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FAMILIES OF FRACTIONAL CAUCHY TRANSFORMS IN THE BALL

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Dedicated to Victor Petrovich Havin on the occasion of his 75th birthday

Abstract. Let B_n denote the unit ball in $\mathbb{C}^n$, $n \geq 1$. Given $\alpha > 0$, let $\mathcal{K}_\alpha(n)$ denote the class of functions defined for $z \in B_n$ by integrating the kernel $(1 - \langle z, \zeta\rangle)^{-\alpha}$ against a complex-valued Borel measure on the sphere $\{\zeta \in \mathbb{C}^n : |\zeta| = 1\}$. The families $\mathcal{K}_\alpha(1)$ of fractional Cauchy transforms have been investigated intensively by several authors. In the paper, various properties of $\mathcal{K}_\alpha(n)$, $n \geq 2$, are studied. In particular, relations between $\mathcal{K}_\alpha(n)$ and other spaces of holomorphic functions in the ball are obtained. Also, pointwise multipliers for the spaces $\mathcal{K}_\alpha(n)$ are investigated.

§1. Introduction

For $n \geq 1$, put $B_n = \{z \in \mathbb{C}^n : |z| < 1\}$. Let $M(n)$ denote the space of complex-valued Borel measures on the sphere ∂B_n.

1.1. Fractional Cauchy transforms. Let $\alpha > 0$. Given a measure $\mu \in M(n)$, its fractional Cauchy transform of order α is defined by the identity

$$K_\alpha[\mu](z) = \int_{\partial B_n} \frac{1}{(1 - \langle z, \zeta\rangle)^\alpha}\, d\mu(\zeta), \quad z \in B_n.$$

Here and in what follows we use the principal branch of the logarithm. Put

$$\mathcal{K}_\alpha(n) = \{K_\alpha[\mu] : \mu \in M(n)\}.$$

1.2. Multipliers. Let $\mathcal{H}ol(B_n)$ denote the space of holomorphic functions in the ball B_n. A function $g \in \mathcal{H}ol(B_n)$ is called a multiplier for the family $\mathcal{K}_\alpha(n)$, $\alpha > 0$, if $fg \in \mathcal{K}_\alpha(n)$ for all $f \in \mathcal{K}_\alpha(n)$. Let $\mathfrak{M}_\alpha(n)$ denote the set of all multipliers for $\mathcal{K}_\alpha(n)$.

1.3. Families $\mathcal{K}_\alpha(n)$ and $\mathfrak{M}_\alpha(n)$ as Banach spaces. For $f \in \mathcal{K}_\alpha(n)$, put

$$\|f\|_{\mathcal{K}_\alpha(n)} = \inf\left\{\|\mu\|_{M(n)} : f = K_\alpha[\mu]\right\}.$$

Standard arguments show that the above infimum is attained, and that $\mathcal{K}_\alpha(n)$ with the norm $\|\cdot\|_{\mathcal{K}_\alpha(n)}$ is a Banach space. Next, assume that $f = K_\alpha[\rho]$, where ρ is a positive measure. For $\mu \in M(n)$, let $f = K_\alpha[\mu]$; then $\|\rho\| = K_\alpha[\rho](0) = K_\alpha[\mu](0) \leq \|\mu\|$. Therefore, $\|f\|_{\mathcal{K}_\alpha(n)} = \|\rho\|_{M(n)}$.

For $g \in \mathfrak{M}_\alpha(n)$, put

$$\|g\|_{\mathfrak{M}_\alpha(n)} = \sup\left\{\|fg\|_{\mathcal{K}_\alpha(n)} : \|f\|_{\mathcal{K}_\alpha(n)} \leq 1\right\}.$$

The closed graph theorem guarantees that $\|g\|_{\mathfrak{M}_\alpha(n)} < \infty$. The space $\mathfrak{M}_\alpha(n)$ with the norm $\|\cdot\|_{\mathfrak{M}_\alpha(n)}$ is a Banach space. Also, note that $\mathfrak{M}_\alpha(n)$ is a Banach algebra.

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1.4. Historical remarks. The families $\mathcal{K}_\alpha(n)$ generalize the classical family $\mathcal{K}_n(n)$ of Cauchy integrals. Investigations of the family $\mathcal{K}_1(1)$ as a Banach space were initiated by Havin in [14, 15]. Further results and references are presented in the monograph [9]. The spaces $\mathcal{K}_\alpha(1)$, $\alpha > 0$, were introduced by MacGregor [20]. Various properties of the families $\mathcal{K}_\alpha(1)$ are collected in [17]. In the present paper we focus on the study of the families $\mathcal{K}_\alpha(n)$ with $n \geq 2$. Certain properties of the spaces $\mathcal{K}_n(n)$, $n \in \mathbb{N}$, were proved in the survey [3]. To the best knowledge of the author, for $n \geq 2$ the families $\mathcal{K}_\alpha(n)$ have not been investigated systematically.

1.5. Related problems. Let σ_n denote the normalized Lebesgue measure on the sphere ∂B_n. For $1 \leq p \leq \infty$ and $0 < \alpha < n$, put

$$\mathcal{K}^p_\alpha(n) = \{K_\alpha[f\sigma_n] \,:\, f \in L^p(\sigma_n)\}\,.$$

The boundary behavior of functions belonging to $\mathcal{K}^p_\alpha(1)$ was investigated in [21], where the families $\mathcal{K}^p_\alpha(1)$ were called Dirichlet type spaces. The families $\mathcal{K}^p_\alpha(n)$, $n \in \mathbb{N}$, were investigated in [26]; see also [1, 19, 8].

1.6. Organization of the paper. Embedding properties for $\mathcal{K}_\alpha(n)$ and $\mathfrak{M}_\alpha(n)$ are studied in §2. Relations between $\mathcal{K}_\alpha(n)$ and the classical spaces of holomorphic functions are obtained in §§3 and 5. Also, in §§4 and 5 we study differentiation operators on the spaces $\mathcal{K}_\alpha(n)$. The boundary behavior of the fractional Cauchy transforms is investigated in §6. Finally, some results about the multiplier spaces $\mathfrak{M}_\alpha(n)$ are obtained in §7.

§2. Embedding properties

2.1. Embedding properties for families of fractional Cauchy transforms. Below we shall use the following lemma.

Lemma 2.1 (see [7, Lemma 1])**.** *Assume that $u, v \in B_1$, $\alpha > 0$, $\beta > 0$. Then*

$$(1-u)^{-\alpha}(1-v)^{-\beta} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}[1-(tu+(1-t)v)]^{-\alpha-\beta}\,dt.$$

Let $\mathcal{P}(n)$ denote the set of all probability measures on the sphere ∂B_n, $n \in \mathbb{N}$. For $n = 1$, the following result was obtained in [7].

Proposition 2.2. *Put $\mathcal{F}_\alpha(n) = \{K_\alpha[\mu] \,:\, \mu \in \mathcal{P}(n)\}$, $n \in \mathbb{N}$.*

(i) *If $\alpha > 0$ and $\beta > 0$, then $\mathcal{F}_\alpha(n) \cdot \mathcal{F}_\beta(n) \subset \mathcal{F}_{\alpha+\beta}(n)$.*
(ii) *If $0 < \alpha < \beta$, then $\mathcal{F}_\alpha(n) \subset \mathcal{F}_\beta(n)$.*

Proof. First, we show how to deduce statement (i) from the following property:

$$(1-\langle z,\zeta\rangle)^{-\alpha}(1-\langle z,\xi\rangle)^{-\beta} \in \mathcal{F}_{\alpha+\beta}(n), \quad z \in B_n, \tag{2.1}$$

for any fixed points $\zeta, \xi \in \partial B_n$. So, let $f \in \mathcal{F}_\alpha(n)$, and let $g \in \mathcal{F}_\beta(n)$. Then, by the definition,

$$f(z)g(z) = \int_{\partial B_n \times \partial B_n} (1-\langle z,\zeta\rangle)^{-\alpha}(1-\langle z,\xi\rangle)^{-\beta}\,d\rho(\zeta,\xi), \quad z \in B_n,$$

where ρ is a probability measure on the product $\partial B_n \times \partial B_n$. We approximate the measure ρ in the weak* topology by probability measures ρ_j, where every ρ_j is a finite sum of atomic charges. The set $\mathcal{F}_{\alpha+\beta}(n)$ is convex; therefore, on the one hand, we have

$$\int_{\partial B_n \times \partial B_n} (1-\langle z,\zeta\rangle)^{-\alpha}(1-\langle z,\xi\rangle)^{-\beta}\,d\rho_j(\zeta,\xi) \in \mathcal{F}_{\alpha+\beta}(n).$$

On the other hand, without loss of generality we may assume that the convergence

$$\int_{\partial B_n \times \partial B_n} (1-\langle z,\zeta\rangle)^{-\alpha}(1-\langle z,\xi\rangle)^{-\beta}\, d\rho_j(\zeta,\xi) \to f(z)g(z)$$

as $j \to \infty$ is uniform on the compact subsets of the ball B_n. It remains to observe that the set $\mathcal{F}_{\alpha+\beta}(n)$ is closed in the topology of uniform convergence on compact subsets of the ball.

Now, we prove property (2.1). Lemma 2.1 guarantees that

$$(1-\langle z,\zeta\rangle)^{-\alpha}(1-\langle z,\xi\rangle)^{-\beta} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}[1-\langle z,w(t)\rangle]^{-\alpha-\beta}\, dt, \tag{2.2}$$

where $z \in B_n$ and $w(t) = t\zeta + (1-t)\xi$. Since

$$\Gamma(\alpha+\beta)\Gamma(\alpha)^{-1}\Gamma(\beta)^{-1}t^{\alpha-1}(1-t)^{\beta-1}\, dt$$

is a probability measure on the interval $[0,1]$, it suffices to show that

$$[1-\langle z,w(t)\rangle]^{-\alpha-\beta} \in \mathcal{F}_{\alpha+\beta}(n), \quad z \in B_n, \tag{2.3}$$

for every fixed parameter $t \in [0,1]$. If $t = 0$ or $t = 1$, then property (2.3) holds true by definition. So, assume that $0 < t < 1$. Then $w(t) \in B_n$. Next, note that $[1-\langle z,\cdot\rangle]^{-\alpha-\beta}$ is a harmonic function in the closed ball $\bar{B}_n$. Hence,

$$\frac{1}{[1-\langle z,w\rangle]^{\alpha+\beta}} = \int_{\partial B_n} \frac{1-|w|^2}{|w-\eta|^{2n}} \frac{1}{[1-\langle z,\eta\rangle]^{\alpha+\beta}}\, d\sigma_n(\eta). \tag{2.4}$$

If a point $w \in B_n$ is fixed, then $(1-|w|^2)|w-\eta|^{-2n}\, d\sigma_n(\eta)$ is a probability measure. Thus, to prove (2.3), it suffices to put $w = w(t)$, $0 < t < 1$. The proof of property (i) is finished.

Let $\beta > \alpha > 0$. Recall that the ball algebra $A(B_n)$ is defined by

$$A(B_n) = \mathcal{H}ol(B_n) \cap C(\overline{B_n}).$$

If $z \in B_n$, then $(1-\langle z,\cdot\rangle)^{-\beta+\alpha} \in \bar{A}(B_n)$. Hence,

$$1 = \int_{\partial B_n} \frac{1}{(1-\langle z,\zeta\rangle)^{\beta-\alpha}}\, d\sigma_n(\zeta), \quad z \in B_n.$$

In other words, $1 \in \mathcal{F}_{\beta-\alpha}(n)$; thus, (i) implies (ii). □

Corollary 2.3. *Let $n \in \mathbb{N}$.*

(i) *If $\alpha > 0$ and $\beta > 0$, then $\mathcal{K}_\alpha(n) \cdot \mathcal{K}_\beta(n) \subset \mathcal{K}_{\alpha+\beta}(n)$.*
(ii) *If $0 < \alpha < \beta$, then $\mathcal{K}_\alpha(n) \subset \mathcal{K}_\beta(n)$.*

Proof. Let $f = K_\alpha[\mu]$ and let $g = K_\beta[\rho]$, where $\mu, \rho \in M(n)$ and $\alpha, \beta > 0$. Considering the Jordan decompositions of the measures μ, ρ and applying part (i) of Proposition 2.2, we obtain properties (i) and (ii). □

2.2. Embedding properties for the multiplier families. The proof of the following lemma will be omitted, because it practically coincides with that given in [29] for the case where $n = \alpha = 1$ (see also [16], where the case of $n = 1$, $\alpha > 0$ was considered).

Lemma 2.4. *Assume that $n \in \mathbb{N}$ and $\alpha > 0$. Then the following properties are equivalent*:

(i) $g \in \mathfrak{M}_\alpha(n)$;
(ii) $g(z)(1-\langle z,\zeta\rangle)^{-\alpha} \in \mathcal{K}_\alpha(n)$ *for all* $\zeta \in \partial B_n$ *and*

$$\sup\left\{ \left\| \frac{g(z)}{(1-\langle z,\zeta\rangle)^\alpha} \right\|_{\mathcal{K}_\alpha(n)} : \zeta \in \partial B_n \right\} < \infty. \tag{2.5}$$

For $n=1$, the following proposition was proved in [16].

Proposition 2.5. *If $n\in\mathbb{N}$ and $0<\alpha<\beta$, then $\mathfrak{M}_\alpha(n)\subset\mathfrak{M}_\beta(n)$.*

Proof. Let $g\in\mathfrak{M}_\alpha(n)$. By Lemma 2.4, there exists a constant $C>0$ and there exist measures $\mu_\zeta\in M(n)$, $\zeta\in\partial B_n$, such that $\|\mu_\zeta\|\le C$ and

$$\frac{g(z)}{(1-\langle z,\zeta\rangle)^\alpha}=\int_{\partial B_n}\frac{1}{(1-\langle z,\xi\rangle)^\alpha}\,d\mu_\zeta(\xi),\quad z\in B_n.$$

Hence,

$$\frac{g(z)}{(1-\langle z,\zeta\rangle)^\beta}=\int_{\partial B_n}\frac{1}{(1-\langle z,\xi\rangle)^\alpha}\frac{1}{(1-\langle z,\zeta\rangle)^{\beta-\alpha}}\,d\mu_\zeta(\xi).$$

By (2.2) and (2.4), for every pair of points $\zeta,\xi\in\partial B_n$, there exists a probability measure $\rho_{\zeta,\xi}\in M(n)$ such that

$$\frac{1}{(1-\langle z,\xi\rangle)^\alpha}\frac{1}{(1-\langle z,\zeta\rangle)^{\beta-\alpha}}=\int_{\partial B_n}\frac{1}{(1-\langle z,\eta\rangle)^\beta}\,d\rho_{\zeta,\xi}(\eta).$$

Therefore,

$$\frac{g(z)}{(1-\langle z,\zeta\rangle)^\beta}=\int_{\partial B_n}\int_{\partial B_n}\frac{1}{(1-\langle z,\eta\rangle)^\beta}\,d\rho_{\zeta,\xi}(\eta)\,d\mu_\zeta(\xi).$$

Fix a point $\zeta\in\partial B_n$. We approximate the measure $\mu=\mu_\zeta$ in the weak* topology by measures μ_k such that $\|\mu_k\|\le C$ and $\mu_k=\sum_{j=1}^{J(k)}a_{j,k}\delta_{\xi_{j,k}}$, $a_{j,k}\in\mathbb{C}$. Let $\lambda=\lambda_\zeta$ denote an accumulation point of the sequence $\lambda_k=\sum_{j=1}^{J(k)}a_{j,k}\rho_{\zeta,\xi_{j,k}}$ in the weak* topology. Then

$$\frac{g(z)}{(1-\langle z,\zeta\rangle)^\beta}=\int_{\partial B_n}\frac{1}{(1-\langle z,\eta\rangle)^\beta}\,d\lambda_\zeta(\eta),$$

where $\|\lambda_\zeta\|\le C$. Thus, Lemma 2.4 guarantees that $g\in\mathfrak{M}_\beta(n)$. □

§3. Fractional Cauchy transforms and Hardy spaces

For $f\in\mathcal{H}ol(B_n)$, $0<p<+\infty$ and $0<r<1$, put

$$M_p(f,r)=\left(\int_{\partial B_n}|f(r\zeta)|^p\,d\sigma_n(\zeta)\right)^{\frac{1}{p}}.$$

Therefore, the classical Hardy space $H^p(B_n)$ is defined by

$$H^p(B_n)=\{f\in\mathcal{H}ol(B_n)\,:\,M_p(f,r)\le C\}.$$

Recall that if f and p are fixed, then the integral mean $M_p(f,r)$ is a monotone increasing function of the variable $r\in(0,1)$.

For $n\in\mathbb{N}$, let $\mathcal{K}_0(n)$ denote the family of all functions f such that

$$f(z)-f(0)=\int_{\partial B_n}\log\frac{1}{1-\langle z,\zeta\rangle}\,d\mu(\zeta),\quad z\in B_n, \tag{3.1}$$

for a measure $\mu\in M(n)$.

Let $f\in\mathcal{K}_\alpha(n)$, $\alpha\ge 0$. In the present section we study the growth of the integral means $M_p(f,r)$ as $r\to 1-$. For $n=1$, the corresponding results were obtained in [11].

The following lemma will be used below in numerous situations. The notation $a(z)\approx b(z)$ means that the quotient $a(z)/b(z)$ has a finite positive limit as $|z|\to 1-$.

Lemma 3.1 (see [23, Proposition 1.4.10])**.** *Suppose $n\in\mathbb{N}$ and $c\in\mathbb{R}$. Put*

$$I_c(z)=\int_{\partial B_n}\frac{d\sigma_n(\zeta)}{|1-\langle z,\zeta\rangle|^{n+c}},\quad z\in\bar{B}_n.$$

(i) *If* $c<0$, *then the function* $I_c(z)$ *is bounded in the closed ball* $\bar{B}_n$.
(ii) *If* $c>0$, *then* $I_c(z)\approx(1-|z|^2)^{-c}$.
(iii) *Finally,*

$$I_0(z)\approx\log\frac{1}{(1-|z|^2)}.$$

Proposition 3.2. *For* $\alpha\geq 0$, *let* $f\in\mathcal{K}_\alpha(n)$. *The following estimates hold true as* $r\to 1-$.

If $\alpha=0$ *and* $0<p<+\infty$, *then*

$$M_p^p(f,r)=\mathcal{O}(1). \tag{3.2}$$

If $0<\alpha\leq n$ *and* $0<p<n/\alpha$, *then*

$$M_p^p(f,r)=\mathcal{O}(1). \tag{3.3}$$

If $0<\alpha\leq n$ *and* $p=n/\alpha$, *then*

$$M_p^p(f,r)=\mathcal{O}\Big(\log\frac{1}{1-r}\Big). \tag{3.4}$$

If $p>n/\alpha$ *and* $p\geq 1$, *then*

$$M_p^p(f,r)=\mathcal{O}\left[\frac{1}{(1-r)^{\alpha p-n}}\right]. \tag{3.5}$$

If $0<p<1$ *and* $\alpha\geq 2n$, *then*

$$M_p^p(f,r)=\mathcal{O}\Big[\frac{1}{(1-r)^{(\alpha-n)p}}\Big]. \tag{3.6}$$

Proof. It is well known that $\mathcal{K}_n(n)\subset H^p(B_n)$ for all $0<p<1$ [23, Theorem 6.2.3]. In other words, estimate (3.3) is fulfilled for $\alpha=n$. Next, let $f=K_\alpha[\mu]$.

Assume that $p\geq 1$. Then the integral Minkowski inequality, Fubini's theorem, and Lemma 3.1 guarantee that

$$\begin{aligned}M_p^p(f,r)=\int_{\partial B_n}|f(r\xi)|^p\,d\sigma_n(\xi)&\leq\int_{\partial B_n}\int_{\partial B_n}\frac{1}{|1-\langle r\xi,\zeta\rangle|^{\alpha p}}\,d\sigma_n(\xi)\,d|\mu|(\zeta)\\&=\begin{cases}\mathcal{O}(1) & \text{if } \alpha p<n,\\ \mathcal{O}\big(\log\frac{1}{1-r}\big) & \text{if } \alpha p=n,\\ \mathcal{O}\big[\frac{1}{(1-r)^{\alpha p-n}}\big] & \text{if } \alpha p>n.\end{cases}\end{aligned}$$

This proves estimates (3.4) and (3.5). Property (3.3) is proved for $p\geq 1$. Recall that $H^q(B_n)\subset H^t(B_n)$ for $q>t>0$. Hence, estimate (3.3) is valid for all $0<p<n/\alpha$.

If $\gamma>0$, then

$$-\log 2\leq\log\frac{1}{|1-\langle r\xi,\zeta\rangle|}\leq C(\gamma)\frac{1}{|1-\langle r\xi,\zeta\rangle|^\gamma},$$

where $0<r<1$ and $\xi,\zeta\in\partial B_n$. Hence, estimate (3.2) is proved for all $p\in(0,+\infty)$.

Now, assume that $0<p<1$ and $\alpha\geq 2n$. Note that $(a+b)^p\leq C(p)(a^p+b^p)$ for $a\geq 0$, $b\geq 0$. Hence, by the Jordan decomposition theorem, we may assume that μ is a probability measure. Therefore, we obtain

$$\begin{aligned}|f(z)|&\leq\int_{\partial B_n}\frac{1}{|1-\langle z,\zeta\rangle|^\alpha}\,d\mu(\zeta)\\&\leq\frac{1}{(1-|z|)^{\alpha-2n}}\int_{\partial B_n}\frac{1}{|1-\langle z,\zeta\rangle|^{2n}}\,d\mu(\zeta)\\&=\frac{1}{(1-|z|)^{\alpha-n}}\int_{\partial B_n}\frac{(1-|z|^2)^n}{|1-\langle z,\zeta\rangle|^{2n}}\,d\mu(\zeta).\end{aligned} \tag{3.7}$$

Note that

$$u(z) = \int_{\partial B_n} \frac{(1-|z|^2)^n}{|1-\langle z,\zeta\rangle|^{2n}}\, d\mu(\zeta)$$

is a positive $\mathcal{M}$-harmonic function in the ball. Thus,

$$\int_{\partial B_n} u(r\zeta)\, d\sigma_n(\zeta) = u(0)$$

for $0 < r < 1$; see [23, Theorem 4.2.4]. Next, since $1/p > 1$, Minkowski's inequality yields

$$\left(\int_{\partial B_n} u^p(r\zeta)\, d\sigma_n(\zeta)\right)^{\frac{1}{p}} \le \int_{\partial B_n} u^{p\cdot\frac{1}{p}}(r\zeta)\, d\sigma_n(\zeta) = u(0) \tag{3.8}$$

for $0 < r < 1$.

Estimates (3.7) and (3.8) imply that

$$M_p^p(f,r) = \int_{\partial B_n} |f(r\zeta)|^p\, d\sigma_n(\zeta) \le \frac{1}{(1-r)^{(\alpha-n)p}} u^p(0) = \frac{(\mu(\partial B_n))^p}{(1-r)^{(\alpha-n)p}}.$$

This proves (3.6). □

Corollary 3.3. *Let $n \in \mathbb{N}$.*

(i) *If $0 \le \alpha \le n$, then $\mathcal{K}_\alpha(n) \subset H^p(B_n)$ for $0 < p < n/\alpha$.*
(ii) *If $f \in H^1(B_n)$, then $f \in \mathcal{K}_n(n)$; moreover, $\|f\|_{\mathcal{K}_n(n)} \le \|f\|_{H^1(B_n)}$.*
(iii) *$\mathcal{K}_0(n) \subset \mathcal{K}_n(n)$.*

Proof. Part (i) is equivalent to estimates (3.2) and (3.3). Next, if $f \in H^1(B_n)$, then $f = K_n[f^*]$, where $f^*(\zeta) = \lim_{r\to 1-} f(r\zeta)$, $\zeta \in \partial B_n$. Therefore, $\|f\|_{\mathcal{K}_n(n)} \le \|f\|_{H^1(B_n)}$. Finally, part (iii) follows from (i) and (ii). □

Part (iii) of Corollary 3.3 will be improved in Corollary 5.5. Also, estimate (3.3) can be refined in terms of the weak Hardy spaces $H^{p,\infty}(B_n)$. Let $p > 0$. By definition, $f \in H^{p,\infty}(B_n)$ if $f \in H^q(B_n)$ for some $q > 0$ and

$$\|f\|_{p,\infty}^p = \sup_{t>0} t^p \sigma_n\{\zeta \in \partial B_n \,:\, |f^*(\zeta)| > t\} < \infty.$$

Proposition 3.4. *If $0 < \alpha \le n$, then $\mathcal{K}_\alpha(n) \subset H^{\frac{n}{\alpha},\infty}(B_n)$.*

Proof. The argument below is due to A. B. Aleksandrov. The case of $\alpha = n$ is well known. So, assume that $p = n/\alpha > 1$; there exists a norm $\|\cdot\|_p^*$ on the space $H^{p,\infty}(B_n)$ such that

$$\|f\|_{p,\infty} \le C_1(p)\|f\|_p^* \le C_2(p)\|f\|_{p,\infty} \tag{3.9}$$

for all $f \in H^{p,\infty}(B_n)$, for some positive constants $C_1(p)$ and $C_2(p)$ (see, e.g., [6]).

Now, assume that $f \in K_\alpha[\mu]$, $\mu \in M(n)$. We must prove that $f \in H^{p,\infty}(B_n)$. Suppose, without loss of generality, that μ is a probability measure. We approximate the measure μ in the weak* topology by probability measures $\mu_k = \sum_{j=1}^{J(k)} a_{k,j}\delta_{\xi_{k,j}}$, where $a_{k,j} > 0$ and $\delta_{\xi_{k,j}}$ denotes the δ-measure at the point $\xi_{k,j} \in \partial B_n$. Applying estimates (3.9), the triangle inequality, and Proposition 5.1.4 in [23], we obtain

$$\begin{aligned}\|K_\alpha[\mu_k]\|_{p,\infty} &\le C_1(p)\|K_\alpha[\mu_k]\|_p^* \le C_1(p)\|(1-z_1)^{-\alpha}\|_p^* \\ &\le C_2(p)\|(1-z_1)^{-\alpha}\|_{p,\infty} \le C(p,n).\end{aligned}$$

Finally, since $\mu_k \to \mu$ in the weak* topology, we see that $f \in H^{p,\infty}(B_n)$. The proof is finished. □

§4. Families $\mathcal{K}_\alpha(n)$ and differentiation

Let $f \in \mathcal{H}ol(B_1)$, and let $\alpha \geq 0$. It is well known that $f \in \mathcal{K}_\alpha(1)$ if and only if $f' \in \mathcal{K}_{\alpha+1}(1)$. In the present section, we prove similar assertions for $f \in \mathcal{H}ol(B_n)$, $n \in \mathbb{N}$.

4.1. Radial derivatives. Let $f \in \mathcal{H}ol(B_n)$. The radial derivative $\mathcal{R}f$ is defined by the identity

$$\mathcal{R}f(z) = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z).$$

Recall that

$$f(z) - f(0) = \int_0^1 \frac{\mathcal{R}f(tz)}{t}\, dt, \quad z \in B_n. \tag{4.1}$$

Proposition 4.1. *Let $f \in \mathcal{H}ol(B_n)$, $n \in \mathbb{N}$. Then $f \in \mathcal{K}_0(n)$ if and only if $\mathcal{R}f \in \mathcal{K}_1(n)$.*

Proof. Suppose that $f \in \mathcal{K}_0(n)$, that is, (3.1) is true. Then

$$\mathcal{R}f(z) = \int_{\partial B_n} \frac{\langle z, \zeta\rangle}{1 - \langle z, \zeta\rangle}\, d\mu(\zeta) = K_1[\rho](z),$$

where $\rho = \mu - \mu(\partial B_n)\sigma_n$.

To prove the reverse implication, assume that

$$\mathcal{R}f(z) = \int_{\partial B_n} \frac{1}{1 - \langle z, \zeta\rangle}\, d\rho(\zeta),$$

where $\rho \in M(n)$. Note that $\mathcal{R}f(0) = 0$; thus, $\rho(\partial B_n) = 0$. Applying identity (4.1) and the property $\rho(\partial B_n) = 0$, we obtain

$$\begin{aligned} f(z) - f(0) &= \int_0^1 \int_{\partial B_n} \frac{1}{t(1 - t\langle z, \zeta\rangle)}\, d\rho(\zeta)\, dt \\ &= \int_0^1 \int_{\partial B_n} \left(\frac{1}{t} + \frac{\langle z, \zeta\rangle}{1 - t\langle z, \zeta\rangle}\right) d\rho(\zeta)\, dt \\ &= \int_{\partial B_n} \int_0^1 \frac{\langle z, \zeta\rangle}{1 - t\langle z, \zeta\rangle}\, dt\, d\rho(\zeta) \\ &= \int_{\partial B_n} \log \frac{1}{1 - \langle z, \zeta\rangle}\, d\rho(\zeta), \end{aligned}$$

as required. □

Proposition 4.2. *Assume that $\alpha > 0$ and $f \in \mathcal{H}ol(B_n)$, $n \in \mathbb{N}$. Then $f \in \mathcal{K}_0(n) + \mathcal{K}_\alpha(n)$ if and only if $\mathcal{R}f \in \mathcal{K}_{\alpha+1}(n)$.*

Proof. Let $g \in \mathcal{K}_0(n)$ and $h \in \mathcal{K}_\alpha(n)$, that is,

$$h(z) = \int_{\partial B_n} \frac{1}{(1 - \langle z, \zeta\rangle)^\alpha}\, d\mu(\zeta), \quad z \in B_n.$$

Straightforward calculations show that

$$\mathcal{R}h(z) = \int_{\partial B_n} \frac{\alpha}{(1 - \langle z, \zeta\rangle)^{\alpha+1}}\, d\mu(\zeta) - \alpha h(z) \in \mathcal{K}_{\alpha+1}(n)$$

because $\alpha h \in \mathcal{K}_\alpha(n) \subset \mathcal{K}_{\alpha+1}(n)$ by part (ii) of Corollary 2.3. On the other hand, Proposition 4.1 guarantees that $\mathcal{R}g \in \mathcal{K}_1(n) \subset \mathcal{K}_{\alpha+1}(n)$.

It is convenient to split the proof of the reverse implication into two steps.

Step 1. Let $\alpha = m \in \mathbb{N}$. By the hypothesis, we have

$$\mathcal{R}f(z) = \int_{\partial B_n} \frac{1}{(1-\langle z,\zeta\rangle)^{m+1}}\, d\mu(\zeta), \quad z \in B_n.$$

Hence,

$$f(z) - f(0) = \int_0^1 \frac{\mathcal{R}f(tz)}{t}\, dt = \int_0^1 \int_{\partial B_n} \frac{1}{t(1-t\langle z,\zeta\rangle)^{m+1}}\, d\mu(\zeta)\, dt, \quad z \in B_n.$$

If $w \in \mathbb{C}$ and $|w| < 1$, then

$$\frac{1}{t(1-tw)^{m+1}} = \frac{1}{t} + \sum_{j=1}^{m+1} \frac{w}{(1-tw)^j}.$$

Since $\mathcal{R}f(0) = 0$, we have $\mu(\partial B_n) = 0$. Putting $w = \langle z,\zeta\rangle$, we obtain

$$\begin{aligned} f(z) - f(0) &= \sum_{j=1}^{m+1} \int_{\partial B_n} \int_0^1 \frac{\langle z,\zeta\rangle}{(1-t\langle z,\zeta\rangle)^j}\, dt\, d\mu(\zeta) \\ &= -\int_{\partial B_n} \log(1-\langle z,\zeta\rangle)\, d\mu(\zeta) + \sum_{j=2}^{m+1} \int_{\partial B_n} \frac{1}{(j-1)(1-\langle z,\zeta\rangle)^{j-1}}\, d\mu(\zeta). \end{aligned}$$

The embeddings $\mathcal{K}_{j-1}(n) \subset \mathcal{K}_m(n)$, $j = 2,\dots,m$, guarantee that $f \in \mathcal{K}_0(n) + \mathcal{K}_m(n)$.

Step 2. Let $\alpha > 0$, $\alpha \notin \mathbb{N}$. Repeating the arguments of Step 1 and reversing the order of integration, we have

$$\begin{aligned} f(z) - f(0) &= \int_0^1 \int_{\partial B_n} \frac{1}{t(1-t\langle z,\zeta\rangle)^{\alpha+1}}\, d\mu(\zeta)\, dt \\ &= \int_{\partial B_n} \int_0^1 \frac{\langle z,\zeta\rangle}{(1-t\langle z,\zeta\rangle)^{\alpha+1}}\, dt\, d\mu(\zeta) + \int_0^1 \frac{1}{t} \int_{\partial B_n} \frac{1}{(1-t\langle z,\zeta\rangle)^{\alpha}}\, d\mu(\zeta)\, dt. \end{aligned}$$

The inner integral in the first summand can be calculated explicitly. So, consider the second summand. Put $[\alpha] = m \in \mathbb{N} \cup \{0\}$. Note that $m+1 > \alpha$ and $\mu(\partial B_n) = 0$. Hence, by part (ii) of Corollary 2.3, there exists a measure $\rho \in M(n)$ such that $K_{m+1}[\rho] = K_\alpha[\mu]$. Also, we have $\rho(\partial B_n) = K_{m+1}[\rho](0) = K_\alpha[\mu](0) = \mu(\partial B_n) = 0$. Therefore, reversing the order of integration once again, we obtain

$$f(z) - f(0) = \frac{1}{\alpha} \int_{\partial B_n} \frac{1}{(1-\langle z,\zeta\rangle)^{\alpha}}\, d\mu(\zeta) + \int_{\partial B_n} \int_0^1 \frac{1}{t(1-t\langle z,\zeta\rangle)^{m+1}}\, dt\, d\rho(\zeta).$$

By the definition, the first summand in the above sum belongs to $\mathcal{K}_\alpha(n)$. If $m = 0$, then the proof of Proposition 4.1 shows that the second summand is in the family $\mathcal{K}_0(n)$. Finally, for $m \in \mathbb{N}$, at Step 1 it was proved that the second summand belongs to $\mathcal{K}_0(n) + \mathcal{K}_m(n) \subset \mathcal{K}_0(n) + \mathcal{K}_\alpha(n)$. □

Note that Proposition 4.2 admits refinement (see Corollary 5.6).

4.2. Fractional differential operators of order 1. By definition, we put $R^1 f = f + \mathcal{R}f$ for $f \in \mathcal{H}ol(B_n)$. The operator R^1 is called the fractional differential operator of order 1. Observe that

$$f(z) = \int_0^1 R^1 f(tz)\, dt, \quad z \in B_n. \tag{4.2}$$

The following proposition can be proved with the help of elementary methods.

Proposition 4.3. *Let $f \in \mathcal{H}ol(B_n)$, $n \in \mathbb{N}$.*

(i) *If $\alpha > 0$, then the property $f \in \mathcal{K}_\alpha(n)$ implies that $R^1 f \in \mathcal{K}_{\alpha+1}(n)$.*
(ii) *If $\alpha \geq 1$, then the property $R^1 f \in \mathcal{K}_{\alpha+1}(n)$ implies that $f \in \mathcal{K}_\alpha(n)$.*

Proof. Let $f \in \mathcal{K}_\alpha(n)$, that is, $f = K_\alpha[\mu]$. Then

$$f + \mathcal{R}f = \alpha K_{\alpha+1}[\mu] + (1-\alpha)f \in \mathcal{K}_{\alpha+1}(n)$$

because $(1-\alpha)f \in \mathcal{K}_\alpha(n) \subset \mathcal{K}_{\alpha+1}(n)$. The proof of part (i) is complete.

We split the proof of part (ii) into two steps.

Step 1. $\alpha = m \in \mathbb{N}$. By assumption, we have

$$R^1 f(tz) = \int_{\partial B_n} \frac{1}{(1-t\langle z,\zeta\rangle)^{m+1}}\, d\mu(\zeta), \quad z \in B_n,\ t \in [0,1].$$

Hence, using (4.2), we obtain

$$f(z) = \int_{\partial B_n} \int_0^1 \frac{1}{(1-t\langle z,\zeta\rangle)^{m+1}}\, dt\, d\mu(\zeta), \quad z \in B_n.$$

Let $I(z,\zeta)$ denote the inner integral. If $\langle z,\zeta\rangle = 0$, then $I(z,\zeta) = 1$. If $\langle z,\zeta\rangle \neq 0$, then

$$I(z,\zeta) = \frac{1}{m}\frac{1-(1-\langle z,\zeta\rangle)^m}{\langle z,\zeta\rangle(1-\langle z,\zeta\rangle)^m} = \frac{1}{m}\sum_{j=1}^m (1-\langle z,\zeta\rangle)^{-j}.$$

Therefore, in both cases we have

$$I(z,\zeta) = \frac{1}{m}\sum_{j=1}^m (1-\langle z,\zeta\rangle)^{-j}.$$

Consequently,

$$f(z) = \frac{1}{m}\sum_{j=1}^m \int_{\partial B_n} \frac{1}{(1-\langle z,\zeta\rangle)^j}\, d\mu(\zeta).$$

Finally, we see that $f \in \mathcal{K}_m(n)$, because $\mathcal{K}_j(n) \subset \mathcal{K}_m(n)$ for $j = 1,\dots,m-1$.

Step 2. $\alpha > 1$, $\alpha \notin \mathbb{N}$. Repeating the arguments of Step 1, we get

$$f(z) = \int_{\partial B_n} \int_0^1 \frac{1}{(1-t\langle z,\zeta\rangle)^{\alpha+1}}\, dt\, d\mu(\zeta), \quad z \in B_n.$$

We represent the inner integral as a sum of two summands and integrate by parts in the second summand, obtaining

$$\begin{aligned}\int_0^1 \frac{1}{(1-t\langle z,\zeta\rangle)^{\alpha+1}}\, dt &= \int_0^1 \frac{1}{(1-t\langle z,\zeta\rangle)^{\alpha}}\, dt + \int_0^1 \frac{t\langle z,\zeta\rangle}{(1-t\langle z,\zeta\rangle)^{\alpha+1}}\, dt \\ &= \frac{1}{\alpha}\frac{1}{(1-\langle z,\zeta\rangle)^\alpha} + \frac{\alpha-1}{\alpha}\int_0^1 \frac{1}{(1-t\langle z,\zeta\rangle)^\alpha}\, dt.\end{aligned}$$

Hence, Fubini's theorem yields

$$f(z) = \frac{1}{\alpha}\int_{\partial B_n} \frac{1}{(1-\langle z,\zeta\rangle)^\alpha}\, d\mu(\zeta) + \frac{\alpha-1}{\alpha}\int_0^1 \int_{\partial B_n} \frac{1}{(1-\langle tz,\zeta\rangle)^\alpha}\, d\mu(\zeta)\, dt.$$

By definition, the first summand in the above sum belongs to $\mathcal{K}_\alpha(n)$.

Consider the second summand. Let $[\alpha] = m \in \mathbb{N}$. Note that $m+1 > \alpha$. Hence, by part (ii) of Corollary 2.3, there exists a measure $\rho \in M(n)$ such that $K_{m+1}[\rho] = K_\alpha[\mu]$. Therefore, the second summand has the following form:

$$\frac{\alpha-1}{\alpha}\int_0^1\int_{\partial B_n}\frac{1}{(1-\langle tz,\zeta\rangle)^{m+1}}\,d\rho(\zeta)\,dt.$$

At Step 1 it was proved that the above function belongs to $\mathcal{K}_m(n) \subset \mathcal{K}_\alpha(n)$. □

To refine Proposition 4.3, we need the following lemma.

Lemma 4.4. *Assume that $\alpha > 0$ and $g(w) = 1-(1-w)^\alpha$, $w \in B_1$. Then there exists a measure $\rho \in M(1)$ such that $(1-w)^{-\alpha}g(w)/w = K_\alpha[\rho](w)$, $w \in B_1$.*

Proof. The definitions imply that $(g(w)/w)'w^2 = g'(w)w - g(w) \in H^1(B_1)$, where $H^1(B_1)$ denotes the Hardy space. Note that $h \in H^1(B_1)$ if $h \in \mathcal{H}ol(B_n)$ and $h(w)w^2 \in H^1(B_1)$. Putting $h(w) = (g(w)/w)'$, we obtain $(g(w)/w)' \in H^1(B_1)$. Hence, $g(w)/w \in \mathfrak{M}_\alpha(1)$ by [16, Theorem 3.5]. Since $(1-w)^{-\alpha} \in \mathcal{K}_\alpha(1)$, the required measure $\rho \in M(1)$ exists by the definition of the multiplier space $\mathfrak{M}_\alpha(1)$. □

Theorem 4.5. *Assume that $\alpha > 0$ and $f \in \mathcal{H}ol(B_n)$, $n \in \mathbb{N}$. Then $f \in \mathcal{K}_\alpha(n)$ if and only if $R^1 f \in \mathcal{K}_{\alpha+1}(n)$.*

Proof. Suppose that $R^1 f = K_{\alpha+1}[\mu]$, where $\mu \in M(n)$. Applying formula (4.2) and reversing the order of integration, we obtain

$$\begin{aligned}\alpha f(z) &= \int_{\partial B_n}\int_0^1 \frac{\alpha\,dt}{(1-t\langle z,\zeta\rangle)^{\alpha+1}}\,d\mu(\zeta)\\ &= \int_{\partial B_n}\frac{1-(1-\langle z,\zeta\rangle)^\alpha}{\langle z,\zeta\rangle(1-\langle z,\zeta\rangle)^\alpha}\,d\mu(\zeta), \quad z \in B_n,\end{aligned}$$

where the integrand is assumed to be equal to α if $\langle z,\zeta\rangle = 0$. We put $w = \langle z,\zeta\rangle$ and apply Lemma 4.4. Reversing the order of integration, we have

$$\begin{aligned}\alpha f(z) &= \int_{\partial B_1}\int_{\partial B_n}(1-\langle z,\lambda\zeta\rangle)^{-\alpha}\,d\mu(\zeta)\,d\rho(\lambda)\\ &= \int_{\partial B_1} K_\alpha[\mu_\lambda](z)\,d\rho(\lambda), \quad z \in B_n,\end{aligned}$$

where $\mu_\lambda(E) = \mu(\bar\lambda E)$ for the Borel sets $E \subset \partial B_n$. Note that $K_\alpha[\mu_\lambda](z)$ is a continuous function of the variable $\lambda \in \partial B_1$. Below we show that the last integral is in the family $\mathcal{K}_\alpha(n)$. Without loss of generality, we may assume that ρ is a probability measure. We approximate the measure ρ in the weak* topology by the probability measures $\rho_k = \sum_{j=1}^{J(k)} a_{k,j}\delta_{\lambda_{k,j}}$, where $\delta_{\lambda_{k,j}}$ denotes the δ-measure at $\lambda_{k,j} \in \partial B_1$. By the definition of weak* convergence, we have

$$\sum_{j=1}^{J(k)} a_{k,j}K_\alpha[\mu_{\lambda_{k,j}}](z) \overset{k\to\infty}{\longrightarrow} \int_{\partial B_1} K_\alpha[\mu_\lambda](z)\,d\rho(\lambda), \quad z \in B_n.$$

Next, observe that the sequence $\left\{\sum_{j=1}^{J(k)} a_{k,j}\mu_{\lambda_{k,j}}\right\}$ is norm bounded. Hence, there exists a subsequence which converges in the weak* topology to a measure $\nu \in M(n)$. Therefore,

$$\sum_{j=1}^{J(k)} a_{k,j}K_\alpha[\mu_{\lambda_{k,j}}](z) \overset{k\to\infty}{\longrightarrow} K_\alpha[\nu](z), \quad z \in B_n.$$

Finally, we obtain $f = K_\alpha[\nu/\alpha]$. Now, it suffices to refer to Proposition 4.3. □

Note that Theorem 4.5 extends to the case of $\alpha = 0$ (see Corollary 5.7).

§5. Families $\mathcal{K}_\alpha(n)$ and Bergman–Sobolev spaces

5.1. Modified operators of fractional differentiation. Consider a pair of parameters $(\beta, t) \in \mathbb{R}^2$ with the property that neither $n-1-\beta$ nor $n-1-\beta+t$ is a strictly negative integer. We define an operator

$$R^{\beta,t} : \mathcal{H}ol(B_n) \to \mathcal{H}ol(B_n)$$

as follows. If

$$f(z) = \sum_{k=0}^{\infty} f_k(z) \tag{5.1}$$

is the homogeneous expansion of $f \in \mathcal{H}ol(B_n)$, then

$$R^{\beta,t} f(z) = \sum_{k=0}^{\infty} \frac{\Gamma(n-\beta)\Gamma(n+k-\beta+t)}{\Gamma(n-\beta+t)\Gamma(n+k-\beta)} f_k(z). \tag{5.2}$$

The inverse of $R^{\beta,t}$, denoted by $R_{\beta,t}$, is given by the formula

$$R_{\beta,t} f(z) = \sum_{k=0}^{\infty} \frac{\Gamma(n-\beta+t)\Gamma(n+k-\beta)}{\Gamma(n-\beta)\Gamma(n+k-\beta+t)} f_k(z).$$

Assume that $\beta < n$, $t > 0$, and identity (5.1) is true. Also, suppose that $r \in [0,1]$ and $z \in B_n$. By the definition of the operator $R^{\beta,t}$, we have

$$\frac{\Gamma(n-\beta+t)}{\Gamma(n-\beta)\Gamma(t)} R^{\beta,t} f(rz) = \sum_{k=0}^{\infty} \frac{r^k f_k(z)}{B(n+k-\beta, t)}.$$

Therefore,

$$\frac{\Gamma(n-\beta+t)}{\Gamma(n-\beta)\Gamma(t)} \int_0^1 R^{\beta,t} f(rz) r^{n-\beta-1}(1-r)^{t-1}\, dr = \sum_{k=0}^{\infty} f_k(z) = f(z). \tag{5.3}$$

Straightforward calculations imply the following lemma.

Lemma 5.1 (cf. [30, Proposition 1.14])**.** *Let $n \in \mathbb{N}$. Suppose that neither $n-1-\beta$ nor $n-1-\beta+t$ is a strictly negative integer. Then*

$$R^{\beta,t}\left(\frac{1}{(1-\langle z,\zeta\rangle)^{n-\beta}}\right) = \frac{1}{(1-\langle z,\zeta\rangle)^{n-\beta+t}}$$

for all points $z \in B_n$ and $\zeta \in \partial B_n$.

5.2. A condition sufficient for membership in $\mathcal{K}_\alpha(n)$ with $\alpha > n$. For $n = 1$, the following fact was proved in [13].

Proposition 5.2. *Suppose $n \in \mathbb{N}$ and $\alpha > n$. If $g \in \mathcal{H}ol(B_n)$ and*

$$\int_0^1 \int_{\partial B_n} |g(r\zeta)|(1-r)^{\alpha-n-1}\, d\sigma_n(\zeta)\, dr = V < +\infty,$$

then $g \in \mathcal{K}_\alpha(n)$ and $\|g\|_{\mathcal{K}_\alpha(n)} \le C(\alpha, n, V) = (\alpha-n)\cdots(\alpha-1)V/(n-1)!$.

Proof. Put $t = \alpha - n$ and $f = R_{0,t} g$. Then $g = R^{0,t} f$ and

$$f(z) = \frac{\Gamma(n+t)}{\Gamma(n)\Gamma(t)} \int_0^1 g(rz) r^{n-1}(1-r)^{t-1}\, dr$$

by (5.3). If $0 < s < 1$, then

$$\begin{aligned}\frac{\Gamma(n)\Gamma(t)}{\Gamma(n+t)}\int_{\partial B_n}|f(s\zeta)|\,d\sigma_n(\zeta) &\le \int_0^1\int_{\partial B_n}|g(sr\zeta)|\,d\sigma_n(\zeta)(1-r)^{t-1}\,dr\\ &\le \int_0^1\int_{\partial B_n}|g(r\zeta)|\,d\sigma_n(\zeta)(1-r)^{t-1}\,dr\\ &= V < +\infty.\end{aligned}$$

Therefore, $f \in H^1(B_n) \subset \mathcal{K}_n(n)$ and $\|f\|_{\mathcal{K}_n(n)} \le \|f\|_{H^1(B_n)} \le C(\alpha, n, V)$. In other words,

$$f(z) = \int_{\partial B_n}\frac{1}{(1-\langle z,\zeta\rangle)^n}\,d\mu(\zeta), \quad z \in B_n,$$

where $\|\mu\|_{M(n)} \le C(\alpha, n, V)$. Thus, with the help of Lemma 5.1, we obtain

$$g(z) = R^{0,t}f(z) = \int_{\partial B_n}\frac{1}{(1-\langle z,\zeta\rangle)^{n+t}}\,d\mu(\zeta), \quad z \in B_n.$$

Hence, $\|g\|_{\mathcal{K}_\alpha(n)} \le C(\alpha, n, V)$. □

5.3. Bergman–Sobolev spaces. Suppose $j \in \mathbb{N}$ and $q > 0$. By definition, the Bergman–Sobolev space $A^1_{q,j}(B_n)$ consists of all functions $f \in \mathcal{H}ol(B_n)$ for which

$$\int_0^1\int_{\partial B_n}|R^j f(r\zeta)|(1-r)^{q-1}\,d\sigma_n(\zeta)\,dr < +\infty,$$

where $R^j = (R^1)^j$. For $n = 1$, the following assertion coincides with [13, Lemma 2].

Proposition 5.3. *Suppose $n \in \mathbb{N}$, $j \in \{1, \dots, n\}$, and $\alpha > n - j$.*

(i) *If $f \in A^1_{\alpha-n+j,j}(B_n)$, then $f \in \mathcal{K}_\alpha(n)$.*
(ii) *Let $\beta > \alpha$. Then $\mathcal{K}_\alpha(n) \subset A^1_{\beta-n+j,j}(B_n)$.*

Proof. We prove part (i). Since $\alpha + j > n$, Proposition 5.2 shows that $R^j f \in \mathcal{K}_{\alpha+j}(n)$. Since $\alpha > 0$, repeated application of Theorem 4.5 guarantees that $f \in \mathcal{K}_\alpha(n)$.

Now, we turn to part (ii). Let $f \in \mathcal{K}_\alpha(n)$, $\alpha > n - j$. By Theorem 4.5, we have $R^j f \in \mathcal{K}_{\alpha+j}(n)$; that is,

$$R^j f(z) = \int_{\partial B_n}\frac{1}{(1-\langle z,\zeta\rangle)^{\alpha+j}}\,d\mu(\zeta), \quad z \in B_n,$$

for some measure $\mu \in M(n)$. Therefore, if $0 \le r < 1$, then

$$\int_{\partial B_n}|R^j f(r\xi)|\,d\sigma_n(\xi) \le \int_{\partial B_n}\int_{\partial B_n}\frac{1}{|1-r\langle\xi,\zeta\rangle|^{\alpha+j}}\,d\sigma_n(\xi)\,d|\mu|(\zeta).$$

Since $\alpha + j > n$, Lemma 3.1 yields

$$\int_{\partial B_n}|R^j f(r\xi)|\,d\sigma_n(\xi) \le C\|\mu\|_{M(n)}(1-r)^{n-\alpha-j}.$$

Now, assume that $\beta > \alpha$. Then the above inequality guarantees that

$$\int_0^1\int_{\partial B_n}|R^j f(r\xi)|(1-r)^{\beta-n+j-1}\,d\sigma_n(\xi)\,dr \le C\|\mu\|\int_0^1(1-r)^{\beta-\alpha-1} < +\infty.$$

In other words, $f \in A^1_{\beta-n+j,j}(B_n)$. □

5.4. Bergman–Sobolev spaces and $\mathcal{K}_0(n)$.

Lemma 5.4. *Let $n \in \mathbb{N}$. Then $\mathcal{K}_0(n) \subset A^1_{\varepsilon,n}(B_n)$ for any $\varepsilon > 0$.*

Proof. Let $f \in \mathcal{K}_0(n)$. Recall that $\mathcal{K}_0(n) \subset \mathcal{K}_n(n)$ by part (iii) of Corollary 3.3. Next, $\mathcal{R}^j f \in \mathcal{K}_j(n) \subset \mathcal{K}_n(n)$ for $j = 1, 2, \dots, n$ by Propositions 4.1, 4.2 and Corollary 2.3. Hence, $R^n f = (I + \mathcal{R})^n f \in \mathcal{K}_n(n)$; that is,

$$R^n f(z) = \int_{\partial B_n} \frac{1}{(1 - \langle z, \zeta\rangle)^n}\, d\mu(\zeta), \quad z \in B_n,$$

for some measure $\mu \in M(n)$. Hence, if $0 \le r < 1$, then

$$\int_{\partial B_n} |R^n f(r\xi)|\, d\sigma_n(\xi) \le \int_{\partial B_n}\int_{\partial B_n} \frac{1}{|1 - r\langle \xi, \zeta\rangle|^n}\, d\sigma_n(\xi)\, d|\mu|(\zeta).$$

By Lemma 3.1, we have

$$\int_{\partial B_n} |R^n f(r\xi)|\, d\sigma_n(\xi) \le C \log \frac{e}{1-r}.$$

The above estimate implies that

$$\int_0^1 \int_{\partial B_n} |R^n f(r\xi)|(1-r)^{\varepsilon-1}\, d\sigma_n(\xi)\, dr \le C \int_0^1 (1-r)^{\varepsilon-1} \log \frac{e}{1-r} < +\infty,$$

as required. □

5.5. An embedding property for $\mathcal{K}_0(n)$. It is well known that $\mathcal{K}_0(1) \subset \mathcal{K}_\alpha(1)$ for all $\alpha > 0$.

Corollary 5.5. *Let $n \in \mathbb{N}$. Then $\mathcal{K}_0(n) \subset \mathcal{K}_\alpha(n)$ for all $\alpha > 0$.*

Proof. Lemma 5.4 guarantees that $\mathcal{K}_0(n) \subset A^1_{\alpha,n}(B_n)$, $\alpha > 0$. It remains to note that $A^1_{\alpha,n}(B_n) \subset \mathcal{K}_\alpha(n)$ by part (i) of Proposition 5.3. □

A. B. Aleksandrov observed that Corollary 5.5 can also be deduced from the embedding $\mathcal{K}_0(1) \subset \mathcal{K}_\alpha(1)$ for all $\alpha > 0$.

5.6. The families $\mathcal{K}_\alpha(n)$, $\alpha \ge 0$, and differentiation. We have the following refinement of Proposition 4.2.

Corollary 5.6. *Suppose $n \in \mathbb{N}$ and $\alpha \ge 0$. Then $f \in \mathcal{K}_\alpha(n)$ if and only if $\mathcal{R}f \in \mathcal{K}_{\alpha+1}(n)$.*

Proof. We apply Propositions 4.1, 4.2 and Corollary 5.5. □

The next corollary refines Theorem 4.5.

Corollary 5.7. *Suppose $n \in \mathbb{N}$ and $\alpha \ge 0$. Then $f \in \mathcal{K}_\alpha(n)$ if and only if $R^1 f \in \mathcal{K}_{\alpha+1}(n)$.*

Proof. The case of $\alpha > 0$ is covered by Theorem 4.5. So, assume that $f \in \mathcal{K}_0(n)$. Corollary 5.5 guarantees that $f \in \mathcal{K}_1(n)$. On the other hand, $\mathcal{R}f \in \mathcal{K}_1(n)$ by Corollary 5.6. Therefore, $R^1 f = f + \mathcal{R}f \in \mathcal{K}_1(n)$.

To prove the reverse implication, assume that $R^1 f \in \mathcal{K}_1(n)$. Since $\mathcal{K}_1(n) \subset \mathcal{K}_2(n)$, we have $R^1 f \in \mathcal{K}_2(n)$. Hence, $f \in \mathcal{K}_1(n)$ by Theorem 4.5. Therefore, $\mathcal{R}f = R^1 f - f \in \mathcal{K}_1(n)$. So, $f \in \mathcal{K}_0(n)$ by Corollary 5.6. □

5.7. Modified Bergman–Sobolev spaces and $\mathcal{K}_\alpha(n)$. Suppose $n \in \mathbb{N}$, $j \in \mathbb{N}$, and $q > 0$. In some applications, it is convenient to use the modified Bergman–Sobolev space $\widetilde{A}^1_{q,j}(B_n)$ in place of $A^1_{q,j}(B_n)$. By definition, the space $\widetilde{A}^1_{q,j}(B_n)$ consists of functions $f \in \mathcal{H}ol(B_n)$ such that

$$\|f\|_{\widetilde{A}^1_{q,j}(B_n)} = \sum_{|m|\le j-1} \left|\frac{\partial^m f}{\partial z^m}(0)\right| + \int_0^1 \int_{\partial B_n} |\mathcal{R}^j f(r\zeta)|(1-r)^{q-1}\, d\sigma_n(\zeta)\, dr < +\infty,$$

where $m = (m_1, \dots, m_n) \in \mathbb{Z}^n_+$ and $|m| = m_1 + \dots + m_n$. The space $\widetilde{A}^1_{q,j}(B_n)$ with the norm $\|\cdot\|_{\widetilde{A}^1_{q,j}(B_n)}$ is a Banach space (cf. [30, Proposition 6.2]).

Proposition 5.8. *Suppose $n \in \mathbb{N}$, $j \in \{1, \dots, n\}$, and $\alpha > n - j$.*

(i) *The embedding $\widetilde{A}^1_{\alpha-n+j,j}(B_n) \subset \mathcal{K}_\alpha(n)$ holds true. If $f \in \widetilde{A}^1_{\alpha-n+j,j}(B_n)$, then $\|f\|_{\mathcal{K}_\alpha(n)} \le C\|f\|_{\widetilde{A}^1_{\alpha-n+j,j}(B_n)}$, where the constant $C > 0$ does not depend on f.*

(ii) *Let $\beta > \alpha$. Then $\mathcal{K}_\alpha(n) \subset \widetilde{A}^1_{\beta-n+j,j}(B_n)$.*

Proof. Assume that $f \in \widetilde{A}^1_{\alpha-n+j,j}(B_n)$. We have

$$\int_0^1 \int_{\partial B_n} |\mathcal{R}^j f(r\zeta)|(1-r)^{\alpha+j-n-1}\, d\sigma_n(\zeta)\, dr < +\infty.$$

Since $\alpha + j > n$, Proposition 5.2 yields $\mathcal{R}^j f \in \mathcal{K}_{\alpha+j}(n)$. Repeated application of Corollary 5.6 shows that $f \in \mathcal{K}_\alpha(n)$.

Now, consider the operator $I : \widetilde{A}^1_{\alpha-n+j,j}(B_n) \to \mathcal{K}_\alpha(n)$ defined by $If = f$. If a functional sequence converges in the space $\widetilde{A}^1_{\alpha-n+j,j}(B_n)$ or in the space $\mathcal{K}_\alpha(n)$, then this sequence converges uniformly on the compact subsets of the ball B_n. Hence, the graph of the operator I is closed. So, by the closed graph theorem,

$$\|f\|_{\mathcal{K}_\alpha(n)} \le C\|f\|_{\widetilde{A}^1_{\alpha-n+j,j}(B_n)}$$

for all $f \in \widetilde{A}^1_{\alpha-n+j,j}(B_n)$. Part (i) is proved.

Part (ii). Let $f \in \mathcal{K}_\alpha(n)$, $\alpha > n - j$. We have $\mathcal{R}^j f \in \mathcal{K}_{\alpha+j}(n)$ by Proposition 4.2. To finish the argument, it suffices to repeat the proof of part (ii) of Proposition 5.3. □

5.8. Fractional Cauchy transforms and inner functions. Assume that $n \in \mathbb{N}$, $j \in \mathbb{N}$, and $q > 0$. The space $\widetilde{A}^2_{q,j}(B_n)$ consists of all functions $f \in \mathcal{H}ol(B_n)$ such that

$$\int_0^1 \int_{\partial B_n} |\mathcal{R}^j f(r\zeta)|^2(1-r)^{q-1}\, d\sigma_n(\zeta)\, dr < +\infty.$$

Lemma 5.9. *Suppose $n \in \mathbb{N}$, $j \in \{1, \dots, n\}$, and $\beta > \alpha > n - j$. Then*

$$H^\infty(B_n) \cap \mathcal{K}_\alpha(n) \subset \widetilde{A}^2_{\beta-n+2j,j}(B_n).$$

Proof. Let $f \in H^\infty(B_n) \cap \mathcal{K}_\alpha(n)$. Since $f \in H^\infty(B_n)$, we have

$$f(z) = \int_{\partial B_n} \frac{f^*(\zeta)}{(1 - \langle z, \zeta\rangle)^n}\, d\sigma_n(\zeta), \quad z \in B_n.$$

Therefore,

$$\mathcal{R}f(z) = \int_{\partial B_n} \frac{n\langle z, \zeta\rangle f^*(\zeta)}{(1 - \langle z, \zeta\rangle)^{n+1}}\, d\sigma_n(\zeta), \quad z \in B_n.$$

Hence, using Lemma 3.1, we obtain

$$|\mathcal{R}f(z)| \le \int_{\partial B_n} \frac{C}{|1-\langle z,\zeta\rangle|^{n+1}}\, d\sigma_n(\zeta) \le \frac{C}{(1-|z|)}, \quad z \in B_n.$$

Arguing similarly, we verify that

$$|\mathcal{R}^j f(z)| \le \frac{C}{(1-|z|)^j}, \quad z \in B_n. \tag{5.4}$$

Since $f \in \mathcal{K}_\alpha(n)$, Proposition 4.2 guarantees that $\mathcal{R}^j f \in \mathcal{K}_{\alpha+j}(n)$. Since $\alpha + j > n$ and $\beta > \alpha$, part (ii) of Proposition 5.8 yields

$$\int_0^1 \int_{\partial B_n} |\mathcal{R}^j f(r\zeta)|(1-r)^{\beta-n+j-1}\, d\sigma_n(\zeta)\, dr < \infty.$$

Hence, applying inequality (5.4), we get

$$\int_0^1 \int_{\partial B_n} |\mathcal{R}^j f(r\zeta)|(1-r)^{j} \cdot |\mathcal{R}^j f(r\zeta)|(1-r)^{\beta-n+j-1}\, d\sigma_n(\zeta)\, dr < +\infty,$$

as required. □

Recall that a nonconstant function $f \in H^\infty(B_n)$ is said to be *inner* if $|f^*| = 1$ σ_n-a.e. On the one hand, the families $\mathcal{K}_\alpha(n)$ with $\alpha \ge n$ contain all inner functions because $H^\infty(B_n) \subset H^1(B_n) \subset \mathcal{K}_n(n) \subset \mathcal{K}_\alpha(n)$. On the other hand, the following assertion is true for $n \ge 2$.

Proposition 5.10. *Let $n \ge 2$. If $0 \le \alpha < n - 1/2$, then the family $\mathcal{K}_\alpha(n)$ contains no inner functions.*

Proof. Let $f \in \mathcal{K}_\alpha(n)$ be an inner function. By Corollary 5.5 and part (ii) of Corollary 2.3, there is no loss of generality in assuming that $\alpha \in (n-1, n-1/2)$.

Suppose $\rho \in [1/2, 1)$ and $f_\rho(\zeta) = f(\rho\zeta)$, $\zeta \in \partial B_n$. For $\zeta \in \partial B_n$ and $\lambda \in B_1$, put $f_\zeta(\lambda) = f(\lambda\zeta)$. Observe that $\mathcal{R}f(\lambda\zeta) = \lambda f'_\zeta(\lambda)$; hence,

$$|f^*(\zeta) - f_\rho(\zeta)| \le 2\int_\rho^1 |\mathcal{R}f(r\zeta)|\, dr \tag{5.5}$$

provided that $f^*(\zeta)$ is well defined. Applying estimate (5.5), Hölder's inequality, and Fubini's theorem, we obtain

$$\begin{aligned}
\|f_\rho - f^*\|^2_{L^2(\partial B_n)} &\le C\int_{\partial B_n}\left(\int_\rho^1 |\mathcal{R}f(r\zeta)|(1-r)^{\frac14}(1-r)^{-\frac14}\, dr\right)^2 d\sigma_n(\zeta)\\
&\le C(1-\rho)^{\frac12}\int_{\partial B_n}\int_\rho^1 |\mathcal{R}f(r\zeta)|^2(1-r)^{\frac12}\, dr\, d\sigma_n(\zeta)\\
&= o(1-\rho)^{\frac12} \quad \text{as } \rho \to 1
\end{aligned}$$

by Lemma 5.9. Tamm [27] proved that no inner function can satisfy the estimate $\|f_\rho - f^*\|^2_{L^2(\partial B_n)} = o(1-\rho)^{\frac12}$ as $\rho \to 1-$ if $n \ge 2$. This contradiction finishes the proof of the proposition. □

§6. Boundary behavior

6.1. Directions of the maximal radial growth. The definition of the norm in the space $\mathcal{K}_\alpha(n)$, $\alpha > 0$, implies that

$$|f(z)| \le \frac{\|f\|_{\mathcal{K}_\alpha(n)}}{(1-|z|)^\alpha}, \quad z \in B_n.$$

Proposition 6.1 shows that this maximal growth is possible for an at most countable set of radial directions. For $n = 1$, this was proved in [11].

Suppose $\xi \in \partial B_n$ and $C > 1$. Recall that the corresponding Korányi domain $D_C(\xi)$ is defined by

$$D_C(\xi) = \{z \in B_n \,:\, |1 - \langle z, \xi\rangle| < C(1-|z|)\}.$$

Proposition 6.1. *Suppose $n \in \mathbb{N}$, $\mu \in M(n)$, $\alpha > 0$, and $f = K_\alpha[\mu]$. If $\xi \in \partial B_n$, then*

$$\lim_{\substack{z\to\xi \\ z\in D_C(\xi)}} (1-\langle z,\xi\rangle)^\alpha f(z) = \mu(\{\xi\})$$

for all $C > 1$.

Proof. Let $\zeta, \xi \in \partial B_n$. We have

$$\lim_{\substack{z\to\xi \\ z\in B_n}} \frac{(1-\langle z,\xi\rangle)^\alpha}{(1-\langle z,\zeta\rangle)^\alpha} = \delta_{\zeta,\xi},$$

where $\delta_{\zeta,\xi} = 0$ for $\zeta \ne \xi$, and $\delta_{\xi,\xi} = 1$. If $z \in D_C(\xi)$, then

$$\left|\frac{1-\langle z,\xi\rangle}{1-\langle z,\zeta\rangle}\right|^\alpha \le \frac{|1-\langle z,\xi\rangle|^\alpha}{(1-|z|)^\alpha} \le C^\alpha.$$

Hence, if $f = K_\alpha[\mu]$, then

$$\lim_{\substack{z\to\xi \\ z\in D_C(\xi)}} (1-\langle z,\xi\rangle)^\alpha f(z) = \lim_{\substack{z\to\xi \\ z\in D_C(\xi)}} \int_{\partial B_n} \frac{(1-\langle z,\xi\rangle)^\alpha}{(1-\langle z,\zeta\rangle)^\alpha}\, d\mu(\zeta) = \mu(\{\xi\})$$

by the dominated convergence theorem. □

6.2. Radial behavior of functions in $\mathcal{K}_\alpha(n)$ with $\alpha > n$. Recall that the Bloch space $\mathfrak{B}(B_n)$ consists of the functions $f \in \mathcal{H}ol(B_n)$ such that

$$|\mathcal{R}f(z)| \le \frac{C}{1-|z|}, \quad z \in B_n,$$

for some constant $C > 0$ (see [30, Chapter 3] for equivalent definitions).

Proposition 6.2. *Suppose $n \in \mathbb{N}$ and $\alpha > n$. Then $\mathfrak{B}(B_n) \subset \mathcal{K}_\alpha(n)$.*

Proof. Let $f \in \mathfrak{B}(B_n)$, that is, $|\mathcal{R}f(r\zeta)| \le C(1-r)^{-1}$ for all $\zeta \in \partial B_n$ and $r \in [0,1)$. The assumption $\alpha > n$ guarantees that

$$\int_0^1 \int_{\partial B_n} |\mathcal{R}f(r\zeta)|(1-r)^{(\alpha+1)-n-1}\, d\sigma_n(\zeta)\, dr < +\infty.$$

Since $\alpha + 1 > n$, we have $\mathcal{R}f \in \mathcal{K}_{\alpha+1}(n)$ by Proposition 5.2. Finally, applying Corollary 5.6, we obtain $f \in \mathcal{K}_\alpha(n)$. □

Corollary 6.3. *Let $n \in \mathbb{N}$. Then there exists a function $f \in \bigcap_{\alpha>n} \mathcal{K}_\alpha(n)$ such that the finite limit $\lim_{r\to 1-} f(r\zeta)$ fails to exist for all $\zeta \in \partial B_n$.*

Proof. Ullrich [28] constructed a function $f \in \mathfrak{B}(B_n)$ with no finite radial limits. It remains to apply Proposition 6.2. □

6.3. Exceptional sets and Hausdorff contents: definitions. Let $0 < \tau \le 1$, and let $\zeta \in \partial B_n$. Put

$$D_{\tau,C}(\zeta) = \{z \in B_n \,:\, |1 - \langle z, \zeta\rangle| < C(1-|z|)^\tau\},$$

where $C > 1$ for $\tau = 1$, and $C > 0$ for $\tau \in (0,1)$. Note that the sets $D_{1,C}(\zeta)$ coincide with the Korányi domains $D_C(\zeta)$, which were considered in Subsection 6.1. Also, note that the order of contact between the set $D_{\tau,C}(\zeta)$ and the sphere ∂B_n increases as the parameter τ decreases.

Consider a function $f : B_n \to \mathbb{C}$. By definition, the set $E_{\tau,C}(f)$ consists of all $\zeta \in \partial B_n$ for which the function $f(z)$ fails to have a limit at the point ζ as z approaches ζ through $D_{\tau,C}(\zeta)$. The sets $E_{1,C}(f)$ are said to be *exceptional*; the sets $E_{\tau,C}(f)$, $0 < \tau < 1$, are *tangentially exceptional*.

To estimate the size of the sets $E = E_{\tau,C}(f)$ for $f \in \mathcal{K}_\alpha(n)$, it is natural to apply the nonisotropic Hausdorff contents, which are defined as follows:

$$H_m(E) = H_m(E, \partial B_n) = \inf\left\{\sum_k \delta_k^m \,:\, E \subset \bigcup_k Q(\zeta_k, \delta_k)\right\}, \tag{6.1}$$

where $\zeta_k \in \partial B_n$ and $Q(\zeta,\delta) = \{\xi \in \partial B_n \,:\, |1 - \langle \zeta, \xi\rangle| < \delta\}$.

6.4. Hardy–Sobolev spaces. Assume that a function $f \in \mathcal{H}ol(B_n)$ has homogeneous expansion $f = \sum_k f_k$. Then the fractional derivative of order $\beta > 0$ is defined by the formula

$$R^\beta f(z) = \sum_k (k+1)^\beta f_k(z), \quad z \in B_n.$$

For $\beta > 0$ and $0 < p < \infty$, the Hardy–Sobolev spaces are defined by

$$H^p_\beta(B_n) = \left\{f \in \mathcal{H}ol(B_n) \,:\, \sup_{0<r<1} \int_{\partial B_n} |R^\beta f(r\zeta)|^p \, d\sigma_n(\zeta) < \infty\right\}.$$

Theorem 6.4 (Ahern and Cohn [1])**.** *Let $n - \beta p \ge 0$. If $f \in H^p_\beta(B_n)$, then*

$$H_{n-\beta p+\varepsilon}(E(f)) = 0$$

for all $\varepsilon > 0$.

Theorem 6.5 (Cascante and Ortega [8, Corollary 2.1]; see also [26])**.** *Assume that $n \in \mathbb{N}$, $0 < p < \infty$, $0 < \beta < n/p$, $0 < \tau < 1$, and $m = (n - \beta p)/\tau$. If $f \in H^p_\beta(B_n)$, then $H_m\left(E_{\tau,C}(f)\right) = 0$ for all $C > 0$.*

6.5. Exceptional sets. By Corollary 6.3, there exists a function $f \in \bigcap_{\alpha>n} \mathcal{K}_\alpha(n)$ such that $E_{1,C}(f) = \partial B_n$ for all $C > 1$. On the other hand, if $0 \le \alpha \le n$, then $\mathcal{K}_\alpha(n) \subset H^p(B_n)$ for all $0 < p < 1$. Hence, if $f \in \mathcal{K}_\alpha(n)$, $0 \le \alpha \le n$, then $\sigma_n(E_{1,C}(f)) = 0$ for all $C > 1$. This observation can be refined in the case where $0 \le \alpha < n$. For the spaces $\mathcal{K}_\alpha(1)$, $0 \le \alpha < 1$, similar refinements in terms of the Bessel capacities were obtained in [12].

Proposition 6.6. *Suppose $n \in \mathbb{N}$ and $\alpha \in (0,n)$. If $f \in \mathcal{K}_\alpha(n)$, then $H_{\alpha+\varepsilon}(E_{1,C}(f)) = 0$ for all $\varepsilon > 0$ and $C > 1$.*

Proof. Let $n - \alpha > \varepsilon > 0$. Then $\mathcal{K}_\alpha(n) \subset A^1_{\alpha+\varepsilon,n}(B_n) \subset H^1_{n-\alpha-\varepsilon}(B_n)$ by Proposition 5.3 and by [4, Theorem 5.12], respectively. It remains to apply Theorem 6.4. □

The following example corresponds to Proposition 6.6.

Proposition 6.7. *Suppose $n \in \mathbb{N}$, $\alpha \in (0,n)$, $0 \le m < \alpha$, $E \subset \partial B_n$ is a compact set, $H_m(E) = 0$, and $C > 1$. Then $E = E_{1,C}(f)$ for some function $f \in \mathcal{K}_\alpha(n)$.*

Proof. Fix $p_0 > 1$ such that $m = n-(n-\alpha)p_0$. We have $H_m(E) = 0$; hence, Theorem 1.2 in [1] guarantees that $E = E_{1,C}(f)$ for some function $f \in H^{p_0}_{n-\alpha}(B_n)$. Next, recall that

$$\mathcal{K}^{p_0}_\alpha(n) = \{K_\alpha[g\sigma_n] \,:\, g \in L^{p_0}(\sigma_n)\}.$$

By [8, Theorem 2.1], we have $H^{p_0}_{n-\alpha}(B_n) = \mathcal{K}^{p_0}_\alpha(n)$. It remains to note that $\mathcal{K}^{p_0}_\alpha(n) \subset \mathcal{K}_\alpha(n)$. □

6.6. Tangentially exceptional sets. For the spaces $\mathcal{K}_\alpha(1)$, $0 \le \alpha < 1$, tangentially exceptional sets were investigated in [10]. In the case of an arbitrary dimension $n \in \mathbb{N}$, the following assertion holds true.

Proposition 6.8. *Suppose $n \in \mathbb{N}$, $\alpha \in (0,n)$, $\tau \in (\alpha/n, 1)$, and $m > \alpha/\tau$. If $f \in \mathcal{K}_\alpha(n)$, then $H_m(E_{\tau,C}(f)) = 0$ for all $C > 0$.*

Proof. We have $\mathcal{K}_\alpha(n) \subset H^1_{n-\alpha-\varepsilon}(B_n)$ for $n - \alpha > \varepsilon > 0$. It remains to apply Theorem 6.5. □

The following example corresponds to Proposition 6.8.

Proposition 6.9. *Suppose $n \in \mathbb{N}$, $\alpha \in (0,n)$, $\tau \in [\alpha/n, 1)$, $m < \alpha/\tau$, and $C > 0$. If $E \subset \partial B_n$ is a compact set and $H_m(E) = 0$, then $E = E_{\tau,C}(f)$ for some function $f \in \mathcal{K}_\alpha(n)$.*

Proof. Since $m < \alpha/\tau$, there exists $p_0 > 1$ such that

$$m = (n - (n-\alpha)p_0)/\tau.$$

Since $p_0 > 1$ and $H_m(E) = 0$, we have $E = E_{\tau,C}(f)$ for some function $f \in \mathcal{K}^{p_0}_\alpha(n)$; see [26, Remark after Corollary 3.11]. It remains to note that $\mathcal{K}^{p_0}_\alpha(n) \subset \mathcal{K}_\alpha(n)$. □

§7. Multipliers

In this section, we study the spaces $\mathfrak{M}_\alpha(n)$ that consist of multipliers for the families $\mathcal{K}_\alpha(n)$. Note that multipliers for the Hardy–Sobolev spaces were investigated in the recent papers [22, 5] and [25].

7.1. Necessary conditions. For $n = 1$, the results of this subsection were obtained in [16].

Lemma 7.1. *Suppose $n \in \mathbb{N}$, $\alpha > 0$, and $f \in \mathfrak{M}_\alpha(n)$. Then $f \in H^\infty(B_n)$; moreover, $\|f\|_{H^\infty(B_n)} \le \|f\|_{\mathfrak{M}_\alpha(n)}$.*

Proof. Fix a point $\zeta \in \partial B_n$ and fix a constant K for which $\|f\|_{\mathfrak{M}_\alpha(n)} < K$. Since $\|(1-\langle\cdot,\zeta\rangle)^{-\alpha}\|_{\mathcal{K}_\alpha(n)} = 1$, there exists a measure $\mu_\zeta \in M(n)$ such that $\|\mu_\zeta\|_{M(n)} < K$ and

$$\frac{f(z)}{(1-\langle z,\zeta\rangle)^\alpha} = \int_{\partial B_n} \frac{1}{(1-\langle z,\xi\rangle)^\alpha}\, d\mu_\zeta(\xi).$$

In other words,

$$f(z) = \int_{\partial B_n} \left(\frac{1-\langle z,\zeta\rangle}{1-\langle z,\xi\rangle}\right)^\alpha d\mu_\zeta(\xi).$$

Put $z = r\zeta$, $0 \le r < 1$. Then

$$|f(r\zeta)| \le \int_{\partial B_n} \left(\frac{1-r}{1-r|\langle\zeta,\xi\rangle|}\right)^\alpha d\mu_\zeta(\xi) \le \|\mu_\zeta\|_{M(n)} < K.$$

Since the point $\zeta \in \partial B_n$ and the constant $K > \|f\|_{\mathfrak{M}_\alpha(n)}$ are arbitrary, we have $\|f\|_{H^\infty(B_n)} \le \|f\|_{\mathfrak{M}_\alpha(n)}$. □

Lemma 7.2. *Suppose $n \in \mathbb{N}$, $\alpha > 0$, and $f \in \mathfrak{M}_\alpha(n)$. Then $f \in \mathcal{K}_\alpha(n)$; moreover, $\|f\|_{\mathcal{K}_\alpha(n)} \le \|f\|_{\mathfrak{M}_\alpha(n)}$.*

Proof. Put $I(z) = 1$ for $z \in B_n$. Recall that $I = K_\alpha[\sigma_n]$. Since σ_n is a positive measure, we have $\|I\|_{\mathcal{K}_\alpha(n)} = \|\sigma_n\| = 1$. The assumption $f \in \mathfrak{M}_\alpha(n)$ implies that $f = fI \in \mathcal{K}_\alpha(n)$. Moreover, $\|f\|_{\mathcal{K}_\alpha(n)} = \|fI\|_{\mathcal{K}_\alpha(n)} \le \|f\|_{\mathfrak{M}_\alpha(n)}\|I\|_{\mathcal{K}_\alpha(n)} = \|f\|_{\mathfrak{M}_\alpha(n)}$. □

Proposition 7.3. *Let $\alpha > 0$. Then there exists a constant C_α such that, for any $f \in \mathfrak{M}_\alpha(n)$ and any $n \in \mathbb{N}$, the radial variation in the direction $\xi \in \partial B_n$ satisfies the estimate $V(f,\xi) \le C_\alpha \|f\|_{\mathfrak{M}_\alpha}$ for all $\xi \in \partial B_n$.*

Proof. Suppose $f \in \mathfrak{M}_\alpha(n)$, $\xi \in \partial B_n$, and $\varepsilon > 0$. By the definition of the space $\mathfrak{M}_\alpha(n)$, there exists a measure μ_ξ such that $\|\mu_\xi\|_{M(n)} \le \|f\|_{\mathfrak{M}_\alpha(n)} + \varepsilon$ and

$$f(z) = \int_{\partial B_n} \frac{(1-\langle z,\xi\rangle)^\alpha}{(1-\langle z,\zeta\rangle)^\alpha}\, d\mu_\xi(\zeta), \quad z \in B_n.$$

Let $\lambda \in B_1$. We put $z = \lambda\xi$ and $f_\xi(\lambda) = f(\lambda\xi)$. Then

$$f_\xi(\lambda) = \int_{\partial B_n} \frac{(1-\lambda)^\alpha}{(1-\lambda\langle\xi,\zeta\rangle)^\alpha}\, d\mu_\xi(\zeta),$$
$$f'_\xi(\lambda) = \alpha \int_{\partial B_n} \frac{(1-\lambda)^{\alpha-1}(\langle\xi,\zeta\rangle - 1)}{(1-\lambda\langle\xi,\zeta\rangle)^{\alpha+1}}\, d\mu_\xi(\zeta).$$

Now, set $\lambda = r$, $0 \le r < 1$. Fubini's theorem guarantees that

$$\int_0^1 |f'_\xi(r)|\, dr \le \alpha \int_{\partial B_n} \int_0^1 \frac{(1-r)^{\alpha-1}|1-\langle\xi,\zeta\rangle|}{|1-r\langle\xi,\zeta\rangle|^{\alpha+1}}\, dr\, d|\mu_\xi|(\zeta). \tag{7.1}$$

Put $w = \langle\xi,\zeta\rangle$ and $b = |1-\langle\xi,\zeta\rangle|$. Observe that

$$|1-rw|^2 = (1-r)(1-r|w|^2) + r|1-w|^2 \ge (1-r)^2 + r^2b^2.$$

Hence,

$$\int_0^1 \frac{(1-r)^{\alpha-1}b}{|1-r\langle\xi,\zeta\rangle|^{\alpha+1}}\, dr \le \int_0^1 \frac{(1-r)^{\alpha-1}b}{((1-r)^2+r^2b^2)^{(\alpha+1)/2}}\, dr := I(\alpha,b).$$

If $b = 0$, then the proof is finished. Otherwise, as was shown in the proof of Theorem 2.6 in [16], we have $I(\alpha,b) \le C_\alpha < \infty$. Therefore, estimate (7.1) implies that $V(f,\xi) \le C_\alpha\|f\|_{\mathfrak{M}_\alpha(n)}$. The proof of the proposition is finished. □

Recall that a bounded nonconstant function $f \in \mathcal{H}ol(B_n)$ is said to be *inner* if $|f^*| = 1$ σ_n-a.e. A complete description of the inner functions belonging to $\mathfrak{M}_1(1)$ was obtained in [18]. For arbitrary $\alpha > 0$, the inner functions in the family $\mathfrak{M}_\alpha(1)$ were investigated in [13]. For $n \ge 2$, the answer to the corresponding question follows from Proposition 7.3.

Corollary 7.4. *Assume that $n \ge 2$ and $\alpha > 0$. If f is an inner function in the ball B_n, then $f \notin \mathfrak{M}_\alpha(n)$.*

Proof. For $n \ge 2$, it is well known that any inner function has no radial limits on a dense subset of the sphere ∂B_n; see [24, §1]. □

7.2. Sufficient conditions. Recall that $A(B_n)$ denotes the ball algebra. If $f \in A(B_n)$, then put $f^* = f|_{\partial B_n}$. The complex modulus of continuity of the function f^* is defined by

$$\omega_{\mathbb{C}}(f^*, \delta) = \sup_{\zeta,\, \xi \in \partial B_n} \{|f^*(\zeta) - f^*(\xi)| \,:\, d(\zeta, \xi) \le \delta\},$$

where $\delta \in (0, 2]$ and $d(\zeta, \xi) = |1 - \langle \zeta, \xi \rangle|$. Note that $d(\zeta, \xi)$ is a quasimetric on the sphere.

Let ν_n denote the Lebesgue measure on B_n normalized by the condition $\nu_n(B_n) = 1$. Suppose that $n \ge 2$. For a Borel function $f : B_1 \to \mathbb{C}$, it is well known that the identity

$$\int_{\partial B_n} f(\langle \xi, \zeta \rangle)\, d\sigma_n(\xi) = (n-1) \int_{B_1} f(z)(1 - |z|^2)^{n-2}\, d\nu_1(z) \tag{7.2}$$

is fulfilled if its left-hand side or right-hand side is well defined.

An analog of the following statement for $n = 1$ was proved in [29].

Proposition 7.5. *Let $\alpha \ge n \ge 2$. Suppose that a function $g \in A(B_n)$ satisfies the following condition*:

$$(n-1) \int_{B_1} \frac{(1 - |z|^2)^{n-2} \omega_{\mathbb{C}}(g^*, |1 - z|)}{|1 - z|^n}\, d\nu_1(z) = C_\omega < +\infty. \tag{7.3}$$

Then $g \in \mathfrak{M}_\alpha(n)$.

Proof. By Lemma 2.4 and Proposition 2.5, it suffices to verify that the function g satisfies condition (2.5) with $\alpha = n$.

Fix a point $\zeta \in \partial B_n$. Then

$$\frac{g(w)}{(1 - \langle w, \zeta \rangle)^n} = \frac{g(w) - g(\zeta)}{(1 - \langle w, \zeta \rangle)^n} + \frac{g(\zeta)}{(1 - \langle w, \zeta \rangle)^n} := h_1(w) + h_2(w), \quad w \in B_n.$$

Observe that $\|h_2\|_{\mathcal{K}_n(n)} \le \|g\|_{A(B_n)}$.

On the other hand, we have $g(\cdot) - g(\zeta) \in H^\infty(B_n)$ and $(1 - \langle \cdot, \zeta \rangle)^{-n} \in H^p(B_n)$ for all $0 < p < 1$. Hence, $h_1 \in H^p(B_n)$ for all $0 < p < 1$. Let $h_1^*(\xi) = \lim_{r \to 1-} h_1(r\xi)$, $\xi \in \partial B_n$, $\xi \ne \zeta$. Formula (7.2) and condition (7.3) show that $\|h^*\|_{L^1(\partial B_n)} \le C_\omega$. Therefore, $h_1 \in H^1(B_n)$; also, we obtain $\|h_1\|_{\mathcal{K}_n(n)} \le \|h_1\|_{H^1(B_n)} \le C_\omega$. This proves (2.5). □

Assume that $0 < \beta < 1$. By definition, the standard Lipschitz space $\Lambda^\beta(\partial B_n)$ consists of all functions $f : \partial B_n \to \mathbb{C}$ such that

$$|f(\zeta) - f(\xi)| \le C_f |\zeta - \xi|^\beta$$

for all $\zeta, \xi \in \partial B_n$. In other words, $\Lambda^\beta(\partial B_n)$ are the Lipschitz spaces with respect to the Euclidean metric on the sphere. Elements of these spaces can be used to construct examples of functions belonging to families $\mathfrak{M}_\alpha(n)$ with $\alpha \ge n$.

Corollary 7.6. *Suppose $n \ge 2$, $0 < \beta < 1/2$, and $f \in \Lambda^\beta(\partial B_n)$. Then $K_n[f] \in \mathfrak{M}_\alpha(n)$ for $\alpha \ge n$.*

Proof. Let $f \in \Lambda^\beta(\partial B_n)$. It is well known that $\omega_{\mathbb{C}}(K_n[f]^*, \delta) = \mathcal{O}(\delta^\beta)$ (see [2]). Hence, condition (7.3) is satisfied. □

Explicit examples of functions belonging to $\mathfrak{M}_\alpha(n)$, $\alpha \ge 0$, can be obtained with the help of the following assertion.

Proposition 7.7. *Suppose $n \in \mathbb{N}$, $j \in \{0, 1, \dots, n\}$, and $\alpha \ge n - j$. Also, suppose that the derivative $\mathcal{R}^j g \in A(B_n)$ satisfies condition* (7.3). *Then $g \in \mathfrak{M}_\alpha(n)$.*

Proof. We argue by induction. As the base of induction, we take the case of $j = 0$. Then the claim is true by Proposition 7.5.

Now, let $k \in \{0, 1, \dots, n-1\}$. Assume that the required implication is proved for $j = k$.

Suppose $j = k+1 \in \{1, \dots, n\}$ and $f \in \mathcal{K}_\alpha(n)$, $\alpha \geq n - j$. We must prove that $fg \in \mathcal{K}_\alpha(n)$. Since $\alpha \geq n - j \geq 0$, Corollary 5.6 guarantees that $fg \in \mathcal{K}_\alpha(n)$ if and only if $\mathcal{R}(fg) \in \mathcal{K}_{\alpha+1}(n)$. Note that

$$\mathcal{R}(fg) = f\mathcal{R}g + g\mathcal{R}f.$$

The function $\mathcal{R}^{j-1}(\mathcal{R}g) \in A(B_n)$ satisfies condition (7.3). We have $\alpha+1 \geq n-(j-1)$; hence, $\mathcal{R}g \in \mathfrak{M}_{\alpha+1}(n)$ by the inductive hypothesis. Since $f \in \mathcal{K}_\alpha(n) \subset \mathcal{K}_{\alpha+1}(n)$, we obtain $f\mathcal{R}g \in \mathcal{K}_{\alpha+1}(n)$ by the definition of the multiplier space.

On the other hand, $f \in \mathcal{K}_\alpha(n)$ if and only if $\mathcal{R}f \in \mathcal{K}_{\alpha+1}(n)$. Next, the function $\mathcal{R}^{j-1}g \in A(B_n)$ satisfies condition (7.3). By the inductive hypothesis, we have $g \in \mathfrak{M}_{\alpha+1}(n)$; thus, $g\mathcal{R}f \in \mathcal{K}_{\alpha+1}(n)$.

Therefore, $\mathcal{R}(fg) \in \mathcal{K}_{\alpha+1}(n)$ and $fg \in \mathcal{K}_\alpha(n)$. The induction step is finished, and the proposition is proved. □

By using the family $\mathcal{K}_0(n)$, other examples of multipliers can be obtained.

Proposition 7.8. *Suppose $n \in \mathbb{N}$, $0 < \alpha < n$, and $g \in \mathcal{K}_0(n) \cap \mathfrak{M}_n(n)$. Then $g \in \mathfrak{M}_\alpha(n)$.*

Proof. We argue by induction. As the base of induction, we take the statement that $g \in \mathfrak{M}_\alpha(n)$ for $\alpha \geq n$, which is true by assumption.

Now, suppose that $j \in \{0, 1, \dots, n-1\}$. Assume that the required implication is valid for $\alpha \geq n - j$. We prove that $g \in \mathfrak{M}_\alpha(n)$ if $\alpha > 0$ and $\alpha \geq n - j - 1$.

Let $f \in \mathcal{K}_\alpha(n)$. We must check that $fg \in \mathcal{K}_\alpha(n)$.

The condition $g \in \mathcal{K}_0(n)$ implies that $\mathcal{R}g \in \mathcal{K}_1(n)$. Since $\alpha > 0$, we obtain $f\mathcal{R}g \in \mathcal{K}_\alpha(n) \cdot \mathcal{K}_1(n) \subset \mathcal{K}_{\alpha+1}(n)$. On the other hand, $\mathcal{R}f \in \mathcal{K}_{\alpha+1}(n)$ because $f \in \mathcal{K}_\alpha(n)$. Next, we have $g \in \mathfrak{M}_{n-j}(n)$ by the inductive hypothesis. Also, $\mathfrak{M}_{n-j}(n) \subset \mathfrak{M}_{\alpha+1}(n)$ because $n - j \leq \alpha + 1$. Thus, $g\mathcal{R}f \in \mathfrak{M}_{\alpha+1}(n) \cdot \mathcal{K}_{\alpha+1}(n) \subset \mathcal{K}_{\alpha+1}(n)$.

Therefore, $\mathcal{R}(fg) = f\mathcal{R}g + g\mathcal{R}f \in \mathcal{K}_{\alpha+1}(n)$ and $fg \in \mathcal{K}_\alpha(n)$ by Corollary 5.6. The induction step is finished. The proof of the proposition is complete. □

Corollary 7.9. *Suppose $n \in \mathbb{N}$ and $0 < \alpha < n$. Also, suppose that a function $g \in \mathcal{K}_0(n) \cap A(B_n)$ satisfies condition (7.3). Then $g \in \mathfrak{M}_\alpha(n)$.*

Proof. It suffices to apply Propositions 7.5 and 7.8. □

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