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THE TRACE OF BV-FUNCTIONS ON AN IRREGULAR SUBSET

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Abstract. Certain basic results on the boundary trace discussed in Maz′ya's monograph on Sobolev spaces are generalized to a wider class of regions. The paper is an extended and supplemented version of a preliminary publication, where some results were presented without proofs or in a weaker form. In Maz′ya's monograph, the boundary trace was defined for regions Ω with finite perimeter, and the main results were obtained under the assumption that normals in the sense of Federer exist almost everywhere on the boundary. Instead, now it is assumed that the region boundary is a countably $(n-1)$-rectifiable set, which is a more general condition.

§1. Introduction

Our purpose in this paper is to generalize all the main results on the boundary trace, as presented in [5, Chapter 6], to a wider class of sets. Chapter 6 in [5] is an extended version of the earlier publication [3]. The present paper is an extended and completed version of our publication [2], where some results were stated without proof or in a weaker form. In [3, 5], the boundary trace was defined for the regions Ω with finite perimeter (in the sense of Caccioppoli–De Giorgi), and the main results were obtained under the additional assumption that normals in the sense of Federer exist almost everywhere on $\partial\Omega$. Instead, now we suppose that $\partial\Omega$ is a countably $(n-1)$-rectifiable set, which is less restrictive. The reader is referred to [5, 4, 10] for the theory of sets of finite perimeter and BV functions.

The analytical tools we use are basically the same as in [3, 5]. The relationship between isoperimetric inequalities and integral inequalities (of Sobolev embedding type) plays an essential role. For the first time, these connections were discovered by V. Maz′ya [6]. Almost all results formulated below are valid not only for regions in $\mathbb{R}^n$ but also for regions in C^1-smooth n-dimensional manifolds. This becomes clear from Corollary 2.

In fact, deep knowledge of geometric measure theory, in particular, of rectifiable currents is not needed. All necessary (very restricted) information from this theory is given below.

Let us explain why our results generalize those in [3]. It is known that the boundary ∂E of a set $E \subset \mathbb{R}^n$ with a finite perimeter consists of two parts. One of them, called the reduced boundary $\partial^* E$, consists of all points at which normals in the sense of Federer exist. It is known that this part is a countably $(n-1)$-rectifiable set. The perimeter $P(E)$ of a set E equals $H_{n-1}(\partial^* E)$, where H_k is k-dimensional Hausdorff measure. Therefore, the requirement that the normals in the sense of Federer exist a.e. on $\partial^*\Omega$ is equivalent to the condition $\partial\Omega = \partial^*\Omega$. For sure, all sets are considered up to sets of $(n-1)$-dimensional Hausdorff measure zero.

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In general, $\partial E \setminus \partial^* E$ consists of two parts, a countably $(n-1)$-rectifiable set and what is called the completely unrectified (irregular) set $\mathrm{Ir}(E)$. The latter may have either finite or infinite $(n-1)$-dimensional Hausdorff measure. The assumption that $\partial\Omega$ is a countably $(n-1)$-rectifiable set means that the set Ir is empty. However, even in this case the countably rectifiable set $\partial\Omega$ can be essentially larger than $\partial^* E$.

This situation can be explained by the following example. Consider an open disk in a plane with a sequence of intervals I_i removed. Suppose that the union of these intervals is closed. The theorems of [3] on boundary traces are not applicable to the resulting region Ω (the intervals do not belong to the reduced boundary), but the boundary of Ω is a countably 1-rectifiable set.

Note by the way that even for a smooth function on Ω, its limits at the points of the intervals I_i from the right and from the left can be different, so that it is reasonable to introduce traces with two different values at some points.

Notation. We denote by $A \triangle B$ the symmetric difference $(A \setminus B) \cup (B \setminus A)$ of A and B. Let H_k denote k-dimensional Hausdorff measure and $\mathrm{Vol}(A)$ the Lebesgue measure of $A \subset \mathbb{R}^n$, or equivalently, its n-dimensional Hausdorff measure.

The dimension $k = n-1$ will play a specific role; for brevity, we denote $H_{n-1} = \mu$. In what follows, the words "almost all", "measurable", etc., will be used with respect to either H_n or $H_{n-1} = \mu$, which will be clear from the context.

Let $B_p(r)$ denote the open ball of radius r centered at p, and $\bar{B}_p(r)$ its closure.

$\Theta_A(p,k)$ denotes the density with respect to the measure H_k of a set A at p, i.e.,

$$\Theta_A(p,k) = \lim_{r\to 0} v_k^{-1} r^{-k} H_k(A \cap B_p(r)),$$

where v_k is the volume of the unit ball in $\mathbb{R}^k$. Note that, in the paper, we mostly use *one-sided* densities rather than densities; see the next section.

Countably rectifiable sets. There are several equivalent definitions of countably (k, H_k)-rectifiable sets. A detailed exposition can be found in Federer's monograph [8, Chapter 3]; more specifically, see Subsections 3.2.19, 3.2.25, 3.2.29 therein.

The following definition is most convenient for our purposes

Definition 1. A measurable set $A \subset \mathbb{R}^n$ is said to be *countably* (k, H_k)*-rectifiable* if there exists a sequence of C^1-smooth k-dimensional surfaces M_i, $i = 1, 2, \dots$, such that $A = \bigcup_{i=0}^{\infty} A_i$, where $\mu(A_0) = 0$ and $A_i \subset M_i$ for $i > 0$. Moreover, the sets A_i can be chosen so that

$$(1) \qquad \Theta_A(p,k) = 1, \qquad \Theta_{(A\setminus A_i)}(p,k) = 0$$

for almost all $p \in A_i$.

We only need the case where $k = n-1$, and so the countably $(n-1, \mu)$-rectifiable sets will be called *countably rectifiable* for short.

For any countably rectifiable set A, there is a so-called approximative tangent $(n-1)$-plane T_pA, which exists a.e. and coincides with the tangent plane to M_i at p. A point at which T_pA exists and, moreover, (1) is true is called a regular point. Thus, almost all (with respect to μ) points of A are regular. We drop the definition of T_pA because we only need the following property: for every sequence of positive numbers $r_j \to 0$, there exist positive numbers $\epsilon_j \to 0$ such that

$$(2) \qquad \lim_{r_j\to 0} r^{1-n} \mu((B_p(r_j) \setminus L_{r_j\epsilon_j}) \cap A) = 0,$$

where L_δ is the δ-neighborhood of T_pA. If ν is a normal to T_pA at p, we say that ν is a normal to A at p.

Functions. As usual, $BV(\Omega)$ means the class of functions locally integrable in Ω and such that their gradients are vector charges. We denote by $\chi(E)$ the characteristic function of E and by $P_\Omega(E)$ the perimeter of E with respect to Ω, i.e., $P_\Omega(E) = \|\chi_E\|_{BV(\Omega)}$. (We use the notation $\|f\|_{BV(\Omega)} = \operatorname{var}\operatorname{grad} f(\Omega)$.) For more details, see [5, 3, 10, 4].

We shall need the Fleming–Rishel formula (see [9])

$$\|f\|_{BV(\Omega)} = \int_{-\infty}^{\infty} P_\Omega(E_t)\,dt, \tag{3}$$

where $f \in BV(\Omega)$ and $E_t = \{x \mid f(x) > t\}$, and also the following formula closely related to (3):

$$\nabla f(E) = \int_{-\infty}^{+\infty} \nabla\chi_{E_t}(E)\,dt, \tag{4}$$

where E is any measurable subset of Ω (see [3, Theorem 14] or [5, Lemma 6.6.5/1]).

Remark 1. We often consider sets E for which $P_\Omega(E) < \infty$. For instance, these can be sets E_t of points where a function f is greater than t. If the considerations are local, then the finite perimeter condition can be replaced by the requirement that a set $E \cap \Omega$ have *locally finite perimeter*, i.e., $P_{\Omega\cap Q}(E) < \infty$ for any bounded region Q.

§2. One-sided densities

Consider a measurable set $E \subset \mathbb{R}^n$. Let ν be a unit vector at a point $x \in \mathbb{R}^n$. Denote $B_x^\nu(r) = B_x(r) \cap \{y \mid (y-x)\nu \ge 0\}$. The limit

$$\Theta_E^\nu(x) = \lim_{r\to 0} 2v_n^{-1} r^{-n} H_n(B_x^\nu(r) \cap E)$$

is called the *one-sided density* of the set E at x with respect to ν.

The upper and lower one-sided densities $\overline{\Theta}_E^\nu(x)$, $\underline{\Theta}_E^\nu(x)$ are defined similarly as upper and lower limits. Now, let x be a regular point of a countably rectifiable set A. Then there are two normals to A at x and, accordingly, it is natural to consider two one-sided densities with respect to A, namely, $\Theta_E^\nu(x)$ and $\Theta_E^{-\nu}(x)$.

Often, we take for A the boundary of Ω, assuming that this boundary is a countably rectifiable set. In such cases, we usually assume that $E \subset \Omega$.

Remark 2. It is easily seen that if a set G is measurable and $\Theta_G^\nu(x) = 1$, then

$$\Theta_E^\nu(x) = \lim_{r\to 0} \frac{H_n(B_x^\nu(r) \cap G \cap E)}{H_n(B_x^\nu(r) \cap G)} = \lim_{r\to 0} 2v_n^{-1} r^{-n} H_n(B_x^\nu(r) \cap G \cap E). \tag{5}$$

The following statement is a simple consequence of the isoperimetric inequality for subsets of a ball.

Lemma 1. *Let E be a measurable set with finite perimeter, and let $Q = \{x \in \mathbb{R}^n \mid \sum x_i^2 < 1,\ a < x_n < 1\}$, where $a \le 1/2$. Then the following isoperimetric inequality holds true*:

$$\min\{H_n(Q \cap E),\, H_n(Q \setminus E)\} \le c_n P_Q(E \cap Q)^{\frac{n}{n-1}}, \tag{6}$$

where $c_n > 0$ depends only on the dimension.

Lemma 2. *Suppose the boundary of a region Ω is a countably rectifiable set. Then $\Theta_\Omega^\nu(x)$ equals either 1 or 0, and $\Theta_{\mathbb{R}^n\setminus\Omega}^\nu(x)$ equals either 1 or 0 at each regular point $x \in \partial\Omega$ and for every normal $\nu(x)$ to $\partial\Omega$.*

Note that $\Theta_\Omega^\nu(x)$ and $\Theta_\Omega^{-\nu}(x)$ are equal either to 1 or to 0, and any combination of 0 and 1 is possible. This can happen even on a set of positive μ-measure.

Proof. Let ν be a normal at a regular point $x \in \partial\Omega$. Consider semiballs $B_i^\nu = B_x^\nu(r_i)$, where $r_i \to +0$ as $i \to \infty$. Denote by C_i the intersection of the $\epsilon_i r_i$-neighborhood of the plane T_x with B_i^ν, $A_i = B_i^\nu \setminus C_i$. It is clear that $\mathrm{Vol}(C_i) < v_{n-1}\epsilon_i r_i^n$. By (2), the inequalities $P_{A_i}(\Omega) \le \mu(A_i \cap \partial\Omega) < \epsilon r_i^{n-1}$ are valid for large i and sufficiently small ϵ_i. Now the lemma follows immediately from the isoperimetric inequality (6) applied to the region A_i and the set $A_i \cap \Omega$. □

Example 1. Consider a sequence of small bubbles (disjoint round balls) $B_{x_i}(r_i)$ located in the open unit ball $B_0(1)$. It is easy to choose these bubbles in such a way that all the points $p \in S_0(1)$ are the limits of some subsequences of bubbles and, moreover, there is no other limit points. Also, suppose that the radii of these balls tend to zero so fast that $\sum_i r_i^{n-1} < \infty$. Define $\Omega = \bigcup B_{x_i}(r_i)$. The boundary of Ω is rectifiable. This set is not connected, but in dimensions $n > 2$, the bubbles can be connected by very thin tubules so that the new set Ω (completed with the tubules) become a region with rectifiable boundary. The sphere $S_0(1)$ belongs to the boundary of Ω. So, almost all points of this sphere are regular points of $\partial\Omega$. However, they do not belong to the reduced boundary of Ω; i.e., the set $S_0(1) \cap \partial^*\Omega$ is empty. Moreover, the bubbles can be chosen in such a way that at every point x of the sphere $S_0(1)$, the condition $\Theta_\Omega^\nu(x) = 0$ is fulfilled for every normal.

We denote by Γ the set of all points $x \in \partial\Omega$ such that $\Theta_\Omega^\nu(x) = 1$ for at least one normal ν. It is not difficult to show that $\partial^*\Omega \subset \Gamma$. Indeed, the vector ν_F is the normal in the sense of Federer if and only if $\Theta_\Omega^{-\nu_F}(x) = 1$ and $\Theta_\Omega^{\nu_F}(x) = 0$.

Remark 3. It is well known that $P(\Omega) = \mu(\partial^*\Omega)$. Recall that if $P(\Omega) < \infty$, then $\mathrm{var}\,\nabla\chi_\Omega(\partial\Omega \setminus \partial^*\Omega) = 0$ and

$$\nabla\chi_\Omega(E) = -\int_E \nu_F(x)\,\mu(dx) \tag{7}$$

for any measurable set $E \subset \partial^*\Omega$; see, e.g., [3, Theorem 6.2.2/1].

Lemma 3. *Any countably rectifiable set A can be equipped with a measurable field ν of* (*unit*) *normals.*

Proof. Up to a subset of measure 0, the set A is located on $(n-1)$-dimensional C^1-smooth manifolds M_i of some countable family. It is easily seen that almost each point $x \in A$ belongs to only one surface M_i. We orient every manifold M_i by a continuous field of normals. Since the approximative tangent plane to A at x coincides with the tangent plane T_xM_i and the intersection $A \cap M_i$ is measurable, we obtain a measurable field of normals to A by choosing normals $\nu(x)$ to M_i in the role of normals to A. □

Remark 4. It is clear that a measurable vector field of unit normals is not unique; there are infinitely many such vector fields. Let us fix some vector field ν as constructed in the proof of Lemma 3. It is not merely measurable: it is located on C^1-smooth surfaces M_i from a chosen family and is continuous along every such surface. Moreover, if a countably rectifiable set A is the boundary of a region Ω, $A = \partial\Omega$, then the vector field ν can be chosen so that, at the points $x \in \partial^*\Omega$, the vectors $\nu(x)$ are directed opposite to normals in the sense of Federer. A vector field having such properties is said to be *standard.*

Lemma 4. *Let A be a countably rectifiable set, ν a measurable field of normals to A, and E a measurable subset of $\mathbb{R}^n$. Then the sets $\{x \in A \mid \Theta_E^\nu(x) = 1\}$ and $\{x \in A \mid \Theta_E^\nu(x) = 0\}$ are measurable.*

Proof. First, assume that the vector field ν is standard and a family of surfaces $\{M_i\}$

is chosen as in Remark 4. The sets $M_i \cap A$ are measurable. The functions $\phi_i^r(x) = 2v_n^{-1}r^{-n}H_n(B_x^\nu(r) \cap E)$ defined on $M_i \cap A$ are continuous. In particular, they are measurable. We extend these functions to A by zero. Their sum $\phi^r = \sum_i \phi_i^r$, defined on A, is also measurable. Therefore, the functions $\underline{\phi}(x) = \liminf_{r\to 0} \phi^r(x)$ and $\overline{\phi}(x) = \limsup_{r\to 0} \phi^r(x)$ are measurable, so that the sets

$$\{x \in A \mid \Theta_E^\nu(x) = 0\} = \{x \in A \mid \underline{\phi}(x) = 0\},$$
$$\{x \in A \mid \Theta_E^\nu(x) = 1\} = \{x \in A \mid \overline{\phi}(x) = 1\}$$

are measurable. The same is true for the field $-\nu$ as well. Now, let $\tilde{\nu}$ be any measurable unit vector field of normals to $\partial\Omega$. Then the sets $\{\nu = \tilde{\nu}\}$ and $\{-\nu = \tilde{\nu}\}$ are measurable, and thereby, the sets $\{x \in A \mid \Theta_E^{\tilde{\nu}}(x) = 0\}$ and $\{x \in A \mid \Theta_E^{\tilde{\nu}}(x) = 1\}$ are also measurable. □

Let A be a countably rectifiable set, let $P(E) < \infty$, and let ν be a normal to A at x. Denote

$$\begin{aligned} &\partial_A^\nu E = \{x \in A \mid \Theta_E^\nu(x) = 1\}, \\ &\partial_A^1 E = (\partial_A^\nu E) \cup (\partial_A^{-\nu} E), \qquad \partial_A^2 E = (\partial_A^\nu E) \cap (\partial_A^{-\nu} E). \end{aligned} \tag{8}$$

Roughly speaking, $\partial_A^1 E$ is the set of points of A such that E "adjoins" A with one-sided density 1 at least from one side, and $\partial_A^2 E$ is the part of A such that E "adjoins" A with one-sided density 1 from both sides.

Observe that

$$\Gamma = \partial_{\partial\Omega}^1 \Omega, \qquad \partial_{\partial\Omega}^\nu E = \partial_\Gamma^\nu E. \tag{9}$$

We shall use Lemma 6.6.3/1 from [5] (or, what is the same, Lemma 13 from [3]). The lemma is about the trace of a characteristic function. Since the notion of the trace will be introduced later, we formulate the lemma in a convenient form.

Lemma 5. *Suppose $P(\Omega) < \infty$, $E \subset \Omega$, and $P_\Omega(E) < \infty$. Then for almost all $x \in \partial^*\Omega$ we have*

$$\chi_{\partial^* E}(x) = \lim_{r\to 0} \frac{\int_{B_x(r)} \chi_E \, dx}{\mathrm{Vol}(B_x(r) \cap \Omega)} = \lim_{r\to 0} \frac{\mathrm{Vol}(B_x(r) \cap E)}{\mathrm{Vol}(B_x(r) \cap \Omega)}. \tag{10}$$

Of course, only the first identity is essential (the second identity is trivial).

Remark 5. In Lemma 5, the condition $E \subset \Omega$ can be dropped if we replace E by $E \cap \Omega$ and the condition $P_\Omega(E) < \infty$ by $P(E) < \infty$.

The following lemma plays a key role in our subsequent considerations.

Lemma 6. *Suppose A is a countably rectifiable set, ν is a measurable field of normals along A, and $P(E) < \infty$. Then the one-sided densities $\Theta_E^\nu(x)$ equal either* 0 *or* 1 *μ-almost everywhere on A.*

Proof. It suffices to prove the lemma for standard normal vector fields and only for regular points of A (see Lemma 3 and Remark 4).

1. First, let A be a C^1-smooth $(n-1)$-dimensional manifold M. Since our statement is local, we may assume that M divides some neighborhood of it, bounded by a smooth hypersurface, into two half-neighborhoods, Ω_1 and Ω_2. Set $E_i = \Omega_i \cap E$, $i = 1, 2$. It is clear that $P(E_i) < \infty$.

Note that $\chi_{\partial^* E_1}(x)$ equals 1 if $x \in \partial^* E_1 \cap M$ and equals 0 if $x \in M \setminus \partial^* E_1$. Therefore, applying Lemma 5 with $E = E_1$ and $\Omega = \Omega_1$ and Remark 2 with $G = \Omega_1$, we see that for almost all points $x \in M$, the one-sided density $\Theta^\nu_{E_1}(x)$ is equal to either 0 or 1, where ν is the normal to M directed to Ω_1. The same is true for E_2 and Ω_2. Finally, since

$$1 \geq \Theta^\nu_E(x) = \Theta^\nu_{E_1}(x) + \Theta^\nu_{E_2}(x),$$

we see that the lemma is proved for $A = M$.

2. Passing to the general case, let $\{M_i\}$ be a family of C^1-smooth submanifolds mentioned in the definition of the standard normal fields. In part 1, the lemma was proved for each M_i. The intersection $A \cap M_i$ is μ-measurable, and the one-sided density at a point depends on ν and E only. Thus, $\Theta^\nu_E(x)$ equals either 0 or 1 almost everywhere on $A \cap M_i$. Since A coincides with the union of the sets $A \cap M_i$ up to a set of measure 0, the lemma is proved. □

Corollary 1. *Let Ω be a region such that its boundary is a countably rectifiable set. If $E \subset \Omega$ and $P(E) < \infty$, then for any (measurable) field ν of normals to $\partial\Omega$, the one-sided densities Θ^ν_E are equal almost everywhere to either 0 or 1.*

Now it is clear that for the reduced boundary of any set E with $P(E) < \infty$, we have

$$A \cap \partial^* E = (\partial^1_A E) \setminus (\partial^2_A E). \tag{11}$$

In particular,

$$\partial^* \Omega = (\partial^1_\Gamma \Omega) \setminus (\partial^2_\Gamma \Omega). \tag{12}$$

Corollary 2. *Let x be a regular point of $\partial\Omega$. Suppose that $\Theta^\nu_{G_1}(x) = \Theta^\nu_{G_2}(x) = 1$ for some sets G_1, G_2. Moreover, assume that there is a family of sets $\mathcal{B}^\nu_x(r)$ such that*

$$B_x(\rho_1(r)) \cap G_1 \subset \mathcal{B}^\nu_x(r) \subset B_x(\rho_2(r)) \cap G_2, \tag{13}$$

where $\rho_2(r) \to 0$ as $r \to 0$. Then

$$\Theta^\nu_E(x) = \lim_{r \to 0} \frac{H_n(\mathcal{B}^\nu_x(r) \cap E)}{H_n(\mathcal{B}^\nu_x(r))} \tag{14}$$

for any set $E \subset \mathbb{R}^n$ with finite perimeter.

This corollary allows us to consider one-sided densities for sets with finite perimeters in any C^1-smooth manifold with a continuous metric tensor. Therefore, the further considerations can be applied not only to $\mathbb{R}^n$ but also to any such manifold.

§3. Trace on a countably rectifiable set

Here we define the trace on a countably rectifiable set for a function defined on Ω. Within this section, we do not require the function to belong to $BV(\Omega)$. Instead, we only suppose that the sets $E_t = \{x \in \Omega \mid f(x) > t\}$ have finite perimeters for almost all t. Such functions are said to be *BV-similar*. (As was mentioned in Remark 1, it would suffice to assume that the sets E_t have locally finite perimeter.)

Let a countably rectifiable set A be contained in the closure $\bar{\Omega}$ of a region Ω. We define the trace[1] $f^\nu(x)$ with respect to normal ν at $x \in \partial^\nu_A \Omega$ for a *BV-similar* function f as follows:

$$f^\nu(x) = \sup\{t \mid x \in \partial^\nu_A E_t\}.$$

We can suppose (this changes nothing) that the supremum is taken only over t such that $P(E_t) < \infty$. Moreover, we agree that $\sup \varnothing = -\infty$.

We emphasize that the trace is defined not everywhere on A. However, if we extend f to all of $\mathbb{R}^n$ (for instance, by a constant) so that $A = \partial^\nu_A(\mathbb{R}^n \setminus A)$, then f^ν is defined on A everywhere.

[1]Our terminology differs from that in [3, 5]. Namely, we use the terms trace and average trace instead of rough trace and trace.

In the case where $x \in \partial^2_A\Omega$, we also define the upper and lower traces by the formulas

$$f^*(x) = \max\{f^\nu(x), f^{-\nu}(x)\}, \quad f_*(x) = \min\{f^\nu(x), f^{-\nu}(x)\}.$$

If $x \in A \cap \partial^*\Omega = (\partial^1_A\Omega) \setminus (\partial^2_A\Omega)$, we put $f^*(x) = f^\nu(x)$, where $-\nu$ is the normal in the sense of Federer. In this case we do not define $f_*(x)$. However, if f is extended to the entire $\mathbb{R}^n$ (for instance, by a constant), then $A = \partial^2_A(\mathbb{R}^n \setminus A) = \partial^1_A(\mathbb{R}^n \setminus A)$, and the upper and lower traces are defined everywhere on A.

It is clear that $f^*(x) = \sup\{t \mid x \in \partial^1_\Gamma E_t\}$, $f_*(x) = \sup\{t \mid x \in \partial^2_\Gamma E_t\}$.

Lemma 7. *Let $A \subset \bar\Omega$ be a countably rectifiable set and ν a measurable field of normals to A. Then, for any BV-similar function f, its trace f^ν on $\partial^\nu_A\Omega$ is measurable, and*

$$\mu(\{x \in \partial^\nu_A\Omega \mid f^\nu(x) \geq t\}) = \mu(\partial^\nu_A E_t) \tag{15}$$

for almost all $t \in \mathbb{R}$.

Remark 6. 1) By analogy with Lemma 7, it can be proved that the traces f^* and f_* are also measurable, and

$$\mu(\{x \in \partial^1_\Gamma\Omega \mid f^*(x) \geq t\}) = \mu(\partial^1_\Gamma E_t), \tag{16}$$

$$\mu(\{x \in \partial^2_\Gamma\Omega \mid f_*(x) \geq t\}) = \mu(\partial^2_\Gamma E_t). \tag{17}$$

2) In fact, instead of (15) we shall prove that

$$\mu(\{x \in \partial^\nu_A\Omega \mid f^\nu(x) \geq t\} \,\triangle\, \partial^\nu_A E_t) = 0$$

for all t except for a countable subset.

3) Note that in (15)–(17) nonstrict inequalities can be replaced by strict ones.

Proof. Denote $B_t = \{x \in \partial^\nu_A\Omega \mid f^\nu(x) \geq t\}$, $Y_t = \partial^\nu_A E_t$, and $X_t = B_t \setminus Y_t$. It is easily seen that $B_t \supset Y_t$. Thus, it remains to prove that $\mu(X_t) = 0$.

The sets Y_t are measurable, and the sets X_t are disjoint. It is not difficult to show that the inclusions $Y_{t_0} \supset Y_{t_1}$ and $Y_{t_0} \cup X_{t_0} \supset Y_{t_1} \cup X_{t_1}$ are valid for $t_0 < t_1$. The latter inclusion implies that $Y_{t_0} \supset X_{t_1}$. So,

$$\Big(\bigcap_{t<t_1} Y_t\Big) \setminus Y_{t_1} \supset X_{t_1}.$$

On the other hand, the sets $(\bigcap_{t<t_1} Y_t) \setminus Y_{t_1}$ are measurable and disjoint. Therefore, $\mu\big((\bigcap_{t<t_1} Y_t) \setminus Y_{t_1}\big) = 0$ for almost all $t_1 \in \mathbb{R}$. It follows that the sets X_t are subsets of some measure zero sets for almost all $t \in \mathbb{R}$. In particular, they are measurable. Hence, the sets B_t are measurable. □

Lemma 8. *Suppose $A \subset \bar\Omega$ is a countably rectifiable set and f a BV-similar function. Then*

$$-f^\nu(x) = (-f)^\nu(x) \tag{18}$$

for almost all $x \in \partial^\nu_A\Omega$.

Proof. Lemma 8 is equivalent to saying that

$$\sup\{t \mid x \in \partial^\nu_A E_t\} = \inf\{t \mid x \in \partial^\nu_A(\Omega \setminus E_t)\}$$

for almost all $x \in A$. This means that

$$\sup\{t \mid \underline{\Theta}^\nu_{E_t}(x) = 1\} = \inf\{t \mid \underline{\Theta}^\nu_{(\Omega\setminus E_t)}(x) = 1\}.$$

In its turn, the latter identity is equivalent to

$$\sup\{t \mid \underline{\Theta}^\nu_{E_t}(x) = 1\} = \inf\{t \mid \overline{\Theta}^\nu_{E_t}(x) = 0\}.$$

We denote by L and R the left and the right sides of the last identity. It is not difficult to show that the functions $\overline{\Theta}^{\nu}_{E_t}(x)$ and $\underline{\Theta}^{\nu}_{E_t}(x)$ are monotone nonincreasing in t. Therefore, $L \leq R$. Consider the set of all points x such that $L(x) < R(x)$. It suffices to prove that the μ-measure of this set is zero.

For this, we choose a countable everywhere dense set $\{t_i\}_{i=1}^{\infty}$ such that $P(E_{t_i}) < \infty$. If $L(x) < R(x)$, then there exists t_i such that $L(x) < t_i < R(x)$. Now our assertion follows from Lemma 6 applied to the set E_{t_i}. □

Corollary 3. *For any BV-similar function f and for almost all $x \in A$, we have*

$$(f^{\nu})^{+} = (f^{+})^{\nu}, \qquad (f^{\nu})^{-} = (f^{-})^{\nu}. \tag{19}$$

Proof. The first formula can be derived directly from the definitions. The second follows from Lemma 8. Indeed,

$$(f^{-})^{\nu} = ((-f)^{+})^{\nu} = ((-f)^{\nu})^{+} = (-(f^{\nu}))^{+} = (f^{\nu})^{-}.$$ □

Lemma 9. *For any BV-similar functions f, g and almost all $x \in A$, we have*

$$(f+g)^{\nu}(x) = f^{\nu}(x) + g^{\nu}(x). \tag{20}$$

Proof. First, we prove that $(f+g)^{\nu}(x) \geq f^{\nu}(x) + g^{\nu}(x)$ for all $x \in \Gamma$. Indeed, let numbers $F < f^{\nu}(x)$ and $G < g^{\nu}(x)$ be such that the sets $E^{f}_{F} = \{x \mid f(x) > F\}$ and $E^{g}_{G} = \{x \mid g(x) > G\}$ have finite perimeters. Then $\Theta^{\nu}_{E^{f}_{F}}(x) = 1$ and $\Theta^{\nu}_{E^{g}_{G}}(x) = 1$.

Denote $W = E^{f+g}_{F+G}$. We have

$$W = \{x \mid f(x) + g(x) > F + G\} \supset E^{f}_{F} \cap E^{g}_{G}.$$

Therefore, $\Theta^{\nu}_{W}(x) = 1$, whence

$$(f+g)^{\nu}(x) = \sup\{t \mid \Theta^{\nu}_{E^{f+g}_{t}} = 1\} \geq F + G.$$

Passing to the limits as $F \to f^{\nu}(x)$ and $G \to g^{\nu}(x)$, we get

$$(f+g)^{\nu}(x) \geq f^{\nu}(x) + g^{\nu}(x).$$

Now we can derive the reverse inequality with the help of Lemma 8. Indeed, for almost all $x \in A$ we have:

$$-(f+g)^{\nu}(x) = ((-f) + (-g))^{\nu}(x) \geq (-f)^{\nu}(x) + (-g)^{\nu}(x) = -f^{\nu}(x) - g^{\nu}(x).$$ □

Lemma 10. *Suppose a function $\phi : \mathbb{R} \to \mathbb{R}$ is monotone increasing and left continuous. If the functions f and $\phi \circ f$ are BV-similar, then*

$$(\phi \circ f)^{\nu}(x) = \phi(f^{\nu}(x)) \tag{21}$$

for almost all $x \in A$.

Proof. The lemma follows easily from the fact that

$$\{x \in \Omega \mid (\phi \circ f(x)) \geq \phi(t)\} = \{x \in \Omega \mid f(x) \geq t\}.$$ □

Remark 7. 1) Suppose that the Hausdorff measure $H_1(\phi^{-1}(E))$ is equal to 0 for any set E of measure 0. Then the statement that $\phi \circ f$ is BV-similar implies that the function f is BV-similar. This assertion is definitely true if (locally) $|\phi(x) - \phi(y)| \geq \mathrm{const}|x - y|$. The last condition is obviously fulfilled if $\phi \in C^1$ and $\phi' \neq 0$.

2) In the lemma, the condition that ϕ is monotone increasing can be replaced by the assumption that the set $\phi^{-1}((t, +\infty))$ is a finite union of intervals and rays for almost all t.

Lemma 11. *If functions f, g, and fg are BV-similar, then*

$$(fg)^{\nu}(x) = f^{\nu}(x)g^{\nu}(x). \tag{22}$$

for almost all $x \in \Gamma$.

Proof. It suffices to prove (22) for $f, g \geq 1$. This follows from Lemma 8, Corollary 3, and the relation $f = (f^+ + 1) - (f^- + 1)$.

In this case, we can use Lemma 9, Lemma 10, and Remark 7 as follows:

$$(fg)^{\nu} = \left(e^{\ln(fg)}\right)^{\nu} = e^{(\ln f + \ln g)^{\nu}} = e^{(\ln f)^{\nu} + (\ln g)^{\nu}} = e^{\ln(f^{\nu}) + \ln(g^{\nu})} = f^{\nu} g^{\nu}. \qquad \square$$

§4. Integral formula for the norm of the trace

Definition 2. We define the norm of the trace on $\partial\Omega$ of a function $f \in BV(\Omega)$ as follows:

$$\|f\|_{\Gamma} = \int_{\partial^*\Omega} |f^*|\, d\mu + \int_{\partial^2_{\Gamma}\Omega} (f^* - f_*)\, d\mu. \tag{23}$$

If $\|f\|_{\Gamma} < \infty$, we say that f has summable trace.

Lemma 12.

$$\|f\|_{\Gamma} = \|f^+\|_{\Gamma} + \|f^-\|_{\Gamma}. \tag{24}$$

Proof. We have

$$\begin{aligned} f^* - f_* &= |f^{\nu} - f^{-\nu}| = |(f^+)^{\nu} - (f^-)^{\nu} - (f^+)^{-\nu} + (f^-)^{-\nu}| \\ &= |(f^+)^{\nu} - (f^+)^{-\nu}| + |(f^-)^{-\nu} - (f^-)^{-\nu}| \\ &= ((f^+)^* - (f^+)_*) + ((f^-)^* - (f^-)_*). \end{aligned} \qquad \square$$

Lemma 13. *Suppose that a function $f \in BV(\Omega)$ is nonnegative and has summable trace on $\partial\Omega$. Moreover, let $\eta\colon \Gamma \to \mathbb{R}^k$, $k \geq 1$, be a measurable and bounded vector-valued function. Then*

$$\int_0^{+\infty} \int_{\Gamma \cap \partial^* E_t} \eta\, d\mu\, dt = \int_{\partial^*\Omega} f^* \eta\, d\mu + \int_{\partial^2_{\Gamma}\Omega} (f^* - f_*)\eta\, d\mu. \tag{25}$$

Proof. Clearly, it suffices to consider only the case where $k = 1$. Define

$$\begin{aligned} &\{x \in \partial^1_{\Gamma}\Omega \mid f^* > t\} = E^1_t, \quad \{x \in \partial^2_{\Gamma}\Omega \mid f^* > t\} = E^2_t, \\ &\{x \in \partial^2_{\Gamma}\Omega \mid f_* > t\} = L^2_t, \quad \{x \in \partial^*\Omega \mid f^* > t\} = E^*_t. \end{aligned}$$

By (11) and Lemma 7, we have

$$\begin{aligned} \int_0^{+\infty} \int_{\Gamma \cap \partial^* E_t} \eta\, d\mu\, dt &= \int_0^{+\infty} \Big(\int_{\Gamma \cap \partial^1_{\Gamma} E_t} \eta\, d\mu - \int_{\Gamma \cap \partial^2_{\Gamma} E_t} \eta\, d\mu \Big)\, dt \\ &= \int_0^{+\infty} \Big(\int_{E^1_t} \eta\, d\mu - \int_{L^2_t} \eta\, d\mu \Big)\, dt \\ &= \int_0^{+\infty} \Big(\int_{E^*_t} \eta\, d\mu + \int_{E^2_t} \eta\, d\mu - \int_{L^2_t} \eta\, d\mu \Big)\, dt \\ &= \int_{\partial^*\Omega} f^* \eta\, d\mu + \int_{\partial^2_{\Gamma}\Omega} (f^* - f_*)\eta\, d\mu. \end{aligned} \qquad \square$$

Corollary 4. *If a function $f \in BV(\Omega)$ is nonnegative, then*

$$\|f\|_\Gamma = \int_0^{+\infty} \mu(\Gamma \cap \partial^* E_t)\, dt. \tag{26}$$

Moreover, f has summable trace if and only if the right-hand side of (26) *is finite.*

Indeed, if $\|f\|_\Gamma < \infty$, then we can obtain (26) by substituting $\eta = 1$ in (25). Now, suppose that the quantity on the right in (26) is finite. Then, to prove (26), it suffices to substitute $\eta = 1$ in the final identities in the proof of Lemma 12 and read them from right to left.

§5. Summability of traces and integral inequalities

In this and the next sections, we are going to show that, actually, all the integral inequalities and other results on traces obtained in [3, 5] can be generalized to the case where the boundary of a region is a countably rectifiable set. As a great variety of integral inequalities were obtained in [5], we restrict ourselves only to key examples.

For a set $A \subset \bar{\Omega}$, denote by τ_A the infimum of the numbers β such that $\mu(\partial^* E \cap \Gamma) \leq \beta\mu(\partial^* E \cap \Omega)$ for all $E \subset \Omega$ satisfying

$$\mathrm{Vol}(A \cap E) + \mu(A \cap \partial^* E) = 0.$$

Note that τ_A goes to infinity as A vanishes. Indeed, we can set $E = \Omega \setminus A$.

The following theorem generalizes Theorem 6.5.3/1 in [5].

Theorem 1. *Suppose the boundary $\partial\Omega$ of a region Ω is a countably rectifiable set and D is a subset of $\bar{\Omega}$. Then for any function $f \in BV(\Omega)$ satisfying $f(A \cap \Omega) = 0$, $f^*(A \cap \Gamma) = 0$, we have*

$$\|f\|_\Gamma \leq \tau_A \|f\|_{BV(\Omega)} \tag{27}$$

and the constant τ_A is sharp.

Proof. We may assume that $\|f\|_{BV(\Omega)} < \infty$. Suppose for a while that $f \geq 0$. Note that $\mathrm{Vol}(A \cap E_t) + \mu(A \cap \partial^* E_t) = 0$ for almost all $t > 0$. Then, by (26) and the definition of τ_A, we have

$$\|f\|_\Gamma = \int_0^{+\infty} \mu(\Gamma \cap \partial^* E_t)\, dt \leq \tau_A \int_0^{+\infty} P_\Omega(E_t)\, dt = \tau_A \|f\|_{BV(\Omega)}. \tag{28}$$

If f is not necessarily nonnegative, we apply Lemma 24 to obtain

$$\|f\|_\Gamma = \|f^+\|_\Gamma + \|f^-\|_\Gamma \leq \tau_A(\|f^+\|_{BV(\Omega)} + \|f^-\|_{BV(\Omega)}) = \tau_A \|f\|_{BV(\Omega)}. \tag{29}$$ □

The next theorem generalizes Theorem 6.5.4/1 in [5].

Theorem 2. *Suppose that the boundary of a region Ω is a countably rectifiable set. Then any function $f \in BV(\Omega)$ satisfies the inequality*

$$\|f\|_\Gamma \leq k(\|f\|_{BV(\Omega)} + \|f\|_{L(\Omega)}) \tag{30}$$

with a constant k independent of f if and only if there exists a constant $\delta > 0$ such that

$$\mu(\partial^* E \cap \partial^* \Omega) \leq k_1 P_\Omega(E) \tag{31}$$

for every measurable set $E \subset \Omega$ with $\operatorname{diam} E \leq \delta$, where the constant k_1 does not depend on E.

To prove the "only if" part, it suffices to put $f = \chi_E$ in (30). The "if" part can be deduced from Theorem 1 with the help of a partition of unity.

Theorem 4 in [3] (or, what is the same, Theorem 6.5.2(1) in [5]) can be naturally generalized to the case of regions with countably rectifiable boundary; it takes the following form.

Theorem 3. *Let the boundary of a region Ω be a countably rectifiable set. Then the inequality*

$$\inf_c \{\|f-c\|_\Gamma\} \le k\|f\|_{BV(\Omega)} \tag{32}$$

is satisfied with a constant k independent of $f \in BV(\Omega)$ if and only if

$$\min\big\{\mu(\Gamma\cap\partial^*E),\ \mu(\Gamma\cap\partial^*(\Omega\setminus E))\big\} \le kP_\Omega(E) \tag{33}$$

for each set $E\subset\Omega$ with finite perimeter.

Proof. First, observe (cf. (11)) that

$$\mu(\Gamma\cap\partial^*E) = \mu(\partial^*\Omega\cap\partial^1_\Gamma E) + \mu(\partial^2_\Gamma\Omega\cap\partial^*E), \tag{34}$$

$$\mu(\Gamma\cap\partial^*(\Omega\setminus E)) = \mu(\partial^*\Omega\setminus\partial^1_\Gamma E) + \mu(\partial^2_\Gamma\Omega\cap\partial^*E). \tag{35}$$

The "only if" part. Suppose $E\subset\Omega$ and $P_\Omega(E)<\infty$. For the characteristic function χ_E of the set E we have

$$\begin{aligned}
kP_\Omega(E) &= k\|\chi_E\|_{BV(\Omega)}\\
&\ge \inf_c\Big\{\int_{\partial^*\Omega}|(\chi_E)^*(x)-c|\,d\mu(x) + \int_{\partial^2_\Gamma\Omega}\big((\chi_E)^*(x)-(\chi_E)_*(x)\big)\,d\mu(x)\Big\}\\
&= \min_c\big\{|1-c|\mu(\partial^*\Omega\cap\partial^1_\Gamma E) + |c|\mu(\partial^*\Omega\setminus\partial^1_\Gamma E) + \mu(\partial^2_\Gamma\Omega\cap\partial^*E)\big\}\\
&= \min\big\{\mu(\partial^*\Omega\cap\partial^1_\Gamma E),\ \mu(\partial^*\Omega\setminus\partial^1_\Gamma E)\big\}) + \mu(\partial^2_\Gamma\Omega\cap\partial^*E)\big\}.
\end{aligned}$$

Combined with (34) and (35), this proves (33).

The "if" part. If $\|f\|_{BV(\Omega)}<\infty$, then $P(E_t)<\infty$ for almost all t. Using (33)–(35) and the Fleming–Rishel formula (3), we get

$$\begin{aligned}
k\|f\|_{BV(\Omega)} &= k\int_{-\infty}^{+\infty} P_\Omega(E_t)\,dt\\
&\ge \int_{-\infty}^{+\infty}\Big(\min\big\{\mu(\partial^*\Omega\cap\partial^1_\Gamma E_t),\mu(\partial^*\Omega\setminus\partial^1_\Gamma E_t)\big\} + \mu(\partial^2_\Gamma\Omega\cap\partial^*E_t)\Big)\,dt.
\end{aligned} \tag{36}$$

Denote $t_0 = \sup\big\{t \mid \mu(\partial^*\Omega\cap\partial^1_\Gamma E_t) \ge \mu(\partial^*\Omega\setminus\partial^1_\Gamma E_t)\big\}$ and observe that $\mu(\partial^*\Omega\cap\partial^1_\Gamma E_t)$ does not increase in t and $\mu(\partial^*\Omega\setminus\partial^1_\Gamma E_t)$ does not decrease in t. Hence, by (26) we obtain

$$\begin{aligned}
k\|f\|_{BV(\Omega)} &\ge \int_{t_0}^{+\infty}\mu(\Gamma\cap\partial^*E_t)\,dt + \int_{-\infty}^{t_0}\mu(\Gamma\cap\partial^*(\Omega\setminus E_t))\,dt\\
&= \|(f-c)^+\|_\Gamma + \|(f-c)^-\|_\Gamma = \|f-c\|_\Gamma.
\end{aligned}$$

So, (33) is true and the theorem is proved. □

§6. Extension of a function in $BV(\Omega)$ to the entire space by a constant

Everywhere in this section we assume that $P(\Omega)<\infty$ and $\partial\Omega$ is a countably rectifiable set.

Let f be a function defined in a region $\Omega\subset\mathbb{R}^n$. Denote by $f_c\colon\mathbb{R}^n\to\mathbb{R}$ the function defined by the condition $f_c(x)=f(x)$ for $x\in\Omega$ and $f_c(x)=c$ for $x\notin\Omega$, where c is a constant.

Lemma 14. *We have*

$$\|f_c\|_{BV(\mathbb{R}^n)} = \|f\|_{BV(\Omega)} + \|f-c\|_\Gamma. \tag{37}$$

Proof. Without loss of generality we may assume that $c=0$; indeed, it suffices to consider $f-c$ in place of f. Identity (24) allows us to assume that $f \geq 0$. As usual, we set $E_t = \{x \in \Omega \mid f_0 > t\}$. Now, by (3) and (26), we have

$$\begin{aligned}\|f_0\|_{BV(\mathbb{R}^n)} &= \int_0^{+\infty} P(\{x \in \mathbb{R}^n \mid f_0 > t\})\,dt \\ &= \int_0^{+\infty} \Big(P_\Omega(E_t) + \mu(\Gamma \cap \partial^* E_t)\Big)\,dt = \|f\|_{BV(\Omega)} + \|f\|_\Gamma. \qquad \square\end{aligned}$$

It may be asked whether it is possible to enlarge Ω by removing $\partial^2_\Gamma\Omega$, and thus reducing our case to that where normals in the sense of Federer exist almost everywhere on $\partial\Omega$. Sometimes this is possible. For instance, let $\Omega = D^2 \setminus \bigcup_{i=1}^\infty I_i$ be the disk with a sequence of intervals removed in such a way that the sum of the lengths of the I_i is finite. Then every $f \in BV(\Omega)$ such that

$$\int_{\bigcup_{i=1}^\infty I_i} (f^* - f_*) < \infty$$

extends to a function $\widetilde{f} \in BV(D^2)$. Unfortunately, a slightly more complicated example shows that this is not necessarily the case.

Example 2. Denote by $K \subset [0,1]$ a Cantor set of positive length. We define a region Ω as follows:

$$\Omega = B_{(0,0)}(2) \setminus \{(x,y) \mid x \in [0,1],\ |y| \leq (\operatorname{dist}(x,K))^2\}. \tag{38}$$

It is not difficult to show that both one-sided densities equal one at all points of the set $K \times \{0\}$, and $\partial^2_\Gamma\Omega$ is merely the set of such points. Nevertheless, it is impossible to enlarge Ω so as to include this set in the region.

§7. Embedding theorems

The following theorem is a direct generalization of Theorem 6.5.7/1 in [5].

Theorem 4. *Suppose that $\partial\Omega$ is a countably μ-rectifiable set. Then for every $f \in BV(\Omega)$ we have*

$$\left[\int_\Omega f^{\frac{n}{n-1}}\,dx\right]^{\frac{n-1}{n}} \leq n c_n^{-\frac{1}{n}} \left\{\|f\|_{BV(\Omega)} + \|f\|_\Gamma\right\}, \tag{39}$$

and the constant $nc_n^{-\frac{1}{n}}$ is sharp.

Proof. By Corollary 3 and Lemma 9, we may assume that $f \geq 0$. Precisely as in Theorem 7 in [3], we get

$$\left[\int_\Omega |f|^{\frac{n}{n-1}}\,dx\right]^{\frac{n-1}{n}} \leq \int_0^{+\infty} H_n(E_t)^{\frac{n-1}{n}}\,dt, \tag{40}$$

where, as usual, $E_t = \{x \in \Omega \mid f(x) > t\}$.

The isoperimetric inequality shows that

$$H_n(E_t)^{\frac{n-1}{n}} \leq n c_n^{-\frac{1}{n}} P_{\mathbb{R}^n}(E_t) = n c_n^{-\frac{1}{n}} \left[P_\Omega(E_t) + \mu(\Gamma \cap \partial^*(E_t))\right]. \tag{41}$$

Now relations (41) and (26) imply that

$$n^{-1}c_n^{\frac{1}{n}}\left[\int_\Omega |f|^{\frac{n}{n-1}}\,dx\right]^{\frac{n-1}{n}} \le \int_{-\infty}^{+\infty} P_\Omega(E_t)\,dt + \int_0^{+\infty} \mu(\Gamma\cap\partial^*(E_t))\,dt$$
$$= \|f\|_{BV(\Omega)} + \|f\|_\Gamma. \qquad \square$$

Note that the multiplicative inequality 6.5.6 in [5] can also be generalized to our case.

§8. The Gauss–Ostrogradskiĭ formula

Theorem 5 (The Gauss–Ostrogradskiĭ formula)**.** *Let the boundary of a region Γ be a countably μ-rectifiable set. Assume that $\partial\Omega$ is equipped with a standard field ν of unit normals and that the trace of a function $f \in BV(\Omega)$ is summable. Then*

$$\nabla f(\Omega) = \int_{\partial^*\Omega} f^\nu(x)\nu(x)\,d\mu(x) + \int_{\partial^2_\Gamma\Omega} \big(f^\nu(x) - f^{-\nu}(x)\big)\nu(x)\,d\mu(x). \tag{42}$$

Proof. It suffices to prove (42) only for nonnegative functions f. Indeed, to prove the theorem in the general case, it suffices to apply (42) to f^+ and f^- and then refer to Corollary 3.

Obviously, the right-hand side of (42) does not depend on a choice of ν. Note that if $f^*(x) \ne f_*(x)$, then the normal to E_t in the sense of Federer at x exists for all $t \in (f_*(x), f^*(x))$ and does not depend on t. Therefore, we may assume that at each such point x, the normal $-\nu(x)$ coincides with the normal to E_t in the sense of Federer for $f_*(x) < t < f^*(x)$. If we choose normals ν in this way, formula (42) can be rewritten in the following form:

$$\nabla f(\Omega) = \int_{\partial^*\Omega} f^*(x)\nu(x)\,d\mu(x) + \int_{\partial^2_\Gamma\Omega} \big(f_*(x) - f^*(x)\big)\nu(x)\,d\mu(x). \tag{43}$$

Obviously, if $P(E) < \infty$, then $\nabla\chi_E(\mathbb{R}^n) = 0$. Applying (4) to the left-hand side of (43), we obtain

$$\nabla f(\Omega) = \int_0^\infty \nabla\chi_{E_t}(\Omega)\,dt = -\int_0^\infty \nabla\chi_{E_t}(\mathbb{R}^n\setminus\Omega)\,dt = -\int_0^\infty \nabla\chi_{E_t}(\Gamma\cap\partial^*E_t)\,dt.$$

On the other hand, by (7) we get

$$\nabla\chi_{E_t}(\Gamma\cap\partial^*E_t) = -\int_{\Gamma\cap\partial^*E_t} \nu_{E_t}(x)\,d\mu(x) = -\int_{\Gamma\cap\partial^*E_t} \nu(x)\,d\mu(x),$$

where ν_{E_t} is the normal to E_t in the sense of Federer. Here the first identity follows from the fact that $\nu_{E_t}(x) = \nu(x)$ for almost all $x \in \Gamma\cap\partial^*E_t$, and the second is true because $\mu(E_t \setminus \bigcup_{\tau>t} E_\tau) = 0$ for almost all $t \in \mathbb{R}$.

Therefore, applying (25) with $\eta = \nu$, we obtain

$$\nabla f(\Omega) = -\int_0^{+\infty} \nabla\chi_{E_t}(\Gamma\cap\partial^*E_t)\,dt = \int_0^{+\infty}\int_{\Gamma\cap\partial^*E_t} \nu(x)\,d\mu(x)$$
$$= \int_\Gamma f^*(x)\nu(x)\,d\mu(x) + \int_{\partial^2_\Gamma\Omega} (f^*(x) - f_*(x))\nu(x)\,d\mu(x).$$

The theorem is proved. $\square$

§9. Average trace of a function in $BV(\Omega)$

Let Ω be a region with countably rectifiable boundary $\partial\Omega$. Suppose that a function $f \in BV(\Omega)$ is integrable in some neighborhood of a point $x \in \Gamma$. We define the upper and lower average traces of f at x with respect to a normal ν as follows:

$$\overline{f}(x,\nu) = \limsup_{r\to 0}\, 2v_n^{-1}r^{-n}\int_{B_r^\nu(x)} f(y)\,dy,$$
$$\underline{f}(x,\nu) = \liminf_{r\to 0}\, 2v_n^{-1}r^{-n}\int_{B_r^\nu(x)} f(y)\,dy.$$

If $\overline{f}(x,\nu) = \underline{f}(x,\nu)$, then their common value is called the *average trace* and is denoted by $\widetilde{f}(x,\nu)$. We start with proving some properties of average traces for nonnegative functions.

Lemma 15. *Suppose that a function $f \in BV(\Omega)$ is nonnegative and locally integrable. Then $\underline{f}(x,\nu) \geq f^\nu(x)$.*

Proof. (Compare with the proof of Lemma 6.6.2/1 in [5].)

Lemma 15 is obviously true if $f^\nu(x) = 0$. Suppose $0 < f^\nu(x)$. Pick $\epsilon > 0$ and choose a number t such that $0 < t < f^\nu(x)$ and $P_\Omega(E_t) < \infty$. Then $x \in \partial_\Gamma^\nu E_t$. This means that $\Theta_E^\nu(x) = 1$. Therefore, there exists $r_0(x) > 0$ such that

$$1-\epsilon < 2v_n^{-n}r^{-n}\operatorname{Vol}(E_t\cap B_r^\nu(x)) \leq 1$$

for $0 < r < r_0(x)$. Since

$$\int_{B_r^\nu(x)} f(y)\,dy = \int_0^\infty \operatorname{Vol}(E_\tau \cap B_r^\nu(x))\,d\tau,$$

we obtain

$$\begin{aligned} 2v_n^{-n}r^{-n}\operatorname{Vol}(B_r^\nu(x))\int_{B_r^\nu(x)} f(y)\,dy &\geq 2v_n^{-n}r^{-n}\int_0^t \operatorname{Vol}(E_\tau\cap B_r^\nu(x))\,d\tau \\ &\geq 2v_n^{-n}r^{-n}\operatorname{Vol}(E_\tau\cap B_r^\nu(x))t \geq (1-\epsilon)t. \end{aligned}$$

Since ϵ is arbitrary, we finish the proof by passing to the limit as $r \to 0$, and then by passing to the limit as $t \to f^\nu(x)$. □

Theorem 6. *If $f \in BV(\Omega)$ and $\|f\|_\Gamma < \infty$, then the average trace $\widetilde{f}(x,\nu)$ of the function f exists and is equal to the trace $f^\nu(x)$ almost everywhere on $\partial_\Gamma^\nu\Omega$.*

If the function is bounded, the proof is unexpectedly simple.

Lemma 16. *If a function $f \in BV(\Omega)$ is bounded, then the average trace $\widetilde{f}(x,\nu)$ of f exists almost everywhere on Γ and coincides with $f^\nu(x)$.*

Proof of the lemma. Let $|f| < C$. By Lemma 15 and relation (20), we have

$$f^\nu(x) = (f+C)^\nu(x) + (-C)^\nu(x) \leq \underline{(f+C)}(x,\nu) - C = \underline{f}(x,\nu).$$

Applying this inequality to $-f$, we obtain

$$(-f)^\nu(x) \leq \underline{(-f)}(x,\nu).$$

Thus, by Lemma 8,

$$f^\nu(x) \geq \overline{f}(x,\nu)$$

for almost all $x \in \Gamma$. The lemma is proved. □

Proof of Theorem 6. As usual, we may assume that $f \geq 0$. We extend $f \in BV(\Omega)$ by zero to $\mathbb{R}^n$. By Lemma 14, the extended function f belongs to $BV(\mathbb{R}^n)$. Suppose that $f \in BV(\Omega)$ is unbounded. We consider the set $E = \{x \in \Omega \mid f(x) > 0\}$ and show that $\widetilde{f}(x,\nu) = 0$ for almost all $x \in \Gamma \setminus \partial^1_\Gamma E$. Recall that almost all points of $\partial\Omega$ are located on C^1-smooth $(n-1)$-dimensional surfaces M_i, and that a standard vector field ν is continuous along each M_i. For $x \in \Gamma \setminus \partial^1_\Gamma E$, we denote by M the surface M_i such that $x \in M_i$. For any point $p \in M$, the surface M divides a small ball centered at p into two open sets, U_1 and U_2. Denote $\widetilde{M} = \partial U_1 \cap \partial U_2 \subset M$. It suffices to check that $\bar{f}(x,\nu) = 0$ at all points $x \in \widetilde{M}$ such that $\Theta^\nu_x(E) = \Theta^{-\nu}_x(E) = 0$. For definiteness, let the normals ν be directed inward of U_1.

It is known that, for U_1 and U_2, the average trace of each function $f \in BV(U_i)$, $i = 1, 2$, is equal to its trace (see [5, Theorem 6.6.2] or [3, Lemma 13]). On the other hand, the trace equals zero at almost all $x \in M \setminus (\partial^*(E \cap U_1) \cap \partial^*(E \cap U_1))$. Therefore, for $i = 1, 2$,

$$0 = \lim_{r\to 0} \frac{\int_{U_i \cap B_r(x)} f\,dx}{\operatorname{Vol}(U_i \cap B_r(x))} = \lim_{r\to 0} 2v_n^{-1} r^{-n} \int_{U_i \cap B_r(x)} f\,dx. \tag{44}$$

Thus,

$$\limsup_{r\to 0} 2v_n^{-1} r^{-n} \int_{B^\nu_r(x)} f\,dx \leq \limsup_{r\to 0} 2v_n^{-1} r^{-n} \int_{B_r(x)} f\,dx = 0. \tag{45}$$

Define

$$f_C(x) = \begin{cases} f(x) & \text{if } f(x) < C, \\ 0 & \text{if } f(x) \geq C, \end{cases} \qquad f^C(x) = \begin{cases} 0 & \text{if } f(x) < C, \\ f(x) & \text{if } f(x) \geq C. \end{cases} \tag{46}$$

Now for almost all $x \in \Gamma \setminus \partial^1_\Gamma E_C$ such that $0 < f^\nu(x) < C$, we have

$$\overline{f}(x,\nu) = \overline{f_C}(x,\nu) + \overline{f^C}(x,\nu) = (f_C)^\nu(x) + (f^C)^\nu(x) = f^\nu(x) + 0. \tag{47}$$

Recalling that $\mu(\bigcap_{t>0} \partial^1_\Gamma E_t) = 0$, we see that the theorem is proved. $\square$

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