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A CLASS OF TOPOGRAPHICAL WAVEGUIDES

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Abstract. In the case of some infinite domains, it is shown that the spectrum of the elasticity theory operator is not purely continuous. This implies the existence of a new class of the so-called topographical waveguides.

§1. Introduction

Volcanic eruptions generate seismic waves of various types. In particular, waves have been observed that spread along "chainlets" of mountains formed by quite firm rocks in a softer, low-speed medium. Thus, a waveguide propagation of seismic waves has been detected.[1] A mathematical description of this type of phenomena is a complicated unsolved problem. However, waveguide propagation of elastic waves has been studied in many papers — either of a computational nature, or done within a "physical" level of rigor, or pure mathematical. In the first place, the paper [1] should be mentioned, in which, "at the theorems level", the so-called topographical waveguides were treated. Such a waveguide is a narrow triangular or rectangular prism "attached" to an elastic half-space (see Figure 1).

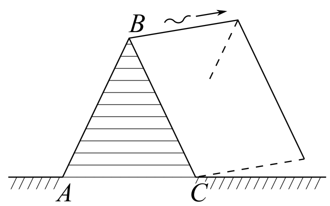

Figure 1

Next, there are papers devoted to the proof of the existence of a wave that runs along the edge of an elastic wedge; they are also in a close relationship with topographical waveguides. The existence of such a wave (whose waveguide essence is clear) was beyond any doubt already after the computational papers [2, 3] and the papers [4, 5] done on a "physical" level of mathematical rigor. However, a rigorous result is due to I. Kamotskiĭ (see [6]), who employed a variational approach to prove the existence of a vector-valued eigenfunction corresponding to this wave mode, for the elasticity theory operator. In [6], a significant role was played by a clever choice of a test function.

Our aim in this paper is to consider new classes of topographical waveguides. We show that in a short-wave situation, an arbitrary elastic prism attached to an elastic half-space is a topographical waveguide provided that the upper dihedral angle ($\angle ABC$ in Figure 1) is acute. This result admits a series of generalizations (see §5). The problem

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reduces to the proof of the existence of a point spectrum for a certain vectorial selfadjoint differential operator.

In the constructions, a key role is played by some analytic tricks borrowed from [6].

§2. Statement of the problem

We assume that Ξ is a domain in $\mathbb{R}^3 = -\infty < x, y, z < +\infty$, described by the inequality $y < f(x)$, where $f(x)$ is a nonnegative continuous function defined for $-\infty < x < +\infty$ and vanishing outside a finite interval $[a, b]$. Moreover, the graph of $f(x)$ for $a \le x \le b$ is assumed to consist of two rectilinear segments $[aM]$ and $[Mb]$ ($f(a) = f(b) = 0$, $M = (x_0, f(x_0))$, $a < x_0 < b$, $f(x_0) > 0$)) forming an acute angle at M. The domain Ξ is a topological product: $\Xi = \Omega \times (-\infty < z < +\infty)$, where Ω is the domain $y < f(x)$ on the (x, y)-plane; see Figure 2.

Let $U(t, x, y, z) = (U_x, U_y, U_z)$ be a solution of the elasticity theory equations in Ξ,

$$\rho U_{tt} = (\lambda + \mu)\operatorname{grad}\operatorname{div} U + \mu \Delta U = 0, \qquad \rho, \lambda, \mu = \text{const}, \ \ \rho > 0, \ \ \mu > 0, \ \ \lambda + \frac{2}{3}\mu > 0, \tag{2.1}$$

that has the form

$$U = e^{-i\omega t + ikz} u(x, y), \quad \omega, k = \text{const}, \ \ \omega > 0, \ \ k > 0, \tag{2.2}$$

and possesses no strain on the boundary $\partial\Xi$. We arrive at the following equation for u in Ω:

$$\rho\omega^2 u = L(ik)u, \tag{2.3}$$

where

$$L(ik)u = -(\lambda + \mu)\operatorname{grad}\operatorname{div} U - \mu\Delta U.$$

On the right in the last identity, the differentiation $\frac{\partial}{\partial z}$ is assumed to be replaced by multiplication by ik.

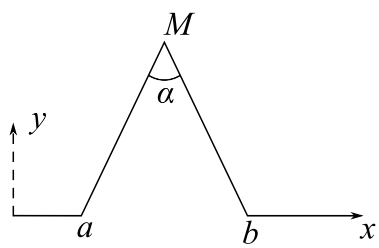

Figure 2

We subject the vector (2.2) to the boundary condition of the absence of strain on $\partial\Omega$ (i. e., for $y = f(x)$). We arrive at the identity

$$N(ik)u = 0, \tag{2.4}$$

where $N(ik)$ is a first order differential operator depending on k.

Later, we shall describe the meaning of the equation and boundary conditions more precisely. We use the notation employed in [6]. In what follows, we put $\rho = 1$, which leads to no loss of generality.

If a function $u \not\equiv 0$ satisfies equation (2.3) and the boundary conditions (2.4), and is square integrable over Ω (i. e., u is a vector-valued eigenfunction of the operator $L(ik)$ that corresponds to the boundary conditions (2.4)), then the vector-valued function (2.2) is called a wave mode. The proof of the existence of wave modes is the subject of the present paper.

We state the problem in more precise terms.[2] On smooth (in $C^\infty(\bar\Omega)$) vector-valued functions u (with values in $\mathbb{C}^3$) having bounded support and satisfying the boundary conditions (2.4), the operator $L(ik)$ is positive and symmetric in the Hilbert space of square integrable vector-valued functions. In what follows, by $L(ik)$ we mean the extension of the above operator up to a Friedrichs selfadjoint operator. We shall prove the existence of a point spectrum for this extension. Namely, the following will be established.

Theorem 1. *For k sufficiently large, $L(ik)$ has at least one eigenvalue on the interval $(0, c_R^2 k^2)$.*

§3. Variational approach to the proof

The strategy of the existence proof for a point spectrum is fairly standard; see [1, 6]. It is well known (see [1]) that the essential spectrum of the operator $L(ik)$ occupies the half-axis $[c_R^2 k^2, +\infty)$, where c_R is the Rayleigh wave velocity.

The lower bound of the spectrum is

$$\inf \Phi(u),\ \Phi(u) := \frac{a_\Omega(ik,u,u)}{(u,u)}, \quad u \in H^1(\Omega),\ \ u \not\equiv 0, \tag{3.1}$$

where a_Ω is the quadratic form of the operator $L(ik)$ and (u,u) is the square of the usual L_2-norm of u. In order to prove the existence of a point spectrum, it suffices to find a function $u^{\rm test}$ with

$$\Phi(u^{\rm test}) < c_R^2 k^2. \tag{3.2}$$

To construct this test function, we place the coordinate system as shown in Figure 3. On the interval $[0,d]$, let the part of $\partial\Omega$ not lying on the x_1-axis be the rectilinear segment described by the equation $x_2 = x_1 \tan\alpha$, $0 < \alpha < \frac{\pi}{2}$ (Figure 3).

We denote by U^R the expression (2.2) that describes the classical Rayleigh wave propagating in the direction of the x_3 axis. Strain is assumed to be absent for $x_2 = 0$. Then

$$U^R = e^{-i\omega t + ikx_3} u^R(x_2), \tag{3.3}$$

where u^R does not depend on x_1 and x_3. Consider the following cutoff function $\eta(x_1)$ defined for $x_1 \geq 0$:

$$\begin{gathered} \eta(x_1) = \begin{cases} 1 & \text{if } \ 0 \le x_1 < d', \\ 0 & \text{if } \ x_1 > d, \end{cases} \\ 0 < d' < d, \quad d, d' = \text{const}, \quad \eta \ge 0, \quad \frac{d\eta}{dx} \le 0, \quad \eta \in C^\infty(\mathbb{R}^+). \end{gathered} \tag{3.4}$$

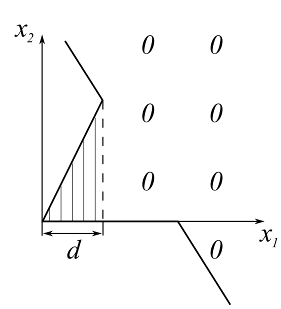

Figure 3

[2]The problems treated in [1] and [6] were introduced similarly.

The test function can be chosen as follows (cf. [6, formula (3.12)]):

$$u_k^{\text{test}}(x_1,x_2) =: \begin{cases} \eta(x_1)u^R(x_2) & \text{in } \Omega_1, \\ 0 & \text{in } \Omega\setminus\Omega_1. \end{cases} \tag{3.5}$$

Here Ω_1 is the triangle

$$0 \le x_1 \le d, \quad 0 \le x_2 \le x_1 \tan\alpha \tag{3.6}$$

(see Figure 3).

Easy considerations to be described in the next section show that

$$\Phi(u_k^{\text{test}}) = c_R^2 k^2 - \operatorname{const} k + O(1), \quad \operatorname{const} > 0, \tag{3.7}$$

as $k \to +\infty$. But (3.7) implies (3.2) for k sufficiently large. This proves the theorem (see the end of §2).

§4. The proof of (3.7)

First, we observe that

$$a_\Omega(ik,u,u) = \int_\Omega a(ik,u,u)\,dx_1\,dx_2, \tag{4.1}$$

where a is an expression quadratic in u and its first derivatives, specifically,

$$a = \lambda \operatorname{div} u\,(\operatorname{div} u)^* + \frac{\mu}{2}\frac{\partial u_l}{\partial x^m}\Big(\frac{\partial u_m}{\partial x^l}\Big)^* + \frac{\mu}{2}\frac{\partial u_l}{\partial x^m}\Big(\frac{\partial u_l}{\partial x^m}\Big)^* \tag{4.2}$$

($\frac{\partial}{\partial x_3}$ must be replaced by multiplication by ik, where the star indicates complex conjugation).

Clearly,

$$\Phi(u_k^{\text{test}}) = \frac{\int_{\Omega_1} a(ik, u_k^{\text{test}}, u_k^{\text{test}})\,dx_1\,dx_2}{\int_{\Omega_1} u_k^{\text{test}} u_k^{*\,\text{test}}\,dx_1\,dx_2}, \qquad u_k^{\text{test}}u_k^{*\,\text{test}} = \sum_{j=2}^{3} u_{k,j}^{\text{test}} u_{k,j}^{*\,\text{test}}. \tag{4.3}$$

This can be transformed to

$$\begin{aligned}\Phi(u_k^{\text{test}}) = c_R^2k^2 &- \frac{\int_0^d \eta^2(x_1)\Big(c_R^2k^2\int_0^{x_1\tan\alpha} u^Ru^{*R}\,dx_2 - \int_0^{x_1\tan\alpha} a(ik,u^R,u^{*R})\,dx_2\Big)\,dx_1}{\int_0^d \eta^2(x_1)\int_0^{x_1\tan\alpha} u^Ru^{*R}\,dx_2\,dx_1} \\ &+ \mu\frac{\int_0^d \big(\frac{d\eta}{dx_1}\big)^2\,dx_1}{\int_0^d \eta^2(x_1)\,dx_1} = A_1 - A_2 + A_3,\end{aligned} \tag{4.4}$$

where μ is the shift module.

The summand A_3 does not depend on k, so $A_3 = O(1)$ as $k \to +\infty$. The expression A_2 is positive because always

$$c_R^2k^2\int_0^N u^Ru^{*R}\,dx_2 > \int_0^N a(ik,u^R,u^R)\,dx_2, \quad N > 0 \tag{4.5}$$

(see [6]). We find the asymptotics of A_2 as $k \to +\infty$. First, recall that, under a due choice of a scalar factor, we have

$$u^R = \alpha_1 e^{-a_1kx_2} + \alpha_2 e^{-a_2kx_2}, \quad a_1, a_2 = \operatorname{const} > 0. \tag{4.6}$$

Here $\alpha_1 = (0, \alpha_{11}, \alpha_{12})$ and $\alpha_2 = (0, \alpha_{21}, \alpha_{22})$ are constant nonzero vectors. From (4.6) it follows that

$$\psi_0(kx_2) := u^R u^{*R} = \sum_{l,m=2}^{3} f_{lm} e^{-\beta_{lm} k x_2}, \tag{4.7}$$

$$k^2 \psi_1(kx_2) := k^2 c_R^2 u^R u^{*R} - a(ik, u^R, u^{*R}) = k^2 \sum_{l,m=2}^{3} g_{lm} e^{-\beta_{lm} k x_2}, \tag{4.8}$$

where the β_{lm} are positive constants and the f_{lm}, g_{lm} are complex constants.

The expression A_2 is a fraction whose numerator is

$$\int_0^d \eta^2(x_1) \int_0^{x_1 \tan\alpha} k^2 \psi_1(kx_2)\, dx_2\, dx_1. \tag{4.9}$$

We find the asymptotics of (4.9) as $k \to +\infty$. Observe that

$$\int_0^{x_1 \tan\alpha} k^2 \psi_1(kx_2)\, dx_2 = k \int_0^{kx_1 \tan\alpha} \psi_1(\xi)\, d\xi = k\psi_2(kx_1). \tag{4.10}$$

By (4.5), here $\psi_2(\xi)$ is a positive function. Since

$$\int_0^\infty k^2 \psi_1(kx_2)\, dx_2 = 0,$$

we have

$$\psi_2(kx_1) = \int_0^{+\infty} \psi_1(\xi)\, d\xi - \int_{kx_1 \tan\alpha}^\infty \psi_1(\xi)\, d\xi = -\int_{kx_1 \tan\alpha}^\infty \psi_1(\xi)\, d\xi.$$

Together with (4.8), this implies that $\psi_2(\xi)$ is exponentially small as $\xi \to +\infty$: $\psi_2(\xi) = O(e^{-\varepsilon\xi})$, $\varepsilon = \text{const} > 0$. Now it is easy to find the asymptotics of (4.9) as $k \to +\infty$. Indeed,

$$\int_0^d \eta^2(x_1) \Big(\int_0^{x_1 \tan\alpha} k^2 \psi_1(kx_2)\, dx_2 \Big)\, dx_1 = \int_0^d \eta^2(x_1) k \psi_2(x_1 k)\, dx_1 = \int_0^{d'} + \int_{d'}^d. \tag{4.11}$$

Since $\psi_2(\xi)$ is exponentially small as $\xi \to +\infty$, the second summand is also exponentially small, and we have

$$\int_0^{d'} k\psi_2(kx_1) = \int_0^{kd'} \psi_2(\xi)\, d\xi = \int_0^{+\infty} - \int_{kd'}^{+\infty} = \int_0^{+\infty} \psi_2(\xi)\, d\xi + O(e^{-k\varepsilon_1}), \qquad \varepsilon_1 = \text{const} > 0. \tag{4.12}$$

Now, we analyze the denominator of the fraction A_2. By (4.7), it can be represented in the form

$$\int_0^d \eta^2(x_1) \Big(\int_0^{x_1 \tan\alpha} u^R u^{*R}\, dx_2 \Big)\, dx_1 = \int_0^d \eta^2(x_1) \Big(\int_0^{x_1 \tan\alpha} \psi_0(kx_2)\, dx_2 \Big)\, dx_1. \tag{4.13}$$

For the inner integral, we have

$$\int_0^{x_1 \tan\alpha} \psi_0(kx_2)\, dx_2 = \frac{1}{k} \int_0^{kx_1 \tan\alpha} \psi_0(\xi)\, d\xi := \frac{1}{k} \psi_3(kx_1), \tag{4.14}$$

where $\psi_3(\xi)$ is a monotone increasing function. By (4.7), the limit $\lim_{\xi \to +\infty} \psi_3(\xi) := \psi_3(+\infty)$ is finite: $0 < \psi_3(+\infty) < +\infty$. We return to (4.13). Taking (4.4) into account, we can write

(4.15)
$$\int_0^d \eta^2(x_1)\Big(\int_0^{x_1\tan\alpha} \psi_0(kx_2)\,dx_2\Big)\,dx_1 = \int_0^{d'} \frac{1}{k}\psi_3(kx_1)\,dx_1 + \int_{d'}^d \eta^2(x_1)\frac{1}{k}\psi_3(kx_1)\,dx_1.$$

Next, $\psi_3(\xi) = \psi_3(+\infty) + O(e^{-\xi\varepsilon_2})$ as $\xi \to +\infty$, $\varepsilon_2 = \text{const} > 0$.

From (4.15), we deduce the following asymptotic relation:

$$\int_0^d \eta^2(x_1)\Big(\int_0^{x_1\tan\alpha} \psi_0(kx_2)\,dx_2\Big)\,dx_1 = \frac{1}{k}\psi_3(+\infty)\int_0^d \eta^2(x_1)\,dx_1 + O\Big(\frac{1}{k^2}\Big). \tag{4.16}$$

Taken together, (4.12) and (4.16) easily give an asymptotic formula for A_2 (see (4.4)):

$$A_2 = k\left(\frac{\int_0^{+\infty}\psi_2(\xi)\,d\xi}{\psi_3(+\infty)\int_0^d \eta^2(x_1)\,dx_1} + O\Big(\frac{1}{k^2}\Big)\right). \tag{4.17}$$

Since $A_3 = O(1)$ as $k \to +\infty$, we arrive at the following asymptotic formula for $\Phi(u_k^{\text{test}})$ (see (3.2)):

$$\Phi(u_k^{\text{test}}) = A_1 - A_2 + A_3 = k^2c_R^2 - k\frac{\int_0^{+\infty}\psi_2(\xi)\,d\xi}{\psi_3(+\infty)\int_0^d \eta^2(x_1)\,dx_1} + O(1). \tag{4.18}$$

For large k, formula (4.18) implies (3.2), which proves Theorem 1 of §2.

§5. Refinements and analogs of the result obtained

Note that the function u_k^{test} vanishes in Ω everywhere outside the dashed "triangle" in Figure 3. Therefore, certain analogs of Theorem 1 remain true for a wide class of domains in $\mathbb{R}^2$. It suffices merely to suppose that the boundary contains a "small angle" such as $\angle ABC$ in Figure 1 and the essential spectrum occupies the half-axis $[k^2c_R^2, +\infty)$. Examples of such domains are shown in Figures 4, 5, and 6.

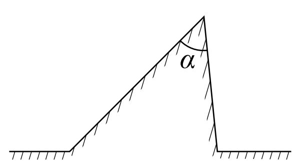

Figure 4

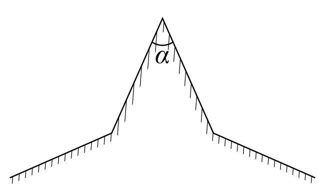

Figure 5

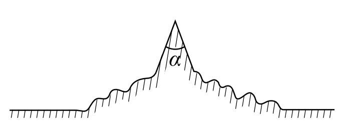

Figure 6

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