

LAGRANGE'S MEAN MOTION PROBLEM

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ABSTRACT. The famous mean motion problem, dating back to Lagrange, is about the existence of the average speed for the amplitude of any exponential polynomial with exponents on the imaginary axis, whenever the variable moves along a horizontal line. This problem was completely solved by B. Jessen and H. Tornehave in *Acta Mathematica*, vol. 77, 1945. Here, we give a simple version of that proof.

Consider an exponential polynomial

$$(1) \quad f(z) = \sum_{j=1}^M c_j e^{i\lambda_j z}, \quad c_j \in \mathbb{C}, \quad \lambda_j \in \mathbb{R}, \quad \lambda_j \neq \lambda_{j'} \quad \text{for } j \neq j'.$$

We set $z = x + iy$, $x, y \in \mathbb{R}$, fix y , and consider the increment $\Delta_{\alpha < x < \beta}$ of the function $\arg^+ f(x + iy)$ on (α, β) , assuming that $\arg^+ f(x + iy)$ is continuous at the points where $f(x + iy) \neq 0$, and

$$\lim_{x \rightarrow x_0+0} \arg^+ f(x + iy) - \lim_{x \rightarrow x_0-0} \arg^+ f(x + iy) = -p\pi$$

at any zero $x_0 \in (\alpha, \beta)$ of multiplicity p . Similarly, the definition of the increment $\Delta_{\alpha < x < \beta}$ of $\arg^- f(x + iy)$ invokes the identity

$$\lim_{x \rightarrow x_0+0} \arg^- f(x + iy) - \lim_{x \rightarrow x_0-0} \arg^- f(x + iy) = p\pi.$$

Lagrange (see [8]) conjectured that the following limits (mean motions) exist for any fixed y :

$$(2) \quad c^+(y) = \lim_{\beta - \alpha \rightarrow \infty} \frac{\Delta_{\alpha < x < \beta} \arg^+ f(x + iy)}{\beta - \alpha}$$

and

$$(3) \quad c^-(y) = \lim_{\beta - \alpha \rightarrow \infty} \frac{\Delta_{\alpha < x < \beta} \arg^- f(x + iy)}{\beta - \alpha}.$$

He proved his conjecture in the case where the absolute value of one of the coefficients in (1) is greater than the sum of the absolute values of the other coefficients. Moreover, if this is the case, then

$$(4) \quad \arg^+ f(x + iy) = c^+ x + O(1), \quad \arg^- f(x + iy) = c^- x + O(1) \quad (x \rightarrow \infty),$$

and the mean motions $c^+(y)$ and $c^-(y)$ are equal. Also, Lagrange showed that formulas (4) hold true in the case where $M = 2$ with arbitrary terms in (1), but in this case c^+ and c^- may be different (for example, this is so for $f(z) = \sin z$ and $y = 0$).

The detailed history of attempts to prove Lagrange's conjecture is presented in the Introduction to the paper [7]. Here we only mention its main stages.

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First, Bohl [1] proved (2) and (3) in the case of three terms in (1). Earlier, Bernstein [3] showed that (4) may fail in this case. Next, Bohr [2] proved (4) with $c^+ = c^-$ under the condition $|f(x + iy)| \geq \kappa > 0$. Also, he extended this assertion to arbitrary almost periodic functions on the real axis. Next, Weyl (see [12, 13]) proved (2) and (3) for arbitrary sums (1). Moreover, in [12] he showed a formula for calculating the mean motion in the case of $\lambda_1, \dots, \lambda_M$ linearly independent over \mathbb{Z} . But Weyl's proof had a flaw. In fact, Weyl proved Lagrange's conjecture for all $y \in \mathbb{R}$ with the exception of points of some discrete sets. Note that, simultaneously, Hartman [5] proved Lagrange's conjecture for the same y where Weyl's proof was correct. Finally, B. Jessen and H. Tornehave [7] found a complete proof of Lagrange's conjecture on the basis of multivariate complex analysis.

In this paper we present their proof with some simplifications and revisions.

The following result will be proved.

Theorem. *For each sum of the form (1), the limits (2) and (3) exist for all $y \in \mathbb{R}$. Moreover, the set $\{y \in \mathbb{R} : c^+(y) \neq c^-(y)\}$ is finite on any bounded interval.*

The proof of the theorem leans upon the following statement.

Lemma 1 (Kronecker–Weyl theorem [11]; see also [9, p. 108]). *Suppose that a function $g(u)$, $u = (u_1, \dots, u_N)$, is 2π -periodic in each of the variables u_1, \dots, u_N and that $g(u)$ is Riemann integrable on $[0, 2\pi]^N$. Let μ_1, \dots, μ_N be real numbers linearly independent over \mathbb{Z} . Then the limit*

$$(5) \quad \lim_{\beta - \alpha \rightarrow \infty} \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} g(\mu_1 t, \dots, \mu_N t) dt$$

exists and is equal to the mean value of the function $g(u)$ over the cube $[0, 2\pi]^N$.

Proof of the lemma. Clearly, we may assume that g is a real-valued function.

If g is a trigonometric polynomial of the form

$$(6) \quad \sum_{k_1, \dots, k_N \in \mathbb{Z}} b_{k_1, \dots, k_N} e^{ik_1 u_1 + \dots + ik_N u_N},$$

then its mean value is equal to the coefficient $b_{0, \dots, 0}$. Since $k_1 \mu^1 + \dots + k_N \mu^N = 0$ only if $k_1 = \dots = k_N = 0$, we see that (5) is also equal to $b_{0, \dots, 0}$.

Furthermore, an arbitrary continuous function g that is 2π -periodic in each of the variables u_1, \dots, u_N can be approximated by polynomials (6) uniformly in $u \in \mathbb{R}^N$, and so we obtain the conclusion of the lemma also in this case.

Finally, for any Riemann integrable function g on $[0, 2\pi]^N$ and any $\varepsilon > 0$ there are continuous functions $g_{\varepsilon}(u) \leq g(u)$ and $g^{\varepsilon}(u) \geq g(u)$ that are 2π -periodic in each variable and such that

$$\begin{aligned} \left(\frac{1}{2\pi}\right)^N \int_{[0, 2\pi]^N} g^{\varepsilon}(u) du_1 \cdots du_N &\leq \left(\frac{1}{2\pi}\right)^N \int_{[0, 2\pi]^N} g(u) du_1 \cdots du_N + \varepsilon, \\ \left(\frac{1}{2\pi}\right)^N \int_{[0, 2\pi]^N} g_{\varepsilon}(u) du_1 \cdots du_N &\geq \left(\frac{1}{2\pi}\right)^N \int_{[0, 2\pi]^N} g(u) du_1 \cdots du_N - \varepsilon. \end{aligned}$$

We get

$$\begin{aligned} \limsup_{\beta - \alpha \rightarrow \infty} \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} g(\mu_1 t, \dots, \mu_N t) dt &\leq \left(\frac{1}{2\pi}\right)^N \int_{[0, 2\pi]^N} g^{\varepsilon}(u) du_1 \cdots du_N, \\ \liminf_{\beta - \alpha \rightarrow \infty} \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} g(\mu_1 t, \dots, \mu_N t) dt &\geq \left(\frac{1}{2\pi}\right)^N \int_{[0, 2\pi]^N} g_{\varepsilon}(u) du_1 \cdots du_N. \end{aligned}$$

Since ε is arbitrarily small, the lemma follows. \square

Proof of the theorem. Let μ^1, \dots, μ^N be a basis of the additive group generated by the numbers $\lambda_1, \dots, \lambda_M$ (for the existence of a basis of a finitely generated group see, e.g., [6, p. 47]). Thus, we have

$$\lambda_j = \sum_{r=1}^N k_{r,j} \mu_r, \quad k_{r,j} \in \mathbb{Z}, \quad 1 \leq j \leq M, \quad 1 \leq r \leq N.$$

For $w = (w_1, \dots, w_N) \in \mathbb{C}^N$, we set

$$F(z, w) = \sum_{j=1}^M c_j \exp \left\{ i \lambda_j z + i \sum_{r=1}^N k_{r,j} w_r \right\}.$$

The function $F(z, u)$ is 2π -periodic in each of the variables u_1, \dots, u_N , and $F(x + iy, \mu_1 t, \dots, \mu_N t) = f(z + t)$. Since the λ_j , $j = 1, \dots, M$, are pairwise different, we see that $F(z, w) \not\equiv 0$ for every fixed w .

First, we prove that the functions

$$(7) \quad \Delta_{-1/2 < x < 1/2} \arg^+ F(x + iy, u)$$

and

$$(8) \quad \Delta_{-1/2 < x < 1/2} \arg^- F(x + iy, u)$$

are uniformly bounded and continuous almost everywhere in $u \in [0, 2\pi]^N$ for every fixed $y = y' \in \mathbb{R}$.

Take $u^{(0)} \in [0, 2\pi]^N$. If $F(x + iy, u^{(0)}) \neq 0$ for all $x \in [-1/2, 1/2]$, then the function $F'_z(x + iy, u)/F(x + iy, u)$ is continuous and uniformly bounded for $x \in [-1/2, 1/2]$ and for u in some neighborhood of the point $u^{(0)}$. Hence, in this case the functions (7) and (8) are equal, uniformly bounded, and continuous in u on this neighborhood.

Now, suppose $F(x^{(1)} + iy', u^{(0)}) = 0$ for some point $x^{(1)} \in [-1/2, 1/2]$. Since $F(z, u^{(0)}) \not\equiv 0$, we can use the Weierstrass preparation theorem (see, e.g., [10, p. 156]) to find $\varepsilon > 0$, $\delta > 0$, and a pseudopolynomial

$$(9) \quad P_1(z, w) = (z - x^{(1)} - iy')^p + a_1(w)(z - x^{(1)} - iy')^{p-1} + \dots + a_p(w)$$

such that

$$(10) \quad F(z, w) = P_1(z, w)F_1(z, w)$$

on the set $\{(z, w) : |w - u^{(0)}| < \varepsilon, |z - x^{(1)} - iy'| < \delta\}$. Here the function $F_1(z, w)$ is holomorphic without zeros, and the coefficients $a_j(w)$ in (9) are holomorphic in the ball $\{w : |w - u^{(0)}| < \varepsilon\}$ and satisfy

$$(11) \quad a_j(u^{(0)}) = 0, \quad j = 1, \dots, p.$$

For ε small, all solutions of the equation $P_1(z, w) = 0$ for $|w - u^{(0)}| < \varepsilon$ belong to the disk $|z - x^{(1)} - iy'| < \delta$. Hence, identity (10) determines the holomorphic function $F_1(z, w)$ on the set $\{(z, w) : z \in \mathbb{C}, |w - u^{(0)}| < \varepsilon\}$.

Let $x^{(2)}$ be another point of the segment $[-1/2, 1/2]$ such that $F(x^{(2)} + iy', u^{(0)}) = 0$. Using the Weierstrass preparation theorem for the function $F_1(z, w)$ in a neighborhood of the point $(x^{(2)} + iy', u^{(0)})$, we get

$$F_1(z, w) = P_2(z, w)F_2(z, w).$$

Here $P_2(z, w)$ is of the form (9) with $x^{(2)}$ instead of $x^{(1)}$. The function $F_2(z, w)$ is holomorphic on the set $\{(z, w) : z \in \mathbb{C}, |w - u^{(0)}| < \varepsilon\}$ and does not vanish on the set

$\{(z, w) : |w - u^{(0)}| < \varepsilon, |z - x^{(2)} - iy'| < \delta\}$. Continuing in the same way, we get a representation

$$(12) \quad F(z, w) = P_1(z, w)P_2(z, w) \cdots P_s(z, w)G(z, w)$$

in which the pseudopolynomials P_j are of the form (9) with various points $x' \in [-1/2, 1/2]$ in place of $x^{(1)}$, their coefficients satisfy (11), and the holomorphic function $G(z, w)$ does not vanish in a neighborhood of the set $\{(z, w) : x \in [-1/2, 1/2], y = y', w = u^{(0)}\}$. Next, since each pseudopolynomial P_j is a product of irreducible pseudopolynomials of the form (9) (see, e.g., [10, p. 156]), we have

$$(13) \quad F(z, w) = Q_1^{q_1}(z, w)Q_2^{q_2}(z, w) \cdots Q_r^{q_r}(z, w)G(z, w)$$

with irreducible pseudopolynomials Q_j of the form (9). Moreover, for any fixed $w \in U$ these pseudopolynomials are products of linear factors, and we can write

$$(14) \quad F(z, w) = (z - b_1(w))^{t_1} \cdots (z - b_k(w))^{t_k}G(z, w)$$

with functions $b_j(w)$, $j = 1, \dots, k$, analytic in U .

Since the increment of the argument of any linear factor along any segment is at most π , the functions (7) and (8) for $y = y'$ are bounded uniformly in the ball $\{u \in \mathbb{R}^N : |u - u^{(0)}| < \varepsilon\}$.

Next, we consider the discriminant $d_P(w)$ of some pseudopolynomial (9). It is holomorphic on the set $\{|w - u^{(0)}| < \varepsilon\}$ and does not vanish identically in the case of an irreducible pseudopolynomial P (see, e.g., [10, p. 162]). The set

$$E = \{u \in [0, 2\pi]^N : |u - u^{(0)}| < \varepsilon, F(-1/2, u)F(1/2, u)d_{Q_1}(u) \cdots d_{Q_r}(u) = 0\}$$

is the intersection of the zero set of a holomorphic function with a real hyperplane, and so it has zero Lebesgue measure. We fix a point $u^{(1)} \in U \setminus E$. All polynomials $Q_m(z, w)$ with w in a neighborhood $U_1 \subset \mathbb{C}^N$ of the point $u^{(1)}$ have only simple zeros. Hence, we get (14) with pairwise different functions $b_j(w)$. Therefore, if all the functions $\Delta_{-1/2 < x < 1/2} \arg^\pm(x + iy - b_n(u))$ are continuous at a point $u \in U_1 \cap \mathbb{R}^N$, then so are the functions (7) and (8). Next, since $F(-1/2, u^{(1)}) \neq 0$ and $F(1/2, u^{(1)}) \neq 0$, we can take U_1 so small that the values of all functions $b_n(w)$ for $w \in U_1$ belong to the set

$$\{z = x + iy : -1/2 < x < 1/2, |y - y'| < \delta\}.$$

Put

$$b'_n(w) = \frac{b_n(w) + \overline{b_n(\bar{w})}}{2}, \quad b''_n(w) = \frac{b_n(w) - \overline{b_n(\bar{w})}}{2i};$$

then $b_n(u) = b'_n(u) + ib''_n(u)$ for $u \in \mathbb{R}^N$.

If $b''_n(w) \neq y$ for $w \in U_1$ and a fixed y , then for any point $u^{(2)} \in (U_1 \setminus \{w : b''_n(w) = y\}) \cap \mathbb{R}^N$ the function $x + iy - b_n(u^{(2)})$ does not vanish for $x \in [-1/2, 1/2]$. Therefore, the functions $\Delta_{-1/2 < x < 1/2} \arg^+(x + iy - b_n(u))$ and $\Delta_{-1/2 < x < 1/2} \arg^-(x + iy - b_n(u))$ are continuous and coincide in U outside the set $\{w : b''_n(w) = y\} \cap \mathbb{R}^N$ of zero measure.

Now, suppose that

$$(15) \quad b''_n(w) \equiv y, \quad w \in U_1.$$

The function $x + iy - b_n(u)$, viewed as a function of x , has a unique simple zero. Since the function $b'_n(u)$ is continuous for all u near $u^{(2)}$, we see that $\Delta_{-1/2 < x < 1/2} \arg^+(x + iy - b_n(u)) \equiv -\pi$ and $\Delta_{-1/2 < x < 1/2} \arg^-(x + iy - b_n(u)) \equiv \pi$ in some neighborhood of $u^{(2)}$ (precisely this case was omitted in [12]).

Since the functions $b''_n(w)$ are analytic, identity (15) on an open set implies that the same is true in the entire domain of $b''_n(w)$. We have finitely many functions $b''_n(w)$; therefore, (15) holds true only at finitely many points $y \in (y' - \delta, y' + \delta)$.

Clearly, the set $[0, 2\pi]^N$ can be covered by a finite number of balls where (12) is true. Thus, we obtain the following assertion.

The functions (7) and (8) are Riemann integrable on the set $[0, 2\pi]^N$; moreover, as functions of u , they coincide almost everywhere whenever y does not belong to some discrete set without finite limit points.

Next, by the lemma, the limits

$$(16) \quad \lim_{\beta-\alpha \rightarrow \infty} \frac{1}{\beta-\alpha} \int_{\alpha}^{\beta} \Delta_{-1/2 < t < 1/2} \arg^+ f(t+x+iy) dx,$$

$$(17) \quad \lim_{\beta-\alpha \rightarrow \infty} \frac{1}{\beta-\alpha} \int_{\alpha}^{\beta} \Delta_{-1/2 < t < 1/2} \arg^- f(t+x+iy) dx$$

exist and coincide for all y off a set without finite limit points. Furthermore, we have already proved that the increments

$$\Delta_{0 < x < \beta} \arg^{\pm} f(t+x+iy) = \Delta_{0 < x < \beta} \arg^{\pm} F(x+iy, \mu_1 t, \dots, \mu_N t)$$

are uniformly bounded in $t \in \mathbb{R}$ and $\beta \leq 1/2$ for each fixed y . Therefore,

$$\left| \arg^{\pm} f(t+iy) - \int_{t-1/2}^{t+1/2} \arg^{\pm} f(x+iy) dx \right| = O(1),$$

and

$$\begin{aligned} & \arg^{\pm} f(\beta+iy) - \arg^{\pm} f(\alpha+iy) \\ &= \int_{\beta-1/2}^{\beta+1/2} \arg^{\pm} f(t+iy) dt - \int_{\alpha-1/2}^{\alpha+1/2} \arg^{\pm} f(t+iy) dt + O(1) \\ &= \int_{\alpha}^{\beta} \Delta_{-1/2 < t < 1/2} \arg^{\pm} f(t+x+iy) dx + O(1). \end{aligned}$$

Consequently, the limits (2) and (3) exist and are equal to (16) and (17), respectively. The theorem is proved. \square

Corollary. For the exponential polynomial (1), the density of zeros

$$D(y) = \lim_{T \rightarrow \infty} (2T)^{-1} \text{card}\{x : |x| < T, f(x+iy) = 0\},$$

counted with multiplicities, exists for each line $y \equiv \text{const}$. Moreover, the set $\{y : D(y) \neq 0\}$ is finite in every bounded interval.

Indeed, the definitions of $\arg^{\pm} f$, $c^{\pm}(y)$ and the theorem imply the existence of the limit and the relation $2\pi D(y) = c^-(y) - c^+(y)$.

Finally, we mention that, in [7], Jessen and Tornehave extended their theorem to a class of functions essentially larger than the exponential polynomials (1). In particular, the theorem is valid for convergent series of the form $\sum_{n=n_0}^{\infty} a_n n^{-s}$. Also, note that Doss [4] found a precise form of the term $O(1)$ in (4) for various classes of functions considered in [7].

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