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TWISTED YANGIANS AND MICKELSSON ALGEBRAS. II

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ABSTRACT. A skew analog for the composition of the Cherednik and Drinfeld functors is introduced for twisted Yangians. The definition is based on the skew Howe duality, and originates from the centralizer construction of twisted Yangians due to Olshanskiĭ. Via the new functor, a correspondence is established between intertwining operators on the tensor products of certain modules over twisted Yangians and the extremal cocycle on the hyperoctahedral group.

§0. INTRODUCTION

This paper is a continuation of our work [KN2], which concerned two known functors. The definition of one of these two functors belongs to V. Drinfeld [D2]. Let $\mathfrak{A}_N$ be the *degenerate affine Hecke algebra* corresponding to the general linear group GL_N over a non-Archimedean local field. This is an associative algebra over the field $\mathbb{C}$, and it contains the symmetric group ring $\mathbb{C}\,\mathfrak{S}_N$ as a subalgebra. Let $\mathrm{Y}(\mathfrak{gl}_n)$ be the *Yangian* of the general linear Lie algebra $\mathfrak{gl}_n$. This is a deformation of the universal enveloping algebra of the polynomial current Lie algebra $\mathfrak{gl}_n[u]$ in the class of Hopf algebras [D1]. It contains the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_n)$ as a subalgebra. There is also a homomorphism of associative algebras $\mathrm{Y}(\mathfrak{gl}_n) \to \mathrm{U}(\mathfrak{gl}_n)$ identical on the subalgebra $\mathrm{U}(\mathfrak{gl}_n) \subset \mathrm{Y}(\mathfrak{gl}_n)$. In [D2], for any $\mathfrak{A}_N$-module M, an action of the algebra $\mathrm{Y}(\mathfrak{gl}_n)$ was defined on the vector space $(M \otimes (\mathbb{C}^n)^{\otimes N})^{\mathfrak{S}_N}_{-}$ of the diagonal skew $\mathfrak{S}_N$-invariants in the tensor product of the vector spaces M and $(\mathbb{C}^n)^{\otimes N}$. Thus, we get a functor from the category of all $\mathfrak{A}_N$-modules to the category of $\mathrm{Y}(\mathfrak{gl}_n)$-modules, the *Drinfeld functor*.

In [KN1] we studied the composition of the Drinfeld functor with another functor, introduced by I. Cherednik [C]. That second functor was also studied by T. Arakawa, T. Suzuki, and A. Tsuchiya [A, AS, AST]. For any module U over the Lie algebra $\mathfrak{gl}_l$, an action of the algebra $\mathfrak{A}_N$ can be defined on the tensor product $U \otimes (\mathbb{C}^l)^{\otimes N}$ of $\mathfrak{gl}_l$-modules. This action of $\mathfrak{A}_N$ commutes with the diagonal action of $\mathfrak{gl}_l$ on the tensor product. This yields a functor from the category of all $\mathfrak{gl}_l$-modules to the category of bimodules over $\mathfrak{gl}_l$ and $\mathfrak{A}_N$, the *Cherednik functor*. By applying the Drinfeld functor to the $\mathfrak{A}_N$-module $M = U \otimes (\mathbb{C}^l)^{\otimes N}$, one turns the vector space

$$(U \otimes (\mathbb{C}^l)^{\otimes N} \otimes (\mathbb{C}^n)^{\otimes N})^{\mathfrak{S}_N}_{-} = U \otimes \Lambda^N(\mathbb{C}^l \otimes \mathbb{C}^n)$$

to a $\mathrm{Y}(\mathfrak{gl}_n)$-module. The action of the associative algebra $\mathrm{Y}(\mathfrak{gl}_n)$ on this vector space commutes with the action of $\mathfrak{gl}_l$. By taking the direct sum of these $\mathrm{Y}(\mathfrak{gl}_n)$-modules over $N = 0, 1, \ldots, n$, we turn the space $U \otimes \Lambda(\mathbb{C}^l \otimes \mathbb{C}^n)$ to a $\mathrm{Y}(\mathfrak{gl}_n)$-module. It is also a $\mathfrak{gl}_l$-module; we denote this bimodule by $\mathcal{E}_l(U)$. We identify the exterior algebra $\Lambda(\mathbb{C}^l \otimes \mathbb{C}^n)$ with the Grassmann algebra $\mathcal{G}(\mathbb{C}^l \otimes \mathbb{C}^n)$, and we denote by $\mathcal{GD}(\mathbb{C}^l \otimes \mathbb{C}^n)$ the ring of $\mathbb{C}$-endomorphisms of $\mathcal{G}(\mathbb{C}^l \otimes \mathbb{C}^n)$. The action of the Yangian $\mathrm{Y}(\mathfrak{gl}_n)$ on its module

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$\mathcal{E}_l(U)$ is then determined by a homomorphism $\alpha_l : \mathrm{Y}(\mathfrak{gl}_n) \to \mathrm{U}(\mathfrak{gl}_l) \otimes \mathcal{GD}(\mathbb{C}^l \otimes \mathbb{C}^n)$; see Proposition 1.2 below.

Now, let $\mathfrak{f}_m$ be either the orthogonal Lie algebra $\mathfrak{so}_{2m}$ or the symplectic Lie algebra $\mathfrak{sp}_{2m}$. Our first objective in the present paper is to define analogs of the functor $\mathcal{E}_l$ and of the homomorphism α_l for the Lie algebra $\mathfrak{f}_m$ instead of $\mathfrak{gl}_l$. The role of the Yangian $\mathrm{Y}(\mathfrak{gl}_n)$ is played here by the *twisted Yangian* $\mathrm{Y}(\mathfrak{g}_n)$, which is a right coideal subalgebra of the Hopf algebra $\mathrm{Y}(\mathfrak{gl}_n)$. Here $\mathfrak{g}_n$ is a Lie subalgebra of $\mathfrak{gl}_n$, orthogonal in the case of $\mathfrak{f}_m = \mathfrak{so}_{2m}$ and symplectic if $\mathfrak{f}_m = \mathfrak{sp}_{2m}$; in the latter case, n must be even. Let the superscript $'$ indicate the transposition in $\mathfrak{gl}_n$ relative to the bilinear form on $\mathbb{C}^n$ preserved by the subalgebra $\mathfrak{g}_n \subset \mathfrak{gl}_n$, so that $\mathfrak{g}_n = \{A \in \mathfrak{gl}_n \,|\, A' = -A\}$. As an associative algebra, $\mathrm{Y}(\mathfrak{g}_n)$ is a deformation of the universal enveloping algebra of the *twisted polynomial current Lie algebra*

$$\{A(u) \in \mathfrak{gl}_n[u] \,|\, A'(u) = -A(-u)\}.$$

Twisted Yangians were introduced by Olshanskiĭ [O2]; their structure was studied in [MNO]. In §2 of the present paper, we introduce a homomorphism $\mathrm{Y}(\mathfrak{g}_n) \to \mathrm{U}(\mathfrak{f}_m) \otimes \mathcal{GD}(\mathbb{C}^m \otimes \mathbb{C}^n)$; see our Propositions 2.3 and 2.4. The image of $\mathrm{Y}(\mathfrak{g}_n)$ under this homomorphism commutes with the image of the algebra $\mathrm{U}(\mathfrak{f}_m)$ under its diagonal embedding (2.7) into the tensor product $\mathrm{U}(\mathfrak{f}_m) \otimes \mathcal{GD}(\mathbb{C}^m \otimes \mathbb{C}^n)$; here we use the homomorphism $\zeta_n : \mathrm{U}(\mathfrak{f}_m) \to \mathcal{GD}(\mathbb{C}^m \otimes \mathbb{C}^n)$ defined by (2.6). The twisted Yangian $\mathrm{Y}(\mathfrak{g}_n)$ contains the universal enveloping algebra $\mathrm{U}(\mathfrak{g}_n)$ as a subalgebra. Also, there is a homomorphism $\pi_n : \mathrm{Y}(\mathfrak{g}_n) \to \mathrm{U}(\mathfrak{g}_n)$ identical on the subalgebra $\mathrm{U}(\mathfrak{g}_n) \subset \mathrm{Y}(\mathfrak{g}_n)$. Our results extend the classical theorem [H] stating that the image of $\mathrm{U}(\mathfrak{f}_m)$ in $\mathcal{GD}(\mathbb{C}^m \otimes \mathbb{C}^n)$ under the homomorphism ζ_n consists of all G_n-invariant elements. Here G_n is either the orthogonal or the symplectic group, so that $\mathfrak{g}_n$ is its Lie algebra; the group G_n acts on $\mathcal{GD}(\mathbb{C}^m \otimes \mathbb{C}^n)$ via its natural action on $\mathbb{C}^n$.

In the present paper we prefer to work with a certain central extension $\mathrm{X}(\mathfrak{g}_n)$ of the algebra $\mathrm{Y}(\mathfrak{g}_n)$, called the *extended twisted Yangian*. Central elements $O^{(1)}, O^{(2)}, \ldots$ of the algebra $\mathrm{X}(\mathfrak{g}_n)$ generating the kernel of the canonical homomorphism $\mathrm{X}(\mathfrak{g}_n) \to \mathrm{Y}(\mathfrak{g}_n)$ are given in §1, together with the definitions of $\mathrm{X}(\mathfrak{g}_n)$ and $\mathrm{Y}(\mathfrak{g}_n)$. There is also a homomorphism $\mathrm{X}(\mathfrak{g}_n) \to \mathrm{X}(\mathfrak{g}_n) \otimes \mathrm{Y}(\mathfrak{gl}_n)$. Using it, we turn the tensor product of any modules over the algebras $\mathrm{X}(\mathfrak{g}_n)$ and $\mathrm{Y}(\mathfrak{gl}_n)$ to another module over $\mathrm{X}(\mathfrak{g}_n)$. Moreover, this homomorphism is a coaction of the Hopf algebra $\mathrm{Y}(\mathfrak{gl}_n)$ on the algebra $\mathrm{X}(\mathfrak{g}_n)$. We define a homomorphism $\beta_m : \mathrm{X}(\mathfrak{g}_n) \to \mathrm{U}(\mathfrak{f}_m) \otimes \mathcal{GD}(\mathbb{C}^m \otimes \mathbb{C}^n)$, which is our analog of the homomorphism α_l; see Proposition 2.3. The image of $\mathrm{X}(\mathfrak{g}_n)$ under β_m commutes with the image of the algebra $\mathrm{U}(\mathfrak{f}_m)$ under its embedding (2.7) to $\mathrm{U}(\mathfrak{f}_m) \otimes \mathcal{GD}(\mathbb{C}^m \otimes \mathbb{C}^n)$. The reason why we work with $\mathrm{X}(\mathfrak{g}_n)$ rather than with $\mathrm{Y}(\mathfrak{g}_n)$ is explained in §2.

The generators of the algebra $\mathrm{X}(\mathfrak{g}_n)$ arise as coefficients of certain series $S_{ij}(u)$ in the variable u, where $i, j = 1, \ldots, n$. We define the homomorphism β_m by applying it to the coefficients, and by giving the resulting series $\beta_m(S_{ij}(u))$ with coefficients in $\mathrm{U}(\mathfrak{f}_m) \otimes \mathcal{GD}(\mathbb{C}^m \otimes \mathbb{C}^n)$ explicitly. Then we define another homomorphism

$$\widetilde{\beta}_m : \mathrm{X}(\mathfrak{g}_n) \to \mathrm{U}(\mathfrak{f}_m) \otimes \mathcal{GD}(\mathbb{C}^m \otimes \mathbb{C}^n),$$

which factors through the canonical homomorphism $\mathrm{X}(\mathfrak{g}_n) \to \mathrm{Y}(\mathfrak{g}_n)$. Thus we obtain the homomorphism $\mathrm{Y}(\mathfrak{g}_n) \to \mathrm{U}(\mathfrak{f}_m) \otimes \mathcal{GD}(\mathbb{C}^m \otimes \mathbb{C}^n)$ mentioned above. Every series $\widetilde{\beta}_m(S_{ij}(u))$ is the product of $\beta_m(S_{ij}(u))$ by a certain series with coefficients in $\mathrm{Z}(\mathfrak{f}_m) \otimes 1$, where $\mathrm{Z}(\mathfrak{f}_m)$ is the center of the algebra $\mathrm{U}(\mathfrak{f}_m)$.

The defining relations of the algebra $\mathrm{X}(\mathfrak{g}_n)$ can be written as the *reflection equation* (1.15) on the $(n \times n)$-matrix $S(u)$ whose (i, j) entry is the series $S_{ij}(u)$. This terminology was introduced by physicists; see, e.g., [KS] and the references therein.

Now, let V be any $\mathfrak{f}_m$-module. Using the homomorphism β_m, we turn the vector space $V \otimes \mathcal{G}(\mathbb{C}^m \otimes \mathbb{C}^n)$ into a bimodule over $\mathfrak{f}_m$ and $\mathrm{X}(\mathfrak{g}_n)$. We denote this bimodule by $\mathcal{F}_m(V)$. The functor $\mathcal{F}_m$ is our analog of the functor $\mathcal{E}_l$ for $\mathfrak{f}_m$ instead of $\mathfrak{gl}_l$. When $m = 0$, we set $\mathcal{F}_0(V) = \mathbb{C}$, so that β_0 is the composition of the canonical homomorphism $\mathrm{X}(\mathfrak{g}_n) \to \mathrm{Y}(\mathfrak{g}_n)$ with the restriction of the counit homomorphism $\mathrm{Y}(\mathfrak{gl}_n) \to \mathbb{C}$ to $\mathrm{Y}(\mathfrak{g}_n)$.

Here we show that the functor $\mathcal{F}_m$ shares the three fundamental properties of the functor $\mathcal{E}_l$ considered in [KN2]. The first of these properties of $\mathcal{E}_l$ concerns *parabolic induction* from the direct sum of Lie algebras $\mathfrak{gl}_m \oplus \mathfrak{gl}_l$ to $\mathfrak{gl}_{m+l}$. Let $\mathfrak{p}$ be the maximal parabolic subalgebra of $\mathfrak{gl}_{m+l}$ containing the direct sum $\mathfrak{gl}_m \oplus \mathfrak{gl}_l$. Let $\mathfrak{q} \subset \mathfrak{gl}_{m+l}$ be the Abelian subalgebra with $\mathfrak{gl}_{m+l} = \mathfrak{q} \oplus \mathfrak{p}$. For any $\mathfrak{gl}_m$-module W, let $W \boxtimes U$ be the $\mathfrak{gl}_{m+l}$-module parabolically induced from the $\mathfrak{gl}_m \oplus \mathfrak{gl}_l$-module $W \otimes U$. This is a module induced from the subalgebra $\mathfrak{p}$. Consider the space $\mathcal{E}_{m+l}(W \boxtimes U)_{\mathfrak{q}}$ of $\mathfrak{q}$-coinvariants of the $\mathfrak{gl}_{m+l}$-module $\mathcal{E}_{m+l}(W \boxtimes U)$. This space is a $\mathrm{Y}(\mathfrak{gl}_n)$-module, which also inherits the action of the Lie algebra $\mathfrak{gl}_m \oplus \mathfrak{gl}_l$. The additive group $\mathbb{C}$ acts on the Hopf algebra $\mathrm{Y}(\mathfrak{gl}_n)$ by automorphisms. Let $\mathcal{E}_l^{-z}(U)$ be the $\mathrm{Y}(\mathfrak{gl}_n)$-module obtained from $\mathcal{E}_l(U)$ by pulling it back through the automorphism of $\mathrm{Y}(\mathfrak{gl}_n)$ corresponding to $z \in \mathbb{C}$. The automorphism itself is denoted by τ_z; see (1.2). Thus, the underlying vector space of the $\mathrm{Y}(\mathfrak{gl}_n)$-module $\mathcal{E}_l^{-z}(U)$ is $U \otimes \mathcal{G}(\mathbb{C}^l \otimes \mathbb{C}^n)$, whereon the action of $\mathrm{Y}(\mathfrak{gl}_n)$ is defined by the composition of two homomorphisms,

$$\mathrm{Y}(\mathfrak{gl}_n) \underset{\tau_z}{\longrightarrow} \mathrm{Y}(\mathfrak{gl}_n) \underset{\alpha_l}{\longrightarrow} \mathrm{U}(\mathfrak{gl}_l) \otimes \mathcal{GD}(\mathbb{C}^l \otimes \mathbb{C}^n). \tag{0.1}$$

Here the target algebra acts on $U \otimes \mathcal{G}(\mathbb{C}^l \otimes \mathbb{C}^n)$ by definition. As a $\mathfrak{gl}_l$-module, $\mathcal{E}_l^{-z}(U)$ coincides with $\mathcal{E}_l(U)$. In [KN2] we proved that the bimodule $\mathcal{E}_{m+l}(W \boxtimes U)_{\mathfrak{q}}$ over $\mathrm{Y}(\mathfrak{gl}_n)$ and $\mathfrak{gl}_m \oplus \mathfrak{gl}_l$ is equivalent to $\mathcal{E}_m(W) \otimes \mathcal{E}_l^m(U)$. We use the comultiplication on $\mathrm{Y}(\mathfrak{gl}_n)$.

Our Theorem 3.1 is an analog of this comultiplicative property of $\mathcal{E}_l$. Take the maximal parabolic subalgebra of the Lie algebra $\mathfrak{f}_{m+l}$ containing the direct sum $\mathfrak{f}_m \oplus \mathfrak{gl}_l$; we do not exclude the case of $m = 0$ here. Using that subalgebra, we determine the $\mathfrak{f}_{m+l}$-module $V \boxtimes U$ parabolically induced from the $\mathfrak{f}_m \oplus \mathfrak{gl}_l$-module $V \otimes U$. Consider the space of coinvariants of the $\mathfrak{f}_{m+l}$-module $\mathcal{F}_{m+l}(V \boxtimes U)$ relative to the nilpotent subalgebra of $\mathfrak{f}_{m+l}$ complementary to our parabolic subalgebra. This space is a bimodule over $\mathfrak{f}_m \oplus \mathfrak{gl}_l$ and $\mathrm{X}(\mathfrak{g}_n)$. We prove that this bimodule is essentially equivalent to the tensor product $\mathcal{F}_m(V) \otimes \mathcal{E}_l^z(U)$ with $z = m - \frac{1}{2}$ for $\mathfrak{f}_m = \mathfrak{so}_{2m}$, and $z = m + \frac{1}{2}$ for $\mathfrak{f}_m = \mathfrak{sp}_{2m}$. More precisely, the underlying vector space of the $\mathrm{X}(\mathfrak{g}_n)$-module $\mathcal{F}_m(V) \otimes \mathcal{E}_l^z(U)$ is

$$V \otimes \mathcal{G}(\mathbb{C}^m \otimes \mathbb{C}^n) \otimes U \otimes \mathcal{G}(\mathbb{C}^l \otimes \mathbb{C}^n), \tag{0.2}$$

whereon the action of $\mathrm{X}(\mathfrak{g}_n)$ is defined by the composition of two homomorphisms,

$$\mathrm{X}(\mathfrak{g}_n) \to \mathrm{X}(\mathfrak{g}_n) \otimes \mathrm{Y}(\mathfrak{gl}_n) \to \mathrm{U}(\mathfrak{f}_m) \otimes \mathcal{GD}(\mathbb{C}^m \otimes \mathbb{C}^n) \otimes \mathrm{U}(\mathfrak{gl}_l) \otimes \mathcal{GD}(\mathbb{C}^l \otimes \mathbb{C}^n).$$

Here the first homomorphism is the coaction of $\mathrm{Y}(\mathfrak{gl}_n)$ on $\mathrm{X}(\mathfrak{g}_n)$, while the second is the tensor product of the homomorphisms $\beta_m : \mathrm{X}(\mathfrak{g}_n) \to \mathrm{U}(\mathfrak{f}_m) \otimes \mathcal{GD}(\mathbb{C}^m \otimes \mathbb{C}^n)$ and

$$\alpha_l\, \tau_{-z} : \mathrm{Y}(\mathfrak{gl}_n) \to \mathrm{U}(\mathfrak{gl}_l) \otimes \mathcal{GD}(\mathbb{C}^l \otimes \mathbb{C}^n);$$

see (0.1). By multiplying the image of $S_{ij}(u) \in \mathrm{X}(\mathfrak{g}_n)[[u^{-1}]]$ under this composition by a certain series with coefficients in the subalgebra

$$1 \otimes 1 \otimes \mathrm{Z}(\mathfrak{gl}_l) \otimes 1 \subset \mathrm{U}(\mathfrak{f}_m) \otimes \mathcal{GD}(\mathbb{C}^m \otimes \mathbb{C}^n) \otimes \mathrm{U}(\mathfrak{gl}_l) \otimes \mathcal{GD}(\mathbb{C}^l \otimes \mathbb{C}^n),$$

we get another homomorphism $\mathrm{X}(\mathfrak{g}_n) \to \mathrm{U}(\mathfrak{gl}_l) \otimes \mathcal{GD}(\mathbb{C}^l \otimes \mathbb{C}^n)$. The latter homomorphism determines another action of $\mathrm{X}(\mathfrak{g}_n)$ on the vector space (0.2). Theorem 3.1 states that this action is equivalent to the action of $\mathrm{X}(\mathfrak{g}_n)$ on the space of coinvariants of $\mathcal{F}_{m+l}(V \boxtimes U)$. Moreover, the actions of the direct summand $\mathfrak{f}_m$ of $\mathfrak{f}_m \oplus \mathfrak{gl}_l$ on $\mathcal{F}_m(V) \otimes \mathcal{E}_l^z(U)$ and on the space of coinvariants of $\mathcal{F}_{m+l}(V \boxtimes U)$ are also equivalent, while the actions of the direct summand $\mathfrak{gl}_l$ differ only by the automorphism (3.6) of the Lie algebra $\mathfrak{gl}_l$. Hence, Theorem 3.1 describes the first fundamental property of the functor $\mathcal{F}_m$.

Now we discuss the second fundamental property of $\mathcal{F}_m$. In [TV], Tarasov and Varchenko established a correspondence between canonical intertwining operators on the l-fold tensor products of certain $\mathrm{Y}(\mathfrak{gl}_n)$-modules, and the *extremal cocycle* on the Weyl group $\mathfrak{S}_l$ of the reductive Lie algebra $\mathfrak{gl}_l$ defined by Zhelobenko [Z]. In [TV], each of l tensor factors was obtained from one of $\mathfrak{gl}_n$-modules $\mathrm{S}^N(\mathbb{C}^n)$ by pulling back through the homomorphism $\mathrm{Y}(\mathfrak{gl}_n) \to \mathrm{U}(\mathfrak{gl}_n)$ and then back through one of the automorphisms $\tau_z : \mathrm{Y}(\mathfrak{gl}_n) \to \mathrm{Y}(\mathfrak{gl}_n)$. Here $\mathrm{S}^N(\mathbb{C}^n)$ is the Nth symmetric power of the vector space $\mathbb{C}^n$, while the homomorphism $\mathrm{Y}(\mathfrak{gl}_n) \to \mathrm{U}(\mathfrak{gl}_n)$ is defined by (1.4). In [KN1] we gave a representation-theoretic explanation of that correspondence from [TV], by employing the theory of Mickelsson algebras [M1, M2] as developed in [KO].

For any $N \in \{1, \ldots, n\}$ and any $z \in \mathbb{C}$, we denote by P_z^N the $\mathrm{Y}(\mathfrak{gl}_n)$-module obtained by pulling back the action of $\mathrm{U}(\mathfrak{gl}_n)$ on the subspace of $\mathcal{G}(\mathbb{C}^n)$ of degree N through the homomorphism $\mathrm{Y}(\mathfrak{gl}_n) \to \mathrm{U}(\mathfrak{gl}_n)$ and then through the automorphism τ_{-z} of $\mathrm{Y}(\mathfrak{gl}_n)$. The action of the algebra $\mathrm{Y}(\mathfrak{gl}_n)$ on P_z^N is defined by the composition of homomorphisms

$$\mathrm{Y}(\mathfrak{gl}_n) \underset{\tau_{-z}}{\longrightarrow} \mathrm{Y}(\mathfrak{gl}_n) \to \mathrm{U}(\mathfrak{gl}_n) \to \mathcal{GD}(\mathbb{C}^n). \tag{0.3}$$

Here the second homomorphism is that defined by (1.4); the algebra $\mathcal{GD}(\mathbb{C}^n)$ acts on $\mathcal{G}(\mathbb{C}^n)$ naturally. Using the functor $\mathcal{E}_l$, in [KN2] we established a correspondence between intertwining operators on the l-fold tensor products of modules of the form P_z^N and the same extremal cocycle on $\mathfrak{S}_l$ as considered in [KN1]. This is an “antisymmetric ” version of the correspondence first established in [TV]. The parameters z corresponding to the l tensor factors are in general position, that is, their differences do not belong to $\mathbb{Z}$. Then each of the tensor products is irreducible as a $\mathrm{Y}(\mathfrak{gl}_n)$-module [NT]. Hence, the intertwining operators between them are unique up to factors from $\mathbb{C}$.

In the present paper we show that the functor $\mathcal{F}_m$ plays a role similar to that of $\mathcal{E}_l$, when the Lie algebra $\mathfrak{gl}_l$ is replaced by $\mathfrak{f}_m$. Namely, we establish a correspondence between intertwining operators of certain $\mathrm{X}(\mathfrak{g}_n)$-modules, and the extremal cocycle on the hyperoctahedral group $\mathfrak{H}_m$ corresponding to the reductive Lie algebra $\mathfrak{f}_m$. Here $\mathfrak{H}_m$ is regarded as the Weyl group of $\mathfrak{f}_m = \mathfrak{sp}_{2m}$, and as an extension of the Weyl group of $\mathfrak{f}_m = \mathfrak{so}_{2m}$ by a Dynkin diagram automorphism. In both cases, the definition of the extremal cocycle is essentially due to Zhelobenko [Z]. However, the original extremal cocycle has been defined on the Weyl group of $\mathfrak{f}_m$, which in the case of $\mathfrak{f}_m = \mathfrak{so}_{2m}$ is only a subgroup of $\mathfrak{H}_m$ of index 2. An extension of the original definition to the entire group $\mathfrak{H}_m$ was given in [KN3]. All necessary details on the extremal cocycle corresponding to $\mathfrak{f}_m$ are also reviewed in §4 below.

The twisted Yangian $\mathrm{Y}(\mathfrak{g}_n)$ is determined by a distinguished involutive automorphism (1.11) of the algebra $\mathrm{Y}(\mathfrak{gl}_n)$. The automorphism (1.11) corresponds to the automorphism

$$A(u) \mapsto -A'(-u)$$

of the Lie algebra $\mathfrak{gl}_n[u]$ if the algebra $\mathrm{Y}(\mathfrak{gl}_n)$ is regarded as a deformation of the universal enveloping algebra of $\mathfrak{gl}_n[u]$. By pulling the $\mathrm{Y}(\mathfrak{gl}_n)$-module P_z^N back through the automorphism (1.11), we get another $\mathrm{Y}(\mathfrak{gl}_n)$-module, denoted by P_z^{-N}. The underlying vector space of P_z^{-N} consists of elements of $\mathcal{G}(\mathbb{C}^n)$ of degree N, whereon the action of $\mathrm{Y}(\mathfrak{gl}_n)$ is defined by the composition of four homomorphisms

$$\mathrm{Y}(\mathfrak{gl}_n) \to \mathrm{Y}(\mathfrak{gl}_n) \underset{\tau_{-z}}{\longrightarrow} \mathrm{Y}(\mathfrak{gl}_n) \to \mathrm{U}(\mathfrak{gl}_n) \to \mathcal{GD}(\mathbb{C}^n).$$

Here the first map is the automorphism (1.11); the other three are the same as in (0.3).

Now take any $\nu_1,\ldots,\nu_m\in\{1,\ldots,n\}$ and any $z_1,\ldots,z_m\in\mathbb{C}$ such that $z_a-z_b\notin\mathbb{Z}$ and $z_a+z_b\notin\mathbb{Z}$ whenever $a\neq b$. If $\mathfrak{f}_m=\mathfrak{sp}_{2m}$, we also assume that $2z_a\notin\mathbb{Z}$ for any a. The hyperoctahedral group $\mathfrak{H}_m$ can be realized as the group of all permutations σ of $-m,\ldots,-1,1,\ldots,m$ such that $\sigma(-c)=-\sigma(c)$ for any c. In §5 of the present paper, we show how the value of the extremal cocycle for the Lie algebra $\mathfrak{f}_m$ at an element $\sigma\in\mathfrak{H}_m$ determines an intertwining operator of $\mathrm{X}(\mathfrak{g}_n)$-modules

$$P^{\nu_m}_{z_m}\otimes\cdots\otimes P^{\nu_1}_{z_1}\to P^{\delta_m\widetilde{\nu}_m}_{\widetilde{z}_m}\otimes\cdots\otimes P^{\delta_1\widetilde{\nu}_1}_{\widetilde{z}_1},\tag{0.4}$$

where

$$\widetilde{\nu}_a=\nu_{|\sigma^{-1}(a)|},\quad \widetilde{z}_a=z_{|\sigma^{-1}(a)|}\quad\text{and}\quad \delta_a=\operatorname{sign}\sigma^{-1}(a)\tag{0.5}$$

for each $a=1,\ldots,m$. The tensor products in (0.4) are those of $\mathrm{Y}(\mathfrak{gl}_n)$-modules. By restricting both tensor products to the subalgebra $\mathrm{Y}(\mathfrak{g}_n)\subset\mathrm{Y}(\mathfrak{gl}_n)$ and by pulling the restrictions back through the canonical homomorphism $\mathrm{X}(\mathfrak{g}_n)\to\mathrm{Y}(\mathfrak{g}_n)$, both tensor products in (0.4) become $\mathrm{X}(\mathfrak{g}_n)$-modules. Thus, the actions of the algebra $\mathrm{X}(\mathfrak{g}_n)$ on both tensor products in (0.4) are obtained by using the composition

$$\mathrm{X}(\mathfrak{g}_n)\to\mathrm{Y}(\mathfrak{g}_n)\to\mathrm{Y}(\mathfrak{gl}_n)\to\mathrm{Y}(\mathfrak{gl}_n)^{\otimes n}.$$

Here the first map is the canonical homomorphism, the second is the embedding defining $\mathrm{Y}(\mathfrak{g}_n)$, while the third is m-fold comultiplication. It was proved in [MN] that, under our assumptions on $z_1,\ldots,z_m$, the two tensor products in (0.4) are irreducible $\mathrm{X}(\mathfrak{g}_n)$-modules equivalent to each other. Hence, an intertwining operator between them is unique up to a factor from $\mathbb{C}$. For our operator, this factor is determined by Proposition 5.9.

To obtain our intertwining operator (0.4), we use the theory of Mickelsson algebras, just as we did in [KN1, KN2]. Our particular Mickelsson algebra is determined by the pair formed by the tensor product $\mathrm{U}(\mathfrak{f}_m)\otimes\mathcal{GD}(\mathbb{C}^m\otimes\mathbb{C}^n)$ and by its subalgebra $\mathrm{U}(\mathfrak{f}_m)$ relative to the embedding (2.7). The extended twisted Yangian $\mathrm{X}(\mathfrak{g}_n)$ appears naturally here, because its image relative to β_m commutes with the image of $\mathrm{U}(\mathfrak{f}_m)$ in the tensor product. Another expression for an intertwining operator (0.4) was given in [N].

In §2 we choose a triangular decomposition (2.17) of the Lie algebra $\mathfrak{f}_m$ into a direct sum of a Cartan subalgebra $\mathfrak{h}$ and two maximal nilpotent subalgebras $\mathfrak{n},\mathfrak{n}'$. For any formal power series $f(u)$ in u^{-1} with coefficients in $\mathbb{C}$ and leading term 1, the assignments (1.17) define an automorphism of the algebra $\mathrm{X}(\mathfrak{g}_n)$. Up to pulling it back through such an automorphism, the source $\mathrm{X}(\mathfrak{g}_n)$-module in (0.4) arises as the space of $\mathfrak{n}$-coinvariants of weight λ for the $\mathfrak{f}_m$-module $\mathcal{F}_m(M_\mu)$, where M_μ is the Verma module over $\mathfrak{f}_m$ with the highest vector of weight μ annihilated by the action of the subalgebra $\mathfrak{n}'\subset\mathfrak{f}_m$. Here the weights λ and μ relative to the Cartan subalgebra $\mathfrak{h}$ are determined by the parameters $\nu_1,\ldots,\nu_m$ and $z_1,\ldots,z_m$ occurring in (0.4). We denote the space of $\mathfrak{n}$-coinvariants of weight λ by $\mathcal{F}_m(M_\mu)^\lambda_{\mathfrak{n}}$. The algebra $\mathrm{X}(\mathfrak{g}_n)$ acts on the latter space, because the action of $\mathrm{X}(\mathfrak{g}_n)$ on $\mathcal{F}_m(M_\mu)$ commutes with that of $\mathfrak{f}_m$. We prove that the above action of the algebra $\mathrm{X}(\mathfrak{g}_n)$ on the source tensor product in (0.4) is equivalent to the action on the vector space of $\mathcal{F}_m(M_\mu)^\lambda_{\mathfrak{n}}$ defined by the composition

$$\mathrm{X}(\mathfrak{g}_n)\to\mathrm{X}(\mathfrak{g}_n)\to\operatorname{End}(\mathcal{F}_m(M_\mu)).\tag{0.6}$$

Here the first map is the automorphism (1.17) with $f(u)^{-1}$ equal to the product (5.24). The second map is the defining homomorphism of the $\mathrm{X}(\mathfrak{g}_n)$-module $\mathcal{F}_m(M_\mu)$.

To get the target $\mathrm{X}(\mathfrak{g}_n)$-module in (0.4), we generalize our definition of the functor $\mathcal{F}_m$. At the beginning of §5, for any sequence $\delta=(\delta_1,\ldots,\delta_m)$ of m elements of the set $\{1,-1\}$, we define a functor $\mathcal{F}_\delta$ with the same source and target categories as the functor $\mathcal{F}_m$. Moreover, for any $\mathfrak{f}_m$-modulef V, the underlying vector spaces of the bimodules

$\mathcal{F}_\delta(V)$ and $\mathcal{F}_m(V)$ are the same, that is, $V\otimes\mathcal{G}(\mathbb{C}^m\otimes\mathbb{C}^n)$. The actions of $\mathfrak{f}_m$ and $\mathrm{X}(\mathfrak{g}_n)$ on $\mathcal{F}_\delta(V)$ are obtained by pushing forward the defining homomorphisms

$$\zeta_n\,:\,\mathrm{U}(\mathfrak{f}_m)\to\mathcal{GD}(\mathbb{C}^m\otimes\mathbb{C}^n)\quad\text{and}\quad\beta_m\,:\,\mathrm{X}(\mathfrak{g}_n)\to\mathrm{U}(\mathfrak{f}_m)\otimes\mathcal{GD}(\mathbb{C}^m\otimes\mathbb{C}^n)$$

through a certain automorphism ϖ of the ring $\mathcal{GD}(\mathbb{C}^m\otimes\mathbb{C}^n)$ depending on δ. Namely, the automorphism ϖ is defined by the assignments (5.1). Thus, to define the functor $\mathcal{F}_\delta$, we use the compositions $\varpi\,\zeta_n$ and $(1\otimes\varpi)\,\beta_m$ instead of the homomorphisms ζ_n and β_m, respectively. In particular, we have $\mathcal{F}_\delta(V)=\mathcal{F}_m(V)$ for the sequence $\delta=(1,\ldots,1)$.

Up to pulling it back through an automorphism of the form (1.17), the target $\mathrm{X}(\mathfrak{g}_n)$-module in (0.4) arises as the space of $\mathfrak{n}$-coinvariants of weight $\sigma\circ\lambda$ for the $\mathfrak{f}_m$-module $\mathcal{F}_\delta(M_{\sigma\circ\mu})$. The sequence $\delta=(\delta_1,\ldots,\delta_m)$ is as defined in (0.5), and the symbol $\circ$ here indicates the *shifted action* of the group $\mathfrak{H}_m$ on the weights of $\mathfrak{h}$. Our Proposition 5.4 states that the action of the algebra $\mathrm{X}(\mathfrak{g}_n)$ on the target tensor product in (0.4) is equivalent to the action on the vector space of $\mathcal{F}_\delta(M_{\sigma\circ\mu})_{\mathfrak{n}}^{\sigma\circ\lambda}$ defined by the composition

$$\mathrm{X}(\mathfrak{g}_n)\to\mathrm{X}(\mathfrak{g}_n)\to\mathrm{End}(\mathcal{F}_m(M_{\sigma\circ\mu})). \tag{0.7}$$

Here the first map is the automorphism (1.17) with $f(u)^{-1}$ equal to the product (5.24). The second map is the defining homomorphism of the $\mathrm{X}(\mathfrak{g}_n)$-module $\mathcal{F}_m(M_{\sigma\circ\mu})$.

In §5 we show that the value of the extremal cocycle for the Lie algebra $\mathfrak{f}_m$ at the element $\sigma\in\mathfrak{H}_m$ determines an intertwining operator of $\mathrm{X}(\mathfrak{g}_n)$-modules

$$\mathcal{F}_m(M_\mu)_{\mathfrak{n}}^{\lambda}\to\mathcal{F}_\delta(M_{\sigma\circ\mu})_{\mathfrak{n}}^{\sigma\circ\lambda}. \tag{0.8}$$

The product (5.24) does not depend on the element $\sigma\in\mathfrak{H}_m$, so that the automorphisms (1.17) of the algebra $\mathrm{X}(\mathfrak{g}_n)$ in (0.6) and (0.7) are the same. Hence, by replacing the source and the target $\mathrm{X}(\mathfrak{g}_n)$-modules by their equivalent modules, we obtain our intertwining operator (0.4). The role played by the functor $\mathcal{F}_m$ in this construction of the operator (0.4) is the second fundamental property of that functor.

The third fundamental property of the functor $\mathcal{E}_l$ considered in [KN2] is its relationship with the *centralizer construction* of the Yangian $\mathrm{Y}(\mathfrak{gl}_n)$ proposed by Olshanskiĭ [O1]. For any two irreducible polynomial modules U and U' over the Lie algebra $\mathfrak{gl}_l$, the results of [O1] provide an action of $\mathrm{Y}(\mathfrak{gl}_n)$ on the vector space

$$\mathrm{Hom}_{\,\mathfrak{gl}_l}(U',U\otimes\mathcal{G}(\mathbb{C}^l\otimes\mathbb{C}^n)). \tag{0.9}$$

Moreover, this action is irreducible. In [KN2] we proved that the same action of $\mathrm{Y}(\mathfrak{gl}_n)$ on the vector space (0.9) is obtained when the target $\mathfrak{gl}_l$-module $U\otimes\mathcal{G}(\mathbb{C}^l\otimes\mathbb{C}^n)$ in (0.9) is regarded as the bimodule $\mathcal{E}_l(U)$ over $\mathrm{Y}(\mathfrak{gl}_n)$ and $\mathfrak{gl}_l$.

There is a centralizer construction of $\mathrm{Y}(\mathfrak{g}_n)$, again due to G. Olshanskiĭ [O2]; see also [MO] and §6 below. That construction served as a motivation for introducing the twisted Yangians. For any irreducible finite-dimensional modules V and V' of the Lie algebra $\mathfrak{f}_m$, the results of [O2] provide an action of the algebra $\mathrm{X}(\mathfrak{g}_n)$ on the vector space

$$\mathrm{Hom}_{\,\mathfrak{f}_m}(V',V\otimes\mathcal{G}(\mathbb{C}^m\otimes\mathbb{C}^n)). \tag{0.10}$$

The group G_n also acts on this vector space, via its natural action on $\mathbb{C}^n$.

If $\mathfrak{g}_n$ is an orthogonal Lie algebra, the space (0.10) is irreducible under the joint action of $\mathrm{X}(\mathfrak{g}_n)$ and G_n. If $\mathfrak{g}_n$ is symplectic, (0.10) is irreducible under the action of the $\mathrm{X}(\mathfrak{g}_n)$ alone. Our Theorem 6.1 states that the action of $\mathrm{X}(\mathfrak{g}_n)$ on (0.10) is essentially the same as the action obtained from the bimodule $\mathcal{F}_m(V)=V\otimes\mathcal{G}(\mathbb{C}^m\otimes\mathbb{C}^n)$ of $\mathrm{X}(\mathfrak{g}_n)$ and $\mathfrak{f}_m$. More precisely, the action of $\mathrm{X}(\mathfrak{g}_n)$ on the vector space (0.10) provided by [O2] can also be obtained from an action of $\mathrm{X}(\mathfrak{g}_n)$ on the target $\mathfrak{f}_m$-module $V\otimes\mathcal{G}(\mathbb{C}^m\otimes\mathbb{C}^n)$ in (0.10). The latter action is not exactly that on $\mathcal{F}_m(V)$, but is defined by the composition

$$\mathrm{X}(\mathfrak{g}_n)\to\mathrm{X}(\mathfrak{g}_n)\underset{\beta_m}{\longrightarrow}\mathrm{U}(\mathfrak{f}_m)\otimes\mathcal{GD}(\mathbb{C}^m\otimes\mathbb{C}^n),$$

where the first map is the automorphism (1.17) with $f(u)$ given by (6.6). The second map is the defining homomorphism of the $\mathrm{X}(\mathfrak{g}_n)$-module $\mathcal{F}_m(V)$. This third property of $\mathcal{F}_m$ was the origin of our definition of that functor. Thus, we have two different descriptions of the same action of $\mathrm{X}(\mathfrak{g}_n)$ on (0.10). Another two still different descriptions of the same action of $\mathrm{X}(\mathfrak{g}_n)$ on the vector space (0.10) were provided in [M] and [N], respectively.

The functor $\mathcal{F}_m$ of the present paper is an "antisymmetric " version of a functor introduced in [KN3]. Here the exterior algebra $\Lambda(\mathbb{C}^m \otimes \mathbb{C}^n)$ replaces the symmetric algebra $\mathrm{S}(\mathbb{C}^m \otimes \mathbb{C}^n)$ in [KN3]. Analogs of the three fundamental properties of $\mathcal{F}_m$ were also given in [KN3].

§1. Twisted Yangians

Let G_n be one of the complex Lie groups O_n and Sp_n. We regard G_n as the subgroup of the general linear Lie group GL_n, preserving a nondegenerate bilinear form $\langle\ ,\ \rangle$ on the vector space $\mathbb{C}^n$. This form is symmetric in the case where $G_n = O_n$, and alternating in the case of $G_n = Sp_n$. In the latter case, n must be even. We always assume that the integer n is positive. Throughout this paper, we shall use the following convention. Whenever the double sign $\pm$ or $\mp$ appears, the upper sign corresponds to the case of $G_n = O_n$, while the lower sign corresponds to the case of $G_n = Sp_n$.

Let i be any of the indices $1, \dots, n$. If i is even, put $\tilde{\imath} = i - 1$. If i is odd and $i < n$, put $\tilde{\imath} = i + 1$. Finally, if $i = n$ and n is odd, put $\tilde{\imath} = i$. Let $e_1, \dots, e_n$ be the vectors of the standard basis in $\mathbb{C}^n$. Choose a bilinear form on $\mathbb{C}^n$ so that for any two basis vectors e_i and e_j we have $\langle e_i, e_j \rangle = \theta_i\, \delta_{\tilde{\imath} j}$, where $\theta_i = 1$ or $\theta_i = (-1)^{i-1}$ in the case of the symmetric or alternating form.

Let $E_{ij} \in \mathrm{End}(\mathbb{C}^n)$ be the standard matrix units. We also regard these matrix units as basis elements of the general linear Lie algebra $\mathfrak{gl}_n$. Let $\mathfrak{g}_n$ be the Lie algebra of the group G_n, so that $\mathfrak{g}_n = \mathfrak{so}_n$ or $\mathfrak{g}_n = \mathfrak{sp}_n$ in the case of the symmetric or alternating form on $\mathbb{C}^n$. The Lie subalgebra $\mathfrak{g}_n \subset \mathfrak{gl}_n$ is spanned by the elements $E_{ij} - \theta_i \theta_j E_{\tilde{\jmath}\tilde{\imath}}$.

Take the *Yangian* $\mathrm{Y}(\mathfrak{gl}_n)$ of the Lie algebra $\mathfrak{gl}_n$. The unital associative algebra $\mathrm{Y}(\mathfrak{gl}_n)$ over $\mathbb{C}$ has a family of generators $T_{ij}^{(1)}, T_{ij}^{(2)}, \dots$, where $i, j = 1, \dots, n$. Defining relations for these generators can be written by using the series

$$T_{ij}(u) = \delta_{ij} + T_{ij}^{(1)} u^{-1} + T_{ij}^{(2)} u^{-2} + \cdots ,$$

where u is a formal parameter. Let v be another formal parameter. Then the defining relations in the associative algebra $\mathrm{Y}(\mathfrak{gl}_n)$ can be written as

$$(u - v)\, [T_{ij}(u), T_{kl}(v)] = T_{kj}(u) T_{il}(v) - T_{kj}(v) T_{il}(u). \tag{1.1}$$

The algebra $\mathrm{Y}(\mathfrak{gl}_n)$ is commutative if $n = 1$. By (1.1), for any $z \in \mathbb{C}$ the assignments

$$\tau_z \,:\, T_{ij}(u) \mapsto T_{ij}(u - z) \tag{1.2}$$

determine an automorphism τ_z of the algebra $\mathrm{Y}(\mathfrak{gl}_n)$. Here each of the formal power series $T_{ij}(u - z)$ in $(u - z)^{-1}$ should be reexpanded in u^{-1}, and every assignment (1.2) is a correspondence between the respective coefficients of series in u^{-1}. Relations (1.1) also show that for any formal power series $g(u)$ in u^{-1} with coefficients in $\mathbb{C}$ and leading term 1, the assignments

$$T_{ij}(u) \mapsto g(u)\, T_{ij}(u) \tag{1.3}$$

determine an automorphism of the algebra $\mathrm{Y}(\mathfrak{gl}_n)$. Using (1.1), one can directly verify that the assignments

$$T_{ij}(u) \mapsto \delta_{ij} + E_{ij} u^{-1} \tag{1.4}$$

determine a homomorphism of unital associative algebras $\mathrm{Y}(\mathfrak{gl}_n) \to \mathrm{U}(\mathfrak{gl}_n)$.

There is an embedding $\mathrm{U}(\mathfrak{gl}_n) \to \mathrm{Y}(\mathfrak{gl}_n)$, defined by the mapping $E_{ij} \mapsto T_{ij}^{(1)}$. So, $\mathrm{Y}(\mathfrak{gl}_n)$ contains the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_n)$ as a subalgebra. The homomorphism (1.4) is identical on the subalgebra $\mathrm{U}(\mathfrak{gl}_n) \subset \mathrm{Y}(\mathfrak{gl}_n)$.

Let $T(u)$ be the $(n \times n)$-matrix whose (i,j)-entry is the series $T_{ij}(u)$. Relations (1.1) can be rewritten by using the *Yang R-matrix* . This is the $(n^2 \times n^2)$-matrix

$$R(u) = u - \sum_{i,j=1}^{n} E_{ij} \otimes E_{ji}, \tag{1.5}$$

where the tensor factors E_{ij} and E_{ji} are regarded as $(n \times n)$-matrices. Note that

$$R(u)\, R(-u) = 1 - u^2. \tag{1.6}$$

Take $(n^2 \times n^2)$-matrices whose entries are series with coefficients in $\mathrm{Y}(\mathfrak{gl}_n)$,

$$T_1(u) = T(u) \otimes 1 \quad \text{and} \quad T_2(v) = 1 \otimes T(v).$$

The collection of relations (1.1) for all possible indices i, j, k, l can be written as

$$R(u-v)\, T_1(u)\, T_2(v) = T_2(v)\, T_1(u)\, R(u-v). \tag{1.7}$$

Using this form of the defining relations together with (1.6), one shows that

$$T(u) \mapsto T(-u)^{-1} \tag{1.8}$$

determines an involutive automorphism of the algebra $\mathrm{Y}(\mathfrak{gl}_n)$. Here each entry of the inverse matrix $T(-u)^{-1}$ is a formal power series in u^{-1} with coefficients in the algebra $\mathrm{Y}(\mathfrak{gl}_n)$, and the assignment (1.8) is as a correspondence between the respective matrix entries.

The Yangian $\mathrm{Y}(\mathfrak{gl}_n)$ is a Hopf algebra over the field $\mathbb{C}$. The comultiplication Δ : $\mathrm{Y}(\mathfrak{gl}_n) \to \mathrm{Y}(\mathfrak{gl}_n) \otimes \mathrm{Y}(\mathfrak{gl}_n)$ is defined by the assignment

$$\Delta \,:\, T_{ij}(u) \mapsto \sum_{k=1}^{n} T_{ik}(u) \otimes T_{kj}(u). \tag{1.9}$$

When taking tensor products of $\mathrm{Y}(\mathfrak{gl}_n)$-modules, we use the comultiplication (1.9). The counit homomorphism $\mathrm{Y}(\mathfrak{gl}_n) \to \mathbb{C}$ is defined by the assignment $T_{ij}(u) \mapsto \delta_{ij}$. The antipodal map $\mathrm{Y}(\mathfrak{gl}_n) \to \mathrm{Y}(\mathfrak{gl}_n)$ is defined by the assignment $T(u) \mapsto T(u)^{-1}$. This map is an antiautomorphism of the associative algebra $\mathrm{Y}(\mathfrak{gl}_n)$. For further details on the Hopf algebra structure on $\mathrm{Y}(\mathfrak{gl}_n)$, see [MNO, Chapter 1].

Let $T'(u)$ be the transpose to the matrix $T(u)$ relative to the form $\langle\ ,\ \rangle$ on $\mathbb{C}^n$. The (i,j)-entry of the matrix $T'(u)$ is $\theta_i \theta_j T_{\tilde{j}\tilde{\imath}}(u)$. Define the $(n^2 \times n^2)$-matrices

$$T'_1(u) = T'(u) \otimes 1 \quad \text{and} \quad T'_2(v) = 1 \otimes T'(v).$$

Note that the Yang R-matrix (1.5) is invariant under applying the transposition relative to $\langle\ ,\ \rangle$ to both tensor factors. Hence, relation (1.7) implies that

$$\begin{gathered} T'_1(u)\, T'_2(v)\, R(u-v) = R(u-v)\, T'_2(v)\, T'_1(u), \\ R(u-v)\, T'_1(-u)\, T'_2(-v) = T'_2(-v)\, T'_1(-u)\, R(u-v). \end{gathered} \tag{1.10}$$

To obtain the latter relation, we have used (1.6). By comparing (1.7) and (1.10), an involutive automorphism of the algebra $\mathrm{Y}(\mathfrak{gl}_n)$ can be defined by the assignment

$$T(u) \mapsto T'(-u). \tag{1.11}$$

This assignment is understood as a correspondence between respective matrix entries.

Now take the product $T'(-u)\, T(u)$. The (i,j)-entry of this matrix is the series

$$\sum_{k=1}^{n} \theta_i \theta_k\, T_{\tilde{k}\tilde{\imath}}(-u)\, T_{kj}(u). \tag{1.12}$$

The *twisted Yangian* corresponding to the form $\langle\ ,\ \rangle$ is the subalgebra of $\mathrm{Y}(\mathfrak{gl}_n)$ generated by the coefficients of all series (1.12). We denote this subalgebra by $\mathrm{Y}(\mathfrak{g}_n)$.

To give defining relations for these generators of $\mathrm{Y}(\mathfrak{g}_n)$, we introduce the *extended twisted Yangian* $\mathrm{X}(\mathfrak{g}_n)$. The unital associative algebra $\mathrm{X}(\mathfrak{g}_n)$ has a family of generators $S_{ij}^{(1)}, S_{ij}^{(2)}, \ldots$, where $i, j = 1, \ldots, n$. Put

$$S_{ij}(u) = \delta_{ij} + S_{ij}^{(1)} u^{-1} + S_{ij}^{(2)} u^{-2} + \cdots$$

and let $S(u)$ be the $(n \times n)$-matrix whose (i,j)-entry is the series $S_{ij}(u)$. Also, we introduce the $(n^2 \times n^2)$-matrix

$$R'(u) = u - \sum_{i,j=1}^{n} \theta_i\, \theta_j\, E_{ij} \otimes E_{\tilde{\imath}\tilde{\jmath}}, \tag{1.13}$$

which is obtained from the Yang R-matrix (1.5) by applying the transposition relative to the form $\langle\ ,\ \rangle$ on $\mathbb{C}^n$ to any of the two tensor factors. Note the relation

$$R'(u)\, R'(n-u) = u(n-u). \tag{1.14}$$

Take $(n^2 \times n^2)$-matrices whose entries are series with coefficients in the algebra $\mathrm{X}(\mathfrak{g}_n)$,

$$S_1(u) = S(u) \otimes 1 \quad \text{and} \quad S_2(v) = 1 \otimes S(v).$$

Defining relations in the algebra $\mathrm{X}(\mathfrak{g}_n)$ can then be written as a single matrix relation

$$R(u-v)\, S_1(u)\, R'(-u-v)\, S_2(v) = S_2(v)\, R'(-u-v)\, S_1(u)\, R(u-v). \tag{1.15}$$

This is equivalent to the collection of relations

$$\begin{aligned}(u^2-v^2)\,[S_{ij}(u), S_{kl}(v)] = {} & (u+v)(S_{kj}(u)\, S_{il}(v) - S_{kj}(v)\, S_{il}(u)) \\ & \mp\, (u-v)(\theta_k \theta_j\, S_{i\tilde{k}}(u)\, S_{\tilde{\jmath}l}(v) - \theta_i \theta_l\, S_{k\tilde{\imath}}(v)\, S_{\tilde{l}j}(u)) \\ & \pm\, \theta_i \theta_j\, (S_{k\tilde{\imath}}(u)\, S_{\tilde{\jmath}l}(v) - S_{k\tilde{\imath}}(v)\, S_{\tilde{\jmath}l}(u)).\end{aligned} \tag{1.16}$$

As in the case of (1.3), this collection of relations shows that, for any formal power series $f(u)$ in u^{-1} with coefficients in $\mathbb{C}$ and leading term 1, the assignments

$$S_{ij}(u) \mapsto f(u)\, S_{ij}(u) \tag{1.17}$$

determine an automorphism of the algebra $\mathrm{X}(\mathfrak{g}_n)$. See [KN3, §1] for the proof of the following statement.

Proposition 1.1. *A homomorphism* $\mathrm{X}(\mathfrak{g}_n) \to \mathrm{Y}(\mathfrak{g}_n)$ *can be defined by assigning*

$$S(u) \mapsto T'(-u)\, T(u). \tag{1.18}$$

By definition, the homomorphism (1.18) is surjective. Next, the algebra $\mathrm{X}(\mathfrak{g}_n)$ has a distinguished family of central elements. Indeed, by dividing each side of identity (1.15) by $S_2(v)$ from the left and from the right and then setting $v = -u$, we get

$$R'(0)\, S_1(u)\, R(2u)\, S_2(-u)^{-1} = S_2(-u)^{-1} R(2u)\, S_1(u)\, R'(0).$$

The rank of the matrix $R'(0)$ equals 1. So the identity last displayed implies the existence of a formal power series $O(u)$ in u^{-1} with coefficients in $\mathrm{X}(\mathfrak{g}_n)$ and leading term 1, and such that

$$R'(0)\, S_1(u)\, R(2u)\, S_2(-u)^{-1} = (2u \mp 1)\, O(u)\, R'(0). \tag{1.19}$$

By [MNO, Theorem 6.3] all coefficients of the series $O(u)$ belong to the center of $\mathrm{X}(\mathfrak{g}_n)$. We write

$$O(u) = 1 + O^{(1)} u^{-1} + O^{(2)} u^{-2} + \cdots .$$

By [MNO, Theorem 6.4], the kernel of the homomorphism (1.18) coincides with the (two-sided) ideal generated by the central elements $O^{(1)}, O^{(2)}, \ldots$ defined as the coefficients of the series $O(u)$. Using (1.6), from (1.19) we deduce the relation $O(u)O(-u) = 1$.

Thus, the twisted Yangian $\mathrm{Y}(\mathfrak{g}_n)$ can be defined as the associative algebra with the generators $S_{ij}^{(1)}, S_{ij}^{(2)}, \ldots$ that satisfy the relation $O(u) = 1$ and the *reflection equation* (1.15). For more details on the definition of the algebra $\mathrm{Y}(\mathfrak{g}_n)$, see [MNO, Chapter 3].

In the present paper we need the algebra $\mathrm{X}(\mathfrak{g}_n)$, which is determined by (1.15) alone, because this algebra admits an analog of the automorphism (1.8) of the Yangian $\mathrm{Y}(\mathfrak{gl}_n)$. Indeed, using (1.15) together with (1.6) and (1.14), we see that the assignment

$$\omega_n \,:\, S(u) \mapsto S(-u-n/2)^{-1} \tag{1.20}$$

determines an involutive automorphism ω_n of $\mathrm{X}(\mathfrak{g}_n)$. However, ω_n does not determine an automorphism of the algebra $\mathrm{Y}(\mathfrak{g}_n)$, because the map ω_n does not preserve the ideal of $\mathrm{X}(\mathfrak{g}_n)$ generated by the elements $O^{(1)}, O^{(2)}, \ldots$; see [MNO, Subsection 6.6]. Note that, by multiplying (1.19) on the right by $S_2(-u)$, the relation $O(u) = 1$ can be rewritten as

$$S'(u) = S(-u) \pm \frac{S(u) - S(-u)}{2u}, \tag{1.21}$$

where $S'(u)$ is the transpose of the matrix $S(u)$ relative to the form $\langle\ ,\ \rangle$ on $\mathbb{C}^n$.

The definition (1.19) of the series $O(u)$ implies that the assignment (1.17) determines an automorphism of the quotient algebra $\mathrm{Y}(\mathfrak{g}_n)$ of $\mathrm{X}(\mathfrak{g}_n)$ if and only if $f(u) = f(-u)$. If $z \neq 0$, the automorphism τ_z of $\mathrm{Y}(\mathfrak{gl}_n)$ does not preserve the subalgebra $\mathrm{Y}(\mathfrak{g}_n) \subset \mathrm{Y}(\mathfrak{gl}_n)$. There is no analog of the automorphism τ_z for the algebra $\mathrm{X}(\mathfrak{g}_n)$.

However, the homomorphism $\mathrm{Y}(\mathfrak{gl}_n) \to \mathrm{U}(\mathfrak{gl}_n)$ defined by (1.4) admits an analog. Namely, we can define a homomorphism $\pi_n \,:\, \mathrm{X}(\mathfrak{g}_n) \to \mathrm{U}(\mathfrak{g}_n)$ by the assignments

$$\pi_n \,:\, S_{ij}(u) \;\mapsto\; \delta_{ij} + \frac{E_{ij} - \theta_i \theta_j E_{\tilde{\jmath}\tilde{\imath}}}{u \pm \frac{1}{2}}. \tag{1.22}$$

This can be proved by using the defining relations (1.16); see [MNO, Proposition 3.11]. Furthermore, the central elements $O^{(1)}, O^{(2)}, \ldots$ of $\mathrm{X}(\mathfrak{g}_n)$ belong to the kernel of π_n. Thus, π_n factors through the homomorphism $\mathrm{X}(\mathfrak{g}_n) \to \mathrm{Y}(\mathfrak{g}_n)$ defined by (1.18).

Next, there is an embedding $\mathrm{U}(\mathfrak{g}_n) \to \mathrm{Y}(\mathfrak{g}_n)$ defined by mapping each element $E_{ij} - \theta_i \theta_j E_{\tilde{\jmath}\tilde{\imath}} \in \mathfrak{g}_n$ to the coefficient of u^{-1} in the series (1.12). Hence, $\mathrm{Y}(\mathfrak{g}_n)$ contains the universal enveloping algebra $\mathrm{U}(\mathfrak{g}_n)$ as a subalgebra. Clearly, the homomorphism $\mathrm{Y}(\mathfrak{g}_n) \to \mathrm{U}(\mathfrak{g}_n)$ corresponding to π_n is the identity map on the subalgebra $\mathrm{U}(\mathfrak{g}_n) \subset \mathrm{Y}(\mathfrak{g}_n)$.

For any positive integer l, consider the vector space $\mathbb{C}^l$ and the corresponding Lie algebra $\mathfrak{gl}_l$. Let $E_{ab} \in \mathrm{End}(\mathbb{C}^l)$ with $a, b = 1, \ldots, l$ be the standard matrix units. Regarding these matrix units as generators of the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_l)$, we introduce the $(l \times l)$-matrix E whose (a, b)-entry is the generator E_{ab}. Denote by E' the $(l \times l)$-matrix whose (a, b)-entry is the generator E_{ba}. Then consider the inverse matrix $(u - E')^{-1}$. Its (a, b)-entry $(u - E')^{-1}_{ab}$ is a formal power series in u^{-1} with the leading term $\delta_{ab}\, u^{-1}$ and with coefficients in the algebra $\mathrm{U}(\mathfrak{gl}_l)$.

Take the tensor product of vector spaces $\mathbb{C}^l \otimes \mathbb{C}^n$. Let x_{ai} with $a = 1, \ldots, l$ and $i = 1, \ldots, n$ be the standard coordinate functions on $\mathbb{C}^l \otimes \mathbb{C}^n$. Consider the *Grassmann algebra* $\mathcal{G}(\mathbb{C}^l \otimes \mathbb{C}^n)$. It is generated by the elements x_{ai} subject to the anticommutation relations $x_{ai}\, x_{bj} = -x_{bj}\, x_{ai}$ for all indices $a, b = 1, \ldots, l$ and $i, j = 1, \ldots, n$. We shall denote the operator of left multiplication by x_{ai} on $\mathcal{G}(\mathbb{C}^l \otimes \mathbb{C}^n)$ by the same symbol. Let ∂_{ai} be the operator of left derivation on $\mathcal{G}(\mathbb{C}^l \otimes \mathbb{C}^n)$ corresponding to the variable x_{ai}, also called the *inner multiplication* in $\mathcal{G}(\mathbb{C}^l \otimes \mathbb{C}^n)$ corresponding to x_{ai}.

The ring of $\mathbb{C}$-endomorphisms of $\mathcal{G}(\mathbb{C}^l \otimes \mathbb{C}^n)$ is generated by all operators x_{ai} and ∂_{ai}; see, e.g., [H, Appendix 2.3]. This ring will be denoted by $\mathcal{GD}(\mathbb{C}^l \otimes \mathbb{C}^n)$. In this ring, we have

$$x_{ai}\,\partial_{bj}+\partial_{bj}\,x_{ai}=\delta_{ab}\,\delta_{ij}. \tag{1.23}$$

Hence, the ring $\mathcal{GD}(\mathbb{C}^l\otimes\mathbb{C}^n)$ is isomorphic to the Clifford algebra corresponding to the direct sum of the vector space $\mathbb{C}^l\otimes\mathbb{C}^n$ with its dual.

The Lie algebra $\mathfrak{gl}_l$ acts on the vector space $\mathcal{G}(\mathbb{C}^l\otimes\mathbb{C}^n)$ so that the generator E_{ab} acts as the operator

$$\sum_{k=1}^{n} x_{ak}\,\partial_{bk}. \tag{1.24}$$

Denote by A_l the tensor product of associative algebras $\mathrm{U}(\mathfrak{gl}_l)\otimes\mathcal{GD}(\mathbb{C}^l\otimes\mathbb{C}^n)$. We have an embedding $\mathrm{U}(\mathfrak{gl}_l)\to\mathrm{A}_l$ defined for $a,b=1,\dots,l$ by the mappings

$$E_{ab}\mapsto E_{ab}\otimes 1+\sum_{k=1}^{n}1\otimes x_{ak}\,\partial_{bk}. \tag{1.25}$$

The following proposition was proved in [KN2, §1]; see also [A, §3].

Proposition 1.2. (i) *A homomorphism* $\alpha_l:\mathrm{Y}(\mathfrak{gl}_n)\to\mathrm{A}_l$ *can be defined by*

$$\alpha_l\,:\,T_{ij}(u)\mapsto\,\delta_{ij}+\sum_{a,b=1}^{l}(u-E')^{-1}_{ab}\otimes x_{ai}\,\partial_{bj}. \tag{1.26}$$

(ii) *The image of* $\mathrm{Y}(\mathfrak{gl}_n)$ *in* A_l *relative to this homomorphism commutes with the image of* $\mathrm{U}(\mathfrak{gl}_l)$ *in* A_l *relative to the embedding* (1.25).

Note that

$$\alpha_l\,:\,T^{(1)}_{ij}\,\mapsto\,\sum_{c=1}^{l}1\otimes x_{ci}\,\partial_{cj}.$$

Hence, the restriction of α_l to the subalgebra $\mathrm{U}(\mathfrak{gl}_n)\subset\mathrm{Y}(\mathfrak{gl}_n)$ corresponds to the natural action of the Lie algebra $\mathfrak{gl}_n$ on $\mathcal{G}(\mathbb{C}^l\otimes\mathbb{C}^n)$.

Denote by $Z(u)$ the trace of the inverse matrix $(u+E)^{-1}$, so that

$$Z(u)=\sum_{c=1}^{l}(u+E)^{-1}_{cc}. \tag{1.27}$$

Then $Z(u)$ is a formal power series in u^{-1} with coefficients in the algebra $\mathrm{U}(\mathfrak{gl}_l)$. It is well known that these coefficients actually belong to the center $\mathrm{Z}(\mathfrak{gl}_l)$ of $\mathrm{U}(\mathfrak{gl}_l)$. Note that the leading term of this series is lu^{-1}.

We choose the Borel subalgebra $\mathfrak{b}$ of the Lie algebra $\mathfrak{gl}_l$ spanned by the elements E_{ab}, where $a\le b$. Let $\mathfrak{t}\subset\mathfrak{b}$ be the Cartan subalgebra of $\mathfrak{gl}_l$ with the basis $(E_{11},\dots,E_{ll})$. Consider the corresponding *Harish-Chandra homomorphism* $\varphi_l:\mathrm{U}(\mathfrak{gl}_l)^{\mathfrak{t}}\to\mathrm{U}(\mathfrak{t})$. By definition, for any $\mathfrak{t}$-invariant element $X\in\mathrm{U}(\mathfrak{gl}_l)$, the difference $X-\varphi_l(X)$ belongs to the left ideal of $\mathrm{U}(\mathfrak{gl}_l)$ generated by the elements E_{ab}, where $a<b$. The restriction of the homomorphism φ_l to $\mathrm{Z}(\mathfrak{gl}_l)\subset\mathrm{U}(\mathfrak{gl}_l)^{\mathfrak{t}}$ is injective. It is well known that

$$1+\varphi_l(Z(u))=\prod_{a=1}^{l}\Big(1+\frac{1}{u+l-a+E_{aa}}\Big); \tag{1.28}$$

see, e.g., [PP, Theorem 3]. For the proof of the next lemma, see [KN3, §1], where the parameter u should be replaced by $-u$.

Lemma 1.3. *For any indices* $a,d=1,\dots,l$, *we have*

$$(u+E)^{-1}_{da}=(1+Z(u))\,(u+l+E')^{-1}_{ad}.$$

Now, let U be a module of the Lie algebra $\mathfrak{gl}_l$. Using the homomorphism (1.26), we can turn the tensor product of $\mathfrak{gl}_l$-modules $U \otimes \mathcal{G}(\mathbb{C}^l \otimes \mathbb{C}^n)$ to a bimodule over $\mathfrak{gl}_l$ and $\mathrm{Y}(\mathfrak{gl}_n)$. This bimodule is denoted by $\mathcal{E}_l(U)$. More generally, for $z \in \mathbb{C}$, denote by $\mathcal{E}_l^z(U)$ the $\mathrm{Y}(\mathfrak{gl}_n)$-module obtained from $\mathcal{E}_l(U)$ via pull-back through the automorphism τ_{-z} of $\mathrm{Y}(\mathfrak{gl}_n)$; see (1.2). It is determined by the homomorphism $\mathrm{Y}(\mathfrak{gl}_n) \to \mathrm{A}_l$ such that

$$T_{ij}(u) \mapsto \delta_{ij} + \sum_{a,b=1}^{l} (u + z - E')^{-1}_{ab} \otimes x_{ai}\, \partial_{bj}$$

for every $i, j = 1, \ldots, n$. As a $\mathfrak{gl}_l$-module, $\mathcal{E}_l^z(U)$ coincides with $\mathcal{E}_l(U)$ by definition. In the next section we shall introduce analogs of the homomorphism (1.25) and of the correspondence $U \mapsto \mathcal{E}_l(U)$ for the twisted Yangian $\mathrm{Y}(\mathfrak{g}_n)$ instead of $\mathrm{Y}(\mathfrak{gl}_n)$.

§2. Howe duality

We shall work with one of the pairs $(\mathfrak{so}_{2m}, O_n)$ and $(\mathfrak{sp}_{2m}, Sp_n)$. The second member of the pair will be the Lie group G_n. The first member will be the Lie algebra $\mathfrak{f}_m$ defined below. These pairs arise in the context of the skew Howe duality; see [H, Subsection 4.3].

Take the even-dimensional vector space $\mathbb{C}^{2m}$. Equip $\mathbb{C}^{2m}$ with a nondegenerate bilinear form, symmetric in the case of $G_n = O_n$, and alternating in the case of $G_n = Sp_n$. Let $\mathfrak{f}_m$ be the subalgebra of the general Lie algebra $\mathfrak{gl}_{2m}$ preserving our bilinear form on $\mathbb{C}^{2m}$. We have $\mathfrak{f}_m = \mathfrak{so}_{2m}$ or $\mathfrak{f}_m = \mathfrak{sp}_{2m}$ (respectively) in the case of a symmetric or an alternating form on $\mathbb{C}^{2m}$.

We label the standard basis vectors of $\mathbb{C}^{2m}$ by the numbers $-m, \ldots, -1, 1, \ldots, m$. Let $E_{ab} \in \mathrm{End}(\mathbb{C}^{2m})$ be the standard matrix units, where the indices a, b run through these numbers. These matrix units will also be viewed as basis elements of $\mathfrak{gl}_{2m}$. Put

$$\varepsilon_{ab} = 1 \quad \text{or} \quad \varepsilon_{ab} = \operatorname{sgn} a \cdot \operatorname{sgn} b \tag{2.1}$$

(respectively) in the case of a symmetric or an alternating form on $\mathbb{C}^{2m}$. Then choose the form on $\mathbb{C}^{2m}$ so that the Lie subalgebra $\mathfrak{f}_m \subset \mathfrak{gl}_{2m}$ is spanned by the elements

$$F_{ab} = E_{ab} - \varepsilon_{ab}\, E_{-b,-a}. \tag{2.2}$$

In the universal enveloping algebra $\mathrm{U}(\mathfrak{f}_m)$ we have the commutation relations

$$[F_{ab}, F_{cd}] = \delta_{cb}\, F_{ad} - \delta_{ad}\, F_{cb} - \varepsilon_{ab}\, \delta_{c,-a}\, F_{-b,d} + \varepsilon_{ab}\, \delta_{-b,d}\, F_{c,-a}. \tag{2.3}$$

Let F be the $(2m \times 2m)$-matrix whose (a, b)-entry is the element F_{ab}. Denote by $F(u)$ the inverse to the matrix $u + F$. Let $F_{ab}(u)$ be the (a, b)-entry of the inverse matrix. Any of these entries may be regarded as a formal power series in u^{-1} with coefficients in the algebra $\mathrm{U}(\mathfrak{f}_m)$. Then

$$F_{ab}(u) = \delta_{ab}\, u^{-1} + \sum_{s=0}^{\infty} \sum_{|c_1|, \ldots, |c_s|=1}^{m} (-1)^{s+1}\, F_{ac_1} F_{c_1c_2} \cdots F_{c_{s-1}c_s} F_{c_sb}\, u^{-s-2}. \tag{2.4}$$

If $s = 0$, the sum over $c_1, \ldots, c_s$ in (2.4) is understood as $-F_{ab}\, u^{-2}$. We denote by $W(u)$ the trace of the matrix $F(u)$, that is,

$$W(u) = \sum_{|c|=1}^{m} F_{cc}(u). \tag{2.5}$$

The coefficients of the series $W(u)$ belong to the center $\mathrm{Z}(\mathfrak{f}_m)$ of the algebra $\mathrm{U}(\mathfrak{f}_m)$.

In what follows, the upper signs in $\pm$ and $\mp$ correspond to the case of a symmetric form on $\mathbb{C}^{2m}$, while the lower signs correspond to the case of an alternating form on $\mathbb{C}^{2m}$. In these cases we also have a symmetric or alternating form on $\mathbb{C}^n$, respectively. Thus, the choice of signs in $\pm$ and $\mp$ here agrees with our general convention on double signs.

Let $F'(u)$ be the transpose to $F(u)$ relative to our bilinear form on $\mathbb{C}^{2m}$, so that the (a,b)-entry $F'_{ab}(u)$ of the matrix $F'(u)$ equals $\varepsilon_{ab}\,F_{-b,-a}(u)$. For the proof of the next proposition, see [KN3, §2], where u should now be replaced by $-u$.

Proposition 2.1. *We have equality of* $(2m\times 2m)$*-matrices:*

$$-F'(u)=\Bigl(W(u)\mp\frac{1}{2u+2m\mp 1}+1\Bigr)\,F(-u-2m\pm 1)\pm\frac{F(u)}{2u+2m\mp 1}.$$

Corollary 2.2. *We have*

$$\Bigl(W(u)\mp\frac{1}{2u+2m\mp 1}+1\Bigr)\Bigl(W(-u-2m\pm 1)\pm\frac{1}{2u+2m\mp 1}+1\Bigr)$$
$$=1-\frac{1}{(2u+2m\mp 1)^2}.$$

On the space $\mathbb{C}^m\otimes\mathbb{C}^n$, we have the coordinate functions x_{ai}, where $a=1,\dots,m$ and $i=1,\dots,n$. Consider the Grassmann algebra $\mathcal{G}(\mathbb{C}^m\otimes\mathbb{C}^n)$ corresponding to this vector space. We shall denote the operator of left multiplication by x_{ai} on $\mathcal{G}(\mathbb{C}^m\otimes\mathbb{C}^n)$ by the same symbol. Let ∂_{ai} be the left derivation on $\mathcal{G}(\mathbb{C}^m\otimes\mathbb{C}^n)$ relative to x_{ai}. There is an action of $\mathfrak{f}_m$ on $\mathcal{G}(\mathbb{C}^m\otimes\mathbb{C}^n)$ that commutes with the natural action of the group G_n. The corresponding homomorphism $\zeta_n:\mathrm{U}(\mathfrak{f}_m)\to\mathcal{GD}(\mathbb{C}^m\otimes\mathbb{C}^n)$ is defined by the following mappings for $a,b=1,\dots,m$:

$$\zeta_n\;:\;F_{ab}\mapsto\;-\delta_{ab}\,n/2+\sum_{k=1}^{n}x_{ak}\,\partial_{bk},$$

(2.6)

$$F_{a,-b}\mapsto\sum_{k=1}^{n}\theta_k\,x_{a\tilde k}\,x_{bk},\quad F_{-a,b}\;\mapsto\sum_{k=1}^{n}\theta_k\,\partial_{ak}\,\partial_{b\tilde k}.$$

Here the homomorphism property can be verified by using relations (2.3). Moreover, the image of the homomorphism ζ_n coincides with the subring of all G_n-invariants in $\mathcal{GD}(\mathbb{C}^m\otimes\mathbb{C}^n)$; see [H, Subsections 3.8.7 and 4.3.3]. Let B_m be the tensor product of associative algebras $\mathrm{U}(\mathfrak{f}_m)\otimes\mathcal{GD}(\mathbb{C}^m\otimes\mathbb{C}^n)$. Take the embedding $\mathrm{U}(\mathfrak{f}_m)\to\mathrm{B}_m$ defined by

$$X\;\mapsto\;X\otimes 1+1\otimes\zeta_n(X)\quad\text{for each}\quad X\in\mathfrak{f}_m. \tag{2.7}$$

Proposition 2.3. (i) *A homomorphism* $\beta_m:\mathrm{X}(\mathfrak{g}_n)\to\mathrm{B}_m$ *can be defined so that the series* $S_{ij}(u)$ *is mapped to the following series with coefficients in the algebra* B_m*:*

$$\delta_{ij}+\sum_{a,b=1}^{m}\bigl(F_{-a,-b}(u\pm\tfrac12-m)\otimes x_{ai}\,\partial_{bj}+F_{-a,b}(u\pm\tfrac12-m)\otimes\theta_j\,x_{ai}\,x_{b\tilde j} \tag{2.8}$$
$$+F_{a,-b}(u\pm\tfrac12-m)\otimes\theta_i\,\partial_{a\tilde i}\,\partial_{bj}+F_{ab}(u\pm\tfrac12-m)\otimes\theta_i\,\theta_j\,\partial_{a\tilde i}\,x_{b\tilde j}\bigr).$$

(ii) *The image of* $\mathrm{X}(\mathfrak{g}_n)$ *in* B_m *relative to this homomorphism commutes with the image of* $\mathrm{U}(\mathfrak{f}_m)$ *in* B_m *relative to the embedding* (2.7)*.*

Proposition 2.3 can be proved by direct calculation using the defining relations (1.16). That calculation is omitted here. In §6 we shall give a more conceptual proof of the proposition. Now, let the indices c and d run through the sequence $-m,\dots,-1,1,\dots,m$. For $c<0$ we put $p_{ci}=x_{-c,i}$ and $q_{ci}=\partial_{-c,i}$. For $c>0$ we put $p_{ci}=\theta_i\,\partial_{c\tilde i}$ and $q_{ci}=\theta_i\,x_{c\tilde i}$. Then our definition of the homomorphism β_m can be written as

$$\beta_m\;:\;S_{ij}(u)\;\mapsto\;\delta_{ij}+\sum_{|c|,|d|=1}^{m}F_{cd}(u\pm\tfrac12-m)\otimes p_{ci}\,q_{dj}, \tag{2.9}$$

as in (1.26). Moreover, by the definition (2.6) we have

$$\zeta_n: \ F_{cd} \mapsto -\delta_{cd}\, n/2 + \sum_{k=1}^{n} q_{ck}\, p_{dk}. \tag{2.10}$$

Using (2.5), we define a formal power series $\overline{W}(u)$ in u^{-1} with coefficients in the center $\mathrm{Z}(\mathfrak{f}_m)$ of the algebra $\mathrm{U}(\mathfrak{f}_m)$ by the equation

$$\Bigl(1 \mp \frac{1}{2u}\Bigr)\, \overline{W}(u) = W(u \pm \tfrac{1}{2} - m).$$

By Corollary 2.2,

$$(\overline{W}(u) + 1)\,(\overline{W}(-u) + 1) = 1.$$

Hence, there is a formal power series $\widetilde{W}(u)$ in u^{-1} with coefficients in $\mathrm{Z}(\mathfrak{f}_m)$, with leading term 1, and such that

$$\widetilde{W}(-u)\, \widetilde{W}(u)^{-1} = 1 + \overline{W}(u). \tag{2.11}$$

The series $\widetilde{W}(u)$ is not unique. But its coefficient at u^{-1} is always $-m$, because the leading term of the series $W(u)$ is $2mu^{-1}$. Let $\widetilde{\beta}_m$ be the homomorphism $\mathrm{X}(\mathfrak{g}_n) \to \mathrm{B}_m$ defined by assigning to $S_{ij}(u)$ the series (2.8) multiplied by

$$\widetilde{W}(u) \otimes 1 \in \mathrm{B}_m[[u^{-1}]]. \tag{2.12}$$

The homomorphism property of $\widetilde{\beta}_m$ follows from part (i) of Proposition 2.3; see also the defining relations (1.16). Part (ii) implies that the image of $\widetilde{\beta}_m$ commutes with the image of $\mathrm{U}(\mathfrak{f}_m)$ in the algebra B_m relative to the embedding (2.7).

Proposition 2.4. *The elements $O^{(1)}, O^{(2)}, \ldots$ of $\mathrm{X}(\mathfrak{g}_n)$ belong to the kernel of $\widetilde{\beta}_m$.*

Proof. Let $\widetilde{S}_{ij}(u)$ denote the product of the series (2.8) and (2.12). Using the equivalent presentation (1.21) of the relation $O(u) = 1$, we see that it suffices to prove the identity

$$\theta_i\, \theta_j\, \widetilde{S}_{\,\widetilde{j}\,\widetilde{\imath}}(u) = \widetilde{S}_{ij}(-u) \pm \frac{\widetilde{S}_{ij}(u) - \widetilde{S}_{ij}(-u)}{2u} \tag{2.13}$$

for any $i, j = 1, \ldots, n$. By the definition of the series $\widetilde{W}(u)$, we have

$$\widetilde{W}(u)\,(1 + W(u \pm \tfrac{1}{2} - m)) = \widetilde{W}(-u) \pm \frac{\widetilde{W}(u) - \widetilde{W}(-u)}{2u}. \tag{2.14}$$

Next, we introduce the $(2m \times 2m)$-matrix

$$\widetilde{F}(u) = \widetilde{W}(u)\, F(u \pm \tfrac{1}{2} - m) \tag{2.15}$$

and its transpose $\widetilde{F}'(u)$ relative to our bilinear form on $\mathbb{C}^{2m}$. By Proposition 2.1,

$$\widetilde{F}'(u) = -\widetilde{F}(-u) \mp \frac{\widetilde{F}(u) - \widetilde{F}(-u)}{2u}. \tag{2.16}$$

Changing the indices i, j in (2.8) to $\tilde{\jmath}, \tilde{\imath}$ (respectively), and multiplying the resulting series by $\theta_i\,\theta_j$, we get

$$
\begin{aligned}
\delta_{ij} + \sum_{a,b=1}^{m} \big(F_{-a,-b}(u \pm \tfrac12 - m) \otimes \theta_i\,\theta_j\,x_{a\tilde{\jmath}}\,\partial_{b\tilde{\imath}} \pm F_{-a,b}(u \pm \tfrac12 - m) \otimes \theta_j\,x_{a\tilde{\jmath}}\,x_{bi}&\\
\pm F_{a,-b}(u \pm \tfrac12 - m) \otimes \theta_i\,\partial_{aj}\,\partial_{b\tilde{\imath}} + F_{ab}(u \pm \tfrac12 - m) \otimes \partial_{aj}\,x_{bi}\big)&\\
= (1 + W(u \pm \tfrac12 + m)) \otimes \delta_{ij}&\\
+ \sum_{a,b=1}^{m} \big(- F_{-b,-a}(u \pm \tfrac12 - m) \otimes \theta_i\,\theta_j\,\partial_{a\tilde{\imath}}\,x_{b\tilde{\jmath}} \mp F_{-b,a}(u \pm \tfrac12 - m) \otimes \theta_j\,x_{ai}\,x_{b\tilde{\jmath}}&\\
\mp F_{b,-a}(u \pm \tfrac12 - m) \otimes \theta_i\,\partial_{a\tilde{\imath}}\,\partial_{bj} - F_{ba}(u \pm \tfrac12 - m) \otimes x_{ai}\,\partial_{bj}\big)&\\
= (1 + W(u \pm \tfrac12 - m)) \otimes \delta_{ij}&\\
- \sum_{a,b=1}^{m} \big(F'_{ab}(u \pm \tfrac12 - m) \otimes \theta_i\,\theta_j\,\partial_{a\tilde{\imath}}\,x_{b\tilde{\jmath}} + F'_{-a,b}(u \pm \tfrac12 - m) \otimes \theta_j\,x_{ai}\,x_{b\tilde{\jmath}}&\\
+ F'_{a,-b}(u \pm \tfrac12 - m) \otimes \theta_i\,\partial_{a\tilde{\imath}}\,\partial_{bj} + F'_{-a,-b}(u \pm \tfrac12 - m) \otimes x_{ai}\,\partial_{bj}\big).&
\end{aligned}
$$

Multiplying the expression in the last three lines by $\widetilde{W}(u) \otimes 1$ and using the definition (2.15), we get

$$
\begin{aligned}
\widetilde{W}(u)\,(1 + W(u \pm \tfrac12 - m)) \otimes \delta_{ij}&\\
- \sum_{a,b=1}^{m} \big(\widetilde{F}'_{ab}(u) \otimes \theta_i\,\theta_j\,\partial_{a\tilde{\imath}}\,x_{b\tilde{\jmath}} + \widetilde{F}'_{-a,b}(u) \otimes \theta_j\,x_{ai}\,x_{b\tilde{\jmath}}&\\
+ \widetilde{F}'_{a,-b}(u) \otimes \theta_i\,\partial_{a\tilde{\imath}}\,\partial_{bj} + \widetilde{F}'_{-a,-b}(u) \otimes x_{ai}\,\partial_{bj}\big).&
\end{aligned}
$$

Now, the required formula (2.13) follows from (2.14) and (2.16). ☐

So, the homomorphism $\widetilde{\beta}_m : \mathrm{X}(\mathfrak{g}_n) \to \mathrm{B}_m$ factors through a homomorphism $\mathrm{Y}(\mathfrak{g}_n) \to \mathrm{B}_m$. This is an analog of the homomorphism (1.26) for the twisted Yangian $\mathrm{Y}(\mathfrak{g}_n)$ instead of $\mathrm{Y}(\mathfrak{gl}_n)$. Recall that

$$\widetilde{W}(u) = 1 - m u^{-1} + \cdots\,,$$

so that

$$
\begin{aligned}
\widetilde{\beta}_m \,:\, S^{(1)}_{ij} \mapsto\ & -m\,\delta_{ij} + \sum_{c=1}^{m} (1 \otimes x_{ci}\,\partial_{cj} + 1 \otimes \theta_i\,\theta_j\,\partial_{c\tilde{\imath}}\,x_{c\tilde{\jmath}})\\
&= \sum_{c=1}^{m} (1 \otimes x_{ci}\,\partial_{cj} - 1 \otimes \theta_i\,\theta_j\,x_{c\tilde{\jmath}}\,\partial_{c\tilde{\imath}}).
\end{aligned}
$$

Thus, for any formal power series $\widetilde{W}(u)$ in u^{-1} that has its coefficients in $\mathrm{Z}(\mathfrak{f}_m)$, has the leading term 1, and satisfies (2.11), the restriction of the homomorphism $\mathrm{Y}(\mathfrak{g}_n) \to \mathrm{B}_m$ to the subalgebra $\mathrm{U}(\mathfrak{g}_n) \subset \mathrm{Y}(\mathfrak{g}_n)$ corresponds to the natural action of the Lie algebra $\mathfrak{g}_n$ on the vector space $\mathcal{G}(\mathbb{C}^m \otimes \mathbb{C}^n)$.

The series $\widetilde{W}(u)$ is not unique, and it will be more convenient for us to work with the homomorphism $\beta_m : \mathrm{X}(\mathfrak{g}_n) \to \mathrm{B}_m$ defined in Proposition 2.3. Using this homomorphism and the action of the Lie algebra $\mathfrak{f}_m$ on $\mathcal{G}(\mathbb{C}^m \otimes \mathbb{C}^n)$ as defined by (2.6), for an arbitrary $\mathfrak{f}_m$-module V, we can turn the tensor product $V \otimes \mathcal{G}(\mathbb{C}^m \otimes \mathbb{C}^n)$ to a bimodule over $\mathfrak{f}_m$ and $\mathrm{X}(\mathfrak{g}_n)$. This bimodule will be denoted by $\mathcal{F}_m(V)$.

Consider the *triangular decomposition* of the Lie algebra $\mathfrak{f}_m$,

$$\mathfrak{f}_m = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}', \tag{2.17}$$

where $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{f}_m$ with the basis $(F_{-m,-m}, \ldots, F_{-1,-1})$. Next, $\mathfrak{n}$ and $\mathfrak{n}'$ are the nilpotent subalgebras of $\mathfrak{f}_m$ spanned by the elements F_{ab} with $a > b$ and $a < b$, respectively; here the indices a, b can be positive or negative. For each $\mathfrak{f}_m$-module V, we denote by $V_{\mathfrak{n}}$ the vector space $V/\mathfrak{n} \cdot V$ of coinvariants of the action of the subalgebra $\mathfrak{n} \subset \mathfrak{f}_m$ on V. The Cartan subalgebra $\mathfrak{h} \subset \mathfrak{f}_m$ acts on the vector space $V_{\mathfrak{n}}$.

Now consider the bimodule $\mathcal{F}_m(V)$. The action of $\mathrm{X}(\mathfrak{g}_n)$ on this bimodule commutes with the action of the Lie algebra $\mathfrak{f}_m$, and hence, with the action of the subalgebra $\mathfrak{n} \subset \mathfrak{f}_m$. Therefore, the space $\mathcal{F}_m(V)_{\mathfrak{n}}$ of coinvariants of the action of $\mathfrak{n}$ is a quotient of the $\mathrm{X}(\mathfrak{g}_n)$-module $\mathcal{F}_m(V)$. Thus, we get a functor from the category of all $\mathfrak{f}_m$-modules to the category of bimodules over $\mathfrak{h}$ and $\mathrm{X}(\mathfrak{g}_n)$,

$$V \mapsto \mathcal{F}_m(V)_{\mathfrak{n}} = (V \otimes \mathcal{G}(\mathbb{C}^m \otimes \mathbb{C}^n))_{\mathfrak{n}}. \tag{2.18}$$

The assignments $E_{ab} \mapsto F_{ab}$ for all $a, b = 1, \ldots, m$ determine a Lie algebra embedding $\mathfrak{gl}_m \to \mathfrak{f}_m$; see relations (2.3). Using this embedding, consider the decomposition

$$\mathfrak{f}_m = \mathfrak{r} \oplus \mathfrak{gl}_m \oplus \mathfrak{r}', \tag{2.19}$$

where $\mathfrak{r}$ and $\mathfrak{r}'$ are the Abelian subalgebras of $\mathfrak{f}_m$ spanned (respectively) by the elements $F_{a,-b}$ and $F_{-a,b}$ for all $a, b = 1, \ldots, m$. For any $\mathfrak{gl}_m$-module U, let V be the $\mathfrak{f}_m$-module *parabolically induced* from the $\mathfrak{gl}_m$-module U. To define V, first we extend the action of the Lie algebra $\mathfrak{gl}_m$ on U to the maximal parabolic subalgebra $\mathfrak{gl}_m \oplus \mathfrak{r}' \subset \mathfrak{f}_m$, so that every element of the summand $\mathfrak{r}'$ acts on U as zero. By definition, V is the $\mathfrak{f}_m$-module induced from the $(\mathfrak{gl}_m \oplus \mathfrak{r}')$-module U. Note that here we have a canonical embedding $U \to V$ of $(\mathfrak{gl}_m \oplus \mathfrak{r}')$-modules; we shall denote by $\bar{u}$ the image of an element $u \in U$ under this embedding. The $\mathfrak{f}_m$-module V determines the bimodule $\mathcal{F}_m(V)$ over $\mathfrak{f}_m$ and $\mathrm{X}(\mathfrak{g}_n)$. The space $\mathcal{F}_m(V)_{\mathfrak{r}}$ of $\mathfrak{r}$-coinvariants is then a bimodule over $\mathfrak{gl}_m$ and $\mathrm{X}(\mathfrak{g}_n)$.

On the other hand, for any $z \in \mathbb{C}$, consider the bimodule $\mathcal{E}_m^z(U)$ over the Lie algebra $\mathfrak{gl}_m$ and over the Yangian $\mathrm{Y}(\mathfrak{gl}_n)$. By restricting the module $\mathcal{E}_m^z(U)$ from the algebra $\mathrm{Y}(\mathfrak{gl}_n)$ to its subalgebra $\mathrm{Y}(\mathfrak{g}_n)$ and then using the homomorphism $\mathrm{X}(\mathfrak{g}_n) \to \mathrm{Y}(\mathfrak{g}_n)$ defined by (1.18), we can regard $\mathcal{E}_m^z(U)$ as a module over the algebra $\mathrm{X}(\mathfrak{g}_n)$ instead of $\mathrm{Y}(\mathfrak{gl}_n)$. This module is determined by the homomorphism $\mathrm{X}(\mathfrak{g}_n) \to \mathrm{A}_m$ such that for any $i, j = 1, \ldots, n$, the series $S_{ij}(u)$ is mapped to

$$\sum_{k=1}^{n} \theta_i \, \theta_k \, \alpha_m\big(T_{\tilde{k}\tilde{\imath}}(-u+z) \, T_{kj}(u+z)\big); \tag{2.20}$$

see (1.12) and (1.26). Now we map $S_{ij}(u)$ to the series (2.20) multiplied by

$$(1 + Z(u - z - m)) \otimes 1 \in \mathrm{A}_m[[u^{-1}]]; \tag{2.21}$$

see (1.27), where the positive integer l must be replaced by m. The latter mapping determines another homomorphism $\mathrm{X}(\mathfrak{g}_n) \to \mathrm{A}_m$. Using it, we turn the vector space $U \otimes \mathcal{G}(\mathbb{C}^m \otimes \mathbb{C}^n)$ of the $\mathrm{X}(\mathfrak{g}_n)$-module $\mathcal{E}_m^z(U)$ to another $\mathrm{X}(\mathfrak{g}_n)$-module, to be denoted by $\widetilde{\mathcal{E}}_m^z(U)$. Next, we define an action of the Lie algebra $\mathfrak{gl}_m$ on $\widetilde{\mathcal{E}}_m^z(U)$ by pulling its action on $\mathcal{E}_m^z(U)$ back through the automorphism

$$E_{ab} \mapsto -\delta_{ab} \, n/2 + E_{ab} \quad \text{for} \quad a, b = 1, \ldots, m. \tag{2.22}$$

Thus, the action of $\mathfrak{gl}_m$ on $\widetilde{\mathcal{E}}_m^z(U)$ is determined by the composition of homomorphisms

$$\mathrm{U}(\mathfrak{gl}_m) \to \mathrm{U}(\mathfrak{gl}_m) \to \mathrm{End}(U \otimes \mathcal{G}(\mathbb{C}^m \otimes \mathbb{C}^n)),$$

where the first map is the automorphism (2.22), while the second map corresponds to the natural action of $\mathfrak{gl}_m$ on $\mathcal{E}_m^z(U)$. The following proposition is a particular case of Theorem 3.1 from the next section.

Proposition 2.5. *For the $\mathfrak{f}_m$-module V parabolically induced from any $\mathfrak{gl}_m$-module U, the bimodule $\mathcal{F}_m(V)_\mathfrak{r}$ over $\mathfrak{gl}_m$ and $\mathrm{X}(\mathfrak{g}_n)$ is equivalent to $\widetilde{\mathcal{E}}^z_m(U)$, where $z = \mp\frac{1}{2}$.*

Now, let u and f range over the vector spaces U and $\mathcal{G}(\mathbb{C}^m \otimes \mathbb{C}^n)$, respectively. In the next section, we shall show that the linear map

$$U \otimes \mathcal{G}(\mathbb{C}^m \otimes \mathbb{C}^n) \to (V \otimes \mathcal{G}(\mathbb{C}^m \otimes \mathbb{C}^n))_\mathfrak{r}$$

defined by mapping $u \otimes f$ to the class of $\bar{u} \otimes f$ in the space of $\mathfrak{r}$-coinvariants, is an equivalence of bimodules $\widetilde{\mathcal{E}}^z_m(U) \to \mathcal{F}_m(V)_\mathfrak{r}$ over $\mathfrak{gl}_m$ and $\mathrm{X}(\mathfrak{g}_n)$.

An element μ of the vector space $\mathfrak{h}^*$ dual to $\mathfrak{h}$ is called a *weight* . A weight μ can be identified with the sequence $(\mu_1, \dots, \mu_m)$ of its *labels*, where

$$\mu_a = \mu(F_{a-m-1,a-m-1}) = -\mu(F_{m-a+1,m-a+1}) \quad \text{for} \quad a = 1, \dots, m.$$

The *Verma module* M_μ of the Lie algebra $\mathfrak{f}_m$ is the quotient of the algebra $\mathrm{U}(\mathfrak{f}_m)$ by the left ideal generated by all elements $X \in \mathfrak{n}'$ and all elements $X - \mu(X)$ with $X \in \mathfrak{h}$. The elements of the Lie algebra $\mathfrak{f}_m$ act on this quotient via left multiplication. The image of the identity element $1 \in \mathrm{U}(\mathfrak{f}_m)$ in this quotient is denoted by 1_μ. Then $X \cdot 1_\mu = 0$ for all $X \in \mathfrak{n}'$, and $X \cdot 1_\mu = \mu(X) \cdot 1_\mu$ for all $X \in \mathfrak{h}$. Let L_μ be the quotient of the Verma module M_μ relative to the maximal proper submodule. This quotient is a simple $\mathfrak{f}_m$-module of the highest weight μ.

For $z \in \mathbb{C}$, we denote by P_z the $\mathrm{Y}(\mathfrak{gl}_n)$-module obtained by pulling the standard action of $\mathrm{U}(\mathfrak{gl}_n)$ on $\mathcal{G}(\mathbb{C}^n)$ back through the homomorphism $\mathrm{Y}(\mathfrak{gl}_n) \to \mathrm{U}(\mathfrak{gl}_n)$ defined by (1.4), and then back through the automorphism τ_{-z} of $\mathrm{Y}(\mathfrak{gl}_n)$. Let $x_1, \dots, x_n$ be the standard generators of $\mathcal{G}(\mathbb{C}^n)$ and let $\partial_1, \dots, \partial_n$ be the corresponding left derivations. From (0.3) it follows that the action of $\mathrm{Y}(\mathfrak{gl}_n)$ on P_z is determined by the homomorphism $\mathrm{Y}(\mathfrak{gl}_n) \to \mathcal{GD}(\mathbb{C}^n)$ such that

$$T_{ij}(u) \mapsto \delta_{ij} + \frac{x_i \partial_j}{u + z}. \tag{2.23}$$

Using the comultiplication (1.9), for any $z_1, \dots, z_m \in \mathbb{C}$ we define the tensor product of $\mathrm{Y}(\mathfrak{gl}_n)$-modules

$$P_{z_m} \otimes \dots \otimes P_{z_1}. \tag{2.24}$$

For $a = 1, \dots, m$, let $\deg_a$ be the linear operator on this tensor product corresponding to evaluation of the total degree in $x_1, \dots, x_n$ in the tensor factor P_{z_a}, i.e., in the ath tensor factor when counting from right to left. By restricting this tensor product of $\mathrm{Y}(\mathfrak{gl}_n)$-modules to the subalgebra $\mathrm{Y}(\mathfrak{g}_n) \subset \mathrm{Y}(\mathfrak{gl}_n)$ and then using the homomorphism $\mathrm{X}(\mathfrak{g}_n) \to \mathrm{Y}(\mathfrak{g}_n)$ defined by (1.18), we can regard the tensor product (2.24) as a module over the extended twisted Yangian $\mathrm{X}(\mathfrak{g}_n)$.

Corollary 2.6. *The bimodule $\mathcal{F}_m(M_\mu)_\mathfrak{n}$ over $\mathfrak{h}$ and $\mathrm{X}(\mathfrak{g}_n)$ is equivalent to the tensor product*

$$P_{\mu_m+z} \otimes P_{\mu_{m-1}+z+1} \otimes \dots \otimes P_{\mu_1+z+m-1} \tag{2.25}$$

pulled back through the automorphism of $\mathrm{X}(\mathfrak{g}_n)$ defined by (1.17), where $f(u)$ equals

$$\prod_{a=1}^{m} \Big(1 + \frac{1}{u - z - m + a - 1 - \mu_a}\Big); \tag{2.26}$$

here $z = \mp\frac{1}{2}$. The element $F_{m-a+1,m-a+1} \in \mathfrak{h}$ acts on (2.25) as the operator

$$-n/2 + \deg_a - \mu_a. \tag{2.27}$$

Proof. We have an embedding of $\mathfrak{gl}_m$ to $\mathfrak{f}_m$ such that $E_{aa} \mapsto F_{aa}$ for $a = 1, \dots, m$. Then the Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{gl}_m$ is identified with the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{f}_m$. Put $\bar{a} = m - a + 1$ for short. If we regard the weight μ as an element of $\mathfrak{t}^*$, then

$$\mu(E_{\bar{a}\bar{a}}) = -\mu_a \quad \text{for} \quad a = 1, \dots, m.$$

Let U be the Verma module of the Lie algebra $\mathfrak{gl}_m$ corresponding to $\mu \in \mathfrak{t}^*$. It is defined as the quotient of the algebra $\mathrm{U}(\mathfrak{gl}_m)$ by the left ideal generated by all elements E_{ab} with $a < b$, and by all elements $E_{aa} - \mu(E_{aa})$. Then the Verma module M_μ of the Lie algebra $\mathfrak{f}_m$ is equivalent to the module V parabolically induced from the $\mathfrak{gl}_m$-module U. Here we use the decomposition (2.19).

Let $\mathfrak{s}$ denote the subalgebra of the Lie algebra $\mathfrak{gl}_m$ spanned by all elements E_{ab} with $a > b$. Using our embedding of $\mathfrak{gl}_m$ to $\mathfrak{f}_m$, we can also regard $\mathfrak{s}$ as a subalgebra of $\mathfrak{f}_m$. The Lie algebra $\mathfrak{n}$ of $\mathfrak{f}_m$ is then spanned by $\mathfrak{r}$ and $\mathfrak{s}$. By Proposition 2.5, the bimodule $\mathcal{F}_m(M_\mu)_\mathfrak{n}$ over $\mathfrak{h}$ and $\mathrm{X}(\mathfrak{g}_n)$ is equivalent to $\widetilde{\mathcal{E}}^z_m(U)_\mathfrak{s}$, where $z = \mp\frac{1}{2}$. To describe the latter bimodule, first we consider the bimodule $\mathcal{E}^z_m(U)_\mathfrak{s}$ over $\mathfrak{t}$ and $\mathrm{Y}(\mathfrak{gl}_n)$. By [KN2, Corollary 2.4], the bimodule $\mathcal{E}^z_m(U)_\mathfrak{s}$ is equivalent to the tensor product of $\mathrm{Y}(\mathfrak{gl}_n)$-modules (2.25), where the element $E_{\bar{a}\bar{a}} \in \mathfrak{t}$ acts as $\deg_a - \mu_a$. After pulling the action of the Lie algebra $\mathfrak{gl}_m$ on $\mathcal{E}^z_m(U)$ back through the automorphism (2.22), the element $E_{\bar{a}\bar{a}} \in \mathfrak{t}$ will act on the tensor product of vector spaces (2.25) as (2.27).

To complete the proof of Corollary 2.6, recall that the action of $\mathrm{X}(\mathfrak{gl}_n)$ on $\widetilde{\mathcal{E}}^z_m(U)$ differs from that on $\mathcal{E}^z_m(U)$ by multiplying the series (2.20) by (2.21). Using (1.28), we see that the series $1 + Z(u - z - m)$ in u^{-1} with the coefficients in $\mathrm{Z}(\mathfrak{gl}_m)$ acts on the Verma module U via scalar multiplication by the series (2.26). $\square$

By definition, the vector spaces of the two equivalent bimodules in Corollary 2.6 are $(M_\mu \otimes \mathcal{G}(\mathbb{C}^m \otimes \mathbb{C}^n))_\mathfrak{n}$ and $\mathcal{G}(\mathbb{C}^n)^{\otimes m}$, respectively. We can define a linear map from the latter vector space to the former by mapping $f_1 \otimes \dots \otimes f_m$ to the class of the element $1_\mu \otimes f$ in the space of $\mathfrak{n}$-coinvariants. Here for any m polynomials $f_1, \dots, f_m$ in the n anticommuting variables $x_1, \dots, x_n$, the polynomial f in the mn anticommuting variables $x_{11}, \dots, x_{mn}$ is defined by setting

$$f(x_{11}, \dots, x_{mn}) = f_1(x_{11}, \dots, x_{1n}) \cdots f_m(x_{m1}, \dots, x_{mn}). \tag{2.28}$$

This provides the bimodule equivalence in Corollary 2.6; see [KN2, Corollary 2.4] and also the remarks made immediately after stating Proposition 2.5 in the present paper.

For any $z \in \mathbb{C}$ denote by P'_z the $\mathrm{Y}(\mathfrak{gl}_n)$-module obtained by pulling P_z back through the automorphism (1.11) of $\mathrm{Y}(\mathfrak{gl}_n)$. In accordance with (2.23), the action of $\mathrm{Y}(\mathfrak{gl}_n)$ on P'_z is determined by the homomorphism $\mathrm{Y}(\mathfrak{gl}_n) \to \mathcal{GD}(\mathbb{C}^n)$,

$$T_{ij}(u) \mapsto \delta_{ij} - \frac{\theta_i \theta_j x_{\tilde{\jmath}} \partial_{\tilde{\imath}}}{u - z}. \tag{2.29}$$

Lemma 2.7. *The* $\mathrm{Y}(\mathfrak{gl}_n)$*-module* P'_z *can also be obtained by pushing the action of* $\mathrm{Y}(\mathfrak{gl}_n)$ *on* P_{-z-1} *forward through the automorphism of* $\mathcal{GD}(\mathbb{C}^n)$ *such that for each* $i = 1, \dots, n$,

$$x_i \mapsto \theta_i \, \partial_{\tilde{\imath}} \quad \textit{and} \quad \partial_i \mapsto \theta_i \, x_{\tilde{\imath}}, \quad i = 1, \dots, n, \tag{2.30}$$

and by pulling the resulting action back through the automorphism (1.3) *of* $\mathrm{Y}(\mathfrak{gl}_n)$*, where*

$$g(u) = 1 - \frac{1}{u - z}. \tag{2.31}$$

Thus, the action of $\mathrm{Y}(\mathfrak{gl}_n)$ on P'_z can also be determined by the composition

$$\mathrm{Y}(\mathfrak{gl}_n) \to \mathrm{Y}(\mathfrak{gl}_n) \underset{\tau_{z+1}}{\longrightarrow} \mathrm{Y}(\mathfrak{gl}_n) \to \mathrm{U}(\mathfrak{gl}_n) \to \mathcal{GD}(\mathbb{C}^n) \to \mathcal{GD}(\mathbb{C}^n).$$

Here the first map is the automorphism (1.3) of $\mathrm{Y}(\mathfrak{gl}_n)$, where the series $g(u)$ is given by (2.31), the last map is the automorphism (2.30) of $\mathcal{GD}(\mathbb{C}^n)$, while the other three maps are defined as in (0.3).

Proof. Applying the automorphism (2.30) to the right-hand side of (2.23) and replacing the parameter z there by $-z-1$, we get

$$\delta_{ij} + \frac{\theta_i\theta_j\partial_{\tilde\imath}\, x_{\tilde\jmath}}{u-z-1} = \delta_{ij} + \frac{\delta_{ij} - \theta_i\theta_j x_{\tilde\jmath}\partial_{\tilde\imath}}{u-z-1} = \frac{u-z}{u-z-1}\Big(\delta_{ij} - \frac{\theta_i\theta_j x_{\tilde\jmath}\partial_{\tilde\imath}}{u-z}\Big),$$

and after multiplying by (2.31), this becomes the right-hand side of (2.29). $\square$

§3. Parabolic induction

The twisted Yangian $\mathrm{Y}(\mathfrak{g}_n)$ is not merely a subalgebra of $\mathrm{Y}(\mathfrak{gl}_n)$, it is also a right coideal of the coalgebra $\mathrm{Y}(\mathfrak{gl}_n)$ relative to the comultiplication (1.9). Indeed, apply this comultiplication to the (i,j)-entry of the $(n\times n)$-matrix $T'(-u)\,T(u)$. We get the sum

$$\begin{aligned}\sum_{k=1}^{n} \theta_i\theta_k\,\Delta(T_{\tilde k\tilde\imath}(-u)\,T_{kj}(u)) &= \sum_{g,h,k=1}^{n} \theta_i\theta_j\,(T_{\tilde k\tilde g}(-u)\otimes T_{\tilde g\tilde\imath}(-u))\,(T_{kh}(u)\otimes T_{hj}(u)) \\ &= \sum_{g,h,k=1}^{n} \theta_g\theta_k\, T_{\tilde k\tilde g}(-u)\,T_{kh}(u)\otimes \theta_i\theta_g\, T_{\tilde g\tilde\imath}(-u)\,T_{hj}(u).\end{aligned}$$

In the last displayed line, by performing summation over $k=1,\dots,n$ in the first tensor factor, we get the (g,h)-entry of the matrix $T'(-u)\,T(u)$. Therefore,

$$\Delta(\mathrm{Y}(\mathfrak{g}_n)) \subset \mathrm{Y}(\mathfrak{g}_n)\otimes \mathrm{Y}(\mathfrak{gl}_n).$$

For the extended twisted Yangian $\mathrm{X}(\mathfrak{g}_n)$, one can define a homomorphism of associative algebras

$$\mathrm{X}(\mathfrak{g}_n) \to \mathrm{X}(\mathfrak{g}_n)\otimes \mathrm{Y}(\mathfrak{gl}_n)$$

by assigning

$$S_{ij}(u) \mapsto \sum_{g,h=1}^{n} S_{gh}(u)\otimes \theta_i\theta_g\, T_{\tilde g\tilde\imath}(-u)\,T_{hj}(u). \tag{3.1}$$

The homomorphism property can be verified directly; see [KN3, §3]. Via the homomorphism (3.1), the tensor product of any modules over the algebras $\mathrm{X}(\mathfrak{g}_n)$ and $\mathrm{Y}(\mathfrak{gl}_n)$ becomes another module over $\mathrm{X}(\mathfrak{g}_n)$.

Furthermore, the homomorphism (3.1) is a *coaction* of the Hopf algebra $\mathrm{Y}(\mathfrak{gl}_n)$ on the algebra $\mathrm{X}(\mathfrak{g}_n)$. Formally, a homomorphism of associative algebras

$$\mathrm{X}(\mathfrak{g}_n) \to \mathrm{X}(\mathfrak{g}_n)\otimes \mathrm{Y}(\mathfrak{gl}_n)\otimes \mathrm{Y}(\mathfrak{gl}_n)$$

can be defined in two different ways: either by using the assignment (3.1) twice, or by using (3.1) and then (1.9). Both ways lead to the same result; see again [KN3, §3].

Now for any positive integer l we consider the general linear Lie algebra $\mathfrak{gl}_{2m+2l}$ and its subalgebra $\mathfrak{f}_{m+l}$. This subalgebra is spanned by the elements F_{ab} with

$$a,b = -m-l,\dots,-1,1,\dots,m+l. \tag{3.2}$$

We extend the notation (2.1) and (2.2) to all such indices a,b and identify $\mathfrak{f}_m$ with the subalgebra of $\mathfrak{f}_{m+l}$ spanned by the elements F_{ab}, where $a,b=-m,\dots,-1,1,\dots,m$. Choose the embedding of the Lie algebra $\mathfrak{gl}_l$ to $\mathfrak{f}_{m+l}$ determined by the mappings

$$E_{ab} \mapsto F_{m+a,m+b} \quad \text{for} \quad a,b=1,\dots,l. \tag{3.3}$$

Let $\mathfrak{q},\mathfrak{q}'$ be the subalgebras of $\mathfrak{f}_{m+l}$ spanned (respectively) by the elements F_{ab},F_{ba}, where

$$a=m+1,\dots,m+l \quad \text{and} \quad b=-m-l,\dots,-1,1,\dots,m;$$

these two subalgebras of $\mathfrak{f}_{m+l}$ are nilpotent. Put $\mathfrak{p}=\mathfrak{f}_m\oplus\mathfrak{gl}_l\oplus\mathfrak{q}'$. Then $\mathfrak{p}$ is a maximal parabolic subalgebra of the reductive Lie algebra $\mathfrak{f}_{m+l}$, and $\mathfrak{f}_{m+l}=\mathfrak{q}\oplus\mathfrak{p}$. We do not exclude the case of $m=0$ here. In this case the nilpotent subalgebras $\mathfrak{q}$ and $\mathfrak{q}'$ of $\mathfrak{f}_{m+l}$ become the Abelian subalgebras $\mathfrak{r}$ and $\mathfrak{r}'$ of the Lie algebra $\mathfrak{f}_l$; see the decomposition (2.19), where the positive integer m must be replaced by l. Note that here the meaning of the symbols $\mathfrak{p}$ and $\mathfrak{q}$ is different from that in §0.

Let V and U be any modules of the Lie algebras $\mathfrak{f}_m$ and $\mathfrak{gl}_l$, respectively. Denote by $V\boxtimes U$ the $\mathfrak{f}_{m+l}$-module *parabolically induced* from the $\mathfrak{f}_m\oplus\mathfrak{gl}_l$-module $V\otimes U$. To define $V\boxtimes U$, first we extend the action of the Lie algebra $\mathfrak{f}_m\oplus\mathfrak{gl}_l$ on $V\otimes U$ to the Lie algebra $\mathfrak{p}$, so that every element of the subalgebra $\mathfrak{q}'\subset\mathfrak{p}$ acts on $V\otimes U$ as zero. By definition, $V\boxtimes U$ is the $\mathfrak{f}_{m+l}$-module induced from the $\mathfrak{p}$-module $V\otimes U$. Note that here we have a canonical embedding $V\otimes U\to V\boxtimes U$ of $\mathfrak{p}$-modules; we denote by $\overline{v\otimes u}$ the image of an element $v\otimes u\in V\otimes U$ under this embedding.

Consider the bimodule $\mathcal{F}_{m+l}(V\boxtimes U)$ over $\mathfrak{f}_{m+l}$ and $\mathrm{X}(\mathfrak{g}_n)$. Here the action of $\mathrm{X}(\mathfrak{gl}_n)$ commutes with the action of the Lie algebra $\mathfrak{f}_{m+l}$, and hence, with the action of the subalgebra $\mathfrak{q}\subset\mathfrak{f}_{m+l}$. Therefore, the vector space $\mathcal{F}_{m+l}(V\boxtimes U)_{\mathfrak{q}}$ of coinvariants of the action of the subalgebra $\mathfrak{q}$ is a quotient of the $\mathrm{X}(\mathfrak{g}_n)$-module $\mathcal{F}_{m+l}(V\boxtimes U)$. Note that the subalgebra $\mathfrak{f}_m\oplus\mathfrak{gl}_l\subset\mathfrak{f}_{m+l}$ also acts on this quotient.

For any $z\in\mathbb{C}$, consider the bimodule $\mathcal{E}_l^z(U)$ over $\mathfrak{gl}_l$ and $\mathrm{Y}(\mathfrak{gl}_n)$, defined as at the end of §1. Also consider the bimodule $\mathcal{F}_m(V)$ over $\mathfrak{f}_m$ and $\mathrm{X}(\mathfrak{g}_n)$. Via the homomorphism (3.1), the tensor product of vector spaces $\mathcal{F}_m(V)\otimes\mathcal{E}_l^z(U)$ becomes a module over $\mathrm{X}(\mathfrak{g}_n)$. This module is determined by the homomorphism $\mathrm{X}(\mathfrak{g}_n)\to\mathrm{B}_m\otimes\mathrm{A}_l$ such that for any $i,j=1,\dots,n$ the series $S_{ij}(u)$ is mapped to

$$\sum_{g,h=1}^{n}\beta_m\big(S_{gh}(u)\big)\otimes\theta_i\theta_g\,\alpha_l\big(T_{\tilde{g}\tilde{\imath}}(-u+z)\,T_{hj}(u+z)\big). \tag{3.4}$$

Now, we map the series $S_{ij}(u)$ to the series (3.4) multiplied by

$$\big(1\otimes 1\big)\otimes\big((1+Z(u-z-l))\otimes 1\big)\in\mathrm{B}_m\otimes\mathrm{A}_l\,[[u^{-1}]]; \tag{3.5}$$

see (1.27). This mapping determines another homomorphism $\mathrm{X}(\mathfrak{g}_n)\to\mathrm{B}_m\otimes\mathrm{A}_l$. Using it, we turn the vector space of the $\mathrm{X}(\mathfrak{g}_n)$-module $\mathcal{F}_m(V)\otimes\mathcal{E}_l^z(U)$ to yet another $\mathrm{X}(\mathfrak{g}_n)$-module, which will be denoted by $\mathcal{F}_m(V)\,\tilde{\otimes}\,\mathcal{E}_l^z(U)$. Define an action of the Lie algebra $\mathfrak{gl}_l$ on the latter $\mathrm{X}(\mathfrak{g}_n)$-module by pulling the action of $\mathfrak{gl}_l$ on $\mathcal{E}_l(U)$ back through the automorphism

$$E_{ab}\,\mapsto\,-\delta_{ab}\,n/2+E_{ab}\quad\text{for}\quad a,b=1,\dots,l. \tag{3.6}$$

The Lie algebra $\mathfrak{f}_m$ acts on the $\mathrm{X}(\mathfrak{g}_n)$-module $\mathcal{F}_m(V)\,\tilde{\otimes}\,\mathcal{E}_l^z(U)$ via the tensor factor $\mathcal{F}_m(V)$. Thus, $\mathcal{F}_m(V)\,\tilde{\otimes}\,\mathcal{E}_l^z(U)$ becomes a bimodule over the direct sum of Lie algebras $\mathfrak{f}_m\oplus\mathfrak{gl}_l$ and over the extended twisted Yangian $\mathrm{X}(\mathfrak{g}_n)$. For $m=0$, the next theorem becomes Proposition 2.5, where the positive integer m must be replaced by l. Here we assume that $\mathcal{F}_0(V)=\mathbb{C}$, so that $\beta_0(S_{ij}(u))=\delta_{ij}$.

Theorem 3.1. *The bimodule* $\mathcal{F}_{m+l}(V\boxtimes U)_{\mathfrak{q}}$ *over* $\mathfrak{f}_m\oplus\mathfrak{gl}_l$ *and* $\mathrm{X}(\mathfrak{g}_n)$ *is equivalent to* $\mathcal{F}_m(V)\,\tilde{\otimes}\,\mathcal{E}_l^z(U)$, *where* $z=m\mp\frac{1}{2}$.

Proof. The remaining part of this section is devoted to the proof of Theorem 3.1. As vector spaces,

$$\mathcal{F}_{m+l}(V \boxtimes U)_{\mathfrak{q}} = (V \boxtimes U \otimes \mathcal{G}(\mathbb{C}^{m+l} \otimes \mathbb{C}^n))_{\mathfrak{q}},$$
$$\mathcal{F}_m(V) \,\widetilde{\otimes}\, \mathcal{E}_l^z(U) = V \otimes \mathcal{G}(\mathbb{C}^m \otimes \mathbb{C}^n) \otimes U \otimes \mathcal{G}(\mathbb{C}^l \otimes \mathbb{C}^n).$$

We can construct a linear map from the latter vector space to the former one by mapping any element $v \otimes f \otimes u \otimes g$ to the class of $\overline{v \otimes u} \otimes f \otimes g$ in the space of $\mathfrak{q}$-coinvariants. Here $v \in V$, $f \in \mathcal{G}(\mathbb{C}^m \otimes \mathbb{C}^n)$ and $u \in U$, $g \in \mathcal{G}(\mathbb{C}^l \otimes \mathbb{C}^n)$, whereas the tensor product $f \otimes g$ is identified with an element of $\mathcal{G}(\mathbb{C}^{m+l} \otimes \mathbb{C}^n)$ in a natural way, which corresponds to the decomposition

$$\mathbb{C}^{m+l} \otimes \mathbb{C}^n = \mathbb{C}^m \otimes \mathbb{C}^n \oplus \mathbb{C}^l \otimes \mathbb{C}^n. \tag{3.7}$$

We shall show that this map establishes an equivalence of bimodules in Theorem 3.1.

The vector space of the $\mathfrak{f}_{m+l}$-module $V \boxtimes U$ can be identified with the tensor product $\mathrm{U}(\mathfrak{q}) \otimes V \otimes U$, where the Lie subalgebra $\mathfrak{q} \subset \mathfrak{f}_{m+l}$ acts via left multiplication on the first tensor factor. Then $\overline{v \otimes u} = 1 \otimes v \otimes u$, so that the tensor product $V \otimes U$ gets identified with the subspace

$$1 \otimes V \otimes U \subset \mathrm{U}(\mathfrak{q}) \otimes V \otimes U. \tag{3.8}$$

On this subspace, every element of the subalgebra $\mathfrak{q}' \subset \mathfrak{f}_{m+l}$ acts as zero, while the two direct summands of the subalgebra $\mathfrak{f}_m \oplus \mathfrak{gl}_l \subset \mathfrak{f}_{m+l}$ act nontrivially only on the tensor factors V and U, respectively. All this determines the action of the Lie algebra $\mathfrak{f}_{m+l}$ on $\mathrm{U}(\mathfrak{q}) \otimes V \otimes U$. Now we view $\mathcal{F}_{m+l}(V \boxtimes U)$ as a $\mathfrak{f}_{m+l}$-module, denoting it by M for short. Then M is the tensor product of two $\mathfrak{f}_{m+l}$-modules,

$$M = (V \boxtimes U) \otimes \mathcal{G}(\mathbb{C}^{m+l} \otimes \mathbb{C}^n) = \mathrm{U}(\mathfrak{q}) \otimes V \otimes U \otimes \mathcal{G}(\mathbb{C}^{m+l} \otimes \mathbb{C}^n). \tag{3.9}$$

The vector spaces of the $\mathrm{X}(\mathfrak{g}_n)$-module $\mathcal{F}_m(V)$ and of the $\mathrm{Y}(\mathfrak{gl}_n)$-module $\mathcal{E}_l^z(U)$ are $V \otimes \mathcal{G}\,(\mathbb{C}^m \otimes \mathbb{C}^n)$ and $U \otimes \mathcal{G}\,(\mathbb{C}^l \otimes \mathbb{C}^n)$, respectively. The action of the Lie algebra $\mathfrak{f}_m$ on the first vector space is defined by (2.6). By pulling back through the automorphism (3.6), the action of the Lie algebra $\mathfrak{gl}_l$ on the second vector space is defined by

$$E_{ab} \,\mapsto\, -\delta_{ab}\, n/2 + E_{ab} \otimes 1 + \sum_{k=1}^{n} 1 \otimes x_{ak}\, \partial_{bk} \quad \text{for} \quad a, b = 1, \ldots, l.$$

We identify the tensor product of these two vector spaces with the vector space

$$V \otimes U \otimes \mathcal{G}\,(\mathbb{C}^m \otimes \mathbb{C}^n) \otimes \mathcal{G}\,(\mathbb{C}^l \otimes \mathbb{C}^n) = V \otimes U \otimes \mathcal{G}\,(\mathbb{C}^{m+l} \otimes \mathbb{C}^n), \tag{3.10}$$

where we use the direct sum decomposition (3.7). We get an action of the direct sum of Lie algebras $\mathfrak{f}_m \oplus \mathfrak{gl}_l$ on the vector space (3.10).

Now we define a linear map

$$\chi \,:\, V \otimes U \otimes \mathcal{G}\,(\mathbb{C}^{m+l} \otimes \mathbb{C}^n) \to M/\,\mathfrak{q} \cdot M$$

by the assignment

$$\chi \,:\, y \otimes x \otimes t \,\mapsto\, 1 \otimes y \otimes x \otimes t + \mathfrak{q} \cdot M$$

for any vectors $y \in V$, $x \in U$ and $t \in \mathcal{G}\,(\mathbb{C}^{m+l} \otimes \mathbb{C}^n)$. The operator χ intertwines the actions of the Lie algebra $\mathfrak{f}_m \oplus \mathfrak{gl}_l$; see the definition (2.6) with m replaced by $m+l$. We show that the operator χ is bijective.

First, consider the action of the Lie subalgebra $\mathfrak{q} \subset \mathfrak{f}_{m+l}$ on the vector space

$$\mathcal{G}(\mathbb{C}^{m+l}) = \mathcal{G}(\mathbb{C}^m) \otimes \mathcal{G}(\mathbb{C}^l);$$

the action is defined by (2.6), where $n=1$ and the integer m is replaced by $m+l$. This vector space admits a descending filtration by the subspaces

$$\bigoplus_{K=N}^{l} \mathcal{G}\,(\mathbb{C}^m)\otimes\mathcal{G}^K(\mathbb{C}^l), \quad \text{where} \quad N=0,1,\ldots,l.$$

Here $\mathcal{G}^K(\mathbb{C}^l)$ stands for the homogeneous subspace of $\mathcal{G}\,(\mathbb{C}^l)$ of degree K. The action of the Lie algebra $\mathfrak{q}$ on $\mathcal{G}(\mathbb{C}^{m+l})$ preserves each of the filtration subspaces and becomes trivial on the associated graded space.

Similarly, for any $n=1,2,\ldots$, the vector space $\mathcal{G}\,(\mathbb{C}^{m+l}\otimes\mathbb{C}^n)$ admits a descending filtration by $\mathfrak{q}$-submodules such that $\mathfrak{q}$ acts trivially on each of the corresponding graded subspaces. The latter filtration induces a filtration of M by $\mathfrak{q}$-submodules such that, on the corresponding graded quotient $\operatorname{gr} M$, the Lie algebra $\mathfrak{q}$ acts via left multiplication on the first tensor factor $\mathrm{U}(\mathfrak{q})$ in (3.9). Therefore, the space $V\otimes U\otimes\mathcal{G}\,(\mathbb{C}^{m+l}\otimes\mathbb{C}^n)$ is isomorphic to the space of coinvariants $(\operatorname{gr} M)_{\mathfrak{q}}$ via the bijective linear map

$$y\otimes x\otimes t\,\mapsto\,1\otimes y\otimes x\otimes t+\mathfrak{q}\cdot(\operatorname{gr} M).$$

Thus, the linear map χ is also bijective. Now it remains to show that the map χ intertwines the actions of the algebra $\mathrm{X}(\mathfrak{g}_n)$.

In this section we shall use the symbol $\equiv$ to indicate equalities in the algebra $\mathrm{U}(\mathfrak{f}_{m+l})$ modulo the left ideal generated by the elements of the subalgebra $\mathfrak{q}'\subset\mathfrak{f}_{m+l}$. Any two elements of $\mathrm{U}(\mathfrak{f}_{m+l})$ related by $\equiv$ act on the subspace (3.8) in the same way. We shall extend the relation $\equiv$ to formal power series in u^{-1} with coefficients in $\mathrm{U}(\mathfrak{f}_{m+l})$, and then to matrices whose entries are such series. Put

$$v=u\pm\tfrac{1}{2}-m-l \quad\text{and}\quad w=-u\pm\tfrac{1}{2}-m-l. \tag{3.11}$$

The definition of the $\mathrm{X}(\mathfrak{g}_n)$-module M involves the $((2m+2l)\times(2m+2l))$-matrix whose (a,b)-entry is $\delta_{ab}\,v+F_{ab}$. The rows and columns of this matrix are labeled by the indices (3.2). In [KN3, §3] we proved that the inverse to this matrix is related by $\equiv$ to the block matrix

$$\begin{bmatrix} H & 0 & 0\\ I & J & 0\\ P & Q & R\end{bmatrix}, \tag{3.12}$$

where the blocks H, P, R are certain matrices of size $l\times l$, while the blocks I, J, Q are certain matrices of sizes $2m\times l$, $2m\times 2m$, and $l\times 2m$, respectively. We label the rows and columns of the blocks by the same indices as in the compound matrix (3.12). For instance, the rows and columns of the $(l\times l)$-matrix R are labeled by $m+1,\ldots,m+l$.

Keeping the notation of §2, let F be the $(2m\times 2m)$-matrix whose (c,d)-entry is F_{cd} for $c,d=-m,\ldots,-1,1,\ldots,m$. Let $F(u)$ be the inverse to the matrix $u+F$. The entries of the matrix $F(u)$ are formal power series in u^{-1} with coefficients in the algebra $\mathrm{U}(\mathfrak{f}_m)$; see (2.4). But now the algebra $\mathrm{U}(\mathfrak{f}_m)$ is regarded as a subalgebra of $\mathrm{U}(\mathfrak{f}_{m+l})$. We denote by $W(u)$ the trace of the matrix $F(u)$, as we did in §2.

Let E denote the $(l\times l)$-matrix whose (a,b)-entry is F_{ab} for $a,b=m+1,\ldots,m+l$. Using our embedding (3.3) of the Lie algebra $\mathfrak{gl}_l$ to $\mathfrak{f}_{m+l}$, we see that this notation agrees with that of §1. But now we use the indices $a,b=m+1,\ldots,m+l$ to label the rows and columns of the matrix E. Let $E(v)$ be the inverse to the matrix $v+E$. Let $E_{ab}(u)$ be the (a,b)-entry of the inverse matrix and $Z(v)$ the trace of the inverse matrix. The coefficients of the formal power series $Z(v)$ in v^{-1} belong to the center of the algebra $\mathrm{U}(\mathfrak{gl}_l)$, which is now regarded as a subalgebra of $\mathrm{U}(\mathfrak{f}_{m+l})$. Next, for any indices $a,b=m+1,\ldots,m+l$, we put $\widetilde{E}_{ab}(v)=(v+l+E')^{-1}_{ba}$. Then, by Lemma 1.3,

$$(1+Z(v))\,\widetilde{E}_{ab}(v)=E_{ab}(v). \tag{3.13}$$

Let $a,b=m+1,\dots,m+l$ and $c,d=-m,\dots,-1,1,\dots,m$. By [KN3, §3], we have

$$
\begin{aligned}
-H_{-b,-a} &= (1+Z(v))\Big(\big(W(v+l)\mp\frac{1}{2u}+1\big)\,\widetilde{E}_{ab}(w)\pm\frac{1}{2u}\,\widetilde{E}_{ab}(v)\Big),\\
-I_{-d,-a} &= \sum_{b>m\geq c\geq -m}\varepsilon_{ad}\,F_{bc}\,(1+Z(v))\Big(\varepsilon_{cd}\,\widetilde{E}_{ab}(w)\,F_{-d,-c}(v+l)\\
&\qquad\qquad\pm\frac{\widetilde{E}_{ab}(w)-\widetilde{E}_{ab}(v)}{2u}\,F_{cd}(v+l)\Big),\\
J_{cd} &= (1+Z(v))\,F_{cd}(v+l),\\
P_{b,-a} &= \sum_{e,f>m}F_{f,-e}\,E_{bf}(v)\,\widetilde{E}_{ae}(w)\\
&\quad\pm\sum_{\substack{e,f>m\\ m\geq c,d\geq -m}}\varepsilon_{ad}\,F_{f,-d}\,F_{ec}\,E_{be}(v)\,\widetilde{E}_{af}(w)\,F_{cd}(v+l),\\
-Q_{ad} &= \sum_{e>m\geq c\geq -m}F_{ec}\,E_{ae}(v)\,F_{cd}(v+l),\qquad R_{ab}=E_{ab}(v).
\end{aligned}
$$

By the definition of the $\mathrm{X}(\mathfrak{g}_n)$-module M, now the action of $\mathrm{X}(\mathfrak{g}_n)$ on the elements of the subspace

$$1\otimes V\otimes U\otimes\mathcal{G}(\mathbb{C}^{m+l}\otimes\mathbb{C}^n)\subset M \tag{3.14}$$

can be described by assigning the following sum of series with coefficients in the algebra $\mathrm{B}_{m+l}=\mathrm{U}(\mathfrak{f}_{m+l})\otimes\mathcal{GD}(\mathbb{C}^{m+l}\otimes\mathbb{C}^n)$ to every series $S_{ij}(u)$:

$$
\begin{aligned}
&\delta_{ij}+\sum_{a,b>m}R_{ab}\otimes\theta_i\,\theta_j\,\partial_{a\tilde{\imath}}\,x_{sb\tilde{\jmath}}+\sum_{a,b>m}H_{-b,-a}\otimes x_{bi}\,\partial_{aj}\\
&\quad+\sum_{a>m\geq d>0}(\,I_{-d,-a}\otimes x_{di}\,\partial_{aj}+I_{d,-a}\otimes\theta_i\,\partial_{d\tilde{\imath}}\,\partial_{aj}\,)\\
&+\sum_{m\geq c,d>0}(J_{-c,-d}\otimes x_{ci}\,\partial_{dj}+J_{-c,d}\otimes\theta_j\,x_{ci}\,x_{d\tilde{\jmath}}+J_{c,-d}\otimes\theta_i\,\partial_{c\tilde{\imath}}\,\partial_{dj}+J_{cd}\otimes\theta_i\,\theta_j\,\partial_{c\tilde{\imath}}\,x_{d\tilde{\jmath}})\\
&\quad+\sum_{a,e>m}P_{e,-a}\otimes\theta_i\,\partial_{e\tilde{\imath}}\,\partial_{aj}\\
&\quad+\sum_{a>m\geq d>0}(Q_{a,-d}\otimes\theta_i\,\partial_{a\tilde{\imath}}\,\partial_{dj}+Q_{ad}\otimes\theta_i\,\theta_j\,\partial_{a\tilde{\imath}}\,x_{d\tilde{\jmath}}\,).
\end{aligned}
\tag{3.15}
$$

Here for $a=1,\dots,m+l$ and $i=1,\dots,n$ we use the standard generators x_{ai} of the Grassmann algebra $\mathcal{G}(\mathbb{C}^{m+l}\otimes\mathbb{C}^n)$. Then ∂_{ai} is the left derivation on $\mathcal{G}(\mathbb{C}^{m+l}\otimes\mathbb{C}^n)$ relative to x_{ai}. The generators x_{ai} with $a\leq m$ and $a>m$ correspond to the first and the second direct summands in (3.7).

Consider the action of $\mathrm{X}(\mathfrak{g}_n)$ on the elements of the subspace (3.14) modulo $\mathfrak{q}\cdot M$, in accordance with the definition (2.6), where m must be replaced by $m+l$. From now till the end of this section, we assume that $a,b,e,f=m+1,\dots,m+l$, while $c,d=1,\dots,m$. The indices g,h and k will run through $1,\dots,n$.

By our description of the block R, the sum displayed in the first of the six lines in (3.15) acts on the elements of the subspace (3.14) as the sum

$$\delta_{ij}+\sum_{a,b}E_{ab}(v)\otimes\theta_i\,\theta_j\,\partial_{a\tilde{\imath}}\,x_{b\tilde{\jmath}}=\delta_{ij}\,(1+Z(v))-\sum_{a,b}E_{ab}(v)\otimes\theta_i\,\theta_j\,x_{b\tilde{\jmath}}\,\partial_{a\tilde{\imath}}. \tag{3.16}$$

By our description of the block H, the sum displayed in the second in (3.15) acts on the elements of (3.14) as the sum over the indices a, b of the expressions

$$-(1+Z(v))\Big(\Big(W(v+l)\mp\frac{1}{2u}+1\Big)\,\widetilde{E}_{ab}(w)\pm\frac{1}{2u}\,\widetilde{E}_{ab}(v)\Big)\otimes x_{bi}\,\partial_{aj}. \tag{3.17}$$

By our description of the block I, the sum in the third line in (3.15) acts on the elements of (3.14) as the sum over the indices a, b, c, d of the expressions

$$\mp F_{b,-c}\,(1+Z(v))\Big(\widetilde{E}_{ab}(w)\,F_{-d,c}(v+l)+\frac{\widetilde{E}_{ab}(w)-\widetilde{E}_{ab}(v)}{2u}\,F_{-c,d}(v+l)\Big)\otimes x_{di}\,\partial_{aj},$$
$$-F_{bc}\,(1+Z(v))\Big(\widetilde{E}_{ab}(w)\,F_{-d,-c}(v+l)\pm\frac{\widetilde{E}_{ab}(w)-\widetilde{E}_{ab}(v)}{2u}\,F_{cd}(v+l)\Big)\otimes x_{di}\,\partial_{aj},$$
$$\mp F_{b-c}\,(1+Z(v))\Big(\widetilde{E}_{ab}(w)\,F_{dc}(v+l)\pm\frac{\widetilde{E}_{ab}(w)-\widetilde{E}_{ab}(v)}{2u}\,F_{-c,-d}(v+l)\Big)\otimes\theta_i\,\partial_{d\tilde{\imath}}\,\partial_{aj},$$
$$-F_{bc}\,(1+Z(v))\Big(\widetilde{E}_{ab}(w)\,F_{d,-c}(v+l)+\frac{\widetilde{E}_{ab}(w)-\widetilde{E}_{ab}(v)}{2u}\,F_{c,-d}(v+l)\Big)\otimes\theta_i\,\partial_{d\tilde{\imath}}\,\partial_{aj}.$$

Here $F_{b,-c}\in\mathfrak{q}$ and $F_{bc}\in\mathfrak{q}$. Hence, modulo $\mathfrak{q}\cdot M$, the expression displayed in the last four lines acts on the elements of (3.14) as the sum over the index k of the expressions

$$\begin{aligned}\Big(&\pm\Big(\widetilde{E}_{ab}(w)\,F_{-d,c}(v-l)+\frac{\widetilde{E}_{ab}(w)-\widetilde{E}_{ab}(v)}{2u}\,F_{-c,d}(v+l)\Big)\otimes\theta_k\,x_{b\tilde{k}}\,x_{ck}\,x_{di}\,\partial_{aj}\\&+\Big(\widetilde{E}_{ab}(w)\,F_{-d,-c}(v+l)\pm\frac{\widetilde{E}_{ab}(w)-\widetilde{E}_{ab}(v)}{2u}\,F_{cd}(v+l)\Big)\otimes x_{bk}\,\partial_{ck}\,x_{di}\,\partial_{aj}\\&\pm\Big(\widetilde{E}_{ab}(w)\,F_{dc}(v+l)\pm\frac{\widetilde{E}_{ab}(w)-\widetilde{E}_{ab}(v)}{2u}\,F_{-c,-d}(v+l)\Big)\otimes\theta_i\,\theta_k\,x_{b\tilde{k}}\,x_{ck}\,\partial_{d\tilde{\imath}}\,\partial_{aj}\\&+\Big(\widetilde{E}_{ab}(w)\,F_{d,-c}(v+l)+\frac{\widetilde{E}_{ab}(w)-\widetilde{E}_{ab}(v)}{2u}\,F_{c,-d}(v+l)\Big)\otimes\theta_i\,x_{bk}\,\partial_{ck}\,\partial_{d\tilde{\imath}}\,\partial_{aj}\Big)\\&\times\ \big((1+Z(v))\otimes 1\big).\end{aligned}\tag{3.18}$$

By our description of the block J, the sum displayed in the fourth line in (3.15) acts on the elements of (3.14) as the sum over c, d of the expressions

$$\begin{aligned}\big((1+Z(v))\otimes 1\big)\,\big(&F_{-c,-d}(v+l)\otimes x_{ci}\,\partial_{dj}+F_{-c,d}(v+l)\otimes\theta_j\,x_{ci}\,x_{d\tilde{\jmath}}\\&+F_{c,-d}(v+l)\otimes\theta_i\,\partial_{c\tilde{\imath}}\,\partial_{dj}+F_{cd}(v+l)\otimes\theta_i\,\theta_j\,\partial_{c\tilde{\imath}}\,x_{d\tilde{\jmath}}\big).\end{aligned}\tag{3.19}$$

By our description of the block P, the sum displayed in the fifth line in (3.15) acts on the elements of the subspace (3.14) as the sum over the indices a, b, e, f of the expressions

$$F_{f,-b}\,E_{ef}(v)\,\widetilde{E}_{ab}(w)\otimes\theta_i\,\partial_{e\tilde{\imath}}\,\partial_{aj}$$

plus the action of the sum over the indices a, b, c, d, e, f of the expressions

$$F_{fd}\,F_{b,-c}\,E_{eb}(v)\,\widetilde{E}_{af}(w)\,F_{-c,-d}(v+l)\otimes\theta_i\,\partial_{e\tilde{\imath}}\,\partial_{aj},$$
$$\pm F_{f,-d}\,F_{b,-c}\,E_{eb}(v)\,\widetilde{E}_{af}(w)\,F_{-c,d}(v+l)\otimes\theta_i\,\partial_{e\tilde{\imath}}\,\partial_{aj},$$
$$+F_{fd}\,F_{bc}\,E_{eb}(v)\,\widetilde{E}_{af}(w)\,F_{c,-d}(v+l)\otimes\theta_i\,\partial_{e\tilde{\imath}}\,\partial_{aj},$$
$$\pm F_{f,-d}\,F_{bc}\,E_{eb}(v)\,\widetilde{E}_{af}(w)\,F_{cd}(v+l)\otimes\theta_i\,\partial_{e\tilde{\imath}}\,\partial_{aj}.$$

Here, modulo $\mathfrak{q}\cdot M$, the expression to be summed over the indices a, b, e, f acts on the elements of the subspace (3.14) as the sum over the index k of the expressions

$$-E_{ef}(v)\,\widetilde{E}_{ab}(w)\otimes\theta_i\,\theta_k\,x_{f\tilde{k}}\,x_{bk}\,\partial_{e\tilde{\imath}}\,\partial_{aj},$$

while the expression to be summed over a,b,c,d,e,f acts as the sum over g,h of the expressions

$$
\begin{aligned}
\big(E_{eb}(v)\,\widetilde{E}_{af}(w)\otimes 1\big)\,\big(\; & F_{-c,-d}(v+l)\otimes\theta_i\,\theta_g\,x_{b\tilde g}\,x_{cg}\,x_{fh}\,\partial_{dh}\,\partial_{e\tilde\imath}\,\partial_{aj}\\
&\pm\, F_{-c,d}(v+l)\otimes\theta_i\,\theta_g\,\theta_h\,x_{b\tilde g}\,x_{cg}\,x_{f\tilde h}\,x_{dh}\,\partial_{e\tilde\imath}\,\partial_{aj}\\
&+\, F_{c,-d}(v+l)\otimes\theta_i\,x_{bg}\,\partial_{cg}\,x_{fh}\,\partial_{dh}\,\partial_{e\tilde\imath}\,\partial_{aj}\\
&\pm\, F_{cd}(v+l)\otimes\theta_i\,\theta_h\,x_{bg}\,\partial_{cg}\,x_{f\tilde h}\,x_{dh}\,\partial_{e\tilde\imath}\,\partial_{aj}\big).
\end{aligned}
$$

We have $\theta_{\tilde k}=\pm\,\theta_k$ for $k=1,\dots,n$. Using the commutation relations in the ring $\mathcal{GD}(\mathbb{C}^{m+l}\otimes\mathbb{C}^n)$, the sum over the index k above equals the sum over k of the expressions

$$
E_{ef}(v)\,\widetilde{E}_{ab}(w)\otimes\theta_i\,\theta_k\,x_{f\tilde k}\,\partial_{e\tilde\imath}\,x_{bk}\,\partial_{aj} \tag{3.20}
$$

plus

$$
\mp\,\delta_{be}\,E_{ef}(v)\,\widetilde{E}_{ab}(w)\otimes x_{f\tilde\imath}\,\partial_{aj}. \tag{3.21}
$$

Similarly, the sum over the indices g,h equals the sum over g,h of the expressions

$$
\begin{aligned}
&\big(F_{-c,-d}(v+l)\otimes x_{cg}\,\partial_{dh}+F_{-c,d}(v+l)\otimes\theta_h\,x_{cg}\,x_{d\tilde h}\\
&\quad+F_{c,-d}(v+l)\otimes\theta_g\,\partial_{c\tilde g}\,\partial_{dh}+F_{cd}(v+l)\otimes\theta_g\,\theta_h\,\partial_{c\tilde g}\,x_{d\tilde h}\,\big)\\
&\quad\times\big(E_{eb}(v)\otimes\theta_i\,\theta_g\,x_{b\tilde g}\,\partial_{e\tilde\imath}\big)\,\big(\widetilde{E}_{af}(w)\otimes x_{fh}\,\partial_{aj}\big)
\end{aligned} \tag{3.22}
$$

plus the sum over k of the expressions

$$
\begin{aligned}
&\big(\delta_{ef}\,E_{eb}(v)\,\widetilde{E}_{af}(w)\otimes 1\big)\big(-F_{-c,-d}(v+l)\otimes\theta_i\,\theta_k\,x_{b\tilde k}\,x_{ck}\,\partial_{d\tilde\imath}\,\partial_{aj}\\
&\qquad\mp F_{-c,d}(v+l)\otimes\theta_k\,x_{b\tilde k}\,x_{ck}\,x_{di}\,\partial_{aj}\\
&\qquad-F_{c,-d}(v+l)\otimes\theta_i\,x_{bk}\,\partial_{ck}\,\partial_{d\tilde\imath}\,\partial_{aj}\mp F_{cd}(v+l)\otimes x_{bk}\,\partial_{ck}\,x_{di}\,\partial_{aj}\big).
\end{aligned} \tag{3.23}
$$

By our description of the block Q, the sum displayed in the last line in (3.15) acts on the elements of (3.14) as the sum over a,b,c,d of the expressions

$$
\begin{aligned}
&-\big(F_{b,-c}\,E_{ab}(v)\,F_{-c,-d}(v+l)+F_{bc}\,E_{ab}(v)\,F_{c,-d}(v+l)\big)\otimes\theta_i\,\partial_{a\tilde\imath}\,\partial_{dj}\\
&-\big(F_{b,-c}\,E_{ab}(v)\,F_{-c,d}(v+l)+F_{bc}\,E_{ab}(v)\,F_{cd}(v+l)\big)\otimes\theta_i\,\theta_j\,\partial_{a\tilde\imath}\,x_{d\tilde\jmath}.
\end{aligned}
$$

Modulo $\mathfrak{q}\cdot M$, the expression in the above two lines acts on the elements of the subspace (3.14) as the sum over k of the expressions

$$
\begin{aligned}
&\big(E_{ab}(v)\otimes 1\big)\big(F_{-c,-d}(v+l)\otimes\theta_i\,\theta_k\,x_{b\tilde k}\,x_{ck}\,\partial_{a\tilde\imath}\,\partial_{dj}+F_{c,-d}(v+l)\otimes\theta_i\,x_{bk}\,\partial_{ck}\,\partial_{a\tilde\imath}\,\partial_{dj}\\
&\qquad+F_{-c,d}(v+l)\otimes\theta_k\,x_{b\tilde k}\,x_{ck}\,\theta_i\,\theta_j\,\partial_{a\tilde\imath}\,x_{d\tilde\jmath}+F_{cd}(v+l)\otimes x_{bk}\,\partial_{ck}\,\theta_i\,\theta_j\,\partial_{a\tilde\imath}\,x_{d\tilde\jmath}\big).
\end{aligned}
$$

Note that this sum over the index k can be rewritten as the sum over k of the expressions

$$
\begin{aligned}
&\big(F_{-c,-d}(v+l)\otimes x_{ck}\,\partial_{dj}+F_{-c,d}(v+l)\otimes\theta_j\,x_{ck}\,x_{d\tilde\jmath}\\
&\quad+F_{c,-d}(v+l)\otimes\theta_k\,\partial_{c\tilde k}\,\partial_{dj}+F_{cd}(v+l)\otimes\theta_k\,\theta_j\partial_{c\tilde k}\,x_{d\tilde\jmath}\big)\\
&\quad\times\big(-E_{ab}(v)\otimes\theta_i\,\theta_k\,x_{b\tilde k}\,\partial_{a\tilde\imath}\big).
\end{aligned} \tag{3.24}
$$

Consider the sum of the expressions (3.23) over the running indices e,f. We add this sum to the expression displayed in the five lines in (3.18). Using the relation

$$
\sum_e\,\widetilde{E}_{eb}(v)\,\widetilde{E}_{ae}(w)=\frac{\widetilde{E}_{ab}(w)-\widetilde{E}_{ab}(v)}{2u} \tag{3.25}
$$

together with (3.13), and performing cancellations, we get the expression

$$\begin{aligned}&\big(\pm F_{-d,c}(v+l)\otimes\theta_k\,x_{b\tilde{k}}\,x_{ck}\,x_{di}\,\partial_{aj}+F_{-d,-c}(v+l)\otimes x_{bk}\,\partial_{ck}\,x_{di}\,\partial_{aj}\\&\qquad+F_{d,-c}(v+l)\otimes\theta_i\,x_{bk}\,\partial_{ck}\,\partial_{d\tilde{\imath}}\,\partial_{aj}\pm F_{dc}(v+l)\otimes\theta_i\,\theta_k\,x_{b\tilde{k}}\,x_{ck}\,\partial_{d\tilde{\imath}}\,\partial_{aj}\big)\\&\quad\times\big((1+Z(v))\,\widetilde{E}_{ab}(w)\otimes 1\big).\end{aligned}$$

After exchanging the running indices c and d, the sum over the index k of the expressions in the last three displayed lines can be rewritten as

$$\delta_{cd}\,(1+Z(v))\,\big(F_{-c,-d}(v+l)+F_{cd}(v+l)\big)\,\widetilde{E}_{ab}(w)\otimes x_{bi}\,\partial_{aj} \tag{3.26}$$

plus the sum over k of the expressions

$$\begin{aligned}&\big(F_{-c,-d}(v+l)\otimes x_{ci}\,\partial_{dk}+F_{-c,d}(v+l)\otimes\theta_k\,x_{ci}\,x_{d\tilde{k}}\\&\qquad+F_{c,-d}(v+l)\otimes\theta_i\,\partial_{c\tilde{\imath}}\,\partial_{dk}+F_{cd}(v+l)\otimes\theta_i\,\theta_k\partial_{c\tilde{\imath}}\,x_{d\tilde{k}}\big)\\&\quad\times\big(-(1+Z(v))\,\widetilde{E}_{ab}(w)\otimes x_{bk}\,\partial_{aj}\big).\end{aligned}\tag{3.27}$$

Again, here we have used the commutation relations in the ring $\mathcal{GD}(\mathbb{C}^{m+l}\otimes\mathbb{C}^n)$.

Now we perform summation over all running indices in the four expressions (3.19), (3.22), (3.24), (3.27) and then take their total. By exchanging the running indices b and f in (3.22), and by replacing the running index k in (3.24), (3.27) by g,h (respectively), the total can be written as the sum over the indices c,d and g,h of the expressions

$$\begin{aligned}&\big((1+Z(v))\otimes 1\big)\big(F_{-c,-d}(v+l)\otimes x_{cg}\,\partial_{dh}+F_{-c,d}(v+l)\otimes\theta_h\,x_{cg}\,x_{d\tilde{h}}\\&\qquad+F_{c,-d}(v+l)\otimes\theta_g\,\partial_{c\tilde{g}}\,\partial_{dh}+F_{cd}(v+l)\otimes\theta_g\,\theta_h\,\partial_{c\tilde{g}}\,x_{d\tilde{h}}\big)\\&\quad\times\Big(\delta_{ig}-\sum_{e,f}\widetilde{E}_{ef}(v)\otimes\theta_i\,\theta_g\,x_{f\tilde{g}}\,\partial_{e\tilde{\imath}}\Big)\Big(\delta_{hj}-\sum_{a,b}\widetilde{E}_{ab}(w)\otimes x_{bh}\,\partial_{aj}\Big).\end{aligned}\tag{3.28}$$

We perform summation in (3.21) over the running indices b,e. Then we replace the running index f by the index b, which becomes free after summation. By adding the resulting sum to the expression (3.17), we get

$$-(1+Z(v))\big(W(v+l)+1\big)\,\widetilde{E}_{ab}(w)\otimes x_{bi}\,\partial_{aj},$$

by (3.13) and (3.25). Performing summation in (3.26) over the running indices c,d and then adding the result to the last displayed expression, we get

$$-(1+Z(v))\,\widetilde{E}_{ab}(w)\otimes x_{bi}\,\partial_{aj}. \tag{3.29}$$

Now we sum over all running indices in the two expressions (3.20), (3.29) and then add the two resulting sums to (3.16). By using (3.13) once again, the total can be written as the sum over the index k of the expressions

$$\begin{aligned}&\big((1+Z(v))\otimes 1\big)\\&\quad\times\Big(\delta_{ik}-\sum_{e,f}\widetilde{E}_{ef}(v)\otimes\theta_i\,\theta_k\,x_{f\tilde{k}}\,\partial_{e\tilde{\imath}}\Big)\Big(\delta_{kj}-\sum_{a,b}\widetilde{E}_{ab}(w)\otimes x_{bk}\,\partial_{aj}\Big).\end{aligned}\tag{3.30}$$

By the definition of the series $\widetilde{E}_{ab}(v)$, as given before (3.13), we have

$$\begin{aligned}\widetilde{E}_{ef}(v)&=(v+l+E')^{-1}_{fe}=-\Big(-u\mp\frac{1}{2}+m-E'\Big)^{-1}_{fe},\\\widetilde{E}_{ab}(w)&=(w+l+E')^{-1}_{ba}=-\Big(u\mp\frac{1}{2}+m-E'\Big)^{-1}_{ba}.\end{aligned}$$

We have also used the definitions (3.11). Hence, the sum of the expressions (3.28) over the indices c,d and g,h plus the sum of the expressions (3.30) over the index k can be rewritten as the sum over the indices g,h of the following series in u^{-1}:

$$
\begin{aligned}
&\Bigl(\Bigl(1+Z\Bigl(u\pm\frac{1}{2}-m-l\Bigr)\Bigr)\otimes 1\Bigr)\\
&\quad\times\Bigl(\delta_{gh}+\sum_{c,d}\Bigl(F_{-c,-d}\Bigl(u\pm\frac{1}{2}-m\Bigr)\otimes x_{cg}\,\partial_{dh}+F_{-c,d}\Bigl(u\pm\frac{1}{2}-m\Bigr)\otimes\theta_h\,x_{cg}\,x_{d\tilde{h}}\\
&\qquad\qquad+F_{c,-d}\Bigl(u\pm\frac{1}{2}-m\Bigr)\otimes\theta_g\,\partial_{c\tilde{g}}\,\partial_{dh}+F_{cd}\Bigl(u\pm\frac{1}{2}-m\Bigr)\otimes\theta_g\,\theta_h\,\partial_{c\tilde{g}}\,x_{d\tilde{h}}\Bigr)\Bigr)\\
&\quad\times\Bigl(\delta_{ig}+\sum_{e,f}\Bigl(-u\mp\frac{1}{2}+m-E'\Bigr)^{-1}_{fe}\otimes\theta_i\,\theta_g\,x_{f\tilde{g}}\,\partial_{e\tilde{\imath}}\Bigr)\\
&\quad\times\Bigl(\delta_{hj}+\sum_{a,b}\Bigl(u\mp\frac{1}{2}+m-E'\Bigr)^{-1}_{ba}\otimes x_{bh}\,\partial_{aj}\Bigr)
\end{aligned}
$$

with coefficients in the algebra $\mathrm{U}(\mathfrak{f}_m\oplus\mathfrak{gl}_l)\otimes\mathcal{GD}(\mathbb{C}^{m+l}\otimes\mathbb{C}^n)$. By mapping the series $S_{ij}(u)$ to this sum, we describe the action of the extended twisted Yangian $\mathrm{X}(\mathfrak{g}_n)$ on the subspace (3.14) modulo $\mathfrak{q}\cdot M$. Comparing this sum with the product of the series (3.4) and (3.5) with $z=m\mp\frac{1}{2}$, we see that the map χ intertwines the actions of $\mathrm{X}(\mathfrak{g}_n)$; we have used (1.26) and (2.8). This completes the proof of Theorem 3.1. □

§4. Zhelobenko operators

Consider the *hyperoctahedral group* $\mathfrak{H}_m$. This is the semidirect product of the symmetric group $\mathfrak{S}_m$ and the Abelian group $\mathbb{Z}_2^m$, where $\mathfrak{S}_m$ acts by permutations of m copies of $\mathbb{Z}_2$. In this section, we assume that $m>0$. The group $\mathfrak{H}_m$ is generated by the elements σ_a with $a=1,\dots,m$. The elements σ_a with indices $a=1,\dots,m-1$ are elementary transpositions generating the symmetric group $\mathfrak{S}_m$, so that $\sigma_a=(a,a+1)$. Then σ_m is the generator of the mth factor $\mathbb{Z}_2$ of $\mathbb{Z}_2^m$. The elements $\sigma_1,\dots,\sigma_m\in\mathfrak{H}_m$ are involutions and satisfy the braid relations

$$
\begin{aligned}
\sigma_a\,\sigma_{a+1}\,\sigma_a&=\sigma_{a+1}\,\sigma_a\,\sigma_{a+1} &&\text{for}\quad a=1,\dots,m-2;\\
\sigma_a\,\sigma_b&=\sigma_b\,\sigma_a &&\text{for}\quad a=1,\dots,b-2;\\
\sigma_{m-1}\,\sigma_m\,\sigma_{m-1}\,\sigma_m&=\sigma_m\,\sigma_{m-1}\,\sigma_m\,\sigma_{m-1}.
\end{aligned}
$$

Note that $\mathfrak{H}_m$ is the Weyl group of the simple Lie algebra $\mathfrak{sp}_{2m}$. Let $\mathfrak{B}_m$ be the braid group corresponding to $\mathfrak{sp}_{2m}$. It is generated by elements $\widetilde{\sigma}_1,\dots,\widetilde{\sigma}_m$ that, by definition, satisfy the above displayed relations, instead of the involutions $\sigma_1,\dots,\sigma_m$, respectively. For any reduced decomposition $\sigma=\sigma_{a_1}\cdots\sigma_{a_K}$ in $\mathfrak{H}_m$ put

$$\widetilde{\sigma}=\widetilde{\sigma}_{a_1}\cdots\widetilde{\sigma}_{a_K}. \tag{4.1}$$

The definition of $\widetilde{\sigma}$ is independent of the choice of a reduced decomposition of σ.

The group $\mathfrak{H}_m$ also contains the Weyl group of the reductive Lie algebra $\mathfrak{so}_{2m}$ as a subgroup of index two. This subgroup $\mathfrak{H}'_m$ is generated by the elementary transpositions $\sigma_1,\dots,\sigma_{m-1}$ and by the involution $\sigma'_m=\sigma_m\,\sigma_{m-1}\,\sigma_m$. Along with the braid relations among $\sigma_1,\dots,\sigma_{m-1}$, we also have braid relations involving σ'_m:

$$
\begin{aligned}
\sigma_a\,\sigma'_m&=\sigma'_m\,\sigma_a &&\text{for}\quad a=1,\dots,m-3,m-1;\\
\sigma_{m-2}\,\sigma'_m\,\sigma_{m-2}&=\sigma'_m\,\sigma_{m-2}\,\sigma'_m.
\end{aligned}
$$

For $m>1$, the braid group of $\mathfrak{so}_{2m}$ is generated by m elements satisfying the same braid relations instead of the m involutions $\sigma_1,\dots,\sigma_{m-1},\sigma'_m$, respectively. When $m=1$, the braid group corresponding to $\mathfrak{f}_m=\mathfrak{so}_2$ consists of the identity element only.

Now, let the indices c, d run through $-m, \dots, -1, 1, \dots, m$. For $c > 0$ we denote $\bar{c} = m + 1 - c$; for $c < 0$ put $\bar{c} = -m - 1 - c$. Consider a representation $\sigma \mapsto \bar{\sigma}$ of the group $\mathfrak{H}_m$ by permutations of $-m, \dots, -1, 1, \dots, m$ such that

$$\bar{\sigma}(c) = \overline{\sigma(\bar{c})} \quad \text{for} \quad \sigma \in \mathfrak{S}_m \tag{4.2}$$

and $\bar{\sigma}_m(c) = -c$ if $|c| = 1$, while $\bar{\sigma}_m(c) = c$ if $|c| > 1$. We can define an action of the braid group $\mathfrak{B}_m$ by automorphisms of the Lie algebra $\mathfrak{f}_m$, by the assignments

$$\widetilde{\sigma} \,:\, F_{cd} \mapsto F_{\bar{\sigma}(c)\bar{\sigma}(d)} \quad \text{for} \quad \sigma \in \mathfrak{S}_m, \tag{4.3}$$

$$\widetilde{\sigma}_m \,:\, F_{cd} \mapsto (\pm 1)^{\delta_{c1} + \delta_{d1}} F_{\bar{\sigma}_m(c)\bar{\sigma}_m(d)}; \tag{4.4}$$

cf. [T]. In accordance with our convention on double signs, the upper sign in $\pm$ corresponds to $\mathfrak{f}_m = \mathfrak{so}_{2m}$, while the lower sign corresponds to $\mathfrak{f}_m = \mathfrak{sp}_{2m}$. The automorphism property can be checked by using relations (2.3); see the proof of statement (i) in Lemma 4.1 below. This action of the group $\mathfrak{B}_m$ on $\mathfrak{f}_m$ extends to an action of $\mathfrak{B}_m$ by automorphisms of the associative algebra $\mathrm{U}(\mathfrak{f}_m)$. Note that if $\mathfrak{f}_m = \mathfrak{so}_{2m}$, then the action of $\mathfrak{B}_m$ on $\mathrm{U}(\mathfrak{f}_m)$ factors through an action of the group $\mathfrak{H}_m$.

Next, an action of the braid group $\mathfrak{B}_m$ by automorphisms of the algebra $\mathcal{GD}(\mathbb{C}^m \otimes \mathbb{C}^n)$ can be defined in the following way. Put

$$\begin{aligned} &\widetilde{\sigma}(x_{ai}) = x_{\bar{\sigma}(a)i} && \text{and} && \widetilde{\sigma}(\partial_{ai}) = \partial_{\bar{\sigma}(a)i} && \text{for} && \sigma \in \mathfrak{S}_m\,, \\ &\widetilde{\sigma}_m(x_{ai}) = x_{ai} && \text{and} && \widetilde{\sigma}_m(\partial_{ai}) = \partial_{ai} && \text{for} && a > 1, \\ &\widetilde{\sigma}_m(x_{1i}) = \theta_i\, \partial_{1\tilde{\imath}} && \text{and} && \widetilde{\sigma}_m(\partial_{1i}) = \theta_i\, x_{1\tilde{\imath}}, \end{aligned} \tag{4.5}$$

where $i = 1, \dots, n$.

Note that in the case where $\mathfrak{f}_m = \mathfrak{so}_{2m}$, the element $\widetilde{\sigma}_m^{\,2} \in \mathfrak{B}_m$ acts on x_{1i} and on ∂_{1i} as the identity, so that the action of $\mathfrak{B}_m$ on $\mathcal{GD}(\mathbb{C}^m \otimes \mathbb{C}^n)$ factors through an action of the group $\mathfrak{H}_m$. But if $\mathfrak{f}_m = \mathfrak{sp}_{2m}$, then the element $\widetilde{\sigma}_m^{\,2}$ acts on x_{1i} and on ∂_{1i} as minus the identity, because $\theta_i\, \theta_{\tilde{\imath}} = -1$ in this case. This is why we use the braid group, rather than the Weyl group $\mathfrak{H}_m$ of the simple Lie algebra $\mathfrak{sp}_{2m}$. Taking the tensor product of the actions of $\mathfrak{B}_m$ on the algebras $\mathrm{U}(\mathfrak{f}_m)$ and $\mathcal{GD}(\mathbb{C}^m \otimes \mathbb{C}^n)$, we get an action of $\mathfrak{B}_m$ by automorphisms of the algebra $\mathrm{B}_m = \mathrm{U}(\mathfrak{f}_m) \otimes \mathcal{GD}(\mathbb{C}^m \otimes \mathbb{C}^n)$.

Lemma 4.1. (i) *The map $\zeta_n : \mathrm{U}(\mathfrak{f}_m) \to \mathcal{GD}(\mathbb{C}^m \otimes \mathbb{C}^n)$ is $\mathfrak{B}_m$-equivariant.*
(ii) *The action of $\mathfrak{B}_m$ on B_m leaves invariant any element of the image of $\mathrm{X}(\mathfrak{g}_n)$ under the homomorphism β_m.*

Proof. We employ the elements p_{ci} and q_{ci} of $\mathcal{GD}(\mathbb{C}^m \otimes \mathbb{C}^n)$, introduced immediately after stating Proposition 2.3. In terms of these elements, the action of $\mathfrak{B}_m$ on the algebra $\mathcal{GD}(\mathbb{C}^m \otimes \mathbb{C}^n)$ can be described by setting

$$\begin{aligned} &\widetilde{\sigma}(p_{ci}) = p_{\bar{\sigma}(c)i} && \text{and} && \widetilde{\sigma}(q_{ci}) = q_{\bar{\sigma}(c)i} && \text{for} && \sigma \in \mathfrak{S}_m\,, \\ &\widetilde{\sigma}_m(p_{ci}) = (\pm 1)^{\delta_{c1}} p_{\bar{\sigma}_m(c)i} && \text{and} && \widetilde{\sigma}_m(q_{ci}) = (\pm 1)^{\delta_{c1}} q_{\bar{\sigma}_m(c)i}, \end{aligned}$$

where $c = -m, \dots, -1, 1, \dots, m$. Statement (i) follows by comparing our definition of the action of $\mathfrak{B}_m$ on $\mathfrak{f}_m$ with the description (2.10) of the homomorphism ζ_n, and (ii) follows similarly, with the help of the description (2.9) of β_m. $\square$

Consider the Cartan subalgebra $\mathfrak{h}$ occurring in the triangular decomposition (2.17). In the notation of this section, our chosen basis of $\mathfrak{h}$ is $(F_{-\bar{a},-\bar{a}} \,|\, a = 1, \dots, m)$. Now, let $(\varepsilon_a \,|\, a = 1, \dots, m) \subset \mathfrak{h}^*$ be the dual basis, so that $\varepsilon_b(F_{-\bar{a},-\bar{a}}) = \delta_{ab}$. For $c < 0$ put $\varepsilon_c = -\varepsilon_{-c}$. Thus, the element $\varepsilon_c \in \mathfrak{h}^*$ is defined for every index $c = -m, \dots, -1, 1, \dots, m$.

Consider the root system of the Lie algebra $\mathfrak{f}_m$ in $\mathfrak{h}^*$. Put

$$\eta_a = \varepsilon_a - \varepsilon_{a+1} \quad \text{for} \quad a = 1, \dots, m - 1.$$

Also, put $\eta_m = \varepsilon_{m-1} + \varepsilon_m$ if $\mathfrak{f}_m = \mathfrak{so}_{2m}$, and $\eta_m = 2\varepsilon_m$ if $\mathfrak{f}_m = \mathfrak{sp}_{2m}$. Then $\eta_1, \dots, \eta_m$ are *simple roots* of $\mathfrak{f}_m$. Denote by Δ^+ the set of *positive roots* of $\mathfrak{f}_m$. These are the weights $\varepsilon_a - \varepsilon_b$ and $\varepsilon_a + \varepsilon_b$, where $1 \le a < b \le m$ if $\mathfrak{f}_m = \mathfrak{so}_{2m}$, and the same weights together with $2\varepsilon_a$, where $1 \le a \le m$ if $\mathfrak{f}_m = \mathfrak{sp}_{2m}$. We assume that in the case where $\mathfrak{f}_m = \mathfrak{so}_2$ the root system of $\mathfrak{f}_m$ is empty. Let ρ be the half-sum of the positive roots of $\mathfrak{f}_m$, so that its sequence of labels $(\rho_1, \dots, \rho_m)$ is $(m-1, \dots, 0)$ if $\mathfrak{f}_m = \mathfrak{so}_{2m}$, and $(m, \dots, 1)$ if $\mathfrak{f}_m = \mathfrak{sp}_{2m}$. For each $a = 1, ..., m-1$, we put

$$E_a = F_{-\bar{a},-\overline{a+1}}\,, \quad F_a = F_{-\overline{a+1},-\bar{a}}\,, \quad H_a = F_{-\bar{a},-\bar{a}} - F_{-\overline{a+1},-\overline{a+1}}. \tag{4.6}$$

Let

$$E_m = F_{-\overline{m-1},\overline{m}}, \quad F_m = F_{\overline{m},-\overline{m-1}}, \quad H_m = F_{-\overline{m-1},-\overline{m-1}} + F_{-\overline{m},-\overline{m}} \tag{4.7}$$

in the case where $\mathfrak{f}_m = \mathfrak{so}_{2m}$ with $m > 1$. In the case where $\mathfrak{f}_m = \mathfrak{sp}_{2m}$, let

$$E_m = F_{-\overline{m},\overline{m}}\,/2, \quad F_m = F_{\overline{m},-\overline{m}}\,/2, \quad H_m = F_{-\overline{m},-\overline{m}}\,. \tag{4.8}$$

For every possible index a, the three elements E_a, F_a, H_a of the Lie algebra $\mathfrak{f}_m$ span a subalgebra isomorphic to $\mathfrak{sl}_2$. They satisfy the commutation relations

$$[E_a, F_a] = H_a, \quad [H_a, E_a] = 2E_a, \quad [H_a, F_a] = -2F_a. \tag{4.9}$$

So far we have denoted by B_m the associative algebra $\mathrm{U}(\mathfrak{f}_m) \otimes \mathcal{GD}(\mathbb{C}^m \otimes \mathbb{C}^n)$. Now we use a different presentation of the same algebra. Namely, from now on until the end of the next section, we regard B_m as the associative algebra generated by the algebras $\mathrm{U}(\mathfrak{f}_m)$ and $\mathcal{GD}(\mathbb{C}^m \otimes \mathbb{C}^n)$ with the cross relations

$$[X, Y] = [\zeta_n(X), Y] \tag{4.10}$$

for any $X \in \mathfrak{f}_m$ and $Y \in \mathcal{GD}(\mathbb{C}^m \otimes \mathbb{C}^n)$. The brackets on the left-hand side of (4.10) denote the commutator in B_m, and the brackets on the right-hand side denote the commutator in the algebra $\mathcal{GD}(\mathbb{C}^m \otimes \mathbb{C}^n)$ embedded in B_m. In particular, we regard $\mathrm{U}(\mathfrak{f}_m)$ as a subalgebra of B_m. An isomorphism of this B_m with the tensor product $\mathrm{U}(\mathfrak{f}_m) \otimes \mathcal{GD}(\mathbb{C}^m \otimes \mathbb{C}^n)$ can be defined by mapping elements $X \in \mathfrak{f}_m$ and $Y \in \mathcal{GD}(\mathbb{C}^m \otimes \mathbb{C}^n)$ of B_m (respectively) to the elements

$$X \otimes 1 + 1 \otimes \zeta_n(X) \quad \text{and} \quad 1 \otimes Y$$

of $\mathrm{U}(\mathfrak{f}_m) \otimes \mathcal{GD}(\mathbb{C}^m \otimes \mathbb{C}^n)$. Here we have used (2.6). The action of the braid group $\mathfrak{B}_m$ on B_m is defined via its isomorphism with $\mathrm{U}(\mathfrak{f}_m) \otimes \mathcal{GD}(\mathbb{C}^m \otimes \mathbb{C}^n)$. Since the map ζ_n is $\mathfrak{B}_m$-equivariant, the same action of $\mathfrak{B}_m$ is obtained by extending the actions of $\mathfrak{B}_m$ from the subalgebras $\mathrm{U}(\mathfrak{f}_m)$ and $\mathcal{GD}(\mathbb{C}^m \otimes \mathbb{C}^n)$ to B_m.

Now consider the following two sets of elements of the algebra $\mathrm{U}(\mathfrak{h}) \subset \mathrm{U}(\mathfrak{f}_m)$:

$$\{\, F_{aa} - F_{bb} + z,\ F_{aa} + F_{bb} + z \mid 1 \le a < b \le m,\ z \in \mathbb{Z}\}\,, \tag{4.11}$$

$$\{\, F_{aa} + z \mid 1 \le a \le m,\ z \in \mathbb{Z}\}. \tag{4.12}$$

In the case where $\mathfrak{f}_m = \mathfrak{so}_{2m}$, we denote by $\overline{\mathrm{U}(\mathfrak{h})}$ the ring of fractions of the commutative algebra $\mathrm{U}(\mathfrak{h})$ relative to the set of denominators (4.11). For $\mathfrak{f}_m = \mathfrak{sp}_{2m}$, we denote by $\overline{\mathrm{U}(\mathfrak{h})}$ the ring of fractions of $\mathrm{U}(\mathfrak{h})$ relative to the union of the sets (4.11) and (4.12). The elements of the ring $\overline{\mathrm{U}(\mathfrak{h})}$ can also be regarded as rational functions on the vector space $\mathfrak{h}^*$. The elements of the subalgebra $\mathrm{U}(\mathfrak{h}) \subset \overline{\mathrm{U}(\mathfrak{h})}$ are then regarded as polynomial functions on $\mathfrak{h}^*$.

Denote by $\bar{\mathrm{B}}_m$ the ring of fractions of B_m relative to the same set of denominators as was used to define the ring of fractions $\overline{\mathrm{U}(\mathfrak{h})}$. But now we regard these denominators as elements of B_m, using the embedding of $\mathfrak{h} \subset \mathfrak{f}_m$ into B_m. The ring $\bar{\mathrm{B}}_m$ is defined due to the following relations in B_m. For $c<0$ put $\varepsilon_c=-\varepsilon_{-c}$. Thus, the element $\varepsilon_c\in\mathfrak{h}^*$ is defined for every $c=-m,\dots,-1,1,\dots,m$. Then for any element $H\in\mathfrak{h}$ we have

$$\begin{aligned}
&[H,F_{cd}]=(\varepsilon_{\bar d}-\varepsilon_{\bar c})(H)F_{cd} &&\text{for}\quad c,d=-m,\dots,-1,1,\dots,m;\\
&[H,x_{ci}]=-\varepsilon_{\bar c}(H)\,x_{ci}\quad\text{and}\quad [H,\partial_{ci}]=\varepsilon_{\bar c}(H)\,\partial_{ci} &&\text{for}\quad c=1,\dots,m.
\end{aligned}$$

So, the ring B_m obeys the Ore condition relative to our set of denominators. Using left multiplication by elements of $\overline{\mathrm{U}(\mathfrak{h})}$, we turn the ring $\bar{\mathrm{B}}_m$ into a $\overline{\mathrm{U}(\mathfrak{h})}$-module.

The ring $\bar{\mathrm{B}}_m$ is also an associative algebra over $\mathbb{C}$. The action of the braid group $\mathfrak{B}_m$ on B_m preserves the set of denominators, so that $\mathfrak{B}_m$ also acts by automorphisms of the algebra $\bar{\mathrm{B}}_m$. Using the elements (4.6) and (4.7) if $\mathfrak{f}_m=\mathfrak{so}_{2m}$, or the elements (4.6) and (4.8) if $\mathfrak{f}_m=\mathfrak{sp}_{2m}$, for every simple root η_a of $\mathfrak{f}_m$ we can define a linear map

$$\xi_a\,:\,\mathrm{B}_m\to\bar{\mathrm{B}}_m$$

by setting

$$\xi_a(Y)=Y+\sum_{s=1}^{\infty}\,(s!\,H_a^{(s)})^{-1}E_a^s\,\widehat{F}_a^s(Y),\tag{4.13}$$

where

$$H_a^{(s)}=(H_a)(H_a-1)\cdots(H_a-s+1)$$

and $\widehat{F}_a$ is the operator of adjoint action corresponding to the element $F_a\in\mathrm{B}_m$,

$$\widehat{F}_a(Y)=[F_a,Y].$$

For a given element $Y\in\mathrm{B}_m$, only finitely many terms of the sum (4.13) differ from zero. In the case where $\mathfrak{f}_m=\mathfrak{so}_2$, there are no roots of $\mathfrak{f}_m$, and no corresponding operators $\mathrm{B}_m\to\bar{\mathrm{B}}_m$. On the other hand, if $\mathfrak{f}_m=\mathfrak{so}_{2m}$ with $m>1$, then, by (4.4),

$$\xi_m\,\widetilde{\sigma}_m=\widetilde{\sigma}_m\,\xi_{m-1},$$

because

$$\widetilde{\sigma}_m\,:\,E_{m-1}\mapsto E_m,\quad F_{m-1}\mapsto F_m,\quad H_{m-1}\mapsto H_m.$$

Let J and $\bar{\mathrm{J}}$ be the right ideals of the algebras B_m and $\bar{\mathrm{B}}_m$ (respectively) generated by all elements of the subalgebra $\mathfrak{n}\subset\mathfrak{f}_m$. The following two properties of the linear operator ξ_a go back to [Z, §2]. For any elements $X\in\mathfrak{h}$ and $Y\in\mathrm{B}_m$,

$$\begin{aligned}
\xi_a(XY)&\in(X+\eta_a(X))\,\xi_a(Y)+\bar{\mathrm{J}},\\
\xi_a(YX)&\in\,\xi_a(Y)(X+\eta_a(X))+\bar{\mathrm{J}}.
\end{aligned}\tag{4.14}$$

See [KN1, §3] for detailed proofs of these two properties. The proofs employ only the commutation relations (4.9), not the actual form of the elements E_a, F_a, H_a.

Property (4.14) allows us to define a linear map $\bar{\xi}_a:\bar{\mathrm{B}}_m\to\bar{\mathrm{J}}\setminus\bar{\mathrm{B}}_m$ by

$$\bar{\xi}_a(X\,Y)=Z\,\xi_a(Y)+\bar{\mathrm{J}}\quad\text{for}\quad X\in\overline{\mathrm{U}(\mathfrak{h})}\quad\text{and}\quad Y\in\mathrm{B}_m,$$

where the element $Z\in\overline{\mathrm{U}(\mathfrak{h})}$ is defined by the relation

$$Z(\mu)=X(\mu+\eta_a)\quad\text{for}\quad\mu\in\mathfrak{h}^*,$$

and both X and Z are regarded as rational functions on $\mathfrak{h}^*$. The backslash in $\bar{\mathrm{J}}\setminus\bar{\mathrm{B}}_m$ indicates that the quotient is taken relative to a *right* ideal of $\bar{\mathrm{B}}_m$. For the proofs of the next two propositions, see [KN3, §4].

Proposition 4.2. *For any simple root η_a of $\mathfrak{f}_m$ we have the inclusion $\widetilde{\sigma}(\bar{\mathrm{J}})\subset\ker\bar{\xi}_a$, where $\sigma=\sigma_a$ unless $\mathfrak{f}_m=\mathfrak{so}_{2m}$ and $a=m$, in which case $\sigma=\sigma'_m$.*

Recall that $\mathfrak{n}'$ denotes the nilpotent subalgebra of $\mathfrak{f}_m$ spanned by all the elements F_{cd} with $c<d$. The relation $F_{cd}=-\,\varepsilon_{cd}\,F_{-d,-c}$ shows that the subalgebra $\mathfrak{n}'$ is also spanned by the elements F_{cd} with $c<d$ and $c<0$. Now, for any $a=1,\dots,m$, denote by $\mathfrak{n}'_a$ the vector subspace of $\mathfrak{f}_m$ spanned by all the elements F_{cd} with $c<d$ and $c<0$, except the element E_a. Let J' be the left ideal of B_m generated by the elements $X-\zeta_n(X)$ with $X\in\mathfrak{n}'$. Under the isomorphism of B_m with $\mathrm{U}(\mathfrak{f}_m)\otimes\mathcal{GD}(\mathbb{C}^m\otimes\mathbb{C}^n)$, for any $X\in\mathfrak{f}_m$ the difference $X-\zeta_n(X)\in\mathrm{B}_m$ is mapped to the element

$$X\otimes 1\in\mathrm{U}(\mathfrak{f}_m)\otimes 1\subset\mathrm{U}(\mathfrak{f}_m)\otimes\mathcal{GD}(\mathbb{C}^m\otimes\mathbb{C}^n). \tag{4.15}$$

Let J'_a be the left ideal of B_m generated by the elements $X-\zeta_n(X)$ with $X\in\mathfrak{n}'_a$, and by the element $E_a\in\mathrm{B}_m$. Denote $\bar{\mathrm{J}}'=\overline{\mathrm{U}(\mathfrak{h})}\mathrm{J}'$ and $\bar{\mathrm{J}}'_a=\overline{\mathrm{U}(\mathfrak{h})}\mathrm{J}'_a$. Then both $\bar{\mathrm{J}}'$ and $\bar{\mathrm{J}}'_a$ are left ideals of the algebra $\bar{\mathrm{B}}_m$.

Proposition 4.3. *For any simple root η_a of $\mathfrak{f}_m$ we have $\bar{\xi}_a(\,\tilde{\sigma}\,(\bar{\mathrm{J}}'_a))\subset\bar{\mathrm{J}}'+\bar{\mathrm{J}}$, where $\sigma=\sigma_a$ unless $\mathfrak{f}_m=\mathfrak{so}_{2m}$ and $a=m$, in which case $\sigma=\sigma'_m$.*

Proposition 4.2 allows us, for any simple root η_a, to define a linear map

$$\check{\xi}_a\,:\,\bar{\mathrm{J}}\setminus\bar{\mathrm{B}}_m\to\bar{\mathrm{J}}\setminus\bar{\mathrm{B}}_m$$

as the composition $\bar{\xi}_a\,\tilde{\sigma}$ applied to the elements of $\bar{\mathrm{B}}_m$ taken modulo $\bar{\mathrm{J}}$. Here the simple reflection $\sigma\in\mathfrak{H}_m$ is chosen as in Proposition 4.2. In their present form, the operators $\check{\xi}_1,\dots,\check{\xi}_m$ on the vector space $\bar{\mathrm{J}}\setminus\bar{\mathrm{B}}_m$ were defined in [KO]. We call them the *Zhelobenko operators*. For the proof of the next proposition, see [KO, §§4 and 6].

Proposition 4.4. *The Zhelobenko operators satisfy the braid relations corresponding to the Lie algebra $\mathfrak{f}_m$. Namely, if $\mathfrak{f}_m=\mathfrak{sp}_{2m}$, then we have*

$$\check{\xi}_a\,\check{\xi}_{a+1}\,\check{\xi}_a=\check{\xi}_{a+1}\,\check{\xi}_a\,\check{\xi}_{a+1}\qquad\textit{for}\quad a=1,\dots,m-2; \tag{4.16}$$

$$\check{\xi}_a\,\check{\xi}_b=\check{\xi}_b\,\check{\xi}_a\qquad\textit{for}\quad a=1,\dots,b-2; \tag{4.17}$$

$$\check{\xi}_{m-1}\,\check{\xi}_m\,\check{\xi}_{m-1}\,\check{\xi}_m=\check{\xi}_m\,\check{\xi}_{m-1}\,\check{\xi}_m\,\check{\xi}_{m-1}.$$

If $\mathfrak{f}_m=\mathfrak{so}_{2m}$ and $m>1$, then we have the same relations (4.16) *and* (4.17) *among $\check{\xi}_1,\dots,\check{\xi}_{m-1}$ as in the case of $\mathfrak{f}_m=\mathfrak{sp}_{2m}$ above, and also the relations*

$$\begin{gathered}\check{\xi}_a\,\check{\xi}_m=\check{\xi}_m\,\check{\xi}_a\quad\textit{for}\quad a=1,\dots,m-3,m-1;\\ \check{\xi}_{m-2}\,\check{\xi}_m\,\check{\xi}_{m-2}=\check{\xi}_m\,\check{\xi}_{m-2}\,\check{\xi}_m.\end{gathered} \tag{4.18}$$

For $\mathfrak{f}_m=\mathfrak{sp}_{2m}$, by using any reduced decomposition of an element $\sigma\in\mathfrak{H}_m$ in terms of the involutions $\sigma_1,\dots,\sigma_m$, we can define a linear operator

$$\check{\xi}_\sigma\,:\,\bar{\mathrm{J}}\setminus\bar{\mathrm{B}}_m\to\bar{\mathrm{J}}\setminus\bar{\mathrm{B}}_m \tag{4.19}$$

in the usual way, as in (4.1). By Proposition 4.4, this definition of $\check{\xi}_\sigma$ is independent of the choice of a reduced decomposition of σ.

When $\mathfrak{f}_m=\mathfrak{sp}_{2m}$, the number of the factors $\sigma_1,\dots,\sigma_m$ in any reduced decomposition $\sigma\in\mathfrak{H}_m$ will be denoted $\ell(\sigma)$. This number is also independent of the choice of a decomposition and is equal to the number of elements in the set

$$\Delta_\sigma=\{\eta\in\Delta^+\,|\,\sigma(\eta)\notin\Delta^+\,\}, \tag{4.20}$$

where Δ^+ denotes the set of positive roots of the Lie algebra $\mathfrak{sp}_{2m}$.

Now suppose that $\mathfrak{f}_m=\mathfrak{so}_{2m}$. Then we can use any reduced decomposition in terms of $\sigma_1,\dots,\sigma_{m-1},\sigma'_m$ to define a linear operator (4.19) for every element $\sigma\in\mathfrak{H}'_m$. Again, this definition is independent of the choice of a reduced decomposition of σ, by Proposition 4.4. It turns out that in this case we can extend the definition of the operator (4.19) to any element $\sigma \in \mathfrak{H}_m$, where $m \geq 1$. Note that in this case the action of the element $\widetilde{\sigma}_m$ on $\bar{\mathrm{B}}_m$ preserves the ideal $\bar{\mathrm{J}}$, and therefore induces a linear operator on the quotient vector space $\bar{\mathrm{J}} \setminus \bar{\mathrm{B}}_m$. This operator will still be denoted by $\widetilde{\sigma}_m$. The extension of the definition of the operators (4.19) to $\sigma \in \mathfrak{H}_m$ is based on the next lemma, which was proved in [KN3, §4].

Lemma 4.5. *If $\mathfrak{f}_m = \mathfrak{so}_{2m}$ and $m > 1$, then the operators $\check{\xi}_1, \dots, \check{\xi}_{m-1}, \widetilde{\sigma}_m$ on $\bar{\mathrm{J}} \setminus \bar{\mathrm{B}}_m$ satisfy the same relations as the m generators of the braid group $\mathfrak{B}_m$, respectively. Also, we have the relation*

$$\check{\xi}_m = \widetilde{\sigma}_m \, \check{\xi}_{m-1} \, \widetilde{\sigma}_m . \tag{4.21}$$

Now, if $\mathfrak{f}_m = \mathfrak{so}_{2m}$ with any $m \geq 1$, take any decomposition of an element $\sigma \in \mathfrak{H}_m$ in terms of the involutions $\sigma_1, \dots, \sigma_m$ such that the number of occurrences of $\sigma_1, \dots, \sigma_{m-1}$ in the decomposition is the minimal possible. For $\mathfrak{f}_m = \mathfrak{so}_{2m}$, the symbol $\ell(\sigma)$ will denote this minimal number. Note that unlike for $\mathfrak{f}_m = \mathfrak{sp}_{2m}$, here we do not count the occurrences of σ_m in the decomposition. All the decompositions of $\sigma \in \mathfrak{H}_m$ with the minimal number of occurrences of $\sigma_1, \dots, \sigma_{m-1}$ can be obtained from each other by using the braid relations among $\sigma_1, \dots, \sigma_m \in \mathfrak{H}_m$ along with the relation $\sigma_m^2 = 1$.

Substituting the operators $\check{\xi}_1, \dots, \check{\xi}_{m-1}, \widetilde{\sigma}_m$ on $\bar{\mathrm{J}} \setminus \bar{\mathrm{B}}_m$ for the involutions $\sigma_1, \dots, \sigma_m$ in such a decomposition of $\sigma \in \mathfrak{H}_m$, we obtain another operator on $\bar{\mathrm{J}} \setminus \bar{\mathrm{B}}_m$. This operator does not depend on the choice of a decomposition, because of the first statement of Lemma 4.5, and because the operator $\widetilde{\sigma}_m^{\,2}$ on the vector space $\bar{\mathrm{J}} \setminus \bar{\mathrm{B}}_m$ is the identity for $\mathfrak{f}_m = \mathfrak{so}_{2m}$, which is the case considered here. Moreover, for $\sigma \in \mathfrak{H}'_m \subset \mathfrak{H}_m$, the operator on $\bar{\mathrm{J}} \setminus \bar{\mathrm{B}}_m$ obtained by the above substitution coincides with the operator (4.19). Indeed, for $\mathfrak{f}_m = \mathfrak{so}_{2m}$, the operator (4.19) was defined by substituting the Zhelobenko operators $\check{\xi}_1, \dots, \check{\xi}_{m-1}, \check{\xi}_m$ for $\sigma_1, \dots, \sigma_{m-1}, \sigma'_m$ in any reduced decomposition of $\sigma \in \mathfrak{H}'_m$. The coincidence of the two operators for $\sigma \in \mathfrak{H}'_m$ now follows from (4.21). Thus, we have extended the definition of the operator (4.19) from $\sigma \in \mathfrak{H}'_m$ to all $\sigma \in \mathfrak{H}_m$.

Note that, for $\mathfrak{f}_m = \mathfrak{so}_{2m}$ and $\sigma \in \mathfrak{H}'_m$, the number $\ell(\sigma)$ is equal to the length of a reduced decomposition of σ in terms of $\sigma_1, \dots, \sigma_{m-1}, \sigma'_m$. Thus, we have also extended the standard length function from the Weyl group $\mathfrak{H}'_m$ of $\mathfrak{so}_{2m}$ to the hyperoctahedral group $\mathfrak{H}_m$. Moreover, for any $\sigma \in \mathfrak{H}_m$, not only for $\sigma \in \mathfrak{H}'_m$, the number $\ell(\sigma)$ equals the number of elements in the set (4.20), where Δ^+ is the set of positive roots of $\mathfrak{so}_{2m}$.

From now on we shall consider $\mathfrak{f}_m = \mathfrak{so}_{2m}$ and $\mathfrak{f}_m = \mathfrak{sp}_{2m}$ simultaneously, working with the operators (4.19) for all elements $\sigma \in \mathfrak{H}_m$. In particular, for $\mathfrak{f}_m = \mathfrak{so}_{2m}$, we shall assume that the operator (4.19) with $\sigma = \sigma_m$ acts as $\widetilde{\sigma}_m$.

The restriction of the action (4.3), (4.4) of the braid group $\mathfrak{B}_m$ on $\mathfrak{f}_m$ to the Cartan subalgebra $\mathfrak{h}$ factors to an action of the hyperoctahedral group $\mathfrak{H}_m$. This is the standard action of the Weyl group of $\mathfrak{f}_m = \mathfrak{sp}_{2m}$. The resulting action of the subgroup $\mathfrak{H}'_m \subset \mathfrak{H}_m$ on $\mathfrak{h}$ is the standard action of the Weyl group of $\mathfrak{f}_m = \mathfrak{so}_{2m}$. The group $\mathfrak{H}_m$ also acts on the dual vector space $\mathfrak{h}^*$, so that $\sigma(\varepsilon_c) = \varepsilon_{\sigma(c)}$ for any $\sigma \in \mathfrak{H}_m$ and any $c = -m, \dots, -1, 1, \dots, m$. Unlike in (4.2), here we use the natural action of the group $\mathfrak{H}_m$ by permutations of $-m, \dots, -1, 1, \dots, m$. Thus, $\sigma_a \in \mathfrak{H}_m$ with $1 \leq a < m$ exchanges $a, a+1$ and also exchanges $-a, -a-1$, while $\sigma_m \in \mathfrak{H}_m$ exchanges $m, -m$. Note that we always have $\sigma(-c) = -\sigma(c)$. If we identify each weight $\mu \in \mathfrak{h}^*$ with the sequence $(\mu_1, \dots, \mu_m)$ of its labels, then

$$\sigma : (\mu_1, \dots, \mu_m) \mapsto (\mu_{\sigma^{-1}(1)}, \dots, \mu_{\sigma^{-1}(m)}) \quad \text{for} \quad \sigma \in \mathfrak{S}_m,$$
$$\sigma_m : (\mu_1, \dots, \mu_m) \mapsto (\mu_1, \dots, \mu_{m-1}, -\mu_m).$$

The *shifted* action of the group $\mathfrak{H}_m$ on the set $\mathfrak{h}^*$ is defined by the assignment

$$\mu \mapsto \sigma \circ \mu = \sigma(\mu + \rho) - \rho \quad \text{for} \quad \sigma \in \mathfrak{H}_m .$$

By regarding the elements of the commutative algebra $\overline{\mathrm{U}(\mathfrak{h})}$ as rational functions on the vector space $\mathfrak{h}^*$, we can also define an action of the group $\mathfrak{H}_m$ on this algebra:

$$(\sigma \circ X)(\mu) = X(\sigma^{-1} \circ \mu) \quad \text{for} \quad X \in \overline{\mathrm{U}(\mathfrak{h})}. \tag{4.22}$$

The next proposition was also proved in [KN3, §4].

Proposition 4.6. *For any $\sigma \in \mathfrak{H}_m$, $X \in \overline{\mathrm{U}(\mathfrak{h})}$, and $Y \in \bar{\mathrm{J}} \setminus \bar{\mathrm{B}}_m$ we have*

$$\begin{aligned} \check{\xi}_\sigma(XY) &= (\sigma \circ X)\, \check{\xi}_\sigma(Y), \\ \check{\xi}_\sigma(YX) &= \check{\xi}_\sigma(Y)(\sigma \circ X). \end{aligned} \tag{4.23}$$

§5. Intertwining operators

Let $\delta = (\delta_1, \ldots, \delta_m)$ be any sequence of m elements from $\{1, -1\}$. The hyperoctahedral group $\mathfrak{H}_m$ acts on the set of all such sequences naturally, so that the generator $\sigma_a \in \mathfrak{H}_m$ with $a < m$ acts on δ as the transposition of δ_a and δ_{a+1}, while the generator $\sigma_m \in \mathfrak{H}_m$ changes the sign of δ_m. Let $\delta_+ = (1, \ldots, 1)$ be the sequence of m elements 1. Given any sequence δ, take the composition of the following automorphisms of the ring $\mathcal{GD}(\mathbb{C}^m \otimes \mathbb{C}^n)$:

$$x_{\bar{a}i} \mapsto \theta_i\, \partial_{\bar{a}\tilde{\imath}} \quad \text{and} \quad \partial_{\bar{a}i} \mapsto \theta_i\, x_{\bar{a}\tilde{\imath}} \quad \text{whenever} \quad \delta_a = -1. \tag{5.1}$$

Here $a \geq 1$ and $i = 1, \ldots, n$. Let ϖ denote this composition. In particular, the automorphism ϖ corresponding to $\delta = (1, \ldots, 1, -1)$ coincides with the action of $\widetilde{\sigma}_m$ on $\mathcal{GD}(\mathbb{C}^m \otimes \mathbb{C}^n)$; see (4.5). For $\mathfrak{f}_m = \mathfrak{so}_{2m}$, the automorphism ϖ is involutive for any δ. But if $\mathfrak{f}_m = \mathfrak{sp}_{2m}$, then the square ϖ^2 acts as follows:

$$x_{\bar{a}i} \mapsto -x_{\bar{a}i} \quad \text{and} \quad \partial_{\bar{a}i} \mapsto -\partial_{\bar{a}i} \quad \text{whenever} \quad \delta_a = -1.$$

For any $\mathfrak{f}_m$-module V, the action of $\mathrm{X}(\mathfrak{g}_n)$ on $\mathcal{F}_m(V) = V \otimes \mathcal{G}(\mathbb{C}^m \otimes \mathbb{C}^n)$ is determined by the homomorphism $\beta_m : \mathrm{X}(\mathfrak{g}_n) \to \mathrm{U}(\mathfrak{f}_m) \otimes \mathcal{GD}(\mathbb{C}^m \otimes \mathbb{C}^n)$; see Proposition 2.3. Next, the action of the Lie algebra $\mathfrak{f}_m$ on the second tensor factor $\mathcal{G}(\mathbb{C}^m \otimes \mathbb{C}^n)$ of $\mathcal{F}_m(V)$ is defined via the homomorphism $\zeta_n : \mathrm{U}(\mathfrak{f}_m) \to \mathcal{GD}(\mathbb{C}^m \otimes \mathbb{C}^n)$; see the definition (2.6). Here any element of the ring $\mathcal{GD}(\mathbb{C}^m \otimes \mathbb{C}^n)$ acts on the vector space $\mathcal{G}(\mathbb{C}^m \otimes \mathbb{C}^n)$ naturally. We can modify the latter action, by making any element $Y \in \mathcal{GD}(\mathbb{C}^m \otimes \mathbb{C}^n)$ act on $\mathcal{G}(\mathbb{C}^m \otimes \mathbb{C}^n)$ via the natural action of $\varpi(Y)$. Then we get another $\mathcal{GD}(\mathbb{C}^m \otimes \mathbb{C}^n)$-module, with the same underlying vector space $\mathcal{G}(\mathbb{C}^m \otimes \mathbb{C}^n)$ for every δ.

For any $\mathfrak{f}_m$-module V, we can now define a bimodule $\mathcal{F}_\delta(V)$ of $\mathfrak{f}_m$ and $\mathrm{X}(\mathfrak{g}_n)$. Its underlying vector space is the same $V \otimes \mathcal{G}(\mathbb{C}^m \otimes \mathbb{C}^n)$ for every δ. The action of $\mathrm{X}(\mathfrak{g}_n)$ on $\mathcal{F}_\delta(V)$ is defined by pushing the homomorphism β_m forward through the automorphism ϖ, applied to $\mathcal{GD}(\mathbb{C}^m \otimes \mathbb{C}^n)$ as to the second tensor factor of the target of β_m. The action of $\mathfrak{f}_m$ on $\mathcal{F}_\delta(V)$ is also defined by pushing the homomorphism ζ_n forward through the automorphism ϖ. Thus, the actions of $\mathrm{X}(\mathfrak{g}_n)$ and $\mathfrak{f}_m$ on the bimodule $\mathcal{F}_\delta(V)$ are determined by the compositions of the homomorphisms

$$\begin{aligned} \mathrm{X}(\mathfrak{g}_n) &\underset{\beta_m}{\longrightarrow} \mathrm{U}(\mathfrak{f}_m) \otimes \mathcal{GD}(\mathbb{C}^m \otimes \mathbb{C}^n) \underset{1 \otimes \varpi}{\longrightarrow} \mathrm{U}(\mathfrak{f}_m) \otimes \mathcal{GD}(\mathbb{C}^m \otimes \mathbb{C}^n), \\ \mathrm{U}(\mathfrak{f}_m) &\underset{1 \otimes \zeta_n}{\longrightarrow} \mathrm{U}(\mathfrak{f}_m) \otimes \mathcal{GD}(\mathbb{C}^m \otimes \mathbb{C}^n) \underset{1 \otimes \varpi}{\longrightarrow} \mathrm{U}(\mathfrak{f}_m) \otimes \mathcal{GD}(\mathbb{C}^m \otimes \mathbb{C}^n), \end{aligned}$$

respectively. Note that here we have $\mathcal{F}_m(V) = \mathcal{F}_{\delta_+}(V)$.

Let $\mu \in \mathfrak{h}^*$ be any weight of $\mathfrak{f}_m$ such that

$$\mu_a - \mu_b \notin \mathbb{Z} \quad \text{and} \quad \mu_a + \mu_b \notin \mathbb{Z} \quad \text{whenever} \quad 1 \leq a < b \leq m. \tag{5.2}$$

In the case where $\mathfrak{f}_m = \mathfrak{sp}_{2m}$, we also suppose that, in addition to (5.2),

$$2\mu_a \notin \mathbb{Z} \quad \text{whenever} \quad 1 \leq a \leq m. \tag{5.3}$$

Our nearest goal is to show how the Zhelobenko operator (4.19) corresponding to an arbitrary element $\sigma \in \mathfrak{H}_m$ determines an $\mathrm{X}(\mathfrak{g}_n)$-intertwining operator

$$\mathcal{F}_m(M_\mu)_\mathfrak{n} \to \mathcal{F}_\delta(M_{\sigma\,\circ\mu})_\mathfrak{n}, \quad \text{where} \quad \delta = \sigma(\delta_+). \tag{5.4}$$

In this section we keep regarding B_m as the associative algebra generated by $\mathrm{U}(\mathfrak{f}_m)$ and $\mathcal{GD}(\mathbb{C}^m \otimes \mathbb{C}^n)$ with the cross relations (4.10). Let I_δ be the left ideal of the algebra B_m generated by the elements $x_{\bar{a}i}$ with $\delta_a = -1$ and the elements $\partial_{\bar{a}i}$ with $\delta_a = 1$. Here $a = 1, \ldots, m$ and $i = 1, \ldots, m$. Note that, in terms of the elements q_{ci} introduced immediately after stating Proposition 2.3, the left ideal I_δ is generated by the elements $q_{-\delta_a \bar{a},i}$, where again $a = 1, \ldots, m$ and $i = 1, \ldots, m$. In particular, the ideal I_{δ_+} is generated by all the left derivations ∂_{ai}. Let $\bar{\mathrm{I}}_\delta$ be the left ideal of $\bar{\mathrm{B}}_m$ generated by the same elements as the ideal I_δ of B_m.

Consider the image of the ideal $\bar{\mathrm{I}}_\delta$ in the quotient space $\bar{\mathrm{J}} \setminus \bar{\mathrm{B}}_m$, i.e., the subspace $\bar{\mathrm{J}} \setminus (\bar{\mathrm{I}}_\delta + \bar{\mathrm{J}})$ in the quotient space $\bar{\mathrm{J}} \setminus \bar{\mathrm{B}}_m$. The image will be denoted occasionally by the same symbol $\bar{\mathrm{I}}_\delta$. In the context of the next proposition, this will cause no confusion.

Proposition 5.1. *For any $\sigma \in \mathfrak{H}_m$ the operator $\check{\xi}_\sigma$ maps the subspace I_{δ_+} to $\mathrm{I}_{\sigma(\delta_+)}$.*

Proof. For any $a = 1, \ldots, m-1$, consider the operator $\widehat{F}_a$ corresponding to the element $F_a \in \mathrm{B}_m$. By (4.6) and also (2.6) and (4.10), for any $Y \in \mathcal{GD}(\mathbb{C}^m \otimes \mathbb{C}^n)$ we have

$$\widehat{F}_a(Y) = -\sum_{k=1}^{n} [x_{\bar{a}k}\, \partial_{\overline{a+1}k}\,, Y].$$

Similarly, in the case where $\mathfrak{f}_m = \mathfrak{sp}_{2m}$, for any $Y \in \mathcal{GD}(\mathbb{C}^m \otimes \mathbb{C}^n)$ we have

$$\widehat{F}_m(Y) = \sum_{k=1}^{n} [x_{\overline{m}\tilde{k}}\, x_{\overline{m}k}\,, Y]\,/2$$

by (4.8). If $\mathfrak{f}_m = \mathfrak{so}_{2m}$, then we do not need to consider the operator $\widehat{F}_m$, because in this case the operator (4.19) corresponding to $\sigma = \sigma_m$ acts on $\bar{\mathrm{J}} \setminus \bar{\mathrm{B}}_m$ as $\widetilde{\sigma}_m$ by our definition.

The above description of the action of $\widehat{F}_a$ with $a < m$ on $\mathcal{GD}(\mathbb{C}^m \otimes \mathbb{C}^n)$ shows that this action preserves each of the two $2n$-dimensional subspaces spanned by the vectors

$$x_{\bar{a}i} \quad \text{and} \quad x_{\overline{a+1}i}, \quad \text{where} \quad i = 1, \ldots, n; \tag{5.5}$$

$$\partial_{\bar{a}i} \quad \text{and} \quad \partial_{\overline{a+1}i}, \quad \text{where} \quad i = 1, \ldots, n. \tag{5.6}$$

This action also maps to zero the $2n$-dimensional subspace spanned by

$$x_{\bar{a}i} \quad \text{and} \quad \partial_{\overline{a+1}i}, \quad \text{where} \quad i = 1, \ldots, n. \tag{5.7}$$

Therefore, for any δ, the operator $\bar{\xi}_a$ with $a < m$ maps the left ideal $\bar{\mathrm{I}}_\delta$ of $\bar{\mathrm{B}}_m$ to the image of $\bar{\mathrm{I}}_\delta$ in $\bar{\mathrm{J}} \setminus \bar{\mathrm{B}}_m$, unless $\delta_a = 1$ and $\delta_{a+1} = -1$. The operator $\check{\xi}_a$ on $\bar{\mathrm{J}} \setminus \bar{\mathrm{B}}_m$ was defined by taking the composition of $\bar{\xi}_a$ and $\widetilde{\sigma}_a$. Hence, $\check{\xi}_a$ with $a < m$ maps the image of $\bar{\mathrm{I}}_\delta$ to the image of $\bar{\mathrm{I}}_{\sigma_a(\delta)}$, unless $\delta_a = -1$ and $\delta_{a+1} = 1$.

For $\mathfrak{f}_m = \mathfrak{sp}_{2m}$, the action of $\widehat{F}_m$ on the vector space $\mathcal{GD}(\mathbb{C}^m \otimes \mathbb{C}^n)$ maps to zero the n-dimensional subspace spanned by the elements

$$x_{\overline{m}i} = x_{1i}, \quad \text{where} \quad i = 1, \ldots, n. \tag{5.8}$$

Therefore, the operator $\bar{\xi}_m$ maps the left ideal $\bar{\mathrm{I}}_\delta$ of $\bar{\mathrm{B}}_m$ to the image of $\bar{\mathrm{I}}_\delta$ in $\bar{\mathrm{J}} \setminus \bar{\mathrm{B}}_m$, unless $\delta_m = 1$. Hence, the operator $\check{\xi}_m$ on $\bar{\mathrm{J}} \setminus \bar{\mathrm{B}}_m$ maps the image of $\bar{\mathrm{I}}_\delta$ to the image of $\bar{\mathrm{I}}_{\sigma_m(\delta)}$, unless $\delta_m = -1$. In the case where $\mathfrak{f}_m = \mathfrak{so}_{2m}$, we only note that $\widetilde{\sigma}_m$ maps the image of $\bar{\mathrm{I}}_\delta$ in $\bar{\mathrm{J}} \setminus \bar{\mathrm{B}}_m$ to the image of $\bar{\mathrm{I}}_{\sigma_m(\delta)}$.

From now on we shall denote the image of the ideal $\bar{\mathrm{I}}_\delta$ in the quotient space $\bar{\mathrm{J}} \setminus \bar{\mathrm{B}}_m$ by the same symbol. Put

$$\widehat{\delta} = \sum_{a=1}^{m} \delta_a \varepsilon_a \in \mathfrak{h}^*.$$

Then for every $\sigma \in \mathfrak{H}_m$ we have $\widehat{\sigma(\delta)} = \sigma(\,\widehat{\delta}\,)$, where on the right-hand side we use the action of the group $\mathfrak{H}_m$ on $\mathfrak{h}^*$. Let $(\ ,\ )$ be the standard bilinear form on $\mathfrak{h}^*$, so that the basis of weights ε_a with $a = 1, \ldots, m$ is orthonormal. The above remarks on the action of the Zhelobenko operators on $\bar{\mathrm{I}}_\delta$ can now be restated as follows:

(5.9) $\quad$ if $\quad (\widehat{\delta}, \varepsilon_a - \varepsilon_{a+1}) \geq 0, \quad$ then $\quad \check{\xi}_a(\,\bar{\mathrm{I}}_\delta) \subset \bar{\mathrm{I}}_{\sigma_a(\delta)} \quad$ for $\quad a = 1, \ldots, m-1\,;$

(5.10) $\quad$ if $\quad (\widehat{\delta}, \varepsilon_m) > 0, \quad$ then $\quad \check{\xi}_m(\,\bar{\mathrm{I}}_\delta) \subset \bar{\mathrm{I}}_{\sigma_m(\delta)} \quad$ for $\quad \mathfrak{f}_m = \mathfrak{sp}_{2m}.$

We shall prove Proposition 5.1 by induction on the length of a reduced decomposition of $\sigma \in \mathfrak{H}_m$ in terms of $\sigma_1, \ldots, \sigma_m$. This number was denoted by $\ell(\sigma)$ in the case where $\mathfrak{f}_m = \mathfrak{sp}_{2m}$, but may be different from the number denoted by $\ell(\sigma)$ in the case of $\mathfrak{f}_m = \mathfrak{so}_{2m}$. Recall that in both cases $\ell(\sigma)$ equals the number of elements in the set (4.20), where Δ^+ is the set of positive roots of $\mathfrak{f}_m$.

If σ is the identity element of $\mathfrak{H}_m$, Proposition 5.1 is tautological. Suppose that for some $\sigma \in \mathfrak{H}_m$ we have

$$\check{\xi}_\sigma(\,\bar{\mathrm{I}}_{\delta_+}) \subset \bar{\mathrm{I}}_{\sigma(\delta_+)}.$$

Take $\sigma_a \in \mathfrak{H}_m$ with $1 \leq a \leq m$ such that $\sigma_a \sigma$ has a longer reduced decomposition in terms of $\sigma_1, \ldots, \sigma_m$ compared to σ. If $\mathfrak{f}_m = \mathfrak{so}_{2m}$ and $a = m$, then $\check{\xi}_{\sigma_m \sigma} = \widetilde{\sigma}_m\, \check{\xi}_\sigma$, and we need the inclusion

$$\widetilde{\sigma}_m(\,\bar{\mathrm{I}}_{\sigma(\delta_+)}) \subset \bar{\mathrm{I}}_{\sigma_m \sigma(\delta_+)}, \tag{5.11}$$

which holds true by the definition of the action of $\mathfrak{H}_m$ on $\bar{\mathrm{J}} \setminus \bar{\mathrm{B}}_m$.

We may exclude the case where $\mathfrak{f}_m = \mathfrak{so}_{2m}$ and $a = m$, and assume that

$$\ell(\sigma_a \sigma) = \ell(\sigma) + 1. \tag{5.12}$$

First, suppose that $a < m$ here. Then we prove the inclusion

$$\check{\xi}_a(\,\bar{\mathrm{I}}_{\sigma(\delta_+)}) \subset \bar{\mathrm{I}}_{\sigma_a \sigma(\delta_+)}.$$

By (5.9), this inclusion will be true if

$$(\widehat{\sigma(\delta_+)}, \varepsilon_a - \varepsilon_{a+1}) = (\sigma(\widehat{\delta}_+), \varepsilon_a - \varepsilon_{a+1}) \geq 0.$$

But condition (5.12) for $a < m$ implies that $\varepsilon_a - \varepsilon_{a+1} \in \sigma(\Delta^+)$. Indeed, since the root $\varepsilon_a - \varepsilon_{a+1}$ of $\mathfrak{f}_m$ is simple, we have $\sigma_a(\eta) \in \Delta^+$ for any $\eta \in \Delta^+$ such that $\eta \neq \varepsilon_a - \varepsilon_{a+1}$. Since $\ell(\sigma)$ and $\ell(\sigma_a \sigma)$ are the numbers of elements in Δ_σ and $\Delta_{\sigma_a \sigma}$ (respectively), here $\varepsilon_a - \varepsilon_{a+1} \in \sigma(\Delta^+)$. So, $\varepsilon_a - \varepsilon_{a+1} = \sigma(\varepsilon_b - \varepsilon_c)$, where $1 \leq b \leq m$ and $1 \leq |c| \leq m$. Thus,

$$(\sigma(\widehat{\delta}_+), \varepsilon_a - \varepsilon_{a+1}) = (\sigma(\widehat{\delta}_+), \sigma(\varepsilon_b - \varepsilon_c)) = (\widehat{\delta}_+, \varepsilon_b - \varepsilon_c) \geq 0.$$

Now suppose that $a = m$. Here we assume that $\mathfrak{f}_m = \mathfrak{sp}_{2m}$. We need the inclusion

$$\check{\xi}_m(\,\bar{\mathrm{I}}_{\sigma(\delta_+)}) \subset \bar{\mathrm{I}}_{\sigma_m \sigma(\delta_+)}.$$

It will be true if

$$(\widehat{\sigma(\delta_+)}, \varepsilon_m) = (\sigma(\widehat{\delta}_+), \varepsilon_m) > 0.$$

But condition (5.12) for $a = m$ implies that $2\varepsilon_m \in \sigma(\Delta^+)$, where Δ^+ is the set of positive roots of $\mathfrak{sp}_{2m}$. Indeed, since the root $2\varepsilon_m$ of $\mathfrak{sp}_{2m}$ is simple, $\sigma_m(\eta) \in \Delta^+$ for any $\eta \in \Delta^+$ such that $\eta \neq 2\varepsilon_m$. Since $\ell(\sigma)$ and $\ell(\sigma_m \sigma)$ are the numbers of elements in Δ_σ and $\Delta_{\sigma_m \sigma}$ (respectively), we have $2\varepsilon_m \in \sigma(\Delta^+)$. So $\varepsilon_m = \sigma(\varepsilon_b)$, where $1 \leq b \leq m$. Thus,

$$(\sigma(\widehat{\delta}_+), \varepsilon_m) = (\sigma(\widehat{\delta}_+), \sigma(\varepsilon_b)) = (\widehat{\delta}_+, \varepsilon_b) > 0.$$

□

Corollary 5.2. *For any $\sigma \in \mathfrak{H}_m$ the operator $\check{\xi}_\sigma$ on $\bar{\mathrm{J}} \setminus \bar{\mathrm{B}}_m$ maps*

$$\bar{\mathrm{J}} \setminus (\bar{\mathrm{J}}' + \bar{\mathrm{I}}_{\delta_+} + \bar{\mathrm{J}}) \quad \textit{to} \quad \bar{\mathrm{J}} \setminus (\bar{\mathrm{J}}' + \bar{\mathrm{I}}_{\sigma(\delta_+)} + \bar{\mathrm{J}}).$$

Proof. We extend the arguments used in the proof of Proposition 5.1. In particular, we shall again use the length of a reduced decomposition of σ in terms of $\sigma_1, \dots, \sigma_m$. If σ is the identity element of $\mathfrak{H}_m$, then the required statement is tautological. Now suppose that the statement of Corollary 5.2 is true for some $\sigma \in \mathfrak{H}_m$. Take any simple reflection $\sigma_a \in \mathfrak{H}_m$ with $1 \leq a \leq m$ such that $\sigma_a\sigma$ has a longer reduced decomposition in terms of $\sigma_1, \dots, \sigma_m$ compared to σ. In the case where $\mathfrak{f}_m = \mathfrak{so}_{2m}$ we may assume that $a < m$, because in that case the required statement for $\sigma_m\sigma$ in place of σ is provided by (5.11).

Thus, we may assume (5.12). With the above assumption on a, we have proved that (5.12) implies

$$(\widehat{\sigma(\delta_+)}, \eta_a) \geq 0. \tag{5.13}$$

Here η_a is the simple root corresponding to σ_a. But (5.13) implies the identity

$$\bar{\mathrm{J}}' + \bar{\mathrm{I}}_{\sigma(\delta_+)} = \bar{\mathrm{J}}'_a + \bar{\mathrm{I}}_{\sigma(\delta_+)} \tag{5.14}$$

of left ideals of $\bar{\mathrm{B}}_m$. Indeed, the two sides of (5.14) differ by elements $Y\zeta_n(E_a)$, where Y ranges over $\bar{\mathrm{B}}_m$. Condition (5.13) implies that $\zeta_n(E_a) \in \bar{\mathrm{I}}_{\sigma(\delta_+)}$; see the definition (2.6) and the arguments at the beginning of the proof of Proposition 5.1. Using Proposition 4.3 and the induction step in our proof of Proposition 5.1, we see that $\check{\xi}_a$ maps

$$\bar{\mathrm{J}} \setminus (\bar{\mathrm{J}}' + \bar{\mathrm{I}}_{\sigma(\delta_+)} + \bar{\mathrm{J}}) = \bar{\mathrm{J}} \setminus (\bar{\mathrm{J}}'_a + \bar{\mathrm{I}}_{\sigma(\delta_+)} + \bar{\mathrm{J}}) \quad \text{to} \quad \bar{\mathrm{J}} \setminus (\bar{\mathrm{J}}' + \bar{\mathrm{I}}_{\sigma_a\sigma(\delta_+)} + \bar{\mathrm{J}}).$$

This constitutes the induction step of our proof of Corollary 5.2. □

Let $\mathrm{I}_{\mu,\delta}$ be the left ideal of the algebra B_m generated by $\mathrm{I}_\delta + \mathrm{J}'$ and by the elements

$$F_{-\bar{a},-\bar{a}} - \zeta_n(F_{-\bar{a},-\bar{a}}) - \mu_a, \quad \text{where} \quad a = 1, \dots, m.$$

Recall that under the isomorphism of the algebra B_m with $\mathrm{U}(\mathfrak{f}_m) \otimes \mathcal{GD}(\mathbb{C}^m \otimes \mathbb{C}^n)$, the difference $X - \zeta_n(X) \in \mathrm{B}_m$ for every $X \in \mathfrak{f}_m$ is mapped to the element (4.15). Denote by $\bar{\mathrm{I}}_{\mu,\delta}$ the subspace $\overline{\mathrm{U}(\mathfrak{h})}\, \mathrm{I}_{\mu,\delta}$ of $\bar{\mathrm{B}}_m$; this is also a left ideal of $\bar{\mathrm{B}}_m$.

Theorem 5.3. *For any element $\sigma \in \mathfrak{H}_m$, the operator $\check{\xi}_\sigma$ on $\bar{\mathrm{J}} \setminus \bar{\mathrm{B}}_m$ maps*

$$\bar{\mathrm{J}} \setminus (\bar{\mathrm{I}}_{\mu,\delta_+} + \bar{\mathrm{J}}) \quad \textit{to} \quad \bar{\mathrm{J}} \setminus (\bar{\mathrm{I}}_{\sigma\circ\mu,\sigma(\delta_+)} + \bar{\mathrm{J}}).$$

Proof. Let κ be a weight of $\mathfrak{f}_m$ with sequence of labels $(\kappa_1, \dots, \kappa_m)$. Suppose that κ satisfies conditions (5.2) instead of μ. For $\mathfrak{f}_m = \mathfrak{sp}_{2m}$, we also suppose that κ satisfies conditions (5.3) instead of μ. Denote by $\tilde{\mathrm{I}}_{\kappa,\delta}$ the left ideal of $\bar{\mathrm{B}}_m$ generated by $\mathrm{I}_\delta + \mathrm{J}'$ and by the elements

$$F_{-\bar{a},-\bar{a}} - \kappa_a, \quad \text{where} \quad a = 1, \dots, m.$$

Proposition 4.6 and Corollary 5.2 imply that the operator $\check{\xi}_\sigma$ on $\bar{\mathrm{J}} \setminus \bar{\mathrm{B}}_m$ maps

$$\bar{\mathrm{J}} \setminus (\tilde{\mathrm{I}}_{\kappa,\delta_+} + \bar{\mathrm{J}}) \quad \text{to} \quad \bar{\mathrm{J}} \setminus (\tilde{\mathrm{I}}_{\sigma\circ\kappa,\sigma(\delta_+)} + \bar{\mathrm{J}}).$$

Now we choose

$$\kappa_a = \mu_a + n/2 \quad \text{for} \quad a = 1, \dots, m. \tag{5.15}$$

Then the conditions on κ stated at the beginning of this proof are satisfied. For every $\sigma \in \mathfrak{H}_m$ we shall prove the following identity of left ideals of $\bar{\mathrm{B}}_m$:

$$\tilde{\mathrm{I}}_{\sigma\circ\kappa,\sigma(\delta_+)} = \bar{\mathrm{I}}_{\sigma\circ\mu,\sigma(\delta_+)}. \tag{5.16}$$

Theorem 5.3 will then follow. Denote $\delta=\sigma(\delta_+)$. By our choice of κ we have

$$\sigma\circ\kappa=\sigma\circ\mu+n\delta/2, \tag{5.17}$$

where the sequence δ is regarded as a weight of $\mathfrak{f}_m$, by identifying the weights with their sequences of labels. Let a run through $1,\dots,m$. If $\delta_a=1$, then, by the definition (2.6),

$$\zeta_n(F_{-\bar a,-\bar a})-n/2=-\sum_{k=1}^{n}x_{\bar a k}\,\partial_{\bar a k}\in \mathrm{I}_\delta\,.$$

If $\delta_a=-1$, then the same definition (2.6) shows that

$$\zeta_n(F_{-\bar a,-\bar a})+n/2=\sum_{k=1}^{n}\partial_{\bar a k}\,x_{\bar a k}\in \mathrm{I}_\delta\,.$$

Hence, relation (5.17) implies (5.16). □

Consider the quotient vector space $\mathrm{B}_m\,/\,\mathrm{I}_{\mu,\delta}$ for any sequence δ. The algebra $\mathrm{U}(\mathfrak{f}_m)$, viewed as a subalgebra of B_m, acts on this quotient via left multiplication. The algebra $\mathrm{X}(\mathfrak{g}_n)$ also acts on this quotient via left multiplication, via the homomorphism $\beta_m:\mathrm{X}(\mathfrak{g}_n)\to\mathrm{B}_m$. Recall that in §2, the target algebra B_m of the homomorphism β_m was defined as $\mathrm{U}(\mathfrak{f}_m)\otimes\mathcal{GD}(\mathbb{C}^m\otimes\mathbb{C}^n)$. Here we use a different presentation of the same algebra, with the help of the cross relations (4.10). In particular, here the image of β_m commutes with the subalgebra $\mathrm{U}(\mathfrak{f}_m)$ of B_m; see statement (ii) in Proposition 2.3. Thus, the vector space $\mathrm{B}_m\,/\,\mathrm{I}_{\mu,\delta}$ becomes a bimodule over $\mathfrak{f}_m$ and $\mathrm{X}(\mathfrak{g}_n)$.

Consider the bimodule $\mathcal{F}_\delta(M_\mu)$ over $\mathfrak{f}_m$ and $\mathrm{X}(\mathfrak{g}_n)$ defined at the beginning of this section. This bimodule is equivalent to $\mathrm{B}_m\,/\,\mathrm{I}_{\mu,\delta}$. Indeed, let Z run through $\mathcal{G}(\mathbb{C}^m\otimes\mathbb{C}^n)$. Then a bijective linear map

$$\mathcal{F}_\delta(M_\mu)\to\mathrm{B}_m\,/\,\mathrm{I}_{\mu,\delta}$$

intertwining the actions of $\mathfrak{f}_m$ and $\mathrm{X}(\mathfrak{g}_n)$ can be defined by mapping the element

$$1_\mu\otimes Z\in M_\mu\otimes\mathcal{G}(\mathbb{C}^m\otimes\mathbb{C}^n)$$

to the image of

$$\varpi^{-1}(Z)\in\mathcal{GD}(\mathbb{C}^m\otimes\mathbb{C}^n)\subset\mathrm{B}_m$$

in the quotient $\mathrm{B}_m\,/\,\mathrm{I}_{\mu,\delta}$. Here the intertwining property follows from the definitions of $\mathcal{F}_\delta(M_\mu)$ and $\mathrm{I}_{\mu,\delta}$. The same mapping determines a bijective linear map

$$\mathcal{F}_\delta(M_\mu)\to\bar{\mathrm{B}}_m/\bar{\mathrm{I}}_{\mu,\delta}. \tag{5.18}$$

In particular, the space $\mathcal{F}_\delta(M_\mu)_{\mathfrak{n}}$ of $\mathfrak{n}$-coinvariants of $\mathcal{F}_\delta(M_\mu)$ is equivalent to the quotient $\bar{\mathrm{J}}\setminus\bar{\mathrm{B}}_m/\bar{\mathrm{I}}_{\mu,\delta}$ as a bimodule over the Cartan subalgebra $\mathfrak{h}\subset\mathfrak{f}_m$ and over $\mathrm{X}(\mathfrak{g}_n)$. But Theorem 5.3 implies that the operator $\check{\xi}_\sigma$ on $\bar{\mathrm{J}}\setminus\bar{\mathrm{B}}_m$ determines a linear map

$$\bar{\mathrm{J}}\setminus\bar{\mathrm{B}}_m/\bar{\mathrm{I}}_{\mu,\delta_+}\to\bar{\mathrm{J}}\setminus\bar{\mathrm{B}}_m/\bar{\mathrm{I}}_{\sigma\circ\mu,\sigma(\delta_+)}. \tag{5.19}$$

The latter map intertwines the actions of $\mathrm{X}(\mathfrak{g}_n)$ on the source and the target vector spaces, because the image of $\mathrm{X}(\mathfrak{g}_n)$ in B_m relative to β_m commutes with the subalgebra $\mathrm{U}(\mathfrak{f}_m)\subset\mathrm{B}_m$; see the definition (4.13). We also use statement (ii) of Lemma 4.1. Recall that $\mathcal{F}_m(V)=\mathcal{F}_{\delta_+}(V)$. Hence, by using the equivalences (5.18) for the sequences $\delta=\delta_+$ and $\delta=\sigma(\delta_+)$, we see that the operator (5.19) becomes the desired $\mathrm{X}(\mathfrak{g}_n)$-intertwining operator (5.4).

As usual, for any $\mathfrak{f}_m$-module V and any element $\lambda\in\mathfrak{h}^*$, let $V^\lambda\subset V$ be the subspace of vectors *of weight* λ, so that any $X\in\mathfrak{h}$ acts on V^λ via multiplication by $\lambda(X)\in\mathbb{C}$. By the property (4.23) of $\check{\xi}_\sigma$, the restriction of our operator (5.4) to the subspace of weight λ is an $\mathrm{X}(\mathfrak{g}_n)$-intertwining operator

$$\mathcal{F}_m(M_\mu)^\lambda_{\mathfrak{n}}\to\mathcal{F}_\delta(M_{\sigma\circ\mu})^{\sigma\circ\lambda}_{\mathfrak{n}},\quad\text{where}\quad\delta=\sigma(\delta_+). \tag{5.20}$$

At the end of §2, we defined the modules P_z and P'_z over the Yangian $\mathrm{Y}(\mathfrak{gl}_n)$. The underlying vector space of these modules is the Grassmann algebra $\mathcal{G}(\mathbb{C}^n)$. This algebra is graded by $0, 1, \ldots, n$. The actions of $\mathrm{Y}(\mathfrak{gl}_n)$ on P_z and P'_z preserve the degree. Now, for any $N = 1, \ldots, n$, denote by P_z^N and P_z^{-N} (respectively) the submodules in P_z and P'_z that consist of the elements of degree N. Note that $\mathrm{Y}(\mathfrak{gl}_n)$ acts on the subspace of P_z of degree zero trivially, that is, via the counit homomorphism $\mathrm{Y}(\mathfrak{gl}_n) \to \mathbb{C}$. That action of $\mathrm{Y}(\mathfrak{gl}_n)$ does not depend on z. It will be convenient to denote by P_z^0 the vector space $\mathbb{C}$ with the trivial action of $\mathrm{Y}(\mathfrak{gl}_n)$.

Denote

$$\nu_a = n/2 + \mu_a - \lambda_a \quad \text{for} \quad a = 1, \ldots, m. \tag{5.21}$$

Suppose that $\nu_1, \ldots, \nu_m \in \{0, 1, \ldots, n\}$; otherwise, the source $\mathrm{X}(\mathfrak{g}_n)$-module in (5.20) would be zero by Corollary 2.6. Under our assumption, Corollary 2.6 implies that the the source $\mathrm{X}(\mathfrak{g}_n)$-module in (5.20) is equivalent to

$$P^{\nu_m}_{\mu_m+z} \otimes P^{\nu_{m-1}}_{\mu_{m-1}+z+1} \otimes \cdots \otimes P^{\nu_1}_{\mu_1+z+m-1} \tag{5.22}$$

pulled back through the automorphism (1.17) of $\mathrm{X}(\mathfrak{g}_n)$, where $f(u)$ is given by (2.26) and $z = \mp\frac{1}{2}$. A more general result is stated as Proposition 5.4 below. The tensor product in (5.22) is that of $\mathrm{Y}(\mathfrak{gl}_n)$-modules. Then we employ the embedding $\mathrm{Y}(\mathfrak{g}_n) \subset \mathrm{Y}(\mathfrak{gl}_n)$ and the homomorphism $\mathrm{X}(\mathfrak{g}_n) \to \mathrm{Y}(\mathfrak{g}_n)$ defined by (1.18). By using the labels $\rho_1, \ldots, \rho_m$ of the half-sum ρ of the positive roots of $\mathfrak{f}_m$, the tensor product (5.22) can be rewritten as

$$P^{\nu_m}_{\mu_m-\frac{1}{2}+\rho_m} \otimes \cdots \otimes P^{\nu_1}_{\mu_1-\frac{1}{2}+\rho_1}. \tag{5.23}$$

In terms of the labels $\rho_1, \ldots, \rho_m$ we can also rewrite the product (2.26) as

$$\prod_{a=1}^{m} \frac{u - \mu_a + \frac{1}{2} - \rho_a}{u - \mu_a - \frac{1}{2} - \rho_a}. \tag{5.24}$$

Now, consider the target $\mathrm{X}(\mathfrak{g}_n)$-module in (5.20). For each $a = 1, \ldots, m$ denote

$$\widetilde{\mu}_a = \mu_{|\sigma^{-1}(a)|}, \quad \widetilde{\nu}_a = \nu_{|\sigma^{-1}(a)|}, \quad \widetilde{\rho}_a = \rho_{|\sigma^{-1}(a)|}.$$

The above description of the source $\mathrm{X}(\mathfrak{g}_n)$-module in (5.20) can be generalized to similar $\mathrm{X}(\mathfrak{g}_n)$-modules depending on an arbitrary element $\sigma \in \mathfrak{H}_m$.

Proposition 5.4. *For $\delta = \sigma(\delta_+)$, the $\mathrm{X}(\mathfrak{g}_n)$-module $\mathcal{F}_\delta(M_{\sigma \circ \mu})_{\mathfrak{n}}^{\sigma \circ \lambda}$ is equivalent to the tensor product*

$$P^{\delta_m \widetilde{\nu}_m}_{\widetilde{\mu}_m-\frac{1}{2}+\widetilde{\rho}_m} \otimes \cdots \otimes P^{\delta_1 \widetilde{\nu}_1}_{\widetilde{\mu}_1-\frac{1}{2}+\widetilde{\rho}_1} \tag{5.25}$$

pulled back through the automorphism (1.17) *of* $\mathrm{X}(\mathfrak{g}_n)$*, where $f(u)$ equals the product* (5.24).

Proof. First, consider the bimodule $\mathcal{F}_m(M_{\sigma \circ \mu})_{\mathfrak{n}}$ of $\mathfrak{h}$ and $\mathrm{X}(\mathfrak{g}_n)$. By Corollary 2.6, this bimodule is equivalent to the tensor product

$$P_{\delta_m \widetilde{\mu}_m - \frac{1}{2} + \delta_m \widetilde{\rho}_m} \otimes \cdots \otimes P_{\delta_1 \widetilde{\mu}_1 - \frac{1}{2} + \delta_1 \widetilde{\rho}_1} \tag{5.26}$$

pulled back through the automorphism (1.17) of $\mathrm{X}(\mathfrak{g}_n)$, where $f(u)$ equals

$$\prod_{a=1}^{m} \frac{u - \delta_a \widetilde{\mu}_a + \frac{1}{2} - \delta_a \widetilde{\rho}_a}{u - \delta_a \widetilde{\mu}_a - \frac{1}{2} - \delta_a \widetilde{\rho}_a}. \tag{5.27}$$

For any $a = 1, \ldots, m$, the element $F_{-\bar{a},-\bar{a}} \in \mathfrak{h}$ acts on the tensor product (5.26) as

$$n/2 - \deg_a + (\sigma \circ \mu)_a,$$

where $\deg_a$ is the degree operator on the ath tensor factor, counting the factors from right to left. It acts on the vector space $\mathcal{G}(\mathbb{C}^n)$ of that tensor factor as the Euler operator

$$\sum_{k=1}^{n} x_k \partial_k \in \mathcal{G}\mathcal{D}(\mathbb{C}^n). \tag{5.28}$$

A bimodule equivalent to $\mathcal{F}_\delta(M_{\sigma\,\circ\mu})_{\mathfrak{n}}$ can be obtained by pushing forward the actions of $\mathfrak{h}$ and $\mathrm{X}(\mathfrak{g}_n)$ on (5.26) through the composition of the automorphisms (2.30), for every tensor factor with number a such that $\delta_a = -1$. Here we number the m tensor factors of (5.26) by $1, \dots, m$ from right to left. Then we also need to pull the resulting $\mathrm{X}(\mathfrak{g}_n)$-module back through the automorphism (1.17), where the series $f(u)$ equals the product (5.27). The automorphism (2.30) maps the element (5.28) to

$$\sum_{k=1}^{n} \partial_{\tilde{k}} x_{\tilde{k}} = n - \sum_{k=1}^{n} x_k \partial_k .$$

Hence, if $\delta_a = -1$, then the element $F_{-\bar{a},-\bar{a}} \in \mathfrak{h}$ acts on the modified tensor product as

$$-\,n/2 + (\sigma \circ \mu)_a + \deg_a .$$

Equating the last displayed expression to $(\sigma \circ \lambda)_a$ and using (5.21) together with the condition $\delta_a = -1$, we get the equation $\deg_a = \widetilde{\nu}_a$. But by Lemma 2.7, pushing forward the $\mathrm{Y}(\mathfrak{gl}_n)$-module

$$P^{\,\widetilde{\nu}_a}_{-\widetilde{\mu}_a - \frac{1}{2} - \widetilde{\rho}_a}$$

through the automorphism (2.30) of $\mathcal{G}\mathcal{D}(\mathbb{C}^n)$ yields the same $\mathrm{Y}(\mathfrak{gl}_n)$-module as pulling

$$P^{-\widetilde{\nu}_a}_{\widetilde{\mu}_a - \frac{1}{2} + \widetilde{\rho}_a}$$

back through the automorphism (1.3) of $\mathrm{Y}(\mathfrak{gl}_n)$, where

$$g(u) = \frac{u - \widetilde{\mu}_a + \frac{1}{2} - \widetilde{\rho}_a}{u - \widetilde{\mu}_a - \frac{1}{2} - \widetilde{\rho}_a}.$$

Thus, the $\mathrm{X}(\mathfrak{g}_n)$-module $\mathcal{F}_\delta(M_{\sigma\,\circ\mu})^{\sigma\,\circ\lambda}_{\mathfrak{n}}$ is equivalent to the tensor product (5.25) pulled back through the automorphism (1.17), where the series $f(u)$ is obtained by multiplying (5.27) by $g(-u)g(u)$ for each index a such that $\delta_a = -1$; see the definition (1.18). But for any element $\sigma \in \mathfrak{H}_m$, the product (5.24) equals

$$\prod_{a=1}^{m} \frac{u - \widetilde{\mu}_a + \frac{1}{2} - \widetilde{\rho}_a}{u - \widetilde{\mu}_a - \frac{1}{2} - \widetilde{\rho}_a}. \tag{5.29}$$

If $\delta_a = -1$, then the factors of (5.27) and (5.29) indexed by a are equal to $g(-u)^{-1}$ and $g(u)$, respectively. If $\delta_a = 1$, then the factors of (5.27) and (5.29) indexed by a coincide. This comparison of (5.27) and (5.29) completes the proof. □

The vector spaces of two equivalent $\mathrm{X}(\mathfrak{g}_n)$-modules in Proposition 5.4 are

$$(M_{\sigma\,\circ\mu} \otimes \mathcal{G}(\mathbb{C}^m \otimes \mathbb{C}^n))^{\sigma\,\circ\lambda}_{\mathfrak{n}} \quad \text{and} \quad \mathcal{G}^{\,\widetilde{\nu}_m}(\mathbb{C}^n) \otimes \dots \otimes \mathcal{G}^{\,\widetilde{\nu}_1}(\mathbb{C}^n),$$

respectively. We can define a linear map from the latter vector space to the former, by mapping $f_1 \otimes \dots \otimes f_m$ to the class of $1_{\sigma\,\circ\mu} \otimes f$ in the space of $\mathfrak{n}$-coinvariants. Here

$$f_1 \in \mathcal{G}^{\,\widetilde{\nu}_m}(\mathbb{C}^n)\,, \dots,\; f_m \in \mathcal{G}^{\,\widetilde{\nu}_1}(\mathbb{C}^n)$$

and $f \in \mathcal{G}(\mathbb{C}^m \otimes \mathbb{C}^n)$ is defined by (2.28). This linear map realizes an equivalence of the $\mathrm{X}(\mathfrak{g}_n)$-modules in Proposition 5.4; see the remarks after our proof of Corollary 2.6.

Thus, for any $\nu_1, \dots, \nu_m \in \{0, 1, \dots, n\}$ we have demonstrated how the Zhelobenko operator $\check{\xi}_\sigma$ on $\bar{\mathrm{J}}\backslash\bar{\mathrm{B}}_m$ determines an intertwining operator between the $\mathrm{X}(\mathfrak{g}_n)$-modules (5.23) and (5.25) pulled back via the automorphism (1.17) of $\mathrm{X}(\mathfrak{g}_n)$, where $f(u)$ is the same product (5.24) for both modules. Hence, this operator also intertwines the $\mathrm{X}(\mathfrak{g}_n)$-modules

$$P^{\,\nu_m}_{\mu_m-\frac{1}{2}+\rho_m}\otimes\cdots\otimes P^{\,\nu_1}_{\mu_1-\frac{1}{2}+\rho_1}\to P^{\,\delta_m\widetilde{\nu}_m}_{\widetilde{\mu}_m-\frac{1}{2}+\widetilde{\rho}_m}\otimes\cdots\otimes P^{\,\delta_1\widetilde{\nu}_1}_{\widetilde{\mu}_1-\frac{1}{2}+\widetilde{\rho}_1}, \tag{5.30}$$

neither of which is now pulled back via the automorphism (1.17). It was proved in [MN] that the two $\mathrm{X}(\mathfrak{g}_n)$-modules in (5.30) are irreducible under our assumptions on μ. Hence, an intertwining operator between them is unique up to a factor from $\mathbb{C}$. For our intertwining operator, this factor is determined by Proposition 5.9 below. Another expression for an intertwining operator of the $\mathrm{X}(\mathfrak{g}_n)$-modules (5.30) was given in [N].

For any $a=1,\ldots,m$ and $s=1,\ldots,n$, we define elements f_{as} and g_{as} of the ring $\mathcal{GD}(\mathbb{C}^m\otimes\mathbb{C}^n)$ as follows. We arrange the indices $1,\ldots,n$ in the sequence

$$1,3,\ldots,n-1,n,\ldots,4,2 \quad\text{or}\quad 1,3,\ldots,n-2,n,n-1,\ldots,4,2 \tag{5.31}$$

when n is even or odd, respectively. The mapping $k\mapsto\widetilde{k}$ reverses the sequence (5.31). We shall write $i\prec j$ if i precedes j in this sequence. Note that then the elements $E_{ij}-\theta_i\theta_jE_{\widetilde{j}\widetilde{i}}\in\mathfrak{gl}_n$ with $i\prec j$ or $i=j$ span a Borel subalgebra of $\mathfrak{g}_n\subset\mathfrak{gl}_n$, while the elements $E_{ii}-E_{\widetilde{i}\widetilde{i}}$ span the corresponding Cartan subalgebra of $\mathfrak{g}_n$. Then f_{as} and g_{as} are defined as the products of the elements x_{ak} and $\partial_{a\widetilde{k}}$ of $\mathcal{GD}(\mathbb{C}^m\otimes\mathbb{C}^n)$, respectively, taken over the first s indices k in the sequence (5.31). For example, if $n\geq 4$, then $f_{a2}=x_{a1}x_{a3}$ and $g_{a2}=\partial_{a2}\partial_{a4}$. If $n=3$, then $f_{a2}=x_{a1}x_{a3}$ but $g_{a2}=\partial_{a2}\partial_{a3}$. We also set $f_{a0}=g_{a0}=1$.

Our proof of Proposition 5.9 will be based on four lemmas below. The proof of the first lemma is quite similar to that of the second and will be omitted.

Lemma 5.5. *For any $a=1,\ldots,m-1$ and $s,t=0,1,\ldots,n$, the operator $\check{\xi}_a$ on $\bar{\mathrm{J}}\setminus\bar{\mathrm{B}}_m$ maps the image in $\bar{\mathrm{J}}\setminus\bar{\mathrm{B}}_m$ of $g_{\bar{a}s}\,g_{\overline{a+1}t}\in\bar{\mathrm{B}}_m$ to the image in $\bar{\mathrm{J}}\setminus\bar{\mathrm{B}}_m$ of the product*

$$\widetilde{\sigma}_a(g_{\bar{a}s}\,g_{\overline{a+1}t})\cdot\begin{cases}\dfrac{H_a-s+t+1}{H_a+1} & \textit{if}\quad s<t,\\[2mm] 1 & \textit{if}\quad s\geq t,\end{cases}$$

plus the images in $\bar{\mathrm{J}}\setminus\bar{\mathrm{B}}_m$ of elements of the left ideal in $\bar{\mathrm{B}}_m$ generated by $\bar{\mathrm{J}}'$ and (5.5).

Lemma 5.6. *For any $a=1,\ldots,m-1$ and $s,t=0,1,\ldots,n$, the operator $\check{\xi}_a$ on $\bar{\mathrm{J}}\setminus\bar{\mathrm{B}}_m$ maps the image in $\bar{\mathrm{J}}\setminus\bar{\mathrm{B}}_m$ of $f_{\bar{a}s}\,f_{\overline{a+1}t}\in\bar{\mathrm{B}}_m$ to the image in $\bar{\mathrm{J}}\setminus\bar{\mathrm{B}}_m$ of the product*

$$\widetilde{\sigma}_a(f_{\bar{a}s}\,f_{\overline{a+1}t})\cdot\begin{cases}\dfrac{H_a+s-t+1}{H_a+1} & \textit{if}\quad s>t,\\[2mm] 1 & \textit{if}\quad s\leq t,\end{cases}$$

plus the images in $\bar{\mathrm{J}}\setminus\bar{\mathrm{B}}_m$ of elements of the left ideal in $\bar{\mathrm{B}}_m$ generated by $\bar{\mathrm{J}}'$ and (5.6).

Proof. By the definitions (2.6) and (4.6), we have

$$\zeta_n(E_a)=-\sum_{k=1}^{n}x_{\overline{a+1}k}\,\partial_{\bar{a}k}\quad\text{and}\quad\zeta_n(F_a)=-\sum_{k=1}^{n}x_{\bar{a}k}\,\partial_{\overline{a+1}k}. \tag{5.32}$$

By (4.5), we also have

$$\widetilde{\sigma}_a(f_{\bar{a}s}\,f_{\overline{a+1}t})=f_{\overline{a+1}s}\,f_{\bar{a}t}.$$

We shall use the symbol $\equiv$ to indicate equality in the vector space $\bar{\mathrm{J}}\setminus\bar{\mathrm{B}}_m$ modulo the subspace that is the image of the left ideal in $\bar{\mathrm{B}}_m$ generated by $\bar{\mathrm{J}}'$ and the elements (5.6).

The element $E_a \in \bar{\mathrm{B}}_m$ belongs to this left ideal. Therefore, the operator $\check{\xi}_a$ maps the image of $f_{\bar{a}s}\, f_{\overline{a+1}t} \in \bar{\mathrm{B}}_m$ in $\bar{\mathrm{J}} \setminus \bar{\mathrm{B}}_m$ to the image in $\bar{\mathrm{J}} \setminus \bar{\mathrm{B}}_m$ of

$$
\begin{aligned}
\xi_a(f_{\overline{a+1}s}\, f_{\bar{a}t}) &= \sum_{r=0}^{\infty} (r!\, H_a^{(r)})^{-1} E_a^r\, \widehat{F}_a^r (f_{\overline{a+1}s}\, f_{\bar{a}t}) \\
&\equiv \sum_{r=0}^{\infty} (r!\, H_a^{(r)})^{-1} \widehat{E}_a^r\, \widehat{F}_a^r (f_{\overline{a+1}s}\, f_{\bar{a}t}).
\end{aligned}
$$

Now we use (4.10) along with (5.32). By the definitions of $f_{\overline{a+1}s}$ and $f_{\bar{a}t}$ we have

$$
\widehat{F}_a(f_{\overline{a+1}s}\, f_{\bar{a}t}) = -\sum_{k=1}^{n} [x_{\bar{a}k}\, \partial_{\overline{a+1}k}, f_{\overline{a+1}s}\, f_{\bar{a}t}].
$$

If $s \leq t$, then every summand above is zero, which proves the lemma in this case. Now suppose that $s > t$. Then, using the proof of [KN2, Proposition 3.7], we obtain

$$
\xi_a(f_{\overline{a+1}s}\, f_{\bar{a}t}) \;\equiv\; \sum_{r=0}^{s-t} \frac{(s-t)\cdots(s-t-r+1)}{H_a \cdots (H_a - r + 1)}\, f_{\overline{a+1}s}\, f_{\bar{a}t}.
$$

Here, the sum of the fractions corresponding to $r = 0, \dots, s-t$ equals

$$
\frac{H_a + 1}{H_a - s + t + 1};
$$

this can easily be proved by induction on the difference $s - t$. Therefore,

$$
\xi_a(f_{\overline{a+1}s}\, f_{\bar{a}t}) \equiv \frac{H_a + 1}{H_a - s + t + 1}\, f_{\overline{a+1}s}\, f_{\bar{a}t} = f_{\overline{a+1}s}\, f_{\bar{a}t}\, \frac{H_a + s - t + 1}{H_a + 1},
$$

as required in the case where $s > t$. Here we have also used the relation

$$
H_a\, f_{\overline{a+1}s}\, f_{\bar{a}t} = f_{\overline{a+1}s}\, f_{\bar{a}t}\, (H_a + s - t)
$$

in the ring B_m, which follows from (4.10), because

$$
\zeta_n(H_a) = \zeta_n(F_{\overline{a+1},\overline{a+1}} - F_{\bar{a}\bar{a}}) = \sum_{k=1}^{n} (x_{\overline{a+1}k}\, \partial_{\overline{a+1}k} - x_{\bar{a}k}\, \partial_{\bar{a}k}). \qquad \square
$$

Lemma 5.7. *For any $a = 1, \dots, m-1$ and $s, t = 0, 1, \dots, n$, the operator $\check{\xi}_a$ on $\bar{\mathrm{J}} \setminus \bar{\mathrm{B}}_m$ maps the image in $\bar{\mathrm{J}} \setminus \bar{\mathrm{B}}_m$ of $f_{\bar{a}s}\, g_{\overline{a+1}t} \in \bar{\mathrm{B}}_m$ to the image in $\bar{\mathrm{J}} \setminus \bar{\mathrm{B}}_m$ of the product*

$$
\widetilde{\sigma}_a(f_{\bar{a}s}\, g_{\overline{a+1}t}) \cdot \begin{cases} \dfrac{H_a + s + t + 1}{H_a + n + 1} & \textit{if} \quad s + t > n, \\ 1 & \textit{if} \quad s + t \leq n, \end{cases}
$$

plus the images in $\bar{\mathrm{J}} \setminus \bar{\mathrm{B}}_m$ of elements of the left ideal in $\bar{\mathrm{B}}_m$ generated by $\bar{\mathrm{J}}'$ and (5.7).

Proof. By (4.5),

$$
\widetilde{\sigma}_a(f_{\bar{a}s}\, g_{\overline{a+1}t}) = f_{\overline{a+1}s}\, g_{\bar{a}t}.
$$

Now we use the symbol $\equiv$ to indicate equality in $\bar{\mathrm{J}} \setminus \bar{\mathrm{B}}_m$ modulo the subspace that is the image of the left ideal in $\bar{\mathrm{B}}_m$ generated by $\bar{\mathrm{J}}'$ and the elements (5.7). The elements $E_a - \zeta_n(E_a)$ and $\zeta_n(F_a)$ of $\bar{\mathrm{B}}_m$ belong to this left ideal; see (5.32). By (4.10), the operator

$\check{\xi}_a$ maps the image $f_{\bar{a}s}\, g_{\overline{a+1}t} \in \bar{\mathrm{B}}_m$ in $\bar{\mathrm{J}} \setminus \bar{\mathrm{B}}_m$ to the image in $\bar{\mathrm{J}} \setminus \bar{\mathrm{B}}_m$ of

$$\begin{aligned}\xi_a(f_{\overline{a+1}s}\, g_{\bar{a}t}) \;&=\; \sum_{r=0}^{\infty} (r!\, H_a^{(r)})^{-1} E_a^r\, \widehat{F}_a^r(f_{\overline{a+1}s}\, g_{\bar{a}t}) \\ &\equiv\; \sum_{r=0}^{\infty} (r!\, H_a^{(r)})^{-1} \zeta_n(E_a)^r \zeta_n(F_a)^r f_{\overline{a+1}s}\, g_{\bar{a}t}.\end{aligned}$$

We have

$$\zeta_n(F_a) f_{\overline{a+1}s}\, g_{\bar{a}t} = -\sum_{k=1}^{n} x_{\bar{a}k}\, \partial_{\overline{a+1}k}\, f_{\overline{a+1}s}\, g_{\bar{a}t}$$

by (5.32). If $s+t \le n$, then every summand in the above displayed sum is zero modulo the left ideal of $\bar{\mathrm{B}}_m$ generated by the elements (5.7), because then there are no factors $x_{\overline{a+1}i}$ of $f_{\overline{a+1}s}$ and $\partial_{\bar{a}i}$ of $g_{\bar{a}t}$ with the same index i. This proves the lemma in this case. Now suppose that $s+t > n$. Then the proof of [KN2, Proposition 3.7] shows that

$$\begin{aligned}\xi_a(f_{\overline{a+1}s}\, g_{\bar{a}t}) &\equiv \sum_{r=0}^{s+t-n} \frac{(s+t-n)\cdots(s+t-n-r+1)}{H_a \cdots (H_a - r + 1)}\, f_{\overline{a+1}s}\, g_{\bar{a}t} \\ &= \frac{H_a + 1}{H_a - s - t + n + 1}\, f_{\overline{a+1}s}\, g_{\bar{a}t} = f_{\overline{a+1}s}\, g_{\bar{a}t}\, \frac{H_a + s + t + 1}{H_a + n + 1},\end{aligned}$$

as required. Here we have also used the following relation in the ring B_m, which follows from (4.10):

$$H_a\, f_{\overline{a+1}s}\, g_{\bar{a}t} = f_{\overline{a+1}s}\, f_{\bar{a}t}\, (H_a + s + t). \qquad \square$$

Lemma 5.8. *If $\mathfrak{f}_m = \mathfrak{sp}_{2m}$, then for any $s = 0, 1, \ldots, n$, the operator $\check{\xi}_m$ on $\bar{\mathrm{J}} \setminus \bar{\mathrm{B}}_m$ maps the image of $f_{\overline{m}s} \in \bar{\mathrm{B}}_m$ in $\bar{\mathrm{J}} \setminus \bar{\mathrm{B}}_m$ to the image in $\bar{\mathrm{J}} \setminus \bar{\mathrm{B}}_m$ of the product*

$$\widetilde{\sigma}_m(f_{\overline{m}s}) \cdot \begin{cases} \dfrac{H_a + s + 1}{H_a + n/2 + 1} & \text{if} \quad s > n/2, \\ 1 & \text{if} \quad s \le n/2, \end{cases}$$

plus the images in $\bar{\mathrm{J}} \setminus \bar{\mathrm{B}}_m$ of elements of the left ideal in $\bar{\mathrm{B}}_m$ generated by $\bar{\mathrm{J}}'$ and (5.8).

Proof. Let $\mathfrak{f}_m = \mathfrak{sp}_{2m}$. Then $\mathfrak{g}_n = \mathfrak{sp}_n$, so that the number n is even. By (4.5), we have

$$\widetilde{\sigma}_m(f_{\overline{m}s}) = g_{\overline{m}s} \quad \text{or} \quad \widetilde{\sigma}_m(f_{\overline{m}s}) = (-1)^{s-n/2}\, g_{\overline{m}s}$$

when $s \le n/2$ or $s > n/2$, respectively. Hence, it suffices to consider the image in $\bar{\mathrm{J}} \setminus \bar{\mathrm{B}}_m$ of the element $\xi_m(g_{\overline{m}s}) \in \bar{\mathrm{B}}_m$, $s = 0, 1, \ldots, n$. By the definitions (2.6) and (4.8),

$$\zeta_n(E_m) = \sum_{k=1}^{n} \theta_k\, \partial_{\overline{m}k}\, \partial_{\overline{m}\tilde{k}}\, /2 \quad \text{and} \quad \zeta_n(F_m) = \sum_{k=1}^{n} \theta_k\, x_{\overline{m}\tilde{k}}\, x_{\overline{m}k}\, /2.$$

Now we let the symbol $\equiv$ indicate equality in $\bar{\mathrm{J}} \setminus \bar{\mathrm{B}}_m$ modulo the subspace that is the image of the left ideal in $\bar{\mathrm{B}}_m$ generated by $\bar{\mathrm{J}}'$ and the elements (5.8). The elements $E_m - \zeta_n(E_m)$ and $\zeta_n(F_m)$ of $\bar{\mathrm{B}}_m$ belong to this left ideal. Therefore, by (4.10),

$$\begin{aligned}\xi_m(g_{\overline{m}s}) &= \sum_{r=0}^{\infty} (r!\, H_m^{(r)})^{-1} E_m^r\, \widehat{F}_m^r(g_{\overline{m}s}) \\ &\equiv \sum_{r=0}^{\infty} (r!\, H_m^{(r)})^{-1} \zeta_n(E_m)^r \zeta_n(F_m)^r g_{\overline{m}s}.\end{aligned}$$

We have

$$\zeta_n(F_m)\, g_{\overline{m}s} = \sum_{k=1}^{n} \theta_k\, x_{\overline{m}\tilde{k}}\, x_{\overline{m}k}\, g_{\overline{m}s}\,/2.$$

If $s \leq n/2$, then every summand in the above sum is zero modulo the left ideal of $\bar{\mathrm{B}}_m$ generated by the elements (5.8), because then for any index k there is no pair of factors $\partial_{\overline{m}k}$ and $\partial_{\overline{m}\tilde{k}}$ in the product $g_{\overline{m}s}$. This proves the lemma in this case. Now suppose that $s > n/2$. Then, using the proof of [KN2, Proposition 3.7] once again, we obtain

$$\begin{aligned}\xi_m(g_{\overline{m}s}) &\equiv \sum_{r=0}^{s-n/2} \frac{(s-n/2)\cdots(s-n/2-r+1)}{H_m\cdots(H_m-r+1)}\, g_{\overline{m}s}\\ &= \frac{H_m+1}{H_m-s+n/2+1}\, g_{\overline{m}s} = g_{\overline{m}s}\, \frac{H_m+s+1}{H_a+n/2+1},\end{aligned}$$

as required. Here we have also used the relation $H_m\, g_{\overline{m}s} = g_{\overline{m}s}\,(H_m+s)$ in the ring B_m, which follows from (4.10), because $\bar{m}=1$ and for $\mathfrak{f}_m = \mathfrak{sp}_{2m}$ we have

$$\zeta_n(H_m) = -\zeta_n(F_{11}) = n/2 - \sum_{k=1}^{n} x_{1k}\,\partial_{1k}$$

by (2.6) and (4.8). $\square$

Now we state Proposition 5.9. We assume that the weight μ satisfies conditions (5.2) and also satisfies conditions (5.3) if $\mathfrak{f}_m = \mathfrak{sp}_{2m}$. Moreover, we assume that $\nu_1,\ldots,\nu_m \in \{0,1,\ldots,n\}$; see the definition (5.21). Let $(\mu_1^*,\ldots,\mu_m^*)$ be the sequence of labels of the weight $\mu+\rho$. Then for each $a=1,\ldots,m$ we have $\mu_a^* = \mu_a + m - a$ if $\mathfrak{f}_m = \mathfrak{so}_{2m}$, and $\mu_a^* = \mu_a + m - a + 1$ if $\mathfrak{f}_m = \mathfrak{sp}_{2m}$. Let $(\lambda_1^*,\ldots,\lambda_m^*)$ be the sequence of labels of $\lambda+\rho$. For each positive root $\eta \in \Delta^+$ we define a number $z_\eta \in \mathbb{C}$:

$$z_\eta = \begin{cases} \dfrac{\lambda_b^* - \lambda_c^*}{\mu_b^* - \mu_c^*} & \text{if} \quad \eta = \varepsilon_b - \varepsilon_c \quad \text{and} \quad \nu_b > \nu_c, \\ \dfrac{\lambda_b^* + \lambda_c^*}{\mu_b^* + \mu_c^*} & \text{if} \quad \eta = \varepsilon_b + \varepsilon_c \quad \text{and} \quad \nu_b + \nu_c > n, \\ \dfrac{\lambda_b^*}{\mu_b^*} & \text{if} \quad \eta = 2\varepsilon_b \qquad \text{and} \quad 2\nu_b > n, \\ 1 & \text{otherwise.} \end{cases}$$

Note that in the first two cases, $1 \leq b < c \leq m$, while in the third case, $1 \leq b \leq m$ and $\mathfrak{f}_m = \mathfrak{sp}_{2m}$. Let v_μ^λ be the image of the product $f_{\bar{1}\nu_1}\cdots f_{\overline{m}\nu_m} \in \bar{\mathrm{B}}_m$ in the quotient vector space $\bar{\mathrm{J}}\setminus\bar{\mathrm{B}}_m/\bar{\mathrm{I}}_{\mu,\delta_+}$. This image is a highest vector relative to the action of the Lie algebra $\mathfrak{g}_n$ on this space: it is annihilated by the elements $E_{ij} - \theta_i\theta_j E_{\tilde{j}\tilde{\imath}} \in \mathfrak{g}_n$ with $i \prec j$.

Proposition 5.9. (i) *The vector* v_μ^λ *is not in the zero coset of* $\bar{\mathrm{J}}\setminus\bar{\mathrm{B}}_m/\bar{\mathrm{I}}_{\mu,\delta_+}$.

(ii) *Under the action of* $\mathfrak{h}$ *on* $\bar{\mathrm{J}}\setminus\bar{\mathrm{B}}_m/\bar{\mathrm{I}}_{\mu,\delta_+}$, *the vector* v_μ^λ *is of weight* λ.

(iii) *For any* $\sigma \in \mathfrak{H}_m$, *the intertwining operator* (5.19) *determined by* $\check{\xi}_\sigma$ *maps the vector* v_μ^λ *to the image in* $\bar{\mathrm{J}}\setminus\bar{\mathrm{B}}_m/\bar{\mathrm{I}}_{\sigma\circ\mu,\sigma(\delta_+)}$ *of* $\tilde{\sigma}(f_{\bar{1}\nu_1}\cdots f_{\overline{m}\nu_m}) \in \bar{\mathrm{B}}_m$ *multiplied by the product*

$$\prod_{\eta\in\Delta_\sigma} z_\eta. \tag{5.33}$$

Proof. Statement (i) of the proposition follows directly from the definition of the ideal $\bar{\mathrm{I}}_{\mu,\delta_+}$. We prove statement (ii). The elements of $\mathfrak{h}$ act on $\bar{\mathrm{J}}\setminus\bar{\mathrm{B}}_m/\bar{\mathrm{I}}_{\mu,\delta_+}$ via left multiplication on $\bar{\mathrm{B}}_m$. Let $\equiv$ indicate equality in $\bar{\mathrm{B}}_m$ modulo the left ideal $\bar{\mathrm{I}}_{\mu,\delta_+}$. Then, by the definition (2.6), for each $a=1,\ldots,m$ in the algebra $\bar{\mathrm{B}}_m$ we have

$$\begin{aligned}F_{-\bar a,-\bar a}\,f_{\bar 1\nu_1}\cdots f_{\overline{m}\nu_m} &= f_{\bar 1\nu_1}\cdots f_{\overline{m}\nu_m}\,F_{-\bar a,-\bar a}-\sum_{k=1}^{n}[x_{\bar a k}\,\partial_{\bar a k}-n/2,f_{\bar 1\nu_1}\cdots f_{\overline{m}\nu_m}]\\ &= f_{\bar 1\nu_1}\cdots f_{\overline{m}\nu_m}\,(F_{-\bar a,-\bar a}-\nu_a)\equiv f_{\bar 1\nu_1}\cdots f_{\overline{m}\nu_m}\,(\zeta_n(F_{-\bar a,-\bar a})+\mu_a-\nu_a)\\ &\equiv x_{\bar 1k}^{\nu_1}\cdots x_{\overline{m}k}^{\nu_m}\,(n/2+\mu_a-\nu_a)=\lambda_a\,f_{\bar 1\nu_1}\cdots f_{\overline{m}\nu_m}.\end{aligned}$$

Thus,

$$F_{-\bar a,-\bar a}\,v_\mu^\lambda\;=\;\lambda_a\,v_\mu^\lambda\quad\text{for}\quad a=1,\ldots,m.$$

Statement (iii) will be proved by induction on the length of a reduced decomposition of σ in terms of $\sigma_1,\ldots,\sigma_m$. If σ is the identity element of $\mathfrak{H}_m$, then the required statement is tautological. Now suppose that statement (iii) is true for some $\sigma\in\mathfrak{H}_m$. Take any simple reflection $\sigma_a\in\mathfrak{H}_m$ with $1\le a\le m$ such that $\sigma_a\sigma$ has a longer reduced decomposition in terms of $\sigma_1,\ldots,\sigma_m$ compared to σ. If $\mathfrak{f}_m=\mathfrak{so}_{2m}$ and $a=m$, then we have $\breve\xi_{\sigma_m\sigma}=\widetilde\sigma_m\,\breve\xi_\sigma$ and $\Delta_{\sigma_m\sigma}=\Delta_\sigma$, so that the induction step is immediate. Now we may assume that $a<m$ in the case where $\mathfrak{f}_m=\mathfrak{so}_{2m}$.

Take the simple root η_a corresponding to the reflection σ_a. Let $\eta=\sigma^{-1}(\eta_a)$. Then $\eta\in\Delta^+$ and

$$\sigma_a\sigma(\eta)=\sigma_a(\eta_a)=-\eta_a\notin\Delta^+.$$

Hence,

$$\Delta_{\sigma_a\sigma}=\Delta_\sigma\sqcup\{\eta\}.$$

Let $\kappa\in\mathfrak{h}^*$ be the weight with the labels (5.15). Using the proof of Theorem 5.3, we see that the following two left ideals of the algebra $\bar{\mathrm{B}}_m$ coincide:

$$\bar{\mathrm{I}}_{(\sigma_a\sigma)\circ\mu,(\sigma_a\sigma)(\delta_+)}=\tilde{\mathrm{I}}_{(\sigma_a\sigma)\circ\kappa,(\sigma_a\sigma)(\delta_+)}.$$

But modulo the second of these two ideals, the element H_a equals

$$\begin{aligned}((\sigma_a\sigma)\circ\kappa)(H_a)&=(\sigma_a\sigma(\kappa+\rho)-\rho)(H_a)=(\kappa+\rho)(\sigma^{-1}\sigma_a(H_a))-\rho(H_a)\\ &=-(\kappa+\rho)(\sigma^{-1}(H_a))-1=-\,(\kappa+\rho)(H_\eta)-1=-\,\frac{2(\kappa+\rho,\eta)}{(\eta,\eta)}-1.\end{aligned}\tag{5.34}$$

Here $H_\eta=\sigma^{-1}(H_a)$ is the coroot corresponding to the root η, and we use the standard bilinear form on $\mathfrak{h}^*$. Using only the definition (5.15), we can rewrite the right-hand side of (5.34) in the form

$$\begin{aligned}-\mu_b^*+\mu_c^*-1&\quad\text{if}\quad \eta=\varepsilon_b-\varepsilon_c,\\ -\mu_b^*-\mu_c^*-n-1&\quad\text{if}\quad \eta=\varepsilon_b+\varepsilon_c,\\ -\mu_b^*-n/2-1&\quad\text{if}\quad \eta=2\varepsilon_b.\end{aligned}$$

Now we shall use (iii) as the induction assumption. Denote $\delta=\sigma(\delta_+)$. Consider five cases.

I. Suppose $\eta=\varepsilon_b-\varepsilon_c$, where $1\le b<c\le m$, while $\sigma(\varepsilon_b)=\varepsilon_a$ and $\sigma(\varepsilon_c)=\varepsilon_{a+1}$. Then $\sigma_a=\varepsilon_a-\varepsilon_{a+1}$ and $\delta_a=\delta_{a+1}=1$. Hence,

$$\widetilde\sigma(f_{\bar 1\nu_1}\cdots f_{\overline{m}\nu_m})=f_{\bar a\nu_b}\,f_{\overline{a+1}\nu_c}Y,$$

where Y is an element of the subalgebra of $\mathcal{GD}(\mathbb{C}^m\otimes\mathbb{C}^n)$ generated by all x_{dk} and ∂_{dk} with $d\ne\bar a,\overline{a+1}$. Here Lemma 5.6 with $s=\nu_b$ and $t=\nu_c$ applies. With these s and t, and with $-\mu_b^*+\mu_c^*-1$ in place of H_a in the fraction displayed in that lemma, the fraction becomes

$$\frac{-\mu_b^*+\mu_c^*-1+\nu_b-\nu_c+1}{-\mu_b^*+\mu_c^*-1+1}=\frac{\lambda_b^*-\lambda_c^*}{\mu_b^*-\mu_c^*}.\tag{5.35}$$

Here the condition $s>t$ in Lemma 5.6 means that $\nu_b>\nu_c$.

II. Suppose $\eta = \varepsilon_b - \varepsilon_c$, where $1 \le b < c \le m$, but $\sigma(\varepsilon_b) = -\varepsilon_{a+1}$ and $\sigma(\varepsilon_c) = -\varepsilon_a$. Then $\sigma_a = \varepsilon_a - \varepsilon_{a+1}$ again, but $\delta_a = \delta_{a+1} = -1$. Hence,

$$\widetilde{\sigma}(f_{\bar{1}\nu_1} \cdots f_{\overline{m}\nu_m}) = g_{\bar{a}\nu_c}\, g_{\overline{a+1}\nu_b} Y,$$

where Y is another element of the subalgebra of $\mathcal{GD}(\mathbb{C}^m \otimes \mathbb{C}^n)$ generated by all x_{dk} and ∂_{dk} with $d \neq \bar{a}, \overline{a+1}$. Now Lemma 5.5 with $s = \nu_c$ and $t = \nu_b$ applies. With these s and t, and with $-\mu_b^* + \mu_c^* - 1$ in place of H_a in the fraction displayed in Lemma 5.5, the fraction becomes the same number (5.35) as in the preceding case, under the same condition $\nu_b > \nu_c$.

III. Suppose $\eta = \varepsilon_b + \varepsilon_c$ and $1 \le b < c \le m$, while $\sigma(\varepsilon_b) = \varepsilon_a$ and $\sigma(\varepsilon_c) = -\varepsilon_{a+1}$. Then $\sigma_a = \varepsilon_a - \varepsilon_{a+1}$ again, but $\delta_a = 1$ and $\delta_{a+1} = -1$. Hence,

$$\widetilde{\sigma}(f_{\bar{1}\nu_1} \cdots f_{\overline{m}\nu_m}) = f_{\bar{a}\nu_b}\, g_{\overline{a+1}\nu_c} Y,$$

where Y is another element of the subalgebra of $\mathcal{GD}(\mathbb{C}^m \otimes \mathbb{C}^n)$ generated by the x_{dk} and ∂_{dk} with $d \neq \bar{a}, \overline{a+1}$. Here Lemma 5.7 with $s = \nu_b$ and $t = \nu_c$ applies. With these s and t, and with $-\mu_b^* - \mu_c^* - n - 1$ in place of H_a in the fraction displayed in that lemma, the fraction becomes the number

$$\frac{-\mu_b^* - \mu_c^* - n - 1 + \nu_b + \nu_c + 1}{-\mu_b^* - \mu_c^* - n - 1 + n + 1} = \frac{\lambda_b^* + \lambda_c^*}{\mu_b^* + \mu_c^*}. \tag{5.36}$$

Here the condition $s + t > n$ in Lemma 5.7 means that $\nu_b + \nu_c > n$.

IV. Suppose $\eta = \varepsilon_b + \varepsilon_c$, where $1 \le b < c \le m$, but $\sigma(\varepsilon_b) = -\varepsilon_{a+1}$ and $\sigma(\varepsilon_c) = \varepsilon_a$. Then $\sigma_a = \varepsilon_a - \varepsilon_{a+1}$ again, but $\delta_a = 1$ and $\delta_{a+1} = -1$. Hence,

$$\widetilde{\sigma}(f_{\bar{1}\nu_1} \cdots f_{\overline{m}\nu_m}) = f_{\bar{a}\nu_c}\, g_{\overline{a+1}\nu_b} Y,$$

where Y is another element of the subalgebra of $\mathcal{GD}(\mathbb{C}^m \otimes \mathbb{C}^n)$ generated by the x_{dk} and ∂_{dk} with $d \neq \bar{a}, \overline{a+1}$. Now Lemma 5.7 with $s = \nu_c$ and $t = \nu_b$ applies. With these s and t, and with $-\mu_b^* - \mu_c^* - n - 1$ in place of H_a in the fraction displayed in that lemma, the fraction becomes the same number (5.36) as in the preceding case, under the same condition $\nu_b + \nu_c > n$.

V. Suppose $\mathfrak{f}_m = \mathfrak{sp}_{2m}$ and $\eta = 2\varepsilon_b$ with $1 \le b \le m$. Then $\sigma(\varepsilon_b) = \varepsilon_m$ and $\sigma_a = \sigma_m$, while $\delta_m = 1$. Hence,

$$\widetilde{\sigma}(f_{\bar{1}\nu_1} \cdots f_{\overline{m}\nu_m}) = f_{\overline{m}\nu_b}\, Y,$$

where Y is now an element of the subalgebra of $\mathcal{GD}(\mathbb{C}^m \otimes \mathbb{C}^n)$ generated by the x_{dk} and ∂_{dk} with $d \neq \bar{m} = 1$. Here Lemma 5.8 with $s = \nu_b$ applies. With this s, and with $-\mu_b^* - n/2 - 1$ in place of H_m in the fraction displayed in that lemma, the fraction becomes

$$\frac{-\mu_b^* - n/2 - 1 + \nu_b + 1}{-\mu_b^* - n/2 - 1 + n/2 + 1} = \frac{\lambda_b^*}{\mu_b^*}.$$

Here the condition $s > n/2$ in Lemma 5.8 means that $2\nu_b > n$.

Using the inductive hypothesis, we see that, in all the five cases above, the intertwining operator

$$\bar{\mathrm{J}} \setminus \bar{\mathrm{B}}_m / \bar{\mathrm{I}}_{\mu,\delta_+} \to \bar{\mathrm{I}}_{(\sigma_a\sigma)\circ\mu,(\sigma_a\sigma)(\delta_+)}$$

determined by $\check{\xi}_{\sigma_a\sigma}$ maps the vector v_μ^λ to the image in $\bar{\mathrm{J}} \setminus \bar{\mathrm{B}}_m / \bar{\mathrm{I}}_{(\sigma_a\sigma)\circ\mu,(\sigma_a\sigma)(\delta_+)}$ of

$$\widetilde{\sigma}_a \widetilde{\sigma}(f_{\bar{1}\nu_1} \cdots f_{\overline{m}\nu_m}) \in \bar{\mathrm{B}}_m$$

multiplied by the product (5.33) over the set Δ_σ and by an extra factor z_η corresponding to the positive root $\eta = \sigma^{-1}(\eta_a)$. This completes the induction step. $\square$

The product (5.33) in Proposition 5.9 does not depend on the choice of a reduced decomposition of $\sigma \in \mathfrak{H}_m$ in terms of $\sigma_1, \ldots, \sigma_m$. Thus, the uniqueness of the intertwining operator (5.30) provides another proof of the independence of our operator (5.20) of the decomposition of σ, not involving Proposition 4.4. Proposition 5.9 also shows that our intertwining operator (5.20) is not zero.

§6. Olshanskiĭ homomorphism

For a positive integer l, take the vector space $\mathbb{C}^{n+l}$. In the case of an alternating form on $\mathbb{C}^n$, we choose l to be even. Let $e_1, \ldots, e_{n+l}$ be the vectors of the standard basis in $\mathbb{C}^{n+l}$. Consider the decomposition $\mathbb{C}^{n+l} = \mathbb{C}^n \oplus \mathbb{C}^l$, where the direct summands $\mathbb{C}^n$ and $\mathbb{C}^l$ are spanned by the vectors $e_1, \ldots, e_n$ and $e_{n+1}, \ldots, e_{n+l}$, respectively. This determines an embedding of the direct sum $\mathfrak{gl}_n \oplus \mathfrak{gl}_l$ of Lie algebras to $\mathfrak{gl}_{n+l}$. As a subalgebra of $\mathfrak{gl}_{n+l}$, the summand $\mathfrak{gl}_n$ is spanned by the matrix units $E_{ij} \in \mathfrak{gl}_{n+l}$, where $i, j = 1, \ldots, n$. The summand $\mathfrak{gl}_l$ is spanned by the matrix units E_{ij}, where $i, j = n+1, \ldots, n+l$.

The subspace $\mathbb{C}^n \subset \mathbb{C}^{n+l}$ comes with the bilinear form chosen in §1. Now we choose a bilinear form on the subspace $\mathbb{C}^l \subset \mathbb{C}^{n+l}$ in a similar way. Namely, let i be any of the indices $n+1, \ldots, n+l$. If $i-n$ is even, then put $\tilde{\imath} = i-1$. If $i-n$ is odd and $i < n+l$, then put $\tilde{\imath} = i+1$. If $i = n+l$ and l is odd, then put $\tilde{\imath} = i$. Next, put $\theta_i = 1$ or $\theta_i = (-1)^{i-n-1}$ in the case of the symmetric or alternating form on $\mathbb{C}^n$. For any basis vectors e_i and e_j of the subspace $\mathbb{C}^l$, put $\langle e_i, e_j \rangle = \theta_i \, \delta_{\tilde{\imath} j}$. We equip the vector space $\mathbb{C}^{n+l}$ with the bilinear form that is the sum of the forms on the direct summands. The forms on $\mathbb{C}^l$ and $\mathbb{C}^{n+l}$ are of the same type (symmetric or alternating) as the form on $\mathbb{C}^n$.

Now we consider the subalgebras $\mathfrak{g}_n, \mathfrak{g}_l$, and $\mathfrak{g}_{n+l}$ of the Lie algebras $\mathfrak{gl}_n, \mathfrak{gl}_l$, and $\mathfrak{gl}_{n+l}$, respectively. We have an embedding of the direct sum $\mathfrak{g}_n \oplus \mathfrak{g}_l$ to the Lie algebra $\mathfrak{g}_{n+l}$, in accordance with our choice of the bilinear forms made above. We also have an embedding of the direct product of Lie groups $G_n \times G_l$ to G_{n+l}. Let C_l denote the subalgebra of G_l-invariants in the universal enveloping algebra $\mathrm{U}(\mathfrak{g}_{n+l})$. Then C_l contains the subalgebra $\mathrm{U}(\mathfrak{g}_n) \subset \mathrm{U}(\mathfrak{g}_{n+l})$. If $\mathfrak{g}_n = \mathfrak{sp}_n$, then C_l coincides with the centralizer of the subalgebra $\mathrm{U}(\mathfrak{sp}_l) \subset \mathrm{U}(\mathfrak{sp}_{n+l})$. If $\mathfrak{g}_n = \mathfrak{so}_n$, then C_l is contained in the centralizer of $\mathrm{U}(\mathfrak{so}_l) \subset \mathrm{U}(\mathfrak{so}_{n+l})$, but may differ from the centralizer.

Take the extended twisted Yangian $\mathrm{X}(\mathfrak{g}_{n+l})$. The subalgebra of $\mathrm{X}(\mathfrak{gl}_{n+l})$ generated by

$$S_{ij}^{(1)}, S_{ij}^{(2)}, \ldots \quad \text{with} \quad i, j = 1, \ldots, n$$

is isomorphic to $\mathrm{X}(\mathfrak{g}_n)$ as an associative algebra; see [MNO, Subsection 3.14]. Thus, we have a natural embedding $\mathrm{X}(\mathfrak{g}_n) \to \mathrm{X}(\mathfrak{g}_{n+l})$; we denote it by ι_l. We also have a surjective homomorphism

$$\pi_{n+l} : \mathrm{X}(\mathfrak{g}_{n+l}) \to \mathrm{U}(\mathfrak{g}_{n+l});$$

see (1.22). Note that the composition $\pi_{n+l} \, \iota_l$ coincides with the homomorphism π_n.

Next, consider the involutive automorphism ω_{n+l} of the algebra $\mathrm{X}(\mathfrak{g}_{n+l})$; see the definition (1.20). The image of the composition of homomorphisms

$$\pi_{n+l} \, \omega_{n+l} \, \iota_l : \mathrm{X}(\mathfrak{g}_n) \to \mathrm{U}(\mathfrak{g}_{n+l}) \tag{6.1}$$

belongs to the subalgebra $\mathrm{C}_l \subset \mathrm{U}(\mathfrak{g}_{n+l})$. Moreover, together with the subalgebra of G_{n+l}-invariants in $\mathrm{U}(\mathfrak{g}_{n+l})$, this image generates C_l. These two results are due to G. Olshanskiĭ [O2]; for their detailed proofs, see [MO, §4]. We shall use the composition of homomorphisms

$$\gamma_l = \pi_{n+l} \, \omega_{n+l} \, \iota_l \, \omega_n$$

and call it the *Olshanskiĭ homomorphism*. The images of the homomorphisms γ_l and (6.1) in $\mathrm{U}(\mathfrak{g}_{n+l})$ coincide. The reason for using the homomorphism γ_l rather than the homomorphism (6.1) will become apparent when we state Theorem 6.1.

An irreducible representation of the group G_n is said to be *polynomial* if it arises as a subrepresentation of some tensor power of the defining representation $\mathbb{C}^n$. In accordance with [W, Subsections V.7 and VI.3], the irreducible polynomial representations of the group G_n are parametrized by all the partitions ν of $N = 0, 1, 2, \ldots$ such that $2\nu'_1 \le n$ in the case of $G_n = Sp_n$, and $\nu'_1 + \nu'_2 \le n$ in the case of $G_n = O_n$. Here ν' is the partition conjugate to ν, while $\nu'_1, \nu'_2, \ldots$ are the parts of ν'. Note that in the case where $G_n = O_n$ we still have $2\nu'_2 \le n$. Denote by W_ν the irreducible polynomial representation of the group G_n corresponding to ν. Let $\nu_1, \nu_2, \ldots$ be the parts of ν.

Let $\bar\nu$ be the weight of the Lie algebra $\mathfrak{f}_m$ with the sequence of labels

$$(n/2 - \nu'_m, \ldots, n/2 - \nu'_1).$$

By the conditions on ν, for $\mathfrak{f}_m = \mathfrak{sp}_{2m}$, the labels $\bar\nu_1, \ldots, \bar\nu_m$ of $\bar\nu$ are integers such that $\bar\nu_1 \ge \cdots \ge \bar\nu_m \ge 0$. For $\mathfrak{f}_m = \mathfrak{so}_{2m}$, either all labels of $\bar\nu$ are integers, or all of them are half-integers. In the case where $\mathfrak{f}_m = \mathfrak{so}_{2m}$, we have $\bar\nu_1 \ge \cdots \ge \bar\nu_{m-1} \ge |\bar\nu_m|$.

Consider $\mathcal{G}(\mathbb{C}^m \otimes \mathbb{C}^n)$ as a bimodule over $\mathfrak{f}_m$ and G_n. Then, by [H, Subsection 3.8.9] when $G_n = Sp_n$, or by [H, Subsection 4.3.5] when $G_n = O_n$, we have a decomposition

$$\mathcal{G}(\mathbb{C}^m \otimes \mathbb{C}^n) = \bigoplus_\nu L_{\bar\nu} \otimes W_\nu, \tag{6.2}$$

where ν ranges over all parameters of the irreducible polynomial representations of G_n such that $\nu_1 \le m$. Here $L_{\bar\nu}$ is the irreducible $\mathfrak{f}_m$-module of the highest weight $\bar\nu$.

Let λ and μ be parameters of any irreducible polynomial representations of the groups G_{n+l} and G_l, respectively. Suppose that $\lambda_1, \mu_1 \le m$. Using the action of the group G_l on W_λ via its embedding to G_{n+l} as the second direct factor of the subgroup $G_n \times G_l$, we consider the vector space

$$\operatorname{Hom}_{G_l}(W_\mu, W_\lambda). \tag{6.3}$$

The subalgebra $\mathrm{C}_l \subset \mathrm{U}(\mathfrak{g}_{n+l})$ acts on this vector space through the action of $\mathrm{U}(\mathfrak{g}_{n+l})$ on W_λ. In the case where $G_n = Sp_n$, the vector space (6.3) is irreducible under the action of the algebra C_l; see [D, Theorem 9.1.12]. If $G_n = O_n$, the C_l-module (6.3) is either irreducible or splits into a direct sum of two irreducible C_l-modules. It is irreducible if W_λ is irreducible as an $\mathfrak{so}_{n+l}$-module, that is, if $2\lambda'_1 \ne n + l$, by [W, Subsection V.9]. Note that for $G_n = O_n$, the condition $2\lambda'_1 \ne n + l$ is sufficient but not necessary for the irreducibility of the C_l-module (6.3); see [N, Subsection 1.7].

In any case, the vector space (6.3) is irreducible under the joint action of the subalgebra $\mathrm{C}_l \subset \mathrm{U}(\mathfrak{g}_{n+l})$ and the subgroup $G_n \subset G_{n+l}$; see again [N, Subsection 1.7]. Hence, the following identifications of bimodules over C_l and G_n are unique up to rescaling of their vector spaces:

$$\begin{aligned} \operatorname{Hom}_{G_l}(W_\mu, W_\lambda) &= \operatorname{Hom}_{G_l}(W_\mu, \operatorname{Hom}_{\mathfrak{f}_m}(L_{\bar\lambda}, \mathcal{G}(\mathbb{C}^m \otimes \mathbb{C}^{n+l}))) \\ &= \operatorname{Hom}_{G_l}(W_\mu, \operatorname{Hom}_{\mathfrak{f}_m}(L_{\bar\lambda}, \mathcal{G}(\mathbb{C}^m \otimes \mathbb{C}^l) \otimes \mathcal{G}(\mathbb{C}^m \otimes \mathbb{C}^n))) \\ &= \operatorname{Hom}_{\mathfrak{f}_m}(L_{\bar\lambda}, L_{\bar\mu} \otimes \mathcal{G}(\mathbb{C}^m \otimes \mathbb{C}^n)). \end{aligned} \tag{6.4}$$

We use the decompositions (6.2) for $n + l$ and l instead of n, and the identification

$$\mathcal{G}(\mathbb{C}^m \otimes \mathbb{C}^{n+l}) = \mathcal{G}(\mathbb{C}^m \otimes \mathbb{C}^l) \otimes \mathcal{G}(\mathbb{C}^m \otimes \mathbb{C}^n) \tag{6.5}$$

of vector spaces. Thus, in (6.4), the labels of the weights $\bar\lambda$ and $\bar\mu$ of $\mathfrak{f}_m$ are (respectively)

$$(n/2 + l/2 - \lambda'_m, \ldots, n/2 + l/2 - \lambda'_1) \quad \text{and} \quad (l/2 - \mu'_m, \ldots, l/2 - \mu'_1).$$

By pulling back via the Olshanskiĭ homomorphism $\gamma_l : \mathrm{X}(\mathfrak{g}_n) \to \mathrm{C}_l$, the vector space (6.3) becomes a module over the extended twisted Yangian $\mathrm{X}(\mathfrak{g}_n)$. Using the above identifications, we see that the vector space (6.4) also becomes a module over $\mathrm{X}(\mathfrak{g}_n)$. But the target $\mathfrak{f}_m$-module $L_{\bar\mu} \otimes \mathcal{G}(\mathbb{C}^m \otimes \mathbb{C}^n)$ in (6.4) coincides with the $\mathfrak{f}_m$-module $\mathcal{F}_m(L_{\bar\mu})$.

Theorem 6.1. *The action of* $\mathrm{X}(\mathfrak{g}_n)$ *on the vector space* (6.4) *via the homomorphism* γ_l *coincides with the action obtained by pulling the action of* $\mathrm{X}(\mathfrak{g}_n)$ *on the bimodule* $\mathcal{F}_m(L_{\bar\mu})$ *back through the homomorphism* (1.17)*, where*

$$f(u) = 1 - m(u - l/2 \pm 1/2)^{-1}. \tag{6.6}$$

Proof. Take the action of the subalgebra $\mathrm{C}_l \subset \mathrm{U}(\mathfrak{gl}_{n+l})$ on the space $\mathcal{G}(\mathbb{C}^m \otimes \mathbb{C}^{n+l})$. The extended twisted Yangian $\mathrm{X}(\mathfrak{g}_n)$ acts on this vector space via the homomorphism $\gamma_l : \mathrm{X}(\mathfrak{g}_n) \to \mathrm{C}_l$. Using the decomposition (6.5), we show that for $i,j = 1,\ldots,n$ the generators $S_{ij}^{(1)}, S_{ij}^{(2)}, \ldots$ of $\mathrm{X}(\mathfrak{g}_n)$ act on this vector space respectively as the coefficients of $u^{-1}, u^{-2}, \ldots$ in the series (2.8) multiplied by the series (6.6).

For any $i,j = 1,\ldots,n+l$, the element $F_{ij} \in \mathrm{U}(\mathfrak{g}_{n+l})$ acts on $\mathcal{G}(\mathbb{C}^m \otimes \mathbb{C}^{n+l})$ as the operator

$$\sum_{c=1}^{m} (x_{ci}\,\partial_{cj} - \theta_i\,\theta_j\,x_{c\tilde{j}}\,\partial_{c\tilde{i}}).$$

Here we use the standard coordinate functions x_{ci} on $\mathbb{C}^m \otimes \mathbb{C}^{n+l}$ with $c = 1,\ldots,m$ and $i = 1,\ldots,n+l$. Then ∂_{ci} is the left derivation on the Grassmann algebra $\mathcal{G}(\mathbb{C}^m \otimes \mathbb{C}^{n+l})$ relative to x_{ci}. The functions x_{ci} with $c \le n$ and $c > n$ correspond to the direct summands $\mathbb{C}^n$ and $\mathbb{C}^l$ of $\mathbb{C}^{n+l}$. Consider the $((n+l)\times(n+l))$-matrix whose (i,j)-entry is

$$\delta_{ij} + (u - l/2 \pm 1/2)^{-1} \sum_{c=1}^{m} (x_{ci}\,\partial_{cj} - \theta_i\,\theta_j\,x_{c\tilde{j}}\,\partial_{c\tilde{i}}).$$

We can write this matrix and its inverse as block matrices

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{bmatrix},$$

where the blocks A,B,C,D and $\widetilde{A},\widetilde{B},\widetilde{C},\widetilde{D}$ are matrices of sizes $n\times n$, $n\times l$, $l\times n$, and $l\times l$, respectively. Now the action of the algebra $\mathrm{X}(\mathfrak{g}_n)$ on the vector space $\mathcal{G}(\mathbb{C}^m\otimes\mathbb{C}^{n+l})$ via the homomorphism $\gamma_l : \mathrm{X}(\mathfrak{gl}_n) \to \mathrm{C}_l$ can be described by assigning the (i,j)-entry of the matrix $\widetilde{A}^{-1}$ to the series $S_{ij}(u)$ with $i,j = 1,\ldots,n$.

We introduce the $((n+l)\times 2m)$-matrix whose (i,c)-entry for $c = -m,\ldots,-1$ is the operator of left multiplication by x_{ci} on $\mathcal{G}(\mathbb{C}^m \otimes \mathbb{C}^{n+l})$. For $c = 1,\ldots,m$, let the (i,c)-entry of this matrix be the operator $\theta_i\,\partial_{c\tilde{i}}$. We write this matrix as

$$\begin{bmatrix} P \\ \bar{P} \end{bmatrix},$$

where the blocks P and $\bar{P}$ are matrices of sizes $n \times 2m$ and $l \times 2m$, respectively. Next, we introduce the $(2m \times (n+l))$-matrix whose (c,j)-entry for $c = -m,\ldots,-1$ is the operator ∂_{cj}. For $c = 1,\ldots,m$, let the (c,j)-entry of this matrix be the operator of left multiplication by $\theta_j\,x_{c\tilde{j}}$. We write this matrix as

$$\begin{bmatrix} Q & \bar{Q} \end{bmatrix},$$

where Q and $\bar{Q}$ are matrices of sizes $2m \times n$ and $2m \times l$, respectively. Then

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = 1 + (u - l/2 \pm 1/2)^{-1} \begin{bmatrix} PQ - m & P\bar{Q} \\ \bar{P}Q & \bar{P}\bar{Q} - m \end{bmatrix},$$

which can also be written as the matrix

$$1 + (u - l/2 \pm 1/2 - m)^{-1} \begin{bmatrix} PQ & P\bar{Q} \\ \bar{P}Q & \bar{P}\bar{Q} \end{bmatrix}$$

multiplied by the series $f(u)$ determined by (6.6). Using a well-known formula for $\widetilde{A}^{\,-1}$, we obtain

$$\begin{aligned}(6.7)\qquad \widetilde{A}^{\,-1} &= A - BD^{-1}C = f(u)\,\bigl(1 + (u - l/2 \pm 1/2 - m)^{-1} PQ \\ &\quad - (u - l/2 \pm 1/2 - m)^{-2}\, P\bar{Q}\,\bigl(1 + (u - l/2 \pm 1/2 - m)^{-1} \bar{P}\bar{Q}\,\bigr)^{-1} \bar{P}Q\,\bigr) \\ &= f(u)\,\bigl(1 + P(u - l/2 \pm 1/2 - m + \bar{Q}\bar{P}\,)^{-1} Q\,\bigr).\end{aligned}$$

Consider the $(2m \times 2m)$-matrix $\bar{Q}\bar{P}$ appearing in the last line. For any indices $a, b = -m, \ldots, -1, 1, \ldots, m$, the (a, b)-entry of this matrix is the operator

$$\delta_{ab}\, l/2 + \bar{\zeta}_l(F_{ab}),$$

where $\bar{\zeta}_l : \mathrm{U}(\mathfrak{f}_m) \to \mathcal{GD}(\mathbb{C}^m \otimes \mathbb{C}^{n+l})$ is the homomorphism corresponding to the action of the Lie algebra $\mathfrak{f}_m$ on $\mathcal{G}(\mathbb{C}^m \otimes \mathbb{C}^{n+l})$ via the tensor factor $\mathcal{G}(\mathbb{C}^m \otimes \mathbb{C}^l)$ in (6.5), similar to the homomorphism (2.6). Namely, for $a, b = 1, \ldots, m$ we have

$$\bar{\zeta}_l(F_{ab}) = -\delta_{ab}\, l/2 \; + \sum_{k=n+1}^{n+l} x_{ak}\, \partial_{bk},$$

$$\bar{\zeta}_l(F_{a,-b}) = \sum_{k=n+1}^{n+l} \theta_k\, x_{a\tilde{k}}\, x_{bk}, \qquad \bar{\zeta}_l(F_{-a,b}) = \sum_{k=n+1}^{n+l} \theta_k\, \partial_{ak}\, \partial_{b\tilde{k}}.$$

Hence, any entry of the $(2m \times 2m)$-matrix

$$(u - l/2 \pm 1/2 - m + \bar{Q}\bar{P})^{-1}$$

can be obtained by applying the homomorphism $\bar{\zeta}_l$ to the corresponding entry of the matrix $F(u \pm \frac{1}{2} - m)$; the last mentioned entries are series in u^{-1} with coefficients in $\mathrm{U}(\mathfrak{f}_m)$. Now we complete the proof by comparing the (i, j)-entry of the $(n \times n)$-matrix (6.7) with the series obtained from (2.8) by replacing $F_{ab}(u \pm \frac{1}{2} - m)$ there by $\bar{\zeta}_l(F_{ab}(u \pm \frac{1}{2} - m))$ for all indices $a, b = -m, \ldots, -1, 1, \ldots, m$. □

Set $\mathrm{C}_0 = \mathrm{U}(\mathfrak{g}_n)$ and $\gamma_0 = \pi_n$. Then Theorem 6.1 remains valid in the case where $l = 0$. In this case we assume that $\mathfrak{g}_l = \{0\}$. Note that our proof of Theorem 6.1 also implies Proposition 2.3, because the kernels of the homomorphisms $\bar{\zeta}_l$ with $l = 0, 1, 2, \ldots$ have only zero intersection. For $\mathfrak{f}_m = \mathfrak{so}_2$, the latter fact follows directly from the definition (2.6). For $\mathfrak{f}_m \neq \mathfrak{so}_2$, all irreducible finite-dimensional $\mathfrak{f}_m$-modules arise from the skew Howe duality.

Let λ and μ be the parameters of any irreducible polynomial representations of G_{n+l} and G_l, respectively. The vector space (6.3) is not zero if and only if

$$(6.8)\qquad \lambda_k \geq \mu_k \quad \text{and} \quad \lambda'_k - \mu'_k \leq n \quad \text{for every} \quad k = 1, 2, \ldots;$$

see [N, Subsection 1.3]. Suppose that $\lambda_1, \mu_1 \leq m$. Then we can identify the vector spaces (6.3) and (6.4). Then the algebra C_l acts on (6.4) irreducibly if $G_n = Sp_n$. If $G_n = O_n$, then (6.4) is irreducible under the joint action of the algebra C_l and the group O_n. In both cases, the G_{n+l}-invariant elements of $\mathrm{U}(\mathfrak{gl}_{n+l})$ act on (6.4) via multiplication by scalars. Then Theorem 6.1 has a corollary, which refers to the action of $\mathrm{X}(\mathfrak{g}_n)$ on the vector space (6.4) inherited from the bimodule $\mathcal{F}_m(L_{\bar{\mu}})$.

Corollary 6.2. *The algebra* $\mathrm{X}(\mathfrak{g}_n)$ *acts on the space* (6.4) *irreducibly if* $G_n = Sp_n$. *If* $G_n = O_n$, *the space* (6.4) *is irreducible under the joint action of* $\mathrm{X}(\mathfrak{g}_n)$ *and* O_n.

Now suppose that $\mathfrak{f}_m \neq \mathfrak{so}_2$. Then any irreducible finite-dimensional module V of $\mathfrak{f}_m$ is equivalent to $L_{\bar{\mu}}$ for some nonnegative integer l and the label μ of some irreducible polynomial representation of the group G_l with $\mu_1 \leq m$. If V' is another irreducible finite-dimensional $\mathfrak{f}_m$-module such that the vector space (0.10) is nonzero, then V' must be equivalent to $L_{\bar\lambda}$ for the label λ of some irreducible polynomial representation of G_{n+l} with $\lambda_1 \le m$. Thus any nonzero vector space (0.10) must be of the form (6.4).

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