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LITTLEWOOD–PALEY INEQUALITY FOR ARBITRARY RECTANGLES IN $\mathbb{R}^2$ FOR $0 < p \le 2$

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Abstract. The one-sided Littlewood–Paley inequality for pairwise disjoint rectangles in $\mathbb{R}^2$ is proved for the L^p-metric, $0 < p \le 2$. This result can be treated as an extension of Kislyakov and Parilov's result (they considered the one-dimensional situation) or as an extension of Journé's result (he considered disjoint parallelepipeds in $\mathbb{R}^n$ but his approach is only suitable for $p \in (1, 2]$). We combine Kislyakov and Parilov's methods with methods "dual" to Journé's arguments.

§1. Statement of the theorem and the history of the problem

Let Δ_m be pairwise disjoint intervals on $\mathbb{R}$. In 1983, Rubio de Francia (see [1]) proved that

$$\Big\| \Big(\sum_m |M_{\Delta_m} f|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{R})} \le C_p \|f\|_{L^p(\mathbb{R})} \tag{1}$$

for $2 \le p < \infty$, where $M_\Delta f = (\widehat{f}\chi_\Delta)^\vee$ is the Fourier multiplier corresponding to a set Δ, and C_p depends on p only. Shortly after that, Journé (see [2]) extended this result to $\mathbb{R}^n$, proving an estimate similar to (1) in the case where f is defined on $\mathbb{R}^n$ and Δ_m are pairwise disjoint parallelepipeds with sides parallel to coordinate axes. Note that the n-dimensional version of (1) cannot be proved by n-fold application of the one-dimensional estimate. For this purpose Journé in [2] described and then used the theory of singular integral operators on product domains. Later, Fernando Soria offered (see [3]) a simpler proof in the case of $\mathbb{R}^2$.

Now we note that the dual version of estimate (1) can be written as

$$\Big\| \sum_m f_m \Big\|_{L^p(\mathbb{R})} \le C_p \big\| \{f_m\} \big\|_{L^p(\mathbb{R}, \ell^2)}, \quad 1 < p \le 2, \tag{2}$$

where $\{f_m\}$ is a collection of functions with $\operatorname{supp} \widehat{f}_m \subset \Delta_m$. In 1984, Bourgain (see [4]) proved that (2) remains true when $p = 1$. He used a more complicated method than that employed by Rubio de Francia. In 2005, Kislyakov and Parilov, acting in the spirit of Rubio de Francia's methods, established (see [5]) that (2) is true for all $0 < p \le 2$ (they considered the unit circle $\mathbb{T}$ rather than $\mathbb{R}$, but this does not play a significant role). In this paper we extend their result (namely, inequality (2) for $0 < p \le 2$) to $\mathbb{R}^2$, proving the following theorem.

Theorem. *Let $\{f_m\}$ be a sequence of functions such that $f_m \in L^1(\mathbb{R}^2)$ and $\operatorname{supp} \widehat{f}_m \subset \Delta_m$, where the Δ_m are disjoint rectangles in $\mathbb{R}^2$ with sides parallel to coordinate axes.*

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Then for $0 < p \le 2$ the following estimate is fulfilled:

$$\Big\| \sum_m f_m \Big\|_{L^p(\mathbb{R}^2)} \le C_p \big\| \{f_m\} \big\|_{L^p(\mathbb{R}^2, \ell^2)}, \tag{3}$$

where C_p does not depend on the functions or the rectangles.

Here the index m ranges over a countable or a finite set. If m ranges over a finite set, we do not show intentionally in our notation that the norm on the right-hand side is calculated in $L^p(\mathbb{R}^2, \ell^2_N)$, $N \in \mathbb{N}$, rather than in $L^p(\mathbb{R}^2, \ell^2)$ (actually, we can assume that finite sequences are supplemented by zeros to become infinite). We shall do similarly in what follows without a special mention. It should also be noted that throughout this paper we use the following definition of a Fourier transform:

$$\widehat{f}(t) = \int_{\mathbb{R}^n} f(\xi) e^{-2\pi i \langle t, \xi \rangle}\, d\xi, \quad f \in L^1(\mathbb{R}^n).$$

Our proof is based on the ideas of [5], but it will be more convenient to refer to the more recent paper [6], because in [6] the case of $\mathbb{R}$ was considered and some arguments were presented with more care. Also, we shall use implicitly the ideas of [3]. In [3], Fernando Soria employed the space $\mathrm{BMO}(\mathbb{R} \times \mathbb{R})$ and Carleson measures on product spaces. But since we work in the dual situation, the Hardy spaces on $\mathbb{R} \times \mathbb{R}$ and the atomic decomposition for them will be used instead. The required theory (i.e., the atomic decomposition for $H^p(\mathbb{R} \times \mathbb{R})$ and singular integral operators acting on these spaces) can be found in Fefferman's paper [7].

Note that Fefferman's theory, as it was presented in [7], is suitable only for the product of two Euclidean spaces $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. However, there is a more complicated theory (see [8]) suitable for the situation where the number of Euclidean factors is arbitrary. That theory allows us to obtain an analog of (3) for disjoint parallelepipeds in $\mathbb{R}^n$. In this case the proof becomes more complex and will be published separately.[1]

§2. Preliminaries: The analytic and the "real" Hardy spaces on $\mathbb{R} \times \mathbb{R}$

All statements presented in this section (except Lemma 1) can be found in [9]. Though in [9] all arguments are for the bidisk and scalar-valued functions, reformulations for $\mathbb{R} \times \mathbb{R}$ can be found at the end of that paper. At the same time, it is clear that all definitions and results from [9] remain valid for ℓ^2-valued functions. Throughout this section, if the range of a function is not indicated explicitly, we assume that it can be either scalar-valued or ℓ^2-valued. We also assume that $0 < p \le 2$.

Definition 1. Let $D = D_1 \times D_2$, where $D_j = \{z_j = x_j + iy_j \mid x_j, y_j \in \mathbb{R};\ y_j > 0\}$. A function f defined and analytic on the domain D belongs to $H^p_A(\mathbb{R}^2)$ if

$$\|f\|^p_{H^p_A} = \sup_{y_1, y_2 > 0} \iint_{\mathbb{R}^2} |f(x_1 + iy_1, x_2 + iy_2)|^p\, dx_1\, dx_2 < \infty.$$

We write $H^p_A(\mathbb{R}^2)$ instead of $H^p_A(D)$, identifying the function f with its limit values as $y = (y_1, y_2) \to 0$, where we pass to the limit in the sense of tempered distributions.

Definition 2. Consider a function φ in the Schwartz class on $\mathbb{R}^2$ such that $\int \varphi = 1$. Let f be a tempered distribution on $\mathbb{R}^2$. Then f belongs to $H^p(\mathbb{R}^2)$ if

$$M_\varphi f(x_1, x_2) = \sup_{\varepsilon_1, \varepsilon_2 > 0} |f * \varphi_{\varepsilon_1, \varepsilon_2}(x_1, x_2)| \in L^p(\mathbb{R}^2),$$

where $\varphi_{\varepsilon_1, \varepsilon_2}(x_1, x_2) = \varepsilon_1^{-1} \varepsilon_2^{-1} \varphi(x_1/\varepsilon_1, x_2/\varepsilon_2)$. Put also $\|f\|_{H^p} = \|M_\varphi f\|_{L^p}$.

[1] Added in translation: see Zapiski Nauchn. Sem. POMI **376** (2010), 88–115; English transl. in J. Math. Sci., to appear.

As in the one-dimensional case, the following relations are fulfilled: $H^p_A(\mathbb{R}^2) \subset H^p(\mathbb{R}^2)$ and $\|f\|_{H^p_A} \asymp \|f\|_{H^p}$ for $f \in H^p_A(\mathbb{R}^2)$. By analogy with H^p_A, we can consider the space of functions analytic in one variable and anti-analytic in the other, or the space of functions anti-analytic in both variables. In fact, H^p is the direct sum of all such spaces. Details can be found in [9].

The following statement will be required later.

Lemma 1. *Let Δ_m be rectangles (maybe, overlapping) in the first quadrant of the plane $\mathbb{R}^2$, and let g_m be functions in $L^1(\mathbb{R}^2)$ with $\operatorname{supp} \widehat{g}_m \subset \Delta_m$. Suppose also that only finitely many g_m's are not equal to zero. Then the function $G = \{g_m\}$ belongs to $H^p_A(\mathbb{R}^2, \ell^2)$ and $\|G\|_{H^p_A(\ell^2)} \le \|G\|_{L^p(\ell^2)}$.*

This statement is trivial from the classical Hardy spaces point of view, but we present its proof here for the sake of completeness.

Proof. For simplicity, we assume that G consists of a single component whose Fourier transform is supported on the rectangle $\Delta = I \times J$. Without loss of generality, we may also assume that G is continuous and bounded. Indeed, the functions G and $(\widehat{G})^\vee$ are equal as distributions, and $(\widehat{G})^\vee$ is a bounded and continuous function (since the function $\widehat{G}$ has compact support). Thus, G is equal a.e. to some continuous and bounded function.

Now we extend the function G to the domain D:

$$G(z_1, z_2) = G(x_1 + iy_1, x_2 + iy_2) = \iint_{\mathbb{R}^2} P_{y_1}(x_1 - t_1) P_{y_2}(x_2 - t_2) G(t_1, t_2)\, dt_1\, dt_2,$$

where $(z_1, z_2) \in D$ and $P_y(x) = cy/(x^2 + y^2)$ is the Poisson kernel (c is a normalizing constant providing the identity $\int_{\mathbb{R}} P_y(x)\, dx = 1$). We must verify that G is analytic in D, that it tends to its boundary values in the sense of tempered distributions as $y = (y_1, y_2) \to 0$, and that the required estimate for the norm $\|G\|_{H^p_A}$ is fulfilled. Since the Poisson kernel is the Fourier transform of the function $e^{-2\pi y|\xi|}$, it follows that

$$G(z_1, z_2) = \iint_{\mathbb{R}^2} \left(e^{2\pi i x_1 \xi_1 - 2\pi y_1 |\xi_1|}\, e^{2\pi i x_2 \xi_2 - 2\pi y_2 |\xi_2|}\right)^\wedge (t_1, t_2)\, G(t_1, t_2)\, dt_1\, dt_2,$$

where the Fourier transformation is applied with respect to $\xi = (\xi_1, \xi_2)$. Now, using the formula $\int \widehat{f}(x) g(x) = \int f(x) \widehat{g}(x)$, we obtain

$$G(z_1, z_2) = \iint_{\Delta} \widehat{G}(\xi_1, \xi_2) e^{2\pi i\, \xi_1 z_1} e^{2\pi i\, \xi_2 z_2}\, d\xi_1\, d\xi_2.$$

Since $\widehat{G}$ is continuous and supported on a compact set and the function $e^{2\pi i \langle \xi, z\rangle}$ is analytic in z on $\mathbb{C}^2$, we see that G is analytic in D and tends to its boundary values. It remains to verify the estimate for the norm.

First, we prove that $G(t_1, \cdot)^\wedge(\xi_2)$ (here we fix the first variable and apply the Fourier transformation with respect to the second) is supported on J for almost all $t_1 \in \mathbb{R}$. Let φ be a function in the Schwartz class that is equal to zero on J. Then

$$0 = \int_{\mathbb{R}} \widehat{G}(\xi_1, \xi_2) \varphi(\xi_2)\, d\xi_2 = \int_{\mathbb{R}} e^{-2\pi i t_1 \xi_1} \int_{\mathbb{R}} G(t_1, \cdot)^\wedge(\xi_2)\, \varphi(\xi_2)\, d\xi_2\, dt_1$$

for any $\xi_1 \in \mathbb{R}$. Since the Fourier transformation is a one-to-one operator, it follows that the inner integral in the right expression is equal to zero for almost all t_1. But

$$\int_{\mathbb{R}} G(t_1, \cdot)^\wedge(\xi_2)\, \varphi(\xi_2)\, d\xi_2 = \int_{\mathbb{R}} G(t_1, t_2) \widehat{\varphi}(t_2)\, dt_2.$$

Thus, since G is continuous and bounded, we see that the integral in question is a continuous function of t_1 and, therefore, is equal to zero for *all* t_1. Hence we obtain our statement about the support of $G(t_1, \cdot)^\wedge$.

Next, consider two functions $G_1 = G(\cdot, x_2 + iy_2)$ and $G_2 = G(x_1, \cdot)$ defined on $\mathbb{R}$. The first function is the Poisson extension of the boundary function G with respect to the second variable, and the second variable (now complex) is fixed. The second function is the boundary function G with the first variable fixed. We have proved that $\operatorname{supp} \widehat{G}_2 \subset J$. By the same argument, we obtain $\operatorname{supp} \widehat{G}_1 \subset I$ (the convolution with the Poisson kernel in the second variable does not interfere). It follows that $G_k \in H^1$, $k = 1, 2$ (where H^1 is the usual Hardy space on $\mathbb{R}$). Indeed, we can prove the analyticity and the convergence to the boundary values much as we did for G (but the simpler one-dimensional case is considered now), and since $\int_{\mathbb{R}} P_y(x)\, dx = 1$, we get the estimate for the H^1-norm. Now, using the subharmonicity of $|G_k(x + iy)|^p$, we easily obtain

$$|G_k(x + iy)|^p \le (P_y * |G_k|^p)(x), \quad k = 1, 2. \tag{4}$$

To get this, we must move away from the boundary (i.e., consider $G_k(\cdot + i\varepsilon)$ instead of G_k), use the maximum principle for harmonic and subharmonic functions, and then pass to the limit in ε. Details can be found in many standard manuals in harmonic analysis, e.g., in [10, III.4.2]. From (4) and the identity $\int_{\mathbb{R}} P_y(x)\, dx = 1$ it follows that $\|G_k\|_{H^p} \le \|G_k\|_{L^p}$, $k = 1, 2$. Applying these two inequalities consecutively to the initial function G, we obtain the desired estimate. □

Note that we can formulate this lemma for any quadrants of the plane $\mathbb{R}^2$, replacing H^p_A with the space of functions analytic or anti-analytic in the corresponding variables.

§3. Preliminaries: The atomic decomposition and bounded operators on $H^p(\mathbb{R}^2)$

In this section, we present the main results of [7] but with a small difference: the atomic decomposition theorem is stated here so that the number of vanishing moments in it (i.e., the number N in Definition 3 below) may be increased arbitrarily (see details below). We assume that $0 < p \le 1$, and that all functions may be either scalar-valued or ℓ^2-valued.

Definition 3. Consider a function α in $L^2(\mathbb{R}^2)$ (scalar-valued or ℓ^2-valued) and a non-negative integer N. We say that α is an (N, p) rectangle atom if the following conditions are satisfied:

(i) $\operatorname{supp} \alpha \subset \Delta$, where $\Delta = I \times J$ is a rectangle with sides parallel to coordinate axes;
(ii) $\|\alpha\|_{L^2} \le |\Delta|^{1/2 - 1/p}$;
(iii) $\int_I \alpha(x_1, x_2^0) x_1^s\, dx_1 = \int_J \alpha(x_1^0, x_2) x_2^s\, dx_2 = 0$ for $0 \le s \le N$ and for all $x_1^0 \in I$ and $x_2^0 \in J$.

Definition 4. Suppose Ω is an open set in $\mathbb{R}^2$ of finite measure, and $\mathcal{M}(\Omega)$ is the set of the maximal dyadic rectangles contained in Ω. A function $a \in L^2(\mathbb{R}^2)$ supported on Ω is called an (N, p) atom if the following conditions are satisfied:

(i) $a = \sum_{\Delta \in \mathcal{M}(\Omega)} \alpha_\Delta$, where each α_Δ is a function supported on a rectangle Δ and satisfying condition (iii) in Definition 3 (i.e., α_Δ is an (N, p) rectangle atom multiplied by a coefficient);
(ii) $\|a\|_{L^2} \le |\Omega|^{1/2 - 1/p}$;
(iii) $\sum_{\Delta \in \mathcal{M}(\Omega)} \|\alpha_\Delta\|_{L^2}^2 \le |\Omega|^{1 - 2/p}$.

Theorem (Atomic decomposition)**.** *Fix $0 < p \le 1$. Then we can choose a number ν_p such that for $N \ge \nu_p$ the following is true: if a function f belongs to $H^p(\mathbb{R}^2)$, then $f = \sum \lambda_k a_k$, where the a_k are (N,p) atoms and the λ_k are scalars such that $\sum |\lambda_k|^p \le C_{p,N}\|f\|^p_{H^p}$.*

In [7], this theorem was formulated and proved for $N = \nu_p$ (more precisely, only the case where $p = 1$ and $N = \nu_p = 0$ was considered in detail there). But to lift this restriction, it suffices to choose an appropriate function ψ that occurs in the definition of the quadratic functional S_ψ: this function must have an appropriate number of vanishing moments (see the corresponding proof in [7]). Using atomic decomposition and Journé's covering lemma, R. Fefferman proved the following theorem (see [7]).

Theorem (R. Fefferman)**.** *Consider an operator T bounded from $L^2(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$ (recall that each of the spaces $L^2(\mathbb{R}^2)$ may consist either of scalar-valued or ℓ^2-valued functions). Let $N \ge \nu_p$. Suppose that for any (N,p) rectangle atom α supported on a rectangle Δ we have*

$$\iint_{(\gamma\cdot\Delta)^c} |(T\alpha)(x_1,x_2)|^p \, dx_1\, dx_2 \le C\gamma^{-\delta} \tag{5}$$

for some fixed $\delta > 0$ and for every $\gamma \ge 2$. Then T is a bounded operator from $H^p(\mathbb{R}^2)$ to $L^p(\mathbb{R}^2)$.

Here and below, $\gamma \cdot \Delta$ denotes the concentric dilation of Δ with coefficient γ.

§4. Proof of the main theorem: The auxiliary operators S and R

Now we prove the main theorem. Since all our estimates will not depend on the number of the rectangles Δ_m, we can assume that there are finitely many of them.

First, consider the special case where all rectangles Δ_m can be obtained from dyadic ones by 8-fold dilation of the sides with preservation of the lower left vertex. This means that, to each Δ_m, we can assign a multi-index $(k,j) = (k_1,k_2,j_1,j_2) \in \mathbb{Z}^4$ such that

$$\Delta_m = \big[j_1 2^{k_1}, (j_1+8)2^{k_1}\big] \times \big[j_2 2^{k_2}, (j_2+8)2^{k_2}\big].$$

Also suppose that $\operatorname{supp} \widehat{f}_m \subset \frac{3}{4}\cdot\Delta_m$. We make a technical remark.

Technical Remark. Any rectangle $R = [a,b]\times[c,d]$ can be placed in a rectangle $\Delta = I \times J$ of the type described. Moreover, this can be done so that $|I| \asymp \big|[a,b]\big|$, $|J| \asymp \big|[c,d]\big|$, and $R \subset \frac{3}{4}\cdot\Delta$.

Indeed, let $I = [j2^k, (j+8)2^k]$, where k satisfies $2^k \le \big|[a,b]\big| < 2^{k+1}$ and $j = \sup\{j \in \mathbb{Z} \mid (j+1)2^k < a\}$. It is clear that $|I| \asymp \big|[a,b]\big|$ and $[a,b] \subset \frac{3}{4}\cdot I$. Choosing J in the same way, we get the required rectangle.

Now we prove the theorem in the special case described above. Let $\mathcal{A}$ denote the set of multi-indices corresponding to the rectangles Δ_m. Let $h = \{h_{k,j}\}_{(k,j)\in\mathbb{Z}^4} \in L^2(\mathbb{R}^2,\ell^2)$. Consider the operator S defined by the formula

$$S(h)(x_1,x_2) = \sum_{(k,j)\in\mathcal{A}} e^{2\pi i\, j_1 2^{k_1}\, x_1} e^{2\pi i\, j_2 2^{k_2}\, x_2} (\Phi_k * h_{k,j})(x_1,x_2),$$

where $\Phi_k(x_1,x_2) = \varphi_{k_1}(x_1)\varphi_{k_2}(x_2)$, and in their turn, the functions φ_n are defined as follows: take a function φ in the Schwartz class on $\mathbb{R}$ such that its Fourier transform is nonnegative, supported on $[0,8]$, and equal to one on $[1,7]$; then put $\varphi_n(t) = 2^n\varphi(2^n t)$.

Lemma 2. *The operator S is bounded from $H^p(\mathbb{R}^2,\ell^2)$ to $L^p(\mathbb{R}^2)$ for $0 < p \le 2$. Moreover, the estimation constant depends on p only.*

We postpone the proof of this lemma. Now we take it for granted and prove the theorem in the special case indicated above. Let $g_{k,j} \equiv 0$ for $(k,j) \notin \mathcal{A}$, and let

$$g_{k,j}(x_1,x_2) = e^{-2\pi i\, j_1 2^{k_1}\, x_1} e^{-2\pi i\, j_2 2^{k_2}\, x_2} f_m(x_1,x_2)$$

for the multi-index $(k,j) \in \mathcal{A}$ corresponding to the rectangle Δ_m. Note that the Fourier transforms of the nonzero functions $g_{k,j}$ are supported on the rectangles

$$\tfrac{3}{4} \cdot \Delta_m - (j_1 2^{k_1}, j_2 2^{k_2}) = \big[2^{k_1}, 7 \cdot 2^{k_1}\big] \times \big[2^{k_2}, 7 \cdot 2^{k_2}\big].$$

By the Plancherel theorem, this implies that $g_{k,j} = \Phi_k * g_{k,j}$. Denote $g = \{g_{k,j}\}_{(k,j)\in\mathbb{Z}^4}$. Taking the above into consideration and using Lemmas 1 and 2, we have

$$\Big\| \sum_m f_m \Big\|_{L^p} = \big\| S(g) \big\|_{L^p} \le C_p \|g\|_{H^p(\ell^2)} \le C'_p \|g\|_{L^p(\ell^2)} = C'_p \big\| \{f_m\} \big\|_{L^p(\ell^2)}.$$

This is the required estimate.

Now we reduce the theorem to the special case considered. The arguments will be similar to those in [6], but, since we consider the two-dimensional case, the computations will be more complicated. We need another auxiliary operator. Let ψ be a function in the Schwartz class on $\mathbb{R}$ such that $\widehat{\psi} \ge 0$ and $\operatorname{supp} \widehat{\psi} \subset [A^{-1}, A]$, where $A > 1$. Put $\psi_n(t) = A^n \psi(A^n t)$, $n \ge 0$. Note that $\operatorname{supp} \widehat{\psi}_n \subset [A^{n-1}, A^{n+1}]$. Let $\Psi_{k_1,k_2}(x_1,x_2) = \psi_{k_1}(x_1)\psi_{k_2}(x_2)$, and let $\{h_m\} \in L^2(\mathbb{R}^2, \ell^2)$; the required operator R is defined by the formula

$$R(h_1, h_2, \dots) = (r(h_1), r(h_2), \dots),$$

where

$$r(h_m) = \big\{ h_m * \Psi_{k_1,k_2} \big\}_{k_1,k_2=0}^{L}.$$

Lemma 3. *The operator R is bounded from $H^p(\mathbb{R}^2, \ell^2)$ to $L^p(\mathbb{R}^2, \ell^2)$ for $0 < p \le 2$. Moreover, the estimation constant does not depend on L.*

The proof of this lemma will also be given later. Now we show the reduction to the special case considered above. First, using dilation, we can assume without loss of generality that the side lengths of the rectangles Δ_m are at least 1. Fix A sufficiently close to 1 and choose a function ψ such that $\sum_{n\ge 0} \widehat{\psi}_n(t) = 1$ for $t \ge 1$. Put $\theta = (1,1)$ and consider the functions $g_m(x) = e^{-2\pi i \langle a_m - \theta, x\rangle} f_m(x)$, where the a_m are the lower left vertices of the rectangles Δ_m. Splitting the supports of $\widehat{g}_m$ by application of R and moving them back, we get the functions $f_{m,k_1,k_2}(x) = e^{2\pi i \langle a_m - \theta, x\rangle} (g_m * \Psi_{k_1,k_2})(x)$ such that $f_m = \sum_{k_1,k_2} f_{m,k_1,k_2}$ (the number of summands is finite: for each m, summation stops when the support of $\widehat{\Psi}_{k_1,k_2}$ leaves Δ_m). Considering the triple sequence $\{f_{m,k_1,k_2}\}$ as a function in $L^p(\mathbb{R}^2, \ell^2)$ and using Lemmas 1 and 3, we have

$$\begin{aligned} \big\| \{f_{m,k_1,k_2}\} \big\|_{L^p(\ell^2)} &= \big\| R(\{g_m\}) \big\|_{L^p(\ell^2)} \\ &\le C_{p,A} \big\| \{g_m\} \big\|_{H^p(\ell^2)} \le C'_{p,A} \big\| \{g_m\} \big\|_{L^p(\ell^2)} = C'_{p,A} \big\| \{f_m\} \big\|_{L^p(\ell^2)}. \end{aligned}$$

Now, let N be a sufficiently large natural number (e.g., $N = 100$). We split the sequence $\{f_{m,k_1,k_2}\}$ into N^2 subsequences in accordance with the residues of k_1 and k_2 modulo N in each group (i.e., for each m). Taking the above into account, we see that it suffices to prove the theorem for each subsequence separately. In such a subsequence, from each group we delete all the functions such that one of their indexes k_1 or k_2 is maximal. Choosing A sufficiently close to 1 and applying the *Technical Remark* about rectangles, we see that the supports of the remaining functions can be placed into rectangles of the same sort as in the special case described above. Now consider the deleted functions. Their Fourier transforms are associated with certain rectangles. We shift the upper right vertices of these rectangles to the point $(-1,-1)$ and split each rectangle once again,

using the method described above (but working with functions anti-analytic in both variables).

§5. Proof of the main theorem: The boundedness of S and R

Now we verify Lemmas 2 and 3. In the paper [11] about the Calderón–Zygmund decomposition on the Hardy spaces $H^p(\mathbb{R}\times\mathbb{R})$, Chang and Fefferman proved the following: if a linear operator T is bounded from $H^1(\mathbb{R}^2)$ to $L^1(\mathbb{R}^2)$ and is bounded from $L^2(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$, then it is bounded from $L^p(\mathbb{R}^2)$ to $L^p(\mathbb{R}^2)$ for all $1<p<2$. Therefore, it suffices to prove Lemmas 2 and 3 for $p=2$ and for $0<p\le 1$. From the Plancherel theorem it follows that S and R are bounded on L^2. To prove boundedness for $0<p\le 1$, it suffices (by Fefferman's theorem) to verify that S and R satisfy estimate (5) for any (N,p) rectangle atom.

Consider S first. We need the following statement.

Lemma 4. *Consider the sequence of functions $\kappa(x,y)=\{\kappa_{k,j}(x,y)\}_{(k,j)\in\mathbb{Z}^2}$, where $\kappa_{k,j}(x,y)=e^{2\pi i\,2^k j\,x}\varphi_k(x-y)$ (recall that the functions φ_k were defined in §4 when we introduced the operator S). Let $N\ge 0$ be an integer. Then for every interval $I\subset\mathbb{R}$, there exists an ℓ^2-valued function $p_I(x,y)$ that is a polynomial of degree not greater than N in y such that for all $\xi=\{\xi_{k,j}\}_{(k,j)\in\mathbb{Z}^2}\in\ell^2$, all $y\in I$, all $\gamma\ge 2$, and all integers $s\ge 1$ we have*

$$\left(\int_{I_s}\left|\langle\kappa(x,y)-p_I(x,y),\xi\rangle_{\ell^2}\right|^2dx\right)^{1/2}\le C_N\,\gamma^{-sB_N}\,|I|^{-1/2}\,|\xi|_{\ell^2},$$

where $B_N>1$ and $B_N\to\infty$ as $N\to\infty$. Here $I^s=\gamma^{s+1}I\setminus\gamma^s I$.

Similar estimates can be found in the treatment of the one-dimensional case (see, e.g., [1]). However, in those estimates the pairs (k,j) did not range over the entire set $\mathbb{Z}^2$, but only over a subset $\mathcal{B}\subset\mathbb{Z}^2$ such that $\sum_{(k,j)\in\mathcal{B}}|\widehat{\varphi}_k(x-2^kj)|^2\le C$. In that situation the sequence $\{\kappa_{k,j}(x,y)\}_{(k,j)\in\mathcal{B}}$ was a kernel of some Calderón–Zygmund operator (the one-dimensional analog of S). The L^2-boundedness of such an operator follows (by the Plancherel theorem) from that very restriction on the set of the pairs (k,j). But Lemma 4 is true without such restrictions, and this will allow us to estimate the norm of S (i.e., to consider the two-dimensional case).

The proof of Lemma 4 will be given in the next section. Now, using this lemma, we prove the boundedness of S. Let $\alpha=\{\alpha_{k,j}\}_{(k,j)\in\mathbb{Z}^4}$ be an arbitrary ℓ^2-valued (N,p) rectangle atom (N is sufficiently large) supported on a rectangle $\Delta=I\times J$ (as above, $(k,j)=(k_1,k_2,j_1,j_2)$). We split the set $(\gamma\cdot\Delta)^c$ occurring in (5) into three parts:

$$\begin{aligned}A_1&=\{(x_1,x_2)\in\mathbb{R}^2\mid x_1\in\gamma\cdot I;\ x_2\notin\gamma\cdot J\},\\ A_2&=\{(x_1,x_2)\in\mathbb{R}^2\mid x_1\notin\gamma\cdot I;\ x_2\in\gamma\cdot J\},\quad\text{and}\\ A_3&=\{(x_1,x_2)\in\mathbb{R}^2\mid x_1\notin\gamma\cdot I;\ x_2\notin\gamma\cdot J\}.\end{aligned}$$

We integrate over each part separately. For A_1 and A_2 we use Lemma 4 with respect to one variable first and then the L^2-boundedness with respect to the other. For A_3, we use Lemma 4 twice, with respect to each variable. The idea is simple but the precise arguments will be fairly bulky.

Consider A_1 first. We introduce some notation. The kernel of S is the following sequence of functions:

$$K(x_1,x_2,y_1,y_2)=\left\{e^{2\pi i\,2^{k_1}j_1\,x_1}\,e^{2\pi i\,2^{k_2}j_2\,x_2}\,\Phi_k(x_1-y_1,x_2-y_2)\right\}_{(k,j)\in\mathcal{A}}.$$

Let $h = \{h_{k_1,j_1}\}_{(k_1,j_1)\in\mathbb{Z}^2} \in L^2(\mathbb{R},\ell^2)$. We define the auxiliary operators S_{k_2,j_2} by the formula

$$S_{k_2,j_2}(h)(x_1) = \sum_{(k_1,j_1)\in\mathbb{Z}^2} \chi_{\mathcal{A}}(k_1,k_2,j_1,j_2)\, e^{2\pi i\, 2^{k_1} j_1\, x_1}\, (\varphi_{k_1} * h_{k_1,j_1})(x_1).$$

Fixing the indices k_2 and j_2, we consider the rectangles $\Delta_m = I_m \times J_m$ corresponding to the multi-indices $(k_1,k_2,j_1,j_2) \in \mathcal{A}$. Since the sides $J_m = [j_2 2^{k_2}, (j_2+8)2^{k_2}]$ of Δ_m coincide and the Δ_m are pairwise disjoint, it follows that the intervals $I_m = [j_1 2^{k_1}, (j_1+8)2^{k_1}]$ are pairwise disjoint. Thus, by the Plancherel theorem, we see that the S_{k_2,j_2} are bounded on L^2. Also denote

$$\xi(y_2,x_1) = \Big\{ S_{k_2,j_2}\Big(\big\{\alpha_{k_1,k_2,j_1,j_2}(\cdot,y_2)\big\}_{(k_1,j_1)\in\mathbb{Z}^2}\Big)(x_1)\Big\}_{(k_2,j_2)\in\mathbb{Z}^2}.$$

Here, for each index (k_2,j_2), we take the components of α corresponding to this index, and then, fixing the second variable, we apply the operator S_{k_2,j_2} to the resulting collection. In this notation we have

$$\begin{aligned} S(\alpha)(x_1,x_2) &= \iint_{\Delta} \big\langle K(x_1,x_2,y_1,y_2), \alpha(y_1,y_2)\big\rangle_{\ell^2}\, dy_1\, dy_2 \\ &= \int_J \sum_{(k_2,j_2)\in\mathbb{Z}^2} e^{2\pi i\, 2^{k_2} j_2\, x_2} \varphi_{k_2}(x_2-y_2) S_{k_2,j_2}\Big(\big\{\alpha_{k_1,k_2,j_1,j_2}(\cdot,y_2)\big\}_{(k_1,j_1)\in\mathbb{Z}^2}\Big)(x_1)\, dy_2 \\ &= \int_J \big\langle \kappa(x_2,y_2) - p_J(x_2,y_2), \xi(y_2,x_1)\big\rangle_{\ell^2}\, dy_2, \end{aligned} \tag{6}$$

where the functions κ and p_J are the same as in Lemma 4, and their components are indexed by the pairs $(k_2,j_2) \in \mathbb{Z}^2$. The last identity results from the fact that for the rectangle atom α the first N moments in each variable vanish. Now, using the Hölder inequality, we have

$$\begin{aligned} \iint_{A_1} \big|(S\alpha)(x_1,x_2)\big|^p dx_1\, dx_2 &= \sum_{s\ge 1} \int_{\gamma\cdot I} \int_{J_s} \big|(S\alpha)(x_1,x_2)\big|^p dx_2\, dx_1 \\ &\le \sum_{s\ge 1} \int_{\gamma\cdot I} |J_s|^{1-p/2} \Big(\int_{J_s} \big|(S\alpha)(x_1,x_2)\big|^2 dx_2\Big)^{p/2} dx_1. \end{aligned}$$

Consider the inner integral raised to the power 1/2, i.e., the $L^2(J_s)$-norm of $(S\alpha)(x_1,\cdot)$. Using formula (6) for $S(\alpha)$, interchanging this $L^2(J_s)$-norm and the integral over J, and applying Lemma 4, we obtain

$$\begin{aligned} \Big(\int_{J_s} \big|(S\alpha)(x_1,x_2)\big|^2 dx_2\Big)^{1/2} &= \Big(\int_{J_s} \Big| \int_J \big\langle \kappa(x_2,y_2) - p_J(x_2,y_2), \xi(y_2,x_1)\big\rangle\, dy_2\Big|^2 dx_2\Big)^{1/2} \\ &\le \int_J \Big(\int_{J_s} \Big|\big\langle \kappa(x_2,y_2) - p_J(x_2,y_2), \xi(y_2,x_1)\big\rangle\Big|^2 dx_2\Big)^{1/2} dy_2 \\ &\le C_N\, \gamma^{-sB_N}\, |J|^{-1/2} \int_J |\xi(y_2,x_1)|_{\ell^2}\, dy_2. \end{aligned}$$

Combining this estimate with the preceding one, using the Hölder inequality again, and interchanging the L^2-norm and the integral, we have

$$\begin{aligned}
&\iint_{A_1} \big|(S\alpha)(x_1,x_2)\big|^p \, dx_2\, dx_1 \\
&\quad \le \sum_{s\ge 1} C_{N,p}\, \gamma^{-spB_N}\, |J_s|^{1-p/2}\, |J|^{-p/2} \int_{\gamma\cdot I} \bigg(\int_J |\xi(y_2,x_1)|_{\ell^2}\, dy_2 \bigg)^p dx_1 \\
&\quad \le \sum_{s\ge 1} C_{N,p}\gamma^{-spB_N} |J_s|^{1-p/2}|J|^{-p/2}|\gamma\cdot I|^{1-p/2} \bigg(\int_J \bigg(\int_{\gamma\cdot I} |\xi(y_2,x_1)|^2_{\ell^2}\, dx_1 \bigg)^{1/2} dy_2 \bigg)^p .
\end{aligned}$$

Recalling the definition of $\xi(y_2,x_1)$ and using the L^2-boundedness of the operators S_{k_2,j_2}, we see that

$$\begin{aligned}
&\int_J \bigg(\int_{\gamma\cdot I} |\xi(y_2,x_1)|^2_{\ell^2}\, dx_1 \bigg)^{1/2} dy_2 \\
&\quad \le \int_J \bigg(\sum_{(k_2,j_2)\in\mathbb{Z}^2} \int_{\mathbb{R}} \Big| \big\{ \alpha_{k_1,k_2,j_1,j_2}(x_1,y_2) \big\}_{(k_1,j_1)\in\mathbb{Z}^2} \Big|^2_{\ell^2} dx_1 \bigg)^{1/2} dy_2 \\
&\quad \le |J|^{1/2} \bigg(\iint_{\mathbb{R}^2} |\alpha(x_1,y_2)|^2_{\ell^2}\, dx_1\, dy_2 \bigg)^{1/2} \le |J|^{1/2}\, (|I|\,|J|)^{1/2-1/p}.
\end{aligned}$$

Putting it all together, we obtain

$$\begin{aligned}
&\iint_{A_1} \big|(S\alpha)(x_1,x_2)\big|^p \, dx_2\, dx_1 \\
&\quad \le C_{N,p} \sum_{s\ge 1} \gamma^{-spB_N}\, |J_s|^{1-p/2}\, |J|^{-p/2}\, |\gamma\cdot I|^{1-p/2}\, |J|^{p-1}\, |I|^{p/2-1} \\
&\quad \le C_{N,p} \sum_{s\ge 1} \gamma^{-spB_N}\, \big(\gamma^s(\gamma-1)\big)^{1-p/2}\, |J|^{1-p}\, \gamma^{1-p/2}\, |I|^{1-p/2}\, |J|^{p-1}\, |I|^{p/2-1} \\
&\quad \le C_{N,p}\gamma^{2-p} \sum_{s\ge 1} \gamma^{s(1-p(1/2+B_N))} \le C_p\gamma^{-\delta}.
\end{aligned}$$

We have obtained the last inequality by choosing N such that B_N is sufficiently large (recall that $B_N \to \infty$ as $N\to\infty$). Our choice depends on p only.

We can treat A_2 in the same way. It remains to consider A_3. Let $\widetilde{\alpha}$ be an ℓ^2-valued (N,p) rectangle atom such that its components with the indices in $\mathcal{A}$ are equal to the corresponding components of α and the other components are equal to zero. We define a sequence ξ of functions by the formula

$$\xi(x_1,y_1,y_2) = \Big\{ \big\langle \kappa(x_1,y_1) - p_I(x_1,y_1), \big\{ \widetilde{\alpha}_{k_1,k_2,j_1,j_2}(y_1,y_2) \big\}_{(k_1,j_1)\in\mathbb{Z}^2} \big\rangle \Big\}_{(k_2,j_2)\in\mathbb{Z}^2}.$$

Here, for each index (k_2,j_2), we take the components of $\widetilde{\alpha}$ corresponding to this index and calculate the inner product of the resulting collection and $\kappa - p_I$, where the functions κ and p_I are the same as in Lemma 4 and their components are indexed by the pairs $(k_1,j_1)\in\mathbb{Z}^2$. Since the first N moments of $\widetilde{\alpha}$ vanish in each variable, it follows that

$$S(\alpha)(x_1,x_2) = \int_J \int_I \big\langle \kappa(x_2,y_2) - p_J(x_2,y_2), \xi(x_1,y_1,y_2) \big\rangle\, dy_1\, dy_2.$$

The components of κ and p_I are indexed by the pairs $(k_2, j_2) \in \mathbb{Z}^2$ this time. Now, using the Hölder inequality and interchanging the L^2-norm and the integral, we obtain

$$\begin{aligned}
&\iint_{A_3} \big|(S\alpha)(x_1,x_2)\big|^p \, dx_2\, dx_1 \\
&\quad = \sum_{s_1,s_2\ge 1} \int_{J_{s_2}} \int_{I_{s_1}} \bigg| \int_J \int_I \big\langle \kappa(x_2,y_2) - p_J(x_2,y_2), \xi(x_1,y_1,y_2)\big\rangle \, dy_1\, dy_2 \bigg|^p \, dx_1\, dx_2 \\
&\quad \le \sum_{s_1,s_2\ge 1} |I_{s_1}|^{1-p/2} |J_{s_2}|^{1-p/2} \bigg(\int_J \int_I \bigg(\int_{I_{s_1}} \int_{J_{s_2}} \big|\langle \cdots \rangle\big|^2 \, dx_2\, dx_1 \bigg)^{1/2} dy_1\, dy_2 \bigg)^p .
\end{aligned}$$

Using Lemma 4 twice, we have

$$\begin{aligned}
&\bigg(\int_{I_{s_1}} \int_{J_{s_2}} \Big|\big\langle \kappa(x_2,y_2) - p_J(x_2,y_2), \xi(x_1,y_1,y_2)\big\rangle\Big|^2 dx_2\, dx_1 \bigg)^{1/2} \\
&\qquad \le C_N\, \gamma^{-s_2 B_N}\, |J|^{-1/2} \bigg(\int_{I_{s_1}} |\xi(x_1,y_1,y_2)|_{\ell^2}^2 \, dx_1 \bigg)^{1/2} \\
&\qquad \le C'_N\, \gamma^{-s_1 B_N}\, |I|^{-1/2}\, \gamma^{-s_2 B_N}\, |J|^{-1/2}\, |\widetilde{\alpha}(y_1,y_2)|_{\ell^2} .
\end{aligned}$$

Using the Hölder inequality once again, we obtain

$$\int_J \int_I |\widetilde{\alpha}(y_1,y_2)|_{\ell^2}\, dy_1\, dy_2 \le |I|^{1/2}\, |J|^{1/2}\, \|\widetilde{\alpha}\|_{L^2} \le |I|^{1-1/p}\, |J|^{1-1/p} .$$

Putting it all together, we have

$$\begin{aligned}
&\iint_{A_3} \big|(S\alpha)(x_1,x_2)\big|^p \, dx_2\, dx_1 \\
&\quad \le C_{N,p} \sum_{s_1,s_2\ge 1} |I_{s_1}|^{1-p/2}\, |J_{s_2}|^{1-p/2}\, \gamma^{-p s_1 B_N}\, \gamma^{-p s_2 B_N}\, |I|^{p/2-1}\, |J|^{p/2-1} \\
&\quad \le C_{N,p} \gamma^{2-p} \sum_{s_1,s_2\ge 1} \gamma^{(s_1+s_2)(1-p(1/2+B_N))} \le C_p \gamma^{-\delta} .
\end{aligned}$$

The last inequality is true for the same reason as in the argument for A_1. This concludes the proof of Lemma 2 (more precisely, we have reduced it to Lemma 4).

Now, consider the operator R. Its kernel Λ is defined by the formula

$$\Lambda(x_1,x_2,y_1,y_2) = \big\{ \Psi_{k_1,k_2}(x_1-y_1, x_2-y_2) \big\}_{k_1,k_2=0}^{L} .$$

This kernel can be treated as an $\mathcal{L}(\ell^2, \ell^2_{\mathbb{N}^3})$-valued function, where the components of vectors in ℓ^2 are indexed by m and the components of vectors in $\ell^2_{\mathbb{N}^3}$ are indexed by (m, k_1, k_2) (see the definition of R). The operator R is simpler than S and can be treated as the tensor square of its one-dimensional analog, i.e., $R = \rho \otimes \rho$, where ρ is the operator from $L^2(\mathbb{R}, \ell^2)$ to $L^2(\mathbb{R}, \ell^2_{\mathbb{N}^2})$ defined by the formula

$$\rho\big(\{h_m\}\big) = \{h_m * \psi_k\}_{m\in\mathbb{N},\, k=0,\dots,L} .$$

The kernel of ρ is defined by

$$\lambda(x,y) = \{\psi_k(x-y)\}_{k=0}^{L} .$$

For the kernel λ, the usual Calderón–Zygmund smoothness condition is fulfilled:

$$|\lambda(x,y) - p_I(x,y)|_{\ell^2} \le \frac{C_N |I|^{N+1}}{|x-y_0|^{N+2}}, \tag{7}$$

where $N \ge 0$, I is an arbitrary finite interval in $\mathbb{R}$, y_0 is the center of I, $y \in I$, $x \notin 2\cdot I$, and $p_I(x,y)$ is an ℓ^2-valued function that is a polynomial of degree not greater than N

in y. The proof can be found in [6] (though this had been known before). We sketch it at the beginning of the next section.

It is well known that a tensor product of singular integral operators is bounded from $H^p(\mathbb{R}^2)$ to $L^p(\mathbb{R}^2)$ for $0 < p < 1$. But since the proof of this fact is hard to find in the literature, we shall give the corresponding argument for R for completeness. The calculation will be similar to the above argument for S but will be much simpler at the same time.

Let $\alpha = \{\alpha_m\}$ be an ℓ^2-valued (N, p) rectangle atom (N is sufficiently large) supported on a rectangle $\Delta = I \times J$. We split the set $(\gamma \cdot \Delta)^c$ from Fefferman's theorem into three parts A_1, A_2, and A_3 (in the same way as above) and integrate $\big|(R\alpha)(x_1, x_2)\big|^p$ over each part separately.

Consider A_1 first. Since the first N moments of the rectangle atom α vanish in each variable, we have

$$(R\alpha)(x_1, x_2) = \int_J \big(\lambda(x_2, y_2) - p_J(x_2, y_2)\big)\, \rho\big(\alpha(\cdot, y_2)\big)(x_1)\, dy_2,$$

where p_J is the polynomial in y as in (7). Next, using (7), we have

$$\begin{aligned}
&\iint_{A_1} \big|(R\alpha)(x_1, x_2)\big|^p\, dx_1\, dx_2 \\
&\quad \le \int_{\gamma\cdot I} \int_{(\gamma\cdot J)^c} \bigg(\int_J \big|\lambda(x_2, y_2) - p_J(x_2, y_2)\big|_{\ell^2} \big|\rho\big(\alpha(\cdot, y_2)\big)(x_1)\big|_{\ell^2}\, dy_2 \bigg)^p dx_2\, dx_1 \\
&\quad \le \int_{(\gamma\cdot J)^c} \frac{|J|^{pN+p}}{|x_2 - y_2^0|^{pN+2p}}\, dx_2 \cdot \int_{\gamma\cdot I} \bigg(\int_J \big|\rho\big(\alpha(\cdot, y_2)\big)(x_1)\big|_{\ell^2}\, dy_2 \bigg)^p dx_1;
\end{aligned}$$

here y_2^0 is the center of J. By a simple calculation, we show that the first factor in the last expression is equal to $C_{N,p}\, |J|^{1-p}\, \gamma^{1-p(N+2)}$. Using the Hölder inequality and interchanging the L^2-norm and the integral, we see that the second factor is majorized by the quantity

$$|\gamma \cdot I|^{1-p/2} \bigg(\int_J \bigg(\int_{\gamma\cdot I} \big|\rho\big(\alpha(\cdot, y_2)\big)(x_1)\big|_{\ell^2}^2\, dx_1 \bigg)^{1/2} dy_2 \bigg)^p,$$

which, in its turn, does not exceed $|\gamma\cdot I|^{1-p/2}\, |J|^{p/2}\, \|\alpha\|_{L^2}^p$ (this follows from the L^2-boundedness of ρ and the Hölder inequality). Putting it all together, we have

$$\begin{aligned}
\iint_{A_1} \big|(R\alpha)(x_1, x_2)\big|^p\, dx_1\, dx_2 &\le C_{N,p}\, |J|^{1-p}\, \gamma^{1-p(N+2)}\, |\gamma \cdot I|^{1-p/2}\, |J|^{p/2}\, (|I|\, |J|)^{p/2-1} \\
&= C_{N,p}\, \gamma^{2-p(N+5/2)} \le C_p \gamma^{-\delta}.
\end{aligned}$$

The set A_2 can be considered in the same way.

We pass to A_3. Since the first N moments of the rectangle atom α vanish, we obtain

$$(R\alpha)(x_1, x_2) = \int_J \int_I \big(\lambda(x_2, y_2) - p_J(x_2, y_2)\big)\, \big(\lambda(x_1, y_1) - p_I(x_1, y_1)\big)\, \alpha(y_1, y_2)\, dy_1\, dy_2.$$

Next, using (7) twice, we have

$$\begin{aligned}
&\iint_{A_3} \big|(R\alpha)(x_1, x_2)\big|^p\, dx_1\, dx_2 \\
&\quad \le \int_{(\gamma\cdot J)^c} \frac{|J|^{pN+p}\, dx_2}{|x_2 - y_2^0|^{pN+2p}} \cdot \int_{(\gamma\cdot I)^c} \frac{|I|^{pN+p}\, dx_1}{|x_1 - y_1^0|^{pN+2p}} \cdot \bigg(\int_J \int_I |\alpha(y_1, y_2)|_{\ell^2}\, dy_1\, dy_2 \bigg)^p;
\end{aligned}$$

here y_1^0 and y_2^0 are the centers of I and J, respectively. Calculating the first two factors and applying the Hölder inequality to the third, we obtain

$$\begin{aligned}\iint_{A_3} \big|(R\alpha)(x_1,x_2)\big|^p\,dx_1\,dx_2 &\le C_{N,p}\,|J|^{1-p}\,\gamma^{1-p(N+2)}\\ &\quad\times |I|^{1-p}\,\gamma^{1-p(N+2)}\cdot |J|^{p/2}\,|I|^{p/2}\|\alpha\|_{L^2}^p\\ &= C_{N,p}\,\gamma^{2-2p(N+2)} \le C_p\gamma^{-\delta}.\end{aligned}$$

Thus, by Fefferman's theorem, the boundedness of R is proved.

§6. The conclusion of the proof

The proof of (7) can be found in [6]. We give its sketch here, because we shall need some intermediate inequalities. Suppose I is an interval, y_0 is its center, $y \in I$, and $x \notin 2\cdot I$. Let $P_{u,k}(t)$ denote the Taylor polynomial of ψ_k at the point u and of degree N. Put $p_k(x,y) = P_{x-y_0,k}(x-y)$ and $p_I(x,y) = \{p_k(x,y)\}_{k=0}^L$. Since the function ψ belongs to the Schwartz class, recalling the definition of the functions ψ_k, we easily obtain

$$|\psi_k(x-y) - p_k(x,y)| \le C_{N,l}\,A^{k(N+2)}\,|y-y_0|^{N+1}\,(1+A^k|x-y_0|)^{-l}, \quad l = 1,2,\dots .$$

Letting $l=0$ and $l=N+3$, we obtain

$$|\psi_k(x-y) - p_k(x,y)| \le C_N\,A^{k(N+2)}\,|I|^{N+1}, \tag{8}$$

$$|\psi_k(x-y) - p_k(x,y)| \le C_N\,A^{-k}\frac{|I|^{N+1}}{|x-y_0|^{N+3}}. \tag{9}$$

Estimate (9) is better than (8) exactly when $A^k \ge \frac{1}{|x-y_0|}$. We split the square of the ℓ^2-norm in (7) into two sums: the first over all k such that $A^k \ge \frac{1}{|x-y_0|}$ and the second over all remaining k. Using (8) and (9), we obtain the required estimate.

In the proofs of (8) and (9), we have used only the fact that ψ belongs to the Schwartz class. Therefore, the same estimates are true for the functions φ_k that occur in the definition of the operator S ($A=2$).

To conclude the proof, it remains to verify Lemma 4. For $N=0$ and $\gamma=2$, a similar statement was established in [1], but there the pairs (k,j) did not range over the entire set $\mathbb{Z}^2$ (though the proof in [1] can be adjusted to our situation). Another statement similar to Lemma 4 was proved in [6] (for all $N\ge 0$ and $\gamma=2$), but a slightly different setting was considered there. Despite those differences, our proof will be almost the same as the corresponding argument in [6].

Using the Taylor polynomials for the functions φ_k, we define the polynomials $p_k(x,y)$ in the same way as we did for the functions ψ_k. As mentioned above, estimates (8) and (9) are true for the functions φ_k for $A=2$. We introduce a function $p_I(x,y)$ by the formula

$$p_I(x,y) = \big\{e^{2\pi i\,2^k j\,x} p_k(x,y)\big\}_{(k,j)\in\mathbb{Z}^2}.$$

Also denote

$$r_{k,s} = \sup_{x\in I_s, y\in I} |\varphi_k(x-y) - p_k(x,y)|.$$

In this notation, we have

$$
\begin{aligned}
&\Big(\int_{I_s} \big|\langle \kappa(x,y) - p_I(x,y), \xi\rangle\big|^2\,dx\Big)^{1/2} \\
&\qquad \le \Big(\int_{I_s} \Big(\sum_{k\in\mathbb{Z}} |\varphi_k(x-y) - p_k(x,y)| \Big|\sum_{j\in\mathbb{Z}} \xi_{k,j}\, e^{2\pi i\, 2^k j\, x}\Big|\Big)^2\,dx\Big)^{1/2} \\
&\qquad \le \sum_k r_{k,s} \Big(\int_{I_s} \Big|\sum_j \xi_{k,j}\, e^{2\pi i\, 2^k j\, x}\Big|^2\,dx\Big)^{1/2} \\
&\qquad \le \Big(\sum_k r_{k,s}\Big)^{1/2} \Big(\sum_k r_{k,s} \int_{I_s} \Big|\sum_j \xi_{k,j}\, e^{2\pi i\, 2^k j\, x}\Big|^2 dx\Big)^{1/2};
\end{aligned}
\tag{10}
$$

here we have used the triangle inequality in L^2 and the Cauchy inequality for sums. Next, by (8) and (9), we obtain

$$
r_{k,s} \le C_N\, 2^{k(N+2)}\, |I|^{N+1}, \quad r_{k,s} \le C_N\, 2^{-k}\, |I|^{-2}\, \gamma^{-(N+3)s}. \tag{11}
$$

The second estimate is stronger than the first exactly when $2^k \ge |I|^{-1}\gamma^{-s}$. Thus,

$$
\begin{aligned}
\sum_k r_{k,s} &\le C_N \sum_{k:2^k<|I|^{-1}\gamma^{-s}} 2^{k(N+2)}\, |I|^{N+1} + C_N \sum_{k:2^k\ge|I|^{-1}\gamma^{-s}} 2^{-k}\, |I|^{-2}\, \gamma^{-(N+3)s} \\
&\le C'_N\, \gamma^{-s(N+2)}\, |I|^{-1}.
\end{aligned}
$$

It remains to estimate the second factor in the last expression in (10). Substituting $t = 2^k x$ and using the Riesz–Fischer theorem, we have

$$
\int_{\gamma^{s+1}\cdot I} \Big|\sum_j \xi_{k,j}\, e^{2\pi i\, 2^k j\, x}\Big|^2 dx \le \begin{cases} C\,\gamma^{s+1}\, |I| \sum_j |\xi_{k,j}|^2 & \text{if } \gamma^{s+1}|I| \ge 2^{-k}, \\ 2^{-k} \sum_j |\xi_{k,j}|^2 & \text{if } \gamma^{s+1}|I| < 2^{-k}. \end{cases}
$$

By these estimates and inequalities (11), we obtain

$$
\begin{aligned}
&\sum_k r_{k,s} \int_{I_s} \Big|\sum_j \xi_{k,j}\, e^{2\pi i\, 2^k j\, x}\Big|^2 dx \\
&\qquad \le C\, |\xi|^2_{\ell^2} \Big(\sum_{k:2^k<|I|^{-1}\gamma^{-s-1}} r_{k,s}\, 2^{-k} + \sum_{k:2^k\ge|I|^{-1}\gamma^{-s-1}} r_{k,s}\, \gamma^{s+1}\, |I|\Big) \\
&\qquad \le C_N\, |\xi|^2_{\ell^2} \Big(\sum_{k:2^k<|I|^{-1}\gamma^{-s-1}} 2^{k(N+1)}\, |I|^{N+1} + \sum_{k:2^k\ge|I|^{-1}\gamma^{-s-1}} 2^{-k}\, \gamma^{-(N+2)s+1}\, |I|^{-1}\Big) \\
&\qquad \le C'_N\, |\xi|^2_{\ell^2}\, \gamma^{-(N+1)s+2}.
\end{aligned}
$$

Putting $B_N = N + 1/2$, we conclude the proof.

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