundary such that U_q is locally bounded on that arc and U_q is not locally bounded at some boundary point $z \neq 0$. Assume that

$$2\rho = \inf_{\zeta \in \Gamma_0} |z - \zeta| > 0$$

and perform the above construction for the point z with a given ρ; as a result, we find points $w \in G'$, $t \in D$ such that for any function $v \in K_q$ there is a path γ in $B(t, \rho)$ that connects the points and the estimate

$$v(\zeta) \le M$$

holds true for all points of the path, where M does not depend on v. Connect the points w and t by a continuous curve γ_0 such that it intersects the boundary of D at a point of the arc Γ_0. Obviously, this curve can be chosen so that the function U_q is locally bounded on it. On the contour composed of the curves γ and γ_0 the function v will be bounded by a constant independent of v. Extending this estimate inside by the maximum principle, we conclude that v is bounded on the interior of the contour. This interior V_0 depends on the function v, but its nonempty subset $V_0 \setminus B(t, \rho)$ does not depend on v. Hence, we see that U_q is bounded on $V_0 \setminus B(t, \rho)$. By the choice of ρ, the latter set contains a boundary arc that is not contained in Γ_0, which contradicts the maximality of Γ_0.

Thus, the function U_q is locally bounded on the set $\partial D \setminus \{0\}$. Let z be an arbitrary point in this set. There exist constants $M, r > 0$ such that

$$U_q(\zeta) \le M \le \ln M(qd(\zeta)), \quad \zeta \in B(z, r).$$

Replacing each function $v \in K_q$ by its harmonic extension to the disk $B(z, r)$, we obtain a right-directed family of functions harmonic in $B(z, r)$, and U_q will be harmonic in $B(z, r)$, as the upper envelope of this family.

Relation (3.3) holds true for the function U_q by its definition.

Lemma 12 is proved. □

To complete the proof of Theorem 5, it remains to show that the set D' is connected. Let $w \in G'$, and let $\epsilon > 0$ be such that

$$U_q(w) \le \ln M(qd(w)) - 2\epsilon.$$

Again, we use a function $f_w(\zeta)$ of the form $1/m_p(\zeta - z_0)^p$, where $p \in \mathbb{N}$, $z_0 \in D$, that satisfies

$$\ln |f_w(\zeta)| \le \ln M(qd(\zeta)), \ \ \zeta \in G, \quad \ln |f_w(w)| \ge \ln M(qd(w)) - \epsilon.$$

Let V be the connected component of the set

$$\{\zeta : U_q(\zeta) < \ln |f_w(\zeta)|\}$$

that contains the point w. Should the singularity point z_0 not belong to V, the harmonic functions U_q and $\ln |f_w|$ would coincide on the boundary V and therefore in V, in particular, at the point w. But this cannot be true because

$$\ln |f_w(w)| \ge \ln M(qd(w)) - \epsilon \ge U_q(w) + \epsilon.$$

Hence, $z_0 \in V$, and this point can be connected with w by a path in $V \subset D'$. This implies that the set D' is connected.

Theorem 5 is proved. □

§4. Localization of the quasianalyticity problem

In this section we prove that the quasianalyticity problem is local: if two domains D_1 and D_2 coincide in a neighborhood of a common boundary point z_0, then the classes $A(\overline{D}_1, \mathcal{M})$ and $A(\overline{D}_2, \mathcal{M})$ are simultaneously quasianalytic or nonquasianalytic at z_0.

By using criteria already known, we can deduce new quasianalyticity criteria from this property under certain restrictions on the domain.

Theorem 6. *Let the sequence (m_n) be regular, and let $z = 0$ be a common boundary point of two bounded Jordan domains D' and D''. If for some $r > 0$ these two domains coincide in the disk $B(0, 2r)$, i.e.,*

$$D' \cap B(0, 2r) = D'' \cap B(0, 2r),$$

then the classes $A(\overline{D}', \mathcal{M})$ and $A(\overline{D}'', \mathcal{M})$ are simultaneously quasianalytic or nonquasianalytic at the point $z = 0$.

Proof. We need an auxiliary lemma.

Lemma 13. *Let $z = 0$ be a common boundary point of two bounded Jordan domains D' and D''. If for some $r > 0$ these two domains coincide in the disk $B(0, 2r)$, i.e.,*

$$D' \cap B(0, 2r) = D'' \cap B(0, 2r),$$

then there exist a positive number p and a Jordan domain D such that

a) *$D \subset D' \cap B(0, 2r) = D'' \cap B(0, 2r)$ and the boundary of D lies on the circle $|z| = 2r$ and on the common part of the boundaries of D' and D'';*

b) *for $\zeta \notin D$ and $|\zeta| \le p$, the distance from ζ to the boundary of D coincides with the distance from ζ to the boundaries of D' and D'':*

$$d(\zeta) := \inf_{z \in D} |z - \zeta| = \inf_{z \in D'} |z - \zeta| = \inf_{z \in D''} |z - \zeta|.$$

Proof. Let D be the connected component of the intersection $D' \cap B(0, 2r)$ that has the point $z = 0$ on its boundary. By assumptions, D coincides with the corresponding part of the intersection $D'' \cap B(0, 2r)$. Obviously, the domain D is simply connected. Let $z = z(t)$, $|t| \le 1$, be a continuous parametrization of the boundary of D' satisfying $z(0) = 0$. We denote by I the set of points $t \in [-1; 1]$ for which $|z(t)| < 2r$. Let $(\alpha; \beta)$ be the largest interval in I containing the point $t = 0$. Put

$$p = \frac{1}{2} \inf\{|z(t)|,\ t \notin (\alpha; \beta)\}.$$

Since the boundary of D is a Jordan curve without self-intersections, it follows that $p > 0$. For any point $\zeta \in B(0, p) \setminus D$, we have

$$d(\zeta) = \inf_{z \in D} |z - \zeta| \le |\zeta| < p < \inf\{|z(t)|,\ t \notin (\alpha; \beta)\},$$

i.e.,

$$d(\zeta) = \inf\{|z(t)|,\ t \in (\alpha; \beta)\} \ge \inf\{|z(t)|,\ t \in [-1; 1]\} = d'(\zeta).$$

Consequently, for $\zeta \in B(0, p) \setminus D$ we have

$$d(\zeta) = d'(\zeta) := \inf_{z \in D'} |z - \zeta|.$$

Since the domains D' and D'' coincide in the disk $B(0, p)$, for ζ as above we have

$$d(\zeta) = d''(\zeta) := \inf_{z \in D''} |z - \zeta|.$$

Lemma 13 is proved. □

Obviously, it suffices to prove simultaneous quasianalyticity for the classes $A(\overline{D}', \mathcal{M})$ and $A(\overline{D}, \mathcal{M})$ at the point $z = 0$. Denote by G, G' the complements of $\overline{D}, \overline{D}'$, respectively, to the extended complex plane, i.e., $G = \overline{\mathbb{C}} \setminus \overline{D}$, $G' = \overline{\mathbb{C}} \setminus \overline{D}'$. Let $d(\zeta), d'(\zeta)$ be the distances from the point ζ to the domains D, D'. These function are defined on the domains G and G', respectively. By Lemma 13,

$$G \cap B(0,p) = G' \cap B(0,p),$$

and also $d(\zeta) = d'(\zeta)$ for $\zeta \in G \cap B(0,p)$.

Since $D \subset D'$, it follows that $d'(\zeta) \geq d(\zeta)$ for $\zeta \in G'$, whence

$$M(qd(\zeta)) \leq M(qd'(\zeta)) \tag{4.1}$$

for any $q \in \mathbb{N}$.

Assume that the class $A(\overline{D}, \mathcal{M})$ is quasianalytic. By Theorem 4, for any $q \in \mathbb{N}$ the relation

$$\sup\{v(z),\ v \in K_q(D)\} = \infty, \quad \zeta \in D,$$

is fulfilled. By property (4.1), we have the inclusion $K_q(D) \subset K_q(D')$, and so

$$\sup\{v(z),\ v \in K_q(D')\} = \infty, \quad \zeta \in D.$$

By Theorem 4, the class $A(\overline{D}', \mathcal{M})$ is also quasianalytic.

Now, assume that the class $A(\overline{D}, \mathcal{M})$ is nonquasianalytic. By Theorem 5, for any $q \in \mathbb{N}$ starting with some q_0, there is a domain D_q containing $\overline{D} \setminus \{0\}$, and a function $h(\zeta)$ harmonic in D_q that is equal to $\ln M(qd(\zeta))$ on the boundary and satisfies

$$\lim_{|\zeta| \longrightarrow 0} \frac{h(\zeta)}{-\ln|\zeta|} = +\infty.$$

Theorem 5 means that, for any fixed $q \in \mathbb{N}$, $q \geq q_0$, the function

$$u(z) = \sup\{v(z),\ v \in K_q(D)\}, \quad z \in D,$$

is well defined and subharmonic on the extended plane except for the point zero, i.e., on $\overline{\mathbb{C}} \setminus \{0\}$, and is harmonic in the domain D_q, containing $\overline{D} \setminus \{0\}$. The domain D' is bounded and the ratio $|\zeta|/d'(\zeta)$ tends to 1 as $|\zeta| \longrightarrow +\infty$. Therefore, there exists a sufficiently large number R such that

$$D' \subset B(0,R), \quad \frac{|\zeta|}{d'(\zeta)} \leq 2 \quad \text{for} \quad |\zeta| \geq R.$$

Let C denote the open annulus bounded by the circles $|z| = p$ and $|z| = R$. Let the Borel measure μ in the domain $\overline{\mathbb{C}} \setminus \{0\}$ be the Riesz measure associated with the subharmonic function $u(\zeta)$, and let μ_0 be the restriction of the measure μ to the ring $\overline{C}$. Since $\overline{C}$ is a compact set in $\overline{\mathbb{C}} \setminus \{0\}$, it follows that $\mu_0(\overline{C}) < \infty$ and the logarithmic potential of the measure μ_0,

$$u_0(\zeta) = \int \ln|\zeta - z|\, d\mu_0(z),$$

is well defined and subharmonic on the entire plane $\mathbb{C}$ and harmonic on $\mathbb{C} \setminus C$. The difference $u(\zeta) - u_0(\zeta)$ is harmonic on the union $(C \cup D_q)$, and, in particular, in the domain

$$\Omega = (C \cup D_q) \cap B(0,R).$$

The boundary of Ω consists of the circle $|\zeta| = R$ and of a part γ lying inside the circle $\overline{B}(0,p)$. By the choice of the numbers p and R, we have

$$\begin{gathered} \ln M(2qd'(\zeta)) \leq \ln M(q|\zeta|), \quad |\zeta| \geq R, \\ \ln M(2qd'(\zeta)) = \ln M(2qd(\zeta)) \leq \ln M(qd(\zeta)), \quad |\zeta| \leq p, \quad \zeta \in G'. \end{gathered} \tag{4.2}$$

Take an arbitrary function $v(\zeta)$ in the class $K_{2q}(D')$. Then v is nonnegative, continuous, and subharmonic on $\overline{\mathbb{C}} \setminus \{0\}$ and satisfies the estimates

$$v(\zeta) = O\Big(\ln \frac{1}{|\zeta|}\Big) \quad \text{as } \zeta \to 0; \quad v(\zeta) \le \ln M(2qd'(\zeta)), \ \zeta \in G'.$$

Using (4.2), from the second inequality we deduce that

$$v(\zeta) \le \ln M(q|\zeta|), \quad |\zeta| \ge R,$$
$$v(\zeta) \le \ln M(qd(\zeta)), \quad |\zeta| \le p, \ \zeta \in G'.$$

Note that the functions $\max(-\ln m_k - k \ln q|\zeta|, 0)$ are subharmonic, nonnegative, and continuous on the extended plane except for zero, and also

$$\max(-\ln m_k - k \ln q|\zeta|, 0) \le \ln M(qd(\zeta)), \quad \zeta \notin D,$$

that is, these functions belong to $K_q(D)$. Hence, by the definition of the function $u(\zeta)$, we have $\ln M(q|\zeta|) \le u(\zeta)$ for all ζ, whence $v(\zeta) \le u(\zeta)$, $|\zeta| \ge R$.

Since

$$B(0,p) \cap G' = B(0,p) \cap G$$

and the boundary of Ω coincides with the boundary of D_q in $B(0,p)$, and on the boundary of D_q the function $u(\zeta)$ equals $\ln M(qd(\zeta))$, we conclude that

$$v(\zeta) \le \ln M(qd(\zeta)) = u(\zeta), \quad \zeta \in \partial\Omega \cap B(0,p).$$

Thus, we have $v(\zeta) \le u(\zeta)$, $\zeta \in \partial\Omega$.

The function $u_0(\zeta)$ is subharmonic in the entire plane, and therefore, is bounded in the disk $B(0,R)$:

$$u_0(\zeta) \le T.$$

Hence, on the boundary of Ω we have $v(\zeta) \le u(\zeta) - u_0(\zeta) + u_0(\zeta) \le u(\zeta) - u_0(\zeta) + T$, $\zeta \in \partial\Omega$.

The function $u(\zeta) - u_0(\zeta)$ is harmonic in Ω, and $v(\zeta)$ is subharmonic in this domain. By the maximum principle,

$$v(\zeta) \le u(\zeta) - u_0(\zeta) + T, \quad \zeta \in \Omega. \tag{4.3}$$

Being subharmonic on $\mathbb{C} \setminus \{0\}$, the function $u(\zeta) - u_0(\zeta)$ is bounded from above in the annulus $\{\frac{p}{2} \le |\zeta| \le R\}$, $u(\zeta) - u_0(\zeta) \le T_1$.

By definition, the function $u(\zeta)$ is nonnegative; hence, the function $-u_0(\zeta)$ is also bounded from above in the same annulus,

$$-u_0(\zeta) \le T_1.$$

Since the function $u_0(\zeta)$ is harmonic in the disk $B(0,p)$, the function $-u_0(\zeta)$ is bounded from above in the disk $B(0, \frac{p}{2})$:

$$-u_0(\zeta) \le T_2.$$

Put $T' = \max(T_1, T_2)$. Then the last two inequalities imply that the function $-u_0(\zeta)$ is bounded by T' in the disk $B(0,R)$. In particular, this estimate is valid for the domain Ω:

$$-u_0(\zeta) \le T', \quad \zeta \in \Omega.$$

Combined with (4.3), this inequality shows that $v(\zeta) \le u(\zeta) + T' + T$, $\zeta \in \Omega$.

Since v is an arbitrary function in $K_{2q}(D')$, the condition of Theorem 4 cannot be satisfied, and the class $A(\overline{D}', \mathcal{M})$ cannot be quasianalytic.

Theorem 6 is proved. □

Theorem 7. *Let D, D_1 be simply connected domains, let Ω be a domain containing the closure $\overline{D}$, and let φ be an analytic function in Ω such that $\varphi(D) \subset D_1$. If the boundary point $w_0 \in \partial D_1$ is the image of a boundary point $z_0 \in \partial D$, i.e., $w_0 = \varphi(z_0)$, then for any sequence $\mathcal{M} = (M_n)$ we have the inclusion*

$$\{f(\varphi(z)),\ f \in A(\overline{D}_1, \mathcal{M})\} \subset A(\overline{D}, \mathcal{M}).$$

Proof. Let $f \in A(\overline{D}_1, \mathcal{M})$. By Dyn′kin's theorem [1] (see Theorem B of the present paper), there exists a continuously differentiable function F on $\mathbb{C}$ such that $F(w) \equiv f(w)$ in D_1 and

$$\left|\frac{\partial F(w)}{\partial \overline{w}}\right| \le \frac{C}{M(Bd_1(w))}, \quad w \in \mathbb{C}, \tag{4.4}$$

where C, B are some positive constants, and $d_1(w)$ denotes the distance from the point $w \notin D_1$ to the boundary of D_1. Let $3r = \inf\{|z - w|,\ z \in \overline{D},\ w \notin \Omega\}$ be the distance from $\overline{D}$ to the boundary of Ω. By the assumptions of the theorem, we have $r > 0$. We denote by Ω' and Ω'' the r-envelope and the $2r$-envelope of the set $\overline{D}$, respectively, i.e., $\Omega' = \bigcup_{z \in D_1} B(z, r)$, $\quad \Omega'' = \bigcup_{z \in D_1} B(z, 2r)$.

We take a smooth Jordan curve Γ in $\Omega'' \setminus \Omega'$ that encloses the set $\overline{D}$ and denote by Ω_1 the interior of the curve. Applying the Borel–Pompeiu formula (see [13]) to the function $g(z) = F(\varphi(z))$ in the domain Ω_1, we obtain

$$g(z) = \frac{1}{2\pi i}\int_\Gamma \frac{g(t)}{t - z}\,dt - \frac{1}{\pi}\int_{\Omega_1} \frac{\partial g}{\partial \overline{t}}\frac{dv(t)}{t - z}.$$

We prove that each term on the right-hand side of this identity belongs to the class $A(\overline{D}, \mathcal{M})$. Put

$$u(z) = \frac{1}{2\pi i}\int_\Gamma \frac{g(t)}{t - z}\,dt, \qquad v(z) = \frac{1}{\pi}\int_{\Omega_1} \frac{\partial g}{\partial \overline{t}}\frac{dv(t)}{t - z}.$$

For $z, \zeta \in D$ and $k, n \in \mathbb{N}$, $n \ge k$, we have

$$\begin{aligned}
&\left|u^{(k)}(\zeta) - \sum_{p=0}^{n-k} u^{(k+p)}(z)\frac{(\zeta - z)^p}{p!}\right| \\
&\qquad = \frac{1}{2\pi}\int_\Gamma |g(t)|\left|\frac{k!}{(t - z)^{k+1}} - \sum_{p=0}^{n-k}\frac{(p + k)!(\zeta - z)^p}{p!(t - z)^{p+k+1}}\right||dt|.
\end{aligned}$$

We apply formula (1.7) and observe that if $z, \zeta \in D$ and $t \in \Gamma$, then $|\zeta - t|$, $|z - t| \ge r$. Therefore,

$$\begin{aligned}
&\left|u^{(k)}(\zeta) - \sum_{p=0}^{n-k} u^{(k+p)}(z)\frac{(\zeta - z)^p}{p!}\right| \\
&\qquad \le |\zeta - z|^{n-k+1}\frac{1}{2\pi r^{n+2}}\max_{t \in \Gamma}|g(t)|\,|\Gamma|\sum_{s=0}^{k}\frac{\binom{k}{s}(n - s)!s!}{(n - k)!} \\
&\qquad = \max_{t \in \Gamma}|g(t)|\,|\Gamma|\,|\zeta - z|^{n-k+1}\frac{k!}{2\pi r^{n+2}}\sum_{s=0}^{k}\binom{n - s}{k - s},
\end{aligned}$$

where $|\Gamma|$ is the length of the curve Γ. The sum of binomial coefficients was evaluated in (1.10); this yields

$$\left|u^{(k)}(\zeta)-\sum_{p=0}^{n-k}u^{(k+p)}(z)\frac{(\zeta-z)^p}{p!}\right| \le \frac{1}{2\pi}\max_{t\in\Gamma}|g(t)|\,|\Gamma|\,|\zeta-z|^{n-k+1}\frac{(n+1)!}{(n-k+1)!r^{n+2}}.$$

Property (1.3) of regular sequences shows that there exists a number $\delta>0$ such that $m_n^{\frac{1}{n}}\ge\delta$, $n=0,1,\dots$, or $M_n\ge n!\delta^n$, $n=0,1,\dots$. Hence,

$$\left|u^{(k)}(\zeta)-\sum_{p=0}^{n-k}u^{(k+p)}(z)\frac{(\zeta-z)^p}{p!}\right| \le \frac{1}{2\pi}\max_{t\in\Gamma}|g(t)|\,|\Gamma|\,|\zeta-z|^{n-k+1}\frac{M_{n+1}}{(n-k+1)!\delta^{n+1}r^{n+2}}.$$

Thus, we have obtained the estimate

$$\sup_{n\ge0,k\le n}\sup_{z,\zeta\in D}\frac{(\delta r)^{n+1}(n-k+1)!}{M_{n+1}|z-\zeta|^{n-k+1}}\left|u^{(k)}(\zeta)-\sum_{p=0}^{n-k}u^{(k+p)}(z)\frac{(\zeta-z)^p}{p!}\right| \le \frac{\max_{t\in\Gamma}|g(t)|\,|\Gamma|}{2\pi r}. \tag{4.5}$$

Next, we work with the function $v(z)$. Since, by the properties of the function $F(w)$, the integral in the definition of $v(z)$ is only taken over the domain $\Omega_1\setminus D$, for $z,\zeta\in D$ and $k,n\in\mathbb{N}$, $n\ge k$, we have

$$\left|v^{(k)}(\zeta)-\sum_{p=0}^{n-k}v^{(k+p)}(z)\frac{(\zeta-z)^p}{p!}\right| \le \frac{1}{\pi}\int_{\Omega_1\setminus D}\left|\frac{\partial g(t)}{\partial\bar t}\right|\left|\frac{k!}{(t-z)^{k+1}}-\sum_{p=0}^{n-k}\frac{(p+k)!(\zeta-z)^p}{p!(t-z)^{p+k+1}}\right|dv(t).$$

Again, we apply formula (1.7) and observe that $|\zeta-t|,|z-t|\ge d(t)$ for $t\in\Omega_1\setminus D$ and $\zeta,z\in D$; this yields

$$\left|v^{(k)}(\zeta)-\sum_{p=0}^{n-k}v^{(k+p)}(z)\frac{(\zeta-z)^p}{p!}\right| \le \frac{1}{\pi}\int_{\Omega_1\setminus D}\left|\frac{\partial g(t)}{\partial\bar t}\right|\frac{|\zeta-z|^{n-k+1}k!}{d(t)^{n+2}}\sum_{s=0}^{n-k}\binom{n-s}{k-s}dv(t).$$

Recalling (1.10), we obtain

$$\left|v^{(k)}(\zeta)-\sum_{p=0}^{n-k}v^{(k+p)}(z)\frac{(\zeta-z)^p}{p!}\right|\le\frac{|\Omega_1|}{\pi}\sup_{\Omega_1\setminus D}\left|\frac{\partial g(t)}{\partial\bar t}\right|\frac{|\zeta-z|^{n-k+1}(n+1)!}{d(t)^{n+2}(n-k+1)!}, \tag{4.6}$$

where $|\Omega_1|$ is the area of the domain Ω_1. Using the definition of the function $g(t)$ and relation (4.4), we get

$$\left|\frac{\partial g(t)}{\partial\bar t}\right|=\left|\frac{\partial F(w)}{\partial\overline{w}}(\varphi(t))\overline{\varphi'(t)}\right|\le\frac{C}{M(Bd_1(\varphi(t)))}\max_{t\in\Omega_1}|\varphi'(t)|. \tag{4.7}$$

Let $t \in \Omega_1 \setminus D$ and $d(t) = |t - t_0|$, where $t_0 \in \partial D$. Then $\varphi(t_0) \in \overline{D}_1$. Also $|t - t_0| < 2r$ and $B(t_0, 2r) \subset \overline{\Omega}''$, and therefore,

$$d_1(\varphi(t)) \leq |\varphi(t) - \varphi(t_0)| \leq \max_{\overline{\Omega}''} |\varphi'(z)|\, |t - t_0| = \max_{\overline{\Omega}''} |\varphi'(z)| d(t).$$

We denote the final quantity $\max_{\overline{\Omega}''} |\varphi'(z)|$ by T. Thus, for $t \in \Omega_1 \setminus D$, we get the estimate

$$d_1(\varphi(t)) \leq T d(t).$$

Substituting this in (4.7), and using the monotonicity of the function $M(x)$, we obtain

$$\Big|\frac{\partial g(t)}{\partial \bar{t}}\Big| = \Big|\frac{\partial F(w)}{\partial \overline{w}}(\varphi(t))\overline{\varphi'(t)}\Big| \leq \frac{TC}{M(BTd(t))}.$$

We plug this in (4.6):

$$\frac{(n-k+1)!}{|\zeta - z|^{n-k+1}}\Big|v^{(k)}(\zeta) - \sum_{p=0}^{n-k} v^{(k+p)}(z)\frac{(\zeta - z)^p}{p!}\Big| \leq \frac{TC|\Omega_1|}{\pi} \sup_{t \in \Omega_1 \setminus D} \frac{1}{M(BTd(t))d(t)^{n+2}}.$$

By the properties (1.5) and (1.2) of regular sequences, we have

$$\begin{aligned} \sup_{t \in \Omega_1 \setminus D} \frac{1}{M(BTd(t))d(t)^{n+2}} &\leq \sup_{x > 0} \frac{1}{M(BTx)x^{n+2}} = (BT)^{n+2} m_{n+2} \\ &\leq BT(BTQ)^{n+1} m_{n+1} = BT(BTQ)^{n+1}\frac{M_{n+1}}{(n+1)!}. \end{aligned}$$

Hence,

$$\frac{(n-k+1)!}{|\zeta - z|^{n-k+1}}\Big|v^{(k)}(\zeta) - \sum_{p=0}^{n-k} v^{(k+p)}(z)\frac{(\zeta - z)^p}{p!}\Big| \leq \frac{BT^2C|\Omega_1|}{\pi}(BTQ)^{n+1}M_{n+1}.$$

Thus, we obtain

$$\begin{aligned} &\sup_{n \geq 0, k \leq n} \sup_{\zeta, z \in D} \frac{(n-k+1)!}{(BTQ)^{n+1}M_{n+1}|\zeta - z|^{n-k+1}}\Big|v^{(k)}(\zeta) - \sum_{p=0}^{n-k} v^{(k+p)}(z)\frac{(\zeta - z)^p}{p!}\Big| \\ &\quad \leq \frac{BT^2C|\Omega_1|}{\pi}. \end{aligned}$$

Since $g(t) \equiv u(t) + v(t)$, combining this estimate with (4.5) yields

$$\sup_{n \geq 0, k \leq n} \sup_{\zeta, z \in D} \frac{P^{n+1}(n-k+1)!}{M_{n+1}|\zeta - z|^{n-k+1}}\Big|g^{(k)}(\zeta) - \sum_{p=0}^{n-k} g^{(k+p)}(z)\frac{(\zeta - z)^p}{p!}\Big| \leq C',$$

where

$$P = \min(\delta r, BTQ), \quad C' = \frac{\max_{t \in \Gamma}|g(t)|\,|\Gamma|}{2\pi r} + \frac{BT^2C|\Omega_1|}{\pi}.$$

Thus, $g(t) \in A(\overline{D}, \mathcal{M})$. Theorem 7 is proved. $\square$

Corollary 1. *Suppose D, D_1 are simply connected domains, the domains Ω, Ω_1 contain the closures $\overline{D}, \overline{D}_1$, respectively, and φ is a conformal mapping of Ω onto Ω_1 such that $\varphi(D) = \varphi(D_1)$. If a boundary point $w_0 \in \partial D_1$ is the image of a boundary point $z_0 \in \partial D$, i.e., $w_0 = \varphi(z_0)$, then for any sequence $\mathcal{M} = (M_n)$, the class $A(\overline{D}_1, \mathcal{M})$ is quasianalytic at the point w_0 if and only if the class $A(\overline{D}, \mathcal{M})$ is quasianalytic at the point z_0.*

Proof. Let the class $A(\overline{D}_1, \mathcal{M})$ be nonquasianalytic at w_0, i.e., there is a nonzero function $f \in A(\overline{D}_1, \mathcal{M})$ that vanishes at w_0 with all its derivatives. By Theorem 7, the function $g(w) = f(\varphi(w))$ belongs to $A(\overline{D}, \mathcal{M})$ and vanishes at z_0 together with all its derivatives. Thus, the class $A(\overline{D}, \mathcal{M})$ also cannot be quasianalytic. The proof of the converse statement is similar. □

Corollary 2. *Let $B' = B'(a, R)$ be the exterior of the disk $B(a, R)$ in the extended complex plane, i.e., $B'(a, R) = \overline{\mathbb{C}} \setminus B(a, R)$. Then, for any point $z_0 \in \partial B'$, a criterion of quasianalyticity for the class $A(\overline{B}', \mathcal{M})$ at the point z_0 is given by the condition*

$$\int_1^\infty \frac{\ln T(r)}{r^{\frac{3}{2}}}\, dr = \infty, \tag{4.8}$$

where

$$T(r) = \sup_{n \geq 0} \frac{r^n}{M_n}$$

is the trace function of the sequence $\mathcal{M}$.

Proof. This follows immediately from Corollary 1, because $\varphi(w) = \frac{R}{w-a}$ is a conformal map of $\overline{\mathbb{C}}$ onto itself and B' is mapped onto the unit disk. □

In what follows we consider domains whose boundary coincides locally with the graph of some function $y = u(x)$, $|x| < \delta$. Denote by $\Omega(u, \delta)$ the supergraph of $u(x)$ on the interval $(-\delta; +\delta)$, i.e.,

$$\Omega(u) = \{z = x + iy : y > u(x),\ |x| < \delta\}.$$

Theorem 8. *Suppose that a Jordan domain D coincides locally with the supergraph of some function $y = u(x)$, $|x| < \delta$, with $u(0) = 0$. This means that for some $r > 0$, the sets $D \cap B(0, r)$ and $\Omega(u, \delta) \cap B(0, r)$ coincide. Suppose that for some $a > 0$ we have*

$$|u(x)| \leq ax^2;$$

then the class $A(\overline{D}, \mathcal{M})$ is quasianalytic at the point $z = 0$ if and only if condition (4.8) *is satisfied.*

Proof. For $|x| \leq \frac{1}{2a}$, we have the inequalities

$$u_+(x) := \frac{1}{2a} - \sqrt{\frac{1}{4a^2} - x^2} = \frac{x^2}{\frac{1}{2a} + \sqrt{\frac{1}{4a^2} - x^2}} \geq ax^2,$$

$$u_-(x) := -\frac{1}{2a} + \sqrt{\frac{1}{4a^2} - x^2} = \frac{-x^2}{\frac{1}{2a} + \sqrt{\frac{1}{4a^2} - x^2}} \leq -ax^2.$$

Hence, if $|x| < \delta_1 := \min(\delta, \frac{1}{2a})$, then

$$u_+(x) \geq u(x), \quad u_-(x) \leq u(x),$$

and the supergraph $\Omega(u_+, \delta_1)$ is contained in the supergraph $\Omega(u, \delta_1)$, and $\Omega(u_-, \delta_1)$ contains $\Omega(u, \delta_1)$. The supergraph $\Omega(u_+, \frac{1}{2a})$ contains the open disk $B(\frac{i}{2a}, \frac{1}{2a})$ and the supergraph $\Omega(u_-, \frac{1}{2a})$ is contained in the complement B' of the closed disk $B(\frac{-i}{2a}, \frac{1}{2a})$. Put $\varepsilon = \min(\delta_1, r)$. Now, the claim follows from the inclusions

$$\begin{aligned} B\Big(\frac{i}{2a}, \frac{1}{2a}\Big) \cap B(0, \varepsilon) &\subset \Omega(u_+) \cap B(0, \varepsilon) \\ &\subset \Omega(u) \cap B(0, \varepsilon) \subset \Omega(u_-) \cap B(0, \varepsilon) \subset B' \cap B(0, \varepsilon) \end{aligned}$$

and from Corollary 2 of Theorem 7.

Theorem 8 is proved. □

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