Алгебра и анализ
Том 22 (2010), № 4

St. Petersburg Math. J.
Vol. 22 (2011), No. 4, Pages 657–668
S 1061-0022(2011)01161-6
Article electronically published on May 3, 2011

HÖLDER FUNCTIONS ARE OPERATOR-HÖLDER

L. N. NIKOL′SKAYA AND YU. B. FARFOROVSKAYA

Abstract. Let A and B be selfadjoint operators in a separable Hilbert space such that $A-B$ is bounded. If a function f satisfies the Hölder condition of order α, $0<\alpha<1$, i.e., $|f(x)-f(y)|\le L|x-y|^\alpha$, then $\|f(A)-f(B)\|\le CL\|A-B\|^\alpha$, where C is a constant, specifically, $C=2^{1-\alpha}+2\pi\sqrt{8}\frac{1}{(1-2^{\alpha-1})^2}$. This result is a consequence of a general inequality in which the norm of $f(A)-f(B)$ is controlled in terms of the continuity modulus of f. Similar results are true for the quasicommutators $f(A)K-Kf(B)$, where K is a bounded operator.

Introduction

In this paper, the following natural question is resolved. Is it true that a function f satisfying a Hölder condition of order α gives rise to an "operator-Hölder" function? In other words, do arbitrary selfadjoint operators on a Hilbert space satisfy the inequality

$$\|f(A)-f(B)\|\le \mathrm{const}\cdot\|A-B\|^\alpha? \tag{1}$$

We mention at once that, for $\alpha=1$, the question was answered in the negative (see [1]). So, it seemed natural to suppose a similar feature for every α. However, for $\alpha<1$ the answer is in the positive. Specifically, if A and B are selfadjoint operators and f belongs to the Hölder class with exponent $\alpha<1$, then

$$\|f(A)-f(B)\|\le CL\|A-B\|^\alpha, \tag{1$'$}$$

where $C=2^{1-\alpha}+2\pi\sqrt{8}\frac{1}{(1-2^{\alpha-1})^2}$.

A similar result was obtained almost simultaneously by Aleksandrov and Peller (see [2]) by different methods. Inequality (1′) is a consequence of a more general estimate in which the norm of $f(A)-f(B)$ is controlled in terms of the modulus of continuity of f (see also [3]). The paper consists of two sections. In §1 the main lemmas are discussed (partially, they were proved in [3] and [4]). Mostly, they yield certain norm estimates for Hadamard–Schur multipliers. The theorems are proved in §2.

§1. Main lemmas

Recall that the *Hadamard product* of two matrices $M=(m_{ij})$ and $A=(a_{ij})$ is the matrix $M\circ A=(m_{ij}a_{ij})$. A matrix M is called an *Hadamard–Schur* multiplier if

$$\|M\|_{\mathcal H}=\sup\{\|M\circ A\|\,\big|\,A:l^2\to l^2,\ \|A\|\le 1\}<\infty.$$

In what follows, finite or infinite matrices are treated as operators on l^2, $\|M\|$ is the operator norm, and $\|M\|_{\mathcal H}$ denotes the multiplier norm. Let φ be a function defined on $\mathbb Z_+$ and taking values in $\mathbb Z_+\cup\{\infty\}$. Put

$$\sigma_\varphi=\{(i,j)\in\mathbb Z_+\times\mathbb Z_+:\varphi(j)\le i\}$$

2010 *Mathematics Subject Classification.* Primary 47A56, 47A60, 47B25.
Key words and phrases. Operator-Hölder functions, Adamar–Schur multipliers.

and denote $n = \operatorname{card}(\varphi(\mathbb{Z}_+))$ (i.e., n is the number of different values of φ; we agree that $n = \infty$ if $\varphi(\mathbb{Z}_+)$ is infinite). Next, let $M_\varphi = \chi_{\sigma_\varphi}$ be the characteristic function (matrix) of σ_φ, i.e., M_φ consists of entries of the form

$$m_{ij} = \begin{cases} 1, & \varphi(j) \le i, \\ 0, & \varphi(j) > i. \end{cases}$$

The following lemma was proved in [4].

Lemma 1. *The matrix M_φ is an Hadamard–Schur multiplier if and only if $n < \infty$. In this case*

$$c \log(n+1) \le \|M_\varphi\|_{\mathcal{H}} \le 1 + \log n,$$

where c is a universal constant.

Remark 2. Lemma 1 is a generalization of the well-known Matsaev triangle projection theorem; see [5].

Remark 3. The claim of Lemma 1 will not change if we take

$$\sigma_\varphi = \{(i,j) \in \mathbb{Z}_+ \times \mathbb{Z}_+ \,:\, \varphi(i) \le j\}.$$

We also note that the lemma remains true for functions defined on $\mathbb{Z}$.

We describe a particular case to be used in what follows. Fix two integers k and r ($r > 0$) and consider the following matrix $M_r = (m_{ij})$ (the "block-triangle projection" multiplier):

1) all entries of M_r outside the vertical strip $k < j \le k + 2^r$ are zero;
2) the strip $(m_{ij})_{k<j\le k+2^r}$ is decomposed in square blocks of size $2^r \times 2^r$ such that
3) every block is the upper (or lower) triangular projection of the square matrix of 1's.

Corollary 4 (to Lemma 1)**.** *We have*

$$\|M_r\|_{\mathcal{H}} \le r + 1.$$

For the proof, it suffices to define φ in the following way. If m is such that $m2^r < i \le (m+1)2^r$, we put $\varphi(i) = i - m2^r$. By Lemma 1, we have $\|M_\varphi\|_{\mathcal{H}} \le r + 1$. Clearly, $\|M_r\|_{\mathcal{H}} \le \|M_\varphi\|_{\mathcal{H}}$.

Obviously, Corollary 4 can be extended to strips of arbitrary length. The next lemma is an elementary generalization of the Bennet lemma [6]. For completeness, we prove this lemma in the vector case. Let $M = (m_{ij})$ be a scalar matrix, and $V = (V_{ij})$ a block matrix consisting of square blocks of the same size. We denote by V_j the block "column matrix", i.e., $V_j = J \circ V$, where $J = (a_{ik})_{i,k\in\mathbb{Z}}$; $a_{ik} = 1$, for $k = j$, $i \in \mathbb{Z}$, and $a_{ik} = 0$ for $k \ne j$.

Lemma 5. $\|M \circ V\| \le \sup_i \left(\sum_j |m_{ij}|^2\right)^{\frac{1}{2}} \sup_j \|V_j\|$.

Proof. Consider the family (P_j) of projections in $l_2(\mathbb{Z})$ corresponding to the block decomposition of the matrix V ($\sum P_j = I$, $P_k P_j = 0$ for $k \ne j$). For $X \in l_2(\mathbb{Z})$, put

$X_j = P_j X$. Then

$$\begin{aligned}\|(M\circ V)X\|^2 &= \sum_i \Big\| \sum_j m_{ij} V_{ij} X_j \Big\|^2 \le \sum_i \Big(\sum_j |m_{ij}|\, \|V_{ij}X_j\| \Big)^2 \\ &\le \sum_i \Big(\sum_j |m_{ij}|^2 \cdot \sum_j \|V_{ij}X_j\|^2 \Big) \le \sup_i \Big(\sum_j |m_{ij}|^2 \Big) \sum_j \sum_i \|V_{ij}X_j\|^2 \\ &= \sup_i \Big(\sum_j |m_{ij}|^2 \Big) \sum_j \|V_j X_j\|^2 \le \sup_i \Big(\sum_j |m_{ij}|^2 \Big) \sum_j \|V_j\|^2 \|X_j\|^2 \\ &\le \sup_i \Big(\sum_j |m_{ij}|^2 \Big) \sup_j \|V_j\|^2 \|X\|^2. \qquad \square\end{aligned}$$

Now we fix a positive number $h > 0$ and put

$$T_h = \begin{pmatrix} \frac{1}{h} & \frac{1}{h+1} & \frac{1}{h+2} & \dots \\ \frac{1}{h+1} & \frac{1}{h} & \frac{1}{h+1} & \dots \\ \frac{1}{h+2} & \frac{1}{h+1} & \frac{1}{h} & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}, \quad H_h = \begin{pmatrix} \frac{1}{h} & \frac{1}{h+1} & \frac{1}{h+2} & \dots \\ \frac{1}{h+1} & \frac{1}{h+2} & \frac{1}{h+3} & \dots \\ \frac{1}{h+2} & \frac{1}{h+3} & \frac{1}{h+4} & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}.$$

The following lemma was proved in [3].

Lemma 6.

$$\|T_h\|_{\mathcal{H}} = \frac{1}{h}, \quad \|H_h\|_{\mathcal{H}} \le \frac{1}{h}.$$

Remark 7 (to Lemma 6). T_h is a Toeplitz matrix:

$$T_h = (m_{ij})_{i,j\ge 1}, \quad m_{ij} = \frac{1}{|j-i|+h} = c_{i-j}, \quad c_n = \frac{1}{h+|n|}, \quad n \in \mathbb{Z}.$$

Consider the matrix $Y_h = (m_{ij})_{i,j\le 0}$ whose entries are defined by the same formulas.

The matrix Y_h has the form

$$Y_h = \begin{pmatrix} \dots & \dots & \dots & \dots \\ \dots & \frac{1}{h} & \frac{1}{h+1} & \frac{1}{h+2} \\ \dots & \frac{1}{h+1} & \frac{1}{h} & \frac{1}{h+1} \\ \dots & \frac{1}{h+2} & \frac{1}{h+1} & \frac{1}{h} \end{pmatrix},$$

and $\|Y_h\|_{\mathcal{H}} = \frac{1}{h}$.

We shall need yet another obvious statement; we formulate it as a lemma.

Lemma 8. *Let $(a_i)_i$ be a bounded sequence, and let $M = (m_{ij})$, where $m_{ij} = a_i$. Then $\|M\|_{\mathcal{H}} \le \sup_i |a_i|$.*

The next lemma was proved in [3].

Lemma 9. *Let A and B be commuting selfadjoint operators in a separable Hilbert space such that $A - B$ is bounded, and let f be a continuous function on the reals. Let ω_f denote the modulus of continuity of f. Then*

$$\|f(A) - f(B)\| \le \omega_f(\|A - B\|). \tag{2}$$

§2. Theorems

Theorem 1. *Let A and B be selfadjoint operators in a separable Hilbert space; suppose $A-B$ is bounded. Next, let f be a continuous function on the real line whose modulus of continuity is ω_f. Then*

$$\|f(A)-f(B)\| \le 2\omega_f\Big(\frac{\|A-B\|}{2}\Big) + 2\pi\sqrt{8}\sum_{r=0}^{\infty}\frac{r+1}{2^r}\omega_f(\|A-B\|2^r).$$

Proof. We split the real line into infinitely many intervals of equal length by points $\{\lambda_i\}$ with $\lambda_{i+1}-\lambda_i = \|A-B\| = d$ and define a piecewise constant function φ as follows: we put $\varphi(t)=\lambda_i$ for $\lambda_i - \frac{d}{2} \le t < \lambda_i + \frac{d}{2}$. Next, consider the operators $A_1=\varphi(A)$ and $B_1=\varphi(B)$. The spectra of A and B are included in the set $\{\lambda_i\}$, and moreover, A commutes with A_1 and B commutes with B_1.

Since $\|A-A_1\| \le \frac{d}{2}$ and $\|B-B_1\| \le \frac{d}{2}$, by Lemma 9 we have

$$\|f(A)-f(A_1)\| \le \omega_f\Big(\frac{d}{2}\Big), \quad \|f(B)-f(B_1)\| \le \omega_f\Big(\frac{d}{2}\Big). \tag{3}$$

Thus,

$$\|f(A)-f(B)\| \le 2\omega_f\Big(\frac{\|A-B\|}{2}\Big) + \|f(A_1)-f(B_1)\|. \tag{4}$$

Our goal is to estimate the norm of $f(A_1)-f(B_1)$. We denote by P_i and Q_j the orthogonal projections onto the eigenspaces of A_1 and B_1 corresponding, respectively, to the points λ_i and λ_j. (If λ_i is not an eigenvalue, we agree that the corresponding projection is zero.) Since $P_iA_1 = A_1P_i = \lambda_iP_i$ and $Q_jB_1 = \lambda_jQ_j$, we have

$$P_iQ_j = \frac{P_i(A_1-B_1)Q_j}{\lambda_i-\lambda_j}, \quad i \ne j.$$

Therefore, $f(A_1)-f(B_1)$ can be represented as follows:

$$f(A_1)-f(B_1) = \sum_{i,j}\frac{f(\lambda_j)-f(\lambda_i)}{\lambda_j-\lambda_i}P_i(A_1-B_1)Q_j. \tag{5}$$

(By definition, $\frac{f(\lambda_j)-f(\lambda_i)}{\lambda_j-\lambda_i} = 0$ if $f(\lambda_j)-f(\lambda_i)=0$.) We observe that formula (5) represents $f(A_1)-f(B_1)$ as a double operator integral (see [7] and also [8]). On the other hand, (5) allows us to view $f(A_1)-f(B_1)$ as an operator matrix $F = M\circ T$, where

$$M = \Big(\frac{f(\lambda_j)-f(\lambda_i)}{\lambda_j-\lambda_i}\Big)_{i,j\in\mathbb{Z}}, \qquad T = (P_i(A_1-B_1)Q_j)_{i,j\in\mathbb{Z}} = (T_{ij})_{i,j\in\mathbb{Z}}.$$

Thus, the problem reduces to estimating the norm of the multiplier M. The passage from an estimation of the norm $\|f(A)-f(B)\|$ to that of the multiplier M will be called *quantization of the spectrum.* □

In the sequel, we consider the upper-triangular projection of the matrix F, to be denoted by $F_{\rm up}$. Since $\lambda_j-\lambda_i = d\cdot(j-i)$, the matrix $F_{\rm up}$ has the form

$$F_{\rm up} = \frac{1}{d}\Big(\frac{f(\lambda_j)-f(\lambda_i)}{j-i}T_{ij}\Big)_{i,j\in\mathbb{Z},\ j>i}$$

(all other entries are equal to zero). Clearly,

$$\|f(A_1)-f(B_1)\| \le 2\|F_{\rm up}\|. \tag{6}$$

We want to represent F_{up} as a sum $F_{\mathrm{up}} = \sum_{r=0}^{\infty} F_r$ of matrices of a special form. The matrices F_r are constructed as follows. Fixing two indices i and j $(j \geq i+1)$, we expand $j-i$ in decreasing powers of 2:

$$j - i = 2^{r_1} + 2^{r_2} + \cdots + 2^{r_l} + 2^r + 2^{r_{l+2}} + \cdots + 2^{r_p} \tag{7}$$

(we assume that the summand 2^r does occur in (7)). Accordingly,

$$\begin{aligned}\frac{f(\lambda_j) - f(\lambda_i)}{j-i} = {} & \frac{f(\lambda_{i+2^{r_1}}) - f(\lambda_i)}{j-i} \\ & + \cdots + \frac{f(\lambda_{i+2^{r_1}+\cdots+2^{r_l}+2^r}) - f(\lambda_{i+2^{r_1}+\cdots+2^{r_l}})}{j-i} \\ & + \cdots + \frac{f(\lambda_{i+2^{r_1}+\cdots+2^{r_p}}) - f(\lambda_{i+2^{r_1}+\cdots+2^{r_{p-1}}})}{j-i}.\end{aligned}$$

By definition, we put

$$F_r = \frac{1}{d}\left(\frac{f(\lambda_{i+2^{r_1}+\cdots+2^{r_l}+2^r}) - f(\lambda_{i+2^{r_1}+\cdots+2^{r_l}})}{j-i} T_{ij}\right)_{j>i,\ i\in\mathbb{Z}}.$$

In more detail, the matrix entries of F_r are of the following nature. For the numbers $i + 2^{r_1} + \cdots + 2^{r_l} + 2^r$ and $i + 2^{r_1} + \cdots + 2^{r_l}$, there is $k \geq 1$ such that

$$i + 2^{r_1} + \cdots + 2^{r_l} + 2^r = i + k2^{r+1} - 2^r, \ \ i + 2^{r_1} + \cdots + 2^{r_l} = i + k2^{r+1} - 2^{r+1}.$$

Thus, for i (the row number) fixed, the nonzero entries of F_r have the form

$$\frac{1}{d} \cdot \frac{f(\lambda_{i+k2^{r+1}-2^r}) - f(\lambda_{i+k2^{r+1}-2^{r+1}})}{j-i} T_{ij}, \quad j > i.$$

Clearly, the numerators of the Hadamard multipliers (for i and k fixed) are the same for all values of j in the interval $i + k2^{r+1} - 2^r \leq j \leq i + k2^{r+1} - 1$ of "length" 2^r. If j satisfies $i + (k-1)2^{r+1} \leq j \leq i + k2^{r+1} - 2^r - 1$, then the summand 2^r does not occur in the dyadic expansion of $j - i$ (i is fixed). Therefore, the corresponding entries of F_r fill strips of width 2^r parallel to the main diagonal. The distance between the strips is also equal to 2^r. For $r = 2$, the nonzero entries of F_r are shown by dots in Figure 1. We split

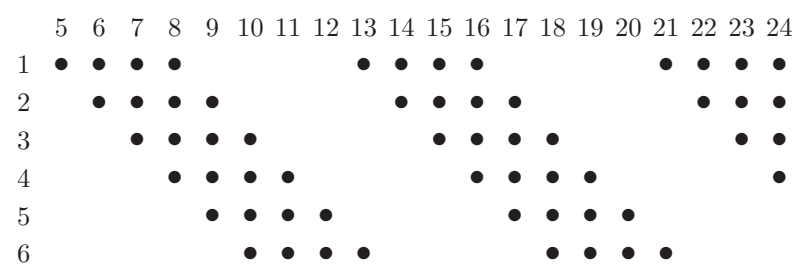

Figure 1

F_r into blocks of height 2^r and width $2 \cdot 2^r$: $F_r = \frac{1}{d}(B_{pk})_{p\in\mathbb{Z},\ k\geq 1}$, where $B_{pk} = (m_{ij})$,

$$p2^r + 1 \leq i \leq (p+1)2^r, \ (p-1)2^r + k2^{r+1} + 1 \leq j \leq (p+1)2^r + 2k2^r.$$

The nonzero entries m_{ij} of B_{pk} are of the form

$$m_{ij} = \frac{f(\lambda_{i+(2k-1)2^r}) - f(\lambda_{i+(2k-2)2^r})}{j-i} P_i T Q_j, \quad i + 2^{r+1}k - 2^r \leq j \leq i + 2^{r+1}k - 1.$$

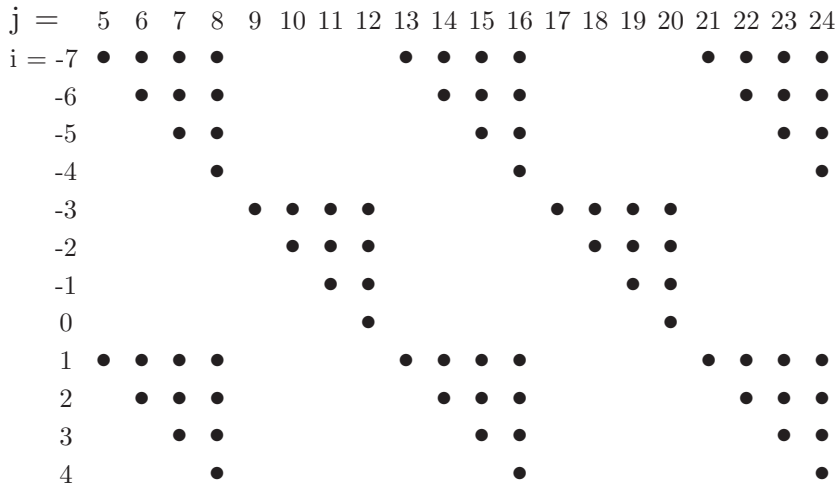

FIGURE 2

Each block can be represented as the union of two blocks $B_{pk}^{(1)}$ and $B_{pk}^{(2)}$ as follows:

$$B_{pk}^{(1)} = (m_{ij}),\ p2^r + 1 \le i \le (p+1)2^r,\ (p-1)2^r + 2k2^r + 1 \le j \le p2^r + 2k2^r,$$

and the nonzero entries m_{ij} of $B_{pk}^{(1)}$ are of the form

$$m_{ij} = \frac{f(\lambda_{i+(2k-1)2^r}) - f(\lambda_{i+(2k-2)2^r})}{j-i} P_i T Q_j,$$
$$p2^r + 1 \le i \le (p+1)2^r, \quad i + 2^r(2k-1) \le j \le p2^r + k2^{r+1}.$$

That is, the block $B_{pk}^{(1)}$ is an upper-triangular square matrix (including the main diagonal) of size $2^r \times 2^r$. The complementary block $B_{pk}^{(2)} = (m_{ij})$,

$$p2^r + 1 \le i \le (p+1)2^r, \quad (p+2k)2^r + 1 \le j \le (p+2k+1)2^r,$$

is also a square matrix of size $2^r \times 2^r$, which is strictly lower-triangular. Its nonzero entries are of the same form as those of $B_{pk}^{(1)}$.

Thus, the matrix F_r is split into square blocks of size $2^r \times 2^r$. We represent it as the sum $F_r = F_r^{(1)} + F_r^{(2)}$ of two block matrices of the following form. To obtain $F_r^{(1)}$, we replace the blocks $B_{pk}^{(2)}$ in F_r with zero blocks of the same size and leave the blocks $B_{pk}^{(1)}$ as they are. Similarly, to obtain $F_r^{(2)}$, we replace the blocks $B_{pk}^{(1)}$ in F_r with zero blocks, leaving the $B_{pk}^{(2)}$ intact. Clearly, $F_r = F_r^{(1)} + F_r^{(2)}$. Our nearest goal is to estimate the norm of the operator $F_r^{(1)}$. We observe that there is a zero block between each two neighboring blocks of $F_r^{(1)}$. We enumerate the nonzero blocks as described below. For fixed $p \in \mathbb{Z}$ (the number of a horizontal strip), the nonzero blocks are situated in vertical strips with numbers $m = p + 2k - 1,\ k \ge 1$. Each of these blocks has upper-triangular form. Their arrangement is clear from Figure 2 ($r = 2$).

Nonzero strips parallel to the main diagonal are enumerated by $k \ge 1$. In each strip (with k fixed), we represent the blocks $B_{pk}^{(1)}$ in the form

$$B_{pk}^{(1)} = B_{m-(2k-1),k}^{(1)} = \frac{1}{2k-1} B_{m-(2k-1),k}^{(1)} \cdot (2k-1) = \frac{1}{2k-1} D_{m-(2k-1),k}.$$

The matrix $F_r^{(1)}$ takes the form

$$\begin{array}{cccccccccccc}
0 & D_{-11} & 0 & \frac{1}{3}D_{-12} & 0 & \frac{1}{5}D_{-13} & 0 & \dots & \dots & \dots & \dots \\
 & 0 & D_{01} & 0 & \frac{1}{3}D_{02} & 0 & \frac{1}{5}D_{03} & 0 & \dots & \dots & \dots \\
 & & 0 & D_{11} & 0 & \frac{1}{3}D_{12} & 0 & \frac{1}{5}D_{13} & 0 & \dots & \dots \\
 & & & 0 & D_{21} & 0 & \frac{1}{3}D_{22} & 0 & \frac{1}{5}D_{23} & 0 & \dots
\end{array}$$

Clearly, it satisfies the assumptions of Lemma 5. Since $\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}$, we have

$$\|F_r^{(1)}\| \le \frac{\pi}{\sqrt{8}} \sup_m \|V_m^{(1)}\|, \tag{8}$$

where $V_m^{(1)}$, $m \in \mathbb{Z}$, is the vertical strip cut out of the matrix $F_r^{(1)}$ and consisting of blocks of size $2^r \times 2^r$; the nonzero blocks among them have been denoted by D_{pk}, $p = m-2k+1$, $k \ge 1$. All blocks below the main diagonal are zero. For fixed $k \ge 1$ and $p = m-2k+1$, the block D_{pk} has the form $D_{pk} = \frac{1}{2^r d}(m_{ij})$, where

$$\begin{aligned}
&m_{ij} = \frac{f(\lambda_{i+(2k-1)2^r}) - f(\lambda_{i+(2k-2)2^r})}{j-i}\,(2k-1)2^r T_{ij}, \\
&p2^r + 1 \le i \le (p+1)2^r, \quad i + k2^{r+1} - 2^r \le j \le p2^r + 2^{r+1}k; \\
&k \ge 1, \quad p = m - 2k - 1,
\end{aligned} \tag{9}$$

and $m_{ij} = 0$ if $(p-1)2^r + 1 + k2^r < j < i + k2^{r+1} - 2^r$.

We represent the matrix $V_m^{(1)}$ as the Hadamard product $V_m^{(1)} = V_m \circ T$, the entries of V_m being the Hadamard multipliers in (9). The quantities $f(\lambda_{i+(2k-1)2^r}) - f(\lambda_{i+(2k-2)2^r})$ do not depend on j. Therefore, Lemma 8 is applicable, yielding

$$\|V_m\|_{\mathcal{H}} \le \frac{1}{d2^r} \sup_{i,k} |f(\lambda_{i+(2k-1)2^r}) - f(\lambda_{i+(2k-2)2^r})| \cdot \|W_m\|_{\mathcal{H}}, \tag{10}$$

where W_m is the vertical strip consisting of triangular blocks G_{pk} whose nonzero entries are equal to $\frac{(2k-1)2^r}{j-i}$:

$$G_{pk} = \Big(\frac{(2k-1)2^r}{j-i}\Big)_{p2^r+1 \le i \le (p+1)2^r,\ i+k2^{r+1}-2^r \le j \le p2^r+2^{r+1}k}, \tag{11}$$

$p = m - 2k + 1$, $k \ge 1$. The entries below the diagonal are zero.

Since the indices of the points $\lambda_{i+(2k-1)2^r}$ and $\lambda_{i+(2k-2)2^r}$ differ by 2^r, (10) leads to the inequality

$$\|V_m\|_{\mathcal{H}} \le \frac{\omega_f(d2^r)}{d2^r} \|W_m\|_{\mathcal{H}}, \quad d = \|A - B\|. \tag{12}$$

We estimate the norm of the multiplier W_m. First, we rewrite the nonzero block G_{pk}, $p = m-2k+1$, in terms of m and k. The nonzero entries $\frac{(2k-1)2^r}{j-i}$ correspond to the following values of i and j:

$$\begin{aligned}
&(m-2k+1)2^r + 1 \le i \le (m-2k+2)2^r, \\
&\quad i + 2^{r+1}k - 2^r \le j \le (m+1)2^r; \quad k \ge 1,\ \ p = m - 2k + 1.
\end{aligned} \tag{13}$$

The entries strictly below the diagonal are zero.

Since

$$\frac{(2k-1)2^r}{j-i} = 1 - \frac{(j-i) - (2k-1)2^r}{j-i},$$

the matrix W_m can be represented as the sum $W_m = I_m + J_m$, where I_m is the multiplier of block-triangular projecting, as was described in Corollary 4 (it is supplemented with zero blocks).

The matrix J_m also has the form of a vertical strip consisting of upper-triangular blocks of size $2^r \times 2^r$, but the entries of the upper triangle are $m_{ij} = \frac{(j-i)-(2k-1)2^r}{j-i}$. Corollary 4 leads to the estimate

$$\|I_m\|_{\mathcal{H}} \le r+1. \tag{14}$$

The nonzero blocks of the matrix J_m (we denote them by J_{mk}) have the form

$$\begin{array}{c} \\ (m-2k+1)2^r+1 \\ (m-2k+1)2^r+2 \\ (m-2k+1)2^r+3 \\ \vdots \\ (m-2k+2)2^r-1 \\ (m-2k+2)2^r \end{array} \begin{array}{c} \begin{array}{cccccc} m2^r+1 & m2^r+2 & m2^r+3 & m2^r+4 & \cdots & (m+1)2^r \end{array} \\ \begin{pmatrix} 0 & \frac{1}{(2k-1)2^r+1} & \frac{2}{(2k-1)2^r+2} & \frac{3}{(2k-1)2^r+3} & \cdots & \frac{2^r-1}{2k2^r-1} \\ 0 & 0 & \frac{1}{(2k-1)2^r+1} & \frac{2}{(2k-1)2^r+2} & \cdots & \frac{2^r-2}{2k2^r-2} \\ 0 & 0 & 0 & \frac{1}{(2k-1)2^r+1} & \cdots & \frac{2^r-3}{2k2^r-3} \\ & & & & & \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{(2k-1)2^r+1} \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \end{array} = J_{mk}.$$

That is, all strips J_m are similar, up to parallel translation along the main diagonal.

Now, we represent the matrix J_{mk} (a nonzero block) as a sum:

$$J_{mk} = \sum_{n=1}^{2^r-1} J_{mkn},$$

where $J_{mkn} = (m_{ij})$,

$$\begin{gathered} (m-2k+1)2^r+1 \le i \le (m-2k+2)2^r, \\ m2^r+1 \le j \le (m+1)2^r. \end{gathered} \tag{15}$$

We have $m_{ij} = \frac{1}{j-i}$ for

$$\begin{gathered} (m-2k+1)2^r+1 \le i \le (m-2k+2)2^r - n, \\ i + k2^{r+1} - 2^r + n \le j \le (m+1)2^r. \end{gathered} \tag{16}$$

For all other values of $i,\ j$, we have $m_{ij} = 0$. In other words,

$$J_{mkn} = \begin{pmatrix} 0 & \dots & 0 & \frac{1}{(2k-1)2^r+n} & \frac{1}{(2k-1)2^r+n+1} & \cdots & \frac{1}{2k\cdot 2^r-1} \\ 0 & \dots & 0 & 0 & \frac{1}{(2k-1)2^r+n} & \cdots & \frac{1}{2k\cdot 2^r-2} \\ \vdots & & \vdots & 0 & 0 & \dots & \vdots \\ \vdots & & \vdots & \vdots & 0 & \dots & \frac{1}{(2k-1)2^r+n} \\ \vdots & & \vdots & \vdots & \vdots & & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & & 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Thus, the entire "block strip" J_m is represented as the sum of 2^r-1 block strips J_{mn}, $J_m = \sum_{n=1}^{2^r-1} J_{mn}$. The nonzero blocks J_{mkn} of the strip J_{mn} are described by formulas (15) and (16). The lowest block corresponds to $k=1$.

We return to Remark 7 (see §1). For $n = 1, \dots, 2^r - 1$, we put $h = \frac{1}{2^r+n}$ in the definition of the multiplier Y_h, and consider the matrices Y_{nr}:

$$Y_{nr} = \begin{pmatrix} \dots & \dots & \dots & \dots \\ \dots & \frac{1}{2^r+n} & \frac{1}{2^r+n+1} & \frac{1}{2^r+n+2} \\ \dots & \frac{1}{2^r+n+1} & \frac{1}{2^r+n} & \frac{1}{2^r+n+1} \\ \dots & \frac{1}{2^r+n+2} & \frac{1}{2^r+n+1} & \frac{1}{2^r+n} \\ \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots \\ \dots & 0 & 0 & 0 \end{pmatrix}$$

(n zero rows are added on the bottom; this does not change the multiplier norm). By Remark 7, $\|Y_{nr}\|_{\mathcal{H}} = \frac{1}{2^r+n}$.

Next, for every $n = 1, \dots, 2^r - 1$, consider the multiplier I_{mn} of upper-triangular projecting (a generalized block-triangular projection), where the matrix I_{mn} is the "vertical strip" of width 2^r that is split into square blocks I_{mkn} of size $2^r \times 2^r$, of which the nonzero blocks are like J_{mkn} with the only difference that all nonzero entries of I_{mkn} are equal to 1. We agree that the lowest block corresponds to the index $k = 1$ in the block representation of J_{mn}. The adjacent blocks are zero. That is, the "strip" I_{mn} has the following form (each block has n zero bottom rows):

$$I_{mn} = \begin{pmatrix} & & & \vdots & & & \\ & & & 0 & & & \\ 0 & \dots & 0 & 1 & 1 & \dots & 1 \\ 0 & \dots & 0 & 0 & 1 & \dots & 1 \\ \vdots & & \vdots & 0 & 0 & \dots & \vdots \\ \vdots & & \vdots & \vdots & 0 & \dots & 1 \\ \vdots & & \vdots & \vdots & \vdots & & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ & & & 0 & & & \\ 0 & \dots & 0 & 1 & 1 & \dots & 1 \\ 0 & \dots & 0 & 0 & 1 & \dots & 1 \\ \vdots & & \vdots & 0 & 0 & \dots & \vdots \\ \vdots & & \vdots & \vdots & 0 & \dots & 1 \\ \vdots & & \vdots & \vdots & \vdots & & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

Moreover, $I_{mn} \circ Y_{nr} = J_{mn}$, whence $\|J_{mn}\|_{\mathcal{H}} \le \|I_{mn}\|_{\mathcal{H}} \cdot \|Y_{mn}\|_{\mathcal{H}}$.

Every nonzero block of the matrix I_{mn} has $2^r - n < 2^r$ nonzero diagonals. Therefore (see Corollary 4),

$$\|J_{mn}\|_{\mathcal{H}} \le (r+1)\frac{1}{2^r+n}.$$

This leads to the estimate

$$\|J_m\|_{\mathcal{H}} \le \sum_{n=1}^{2^r-1} \frac{r+1}{2^r+n} \le (r+1)\frac{2^r-1}{2^r+1} \le r+1. \tag{17}$$

Taking (14) into account, we obtain

$$\|W_m\|_{\mathcal{H}} \le 2(r+1)$$

(because $W_m = I_m + J_m$). Thus, (12) turns into

$$\|V_m\|_{\mathcal{H}} \le \frac{2(r+1)\omega_f(d2^r)}{d2^r}. \tag{18}$$

Consequently, the norm of the vertical strip $V_m^{(1)} = V_m \circ T$ cut out of the matrix $F_r^{(1)}$ (see (8)) admits the following control in terms of the continuity modulus of f:

$$\|V_m^{(1)}\| \le \frac{2(r+1)\omega_f(d2^r)}{d2^r}\|T\| \le \frac{4(r+1)\omega_f(d2^r)}{2^r}$$

(because $\|T\| = \|A_1 - B_1\| \le 2\|A - B\| = 2d$).

By (8), we obtain

$$\|F_r^{(1)}\| \le \frac{\pi}{\sqrt{8}} \cdot \frac{4(r+1)\omega_f(\|A-B\|2^r)}{2^r}. \tag{19}$$

It remains to estimate the norm of $F_r^{(2)}$. The corresponding blocks $B_{pk}^{(2)}$ lie in the same vertical strips, but are shifted by 2^r and have lower-triangular form (without the main diagonal). We represent the block $B_{pk}^{(2)}$ in the form

$$B_{pk}^{(2)} = \frac{1}{2k} D_{m-2k,k}, \quad k \ge 1, \ \ p = m - 2k.$$

Since $\sum_{k=1}^{\infty} \frac{1}{(2k)^2} = \frac{\pi^2}{24}$, Lemma 5 implies the inequality

$$\|F_r^{(2)}\| \le \frac{\pi}{\sqrt{24}} \sup_m \|V_m^{(2)}\| < \frac{\pi}{\sqrt{8}} \sup_m \|V_m^{(2)}\|,$$

where $V_m^{(2)}$ is a strip similar to $V_m^{(1)}$.

The new blocks G_{pk} will be of lower-triangular form with the nonzero entries

$$\frac{2k \cdot 2^r}{j-i} = \frac{2k \cdot 2^r - (j-i)}{j-i} + 1,$$

and the new blocks J_{mk} of the new strip J_m will look like this:

$$J_{mk} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ \frac{1}{2k\cdot 2^r-1} & 0 & 0 & \dots & 0 \\ \frac{2}{2k\cdot 2^r-2} & \frac{1}{2k\cdot 2^r-1} & 0 & \dots & 0 \\ \vdots & \vdots & & & \\ \frac{2^r-1}{(2k-1)\cdot 2^r+1} & \frac{2^r-2}{(2k-1)\cdot 2^r+2} & \dots & \frac{1}{2k\cdot 2^r-1} & 0 \end{pmatrix}.$$

The norm of the new multiplier corresponding to J_{mn} is estimated as follows: $\|J_{mn}\|_{\mathcal{H}} \le \frac{r+1}{2^r+1}$, independently of n (see Corollary 4 and Lemma 7). Thus, the operator $F_r^{(2)}$ obeys (18). Consequently,

$$\|F_r\| \le \pi\sqrt{8} \cdot \frac{r+1}{2^r}\omega_f(\|A-B\|2^r). \tag{20}$$

Finally, by (4) and (6) we obtain

$$\|f(A)-f(B)\| \le 2\omega_f\Big(\frac{\|A-B\|}{2}\Big) + 2\pi\sqrt{8}\sum_{r=0}^{\infty}\frac{r+1}{2^r}\,\omega_f(\|A-B\|2^r). \tag{21}$$

If f belongs to the Hölder class of order α, $0<\alpha<1$, its continuity modulus satisfies the inequality $\omega_f(\|A-B\|2^r) \le L\|A-B\|^\alpha 2^{\alpha r}$. Therefore, the following theorem holds.

Theorem 2. *Let A and B be selfadjoint operators on a separable Hilbert space; suppose $A-B$ is bounded. If f is a Hölder function with $0<\alpha<1$, then*

$$\|f(A)-f(B)\| \le CL\|A-B\|^\alpha, \tag{22}$$

where C is a universal constant, $C = 2^{1-\alpha} + 2\pi\sqrt{8}\frac{1}{(1-2^{\alpha-1})^2}$.

Remark on quasicommutators. Almost without changes, the proofs of Theorems 1 and 2 can be extended to the case of quasicommutators.

A *quasicommutator* is an operator of the form $f(A)K - Kf(B)$, where K is a nonzero bounded operator, A and B are selfadjoint operators, and f is a continuous function on the reals (its modulus of continuity is denoted by ω_f). We assume that the operator $AK-KB$ is bounded.

We split the real line by points $\{\lambda_i\}$ in such a way that $\lambda_{i+1}-\lambda_i = \frac{\|AK-KB\|}{\|K\|} = d$. We have

$$\|f(A)K - Kf(B)\| \le 2\|K\|\omega_f\Big(\frac{d}{2}\Big) + \|f(A_1)K - Kf(B_1)\|, \tag{4'}$$

$$\|f(A_1)K - Kf(B_1)\| = \sum_{i,j}\frac{f(\lambda_j)-f(\lambda_i)}{\lambda_j-\lambda_i}P_i(A_1K-KB_1)Q_j, \tag{5'}$$

where A_1 and B_1 are the same operators as before.

The matrix T in the representation $F = M\circ T$ has the form

$$T = (P_i(A_1K-KB_1)Q_j)_{i,j\in\mathbb{Z}} = (T_{ij})_{i,j\in\mathbb{Z}}.$$

The multiplier M remains the same as before. Thus, the preceding calculations lead to inequality (18), whence

$$\|V_m^{(1)}\| \le \frac{2(r+1)\omega_f(d2^r)}{d2^r}\|T\|.$$

Since

$$\begin{aligned}\|T\| &\le \|K\|\|A-A_1\| + \|K\|\|B-B_1\| + \|AK-KB\|\\ &\le \|K\|\cdot\frac{d}{2} + \|K\|\cdot\frac{d}{2} + \frac{\|AK-KB\|}{\|K\|}\|K\| \le 2d\|K\|,\end{aligned}$$

we have

$$\|V_m^{(1)}\| \le \frac{4(r+1)\omega_f(d2^r)}{2^r}\|K\|$$

and

$$\|F_r\| \le \frac{\pi\sqrt{8}(r+1)\omega_f\Big(\frac{\|AK-KB\|}{\|K\|}2^r\Big)}{2^r}\cdot\|K\|.$$

Finally,

$$\begin{aligned}&\|f(A)K-Kf(B)\|\\ &\quad\le 2\|K\|\omega_f\Big(\frac{\|AK-KB\|}{2\|K\|}\Big) + 2\pi\sqrt{8}\|K\|\sum_{r=0}^{\infty}\frac{r+1}{2^r}\omega_f\Big(\frac{\|AK-KB\|}{\|K\|}2^r\Big).\end{aligned}$$

If f is Hölder continuous of order α, we have

$$\begin{aligned}\|f(A)K - Kf(B)\| &\le 2L\|K\|\frac{\|AK-KB\|^\alpha}{2^\alpha\|K\|^\alpha}\\ &\qquad + 2\pi\sqrt{8}L\sum_{r=0}^{\infty}\frac{r+1}{2^r}\cdot\frac{\|AK-KB\|^\alpha}{\|K\|^\alpha}\cdot 2^{r\alpha}\cdot\|K\|\\ &= L\|K\|^{1-\alpha}\cdot\|AK-KB\|^\alpha\left(2^{1-\alpha}+2\pi\sqrt{8}\sum_{r=0}^{\infty}\frac{r+1}{2^{r(1-\alpha)}}\right)\\ &= C\cdot L\|K\|^{1-\alpha}\|AK-KB\|^\alpha,\end{aligned}$$

where

$$C = 2^{1-\alpha} + 2\pi\sqrt{8}\frac{1}{(1-2^{\alpha-1})^2}.$$

References

[1] Yu. B. Farforovskaya, *The connection of the Kantorovich–Rubinshteĭn metric for spectral resolutions of selfadjoint operators with functions of operators*, Vestnik Leningrad. Univ. Mat. Mekh. Astronom. **1968**, vyp. 4, 94–97; English transl. in Vestnik Leningrad. Univ. Math. **1** (1974). MR0238103 (38:6379)

[2] A. Aleksandrov and V. Peller, *Functions of perturbed operators*, C. R. Math. Acad. Sci. Paris **347** (2009), no. 9–10, 483–488. MR2576894 (2010m:47019)

[3] Yu. B. Farforovskaya and L. Nikolskaya, *Modulus of continuity of operator functions*, Algebra i Analiz **20** (2008), no. 3, 224–242; English transl., St. Petersburg Math. J. **20** (2009), no. 3, 493–506. MR2454458 (2009i:47047)

[4] ———, *Toeplitz and Hankel matrices as Hadamard–Schur multipliers*, Algebra i Analiz **15** (2003), no. 6, 141–160; English transl., St. Petersburg Math. J. **15** (2004), no. 6, 915–928. MR2044634 (2005a:47050)

[5] V. I. Matsaev, *A class of completely continuous operators*, Dokl. Akad. Nauk SSSR **139** (1961), no. 3, 548–551; English transl. in Soviet Math. Dokl. **2** (1961). MR0131769 (24:A1617)

[6] G. Bennett, *Schur multipliers*, Duke Math. J. **44** (1977), 603–639. MR0493490 (58:12490)

[7] M. Sh. Birman and M. Z. Solomyak, *Double Stieltjes operator integrals*, Probl. Mat. Fiz., No. 1, Leningrad. Univ., Leningrad, 1966, pp. 33–67. (Russian) MR0209872 (35:767b)

[8] ———, *Double operator integrals in a Hilbert space*, Integral Equations Operator Theory **47** (2003), no. 2, 131–168. MR2002663 (2004f:47029)

Institut Mathématique de Bordeaux, Université Bordeaux-1, 351 Libération, 33405 Talence, France

E-mail address: `andreeva@math.u-bordeaux1.fr`

Division of Mathematics, M. A. Bonch-Bruevich St. Petersburg University of Telecommunication, Naberezhnaya Moĭki 61, St. Petersburg 191186, Russia

E-mail address: `rabk@sut.ru`

Received 15/NOV/2010

Translated by S. KISLYAKOV