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THE KREIN DIFFERENTIAL SYSTEM AND INTEGRAL OPERATORS OF RANDOM MATRIX THEORY

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Abstract. Earlier, the Krein differential system has been studied under certain regularity conditions. In this paper, some cases are treated where these conditions are not fulfilled. Examples related to random matrix theory are studied.

Introduction

Following [7] and [8], we consider the operator

$$S_r f = f(x) + \int_0^r k(x-t)f(t)\,dt, \quad 0 < r < \infty. \tag{0.1}$$

We assume that the operator S_r is positive definite and invertible, and that the function $k(t)$ is continuous and

$$k(t) = \overline{k(-t)}, \quad -r \le t \le r. \tag{0.2}$$

Then there exists a Hermitian resolvent $\Gamma_r(t,s) = \overline{\Gamma_r(s,t)}$ satisfying the relation

$$\Gamma_r(t,s) + \int_0^r k(t-u)\Gamma_r(u,s)\,du = -k(t-s), \quad 0 \le s,t \le r. \tag{0.3}$$

We set

$$P_1(r,z) = e^{irz}\left(1 - \int_0^r \Gamma_r(s,0)e^{-isz}\,ds\right), \tag{0.4}$$

$$P_2(r,z) = 1 - \int_0^r \Gamma_r(0,s)e^{isz}\,ds. \tag{0.5}$$

In [7], Krein deduced the differential system

$$\frac{dP_1(r,z)}{dr} = izP_1(r,z) - \overline{A(r)}P_2(r,z), \quad \frac{dP_2(r,z)}{dr} = -A(r)P_1(r,z), \tag{0.6}$$

where

$$A(r) = \Gamma_r(0,r), \tag{0.7}$$

and proved [7] that there exists a monotone nondecreasing function $\sigma(\lambda)$ (spectral function) such that the operator

$$(Uf)(z) = \int_0^\infty f(r)P_1(r,z)\,dr, \quad -\infty < z < \infty, \tag{0.8}$$

is an isometry from $L^2(0,\infty)$ to $L^2_\sigma(-\infty,\infty)$.

The Krein system has been studied in the most detailed way, see [5, 7, 8, 17, 18], in the case where at least one of the following conditions is fulfilled.

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The first condition:

$$\int_0^\infty |k(t)|\,dt < \infty. \tag{0.9}$$

The second condition: the integral

$$\int_{-\infty}^\infty \frac{\log \sigma'(u)}{1+u^2}\,du \tag{0.10}$$

converges, where $\sigma(u)$ is the spectral function of system (0.6).

The third condition:

$$\int_0^\infty |A(t)|\,dt < \infty. \tag{0.11}$$

In the present paper we consider the case where $A(r)$ is a continuous function and

$$A(r) = C + B(r), \quad C \neq 0, \quad \int_0^\infty |B(t)|\,dt < \infty. \tag{0.12}$$

It is obvious that in the case of (0.12) the third condition (0.11) is not fulfilled. Below, we prove that $\sigma'(u) = 0$ almost everywhere for $|u| < 2|C|$. Hence, the second condition (0.10) also fails.

We note that the radial Dirac equation [14, 17] can be reduced to the form (0.6). In this case, the constant C in (0.12) has the meaning of the mass. In the case of (0.12) we investigate the Weyl–Titchmarsh function $v(z)$ and the corresponding scattering function $s(u)$.

To illustrate the general theory we consider three important examples.

Example 0.1. Let the potential $A(x)$ be constant, i.e.,

$$A(r) = C, \quad C \neq 0. \tag{0.13}$$

Example 0.2. The kernel $k(x)$ of the operator S_r has the form

$$k(x) = -\frac{\sin \pi x}{\pi x}. \tag{0.14}$$

Example 0.3. The kernel $k(x)$ of the operator S_r has the form

$$k(x) = -\mu \frac{\sin \pi x}{\pi x}, \quad 0 < \mu < 1. \tag{0.15}$$

In the cases of (0.14) and (0.15) the first condition (0.9) fails. We shall show that if (0.14) is true, then the corresponding potential $A(r)$ satisfies (0.12). We note that the operators S_r with the kernels (0.14) and (0.15) play an important role in random matrix theory, see [4, 19, 20]. The operator S_r satisfies simultaneously the two operator identities obtained in [15, 17] and generates two canonical differential systems.

§1. Spectral and scattering data

Our aim is to deduce the relationship between the spectral and scattering data in the case of (0.12). (The corresponding result for the Schrödinger equation is well known (see [2]).) We denote by $P_1(r,z)$, $P_2(r,z)$ the solution of (0.6) that satisfies

$$P_1(0,z) = P_2(0,z) = 1. \tag{1.1}$$

Together with $P_1(r,z), P_2(r,z)$ we introduce the solution $\widehat{P}_1(r,z), \widehat{P}_2(r,z)$:

$$\widehat{P}_1(0,z) = -\widehat{P}_2(0,z) = 1/2. \tag{1.2}$$

The spectral data $\sigma(\lambda)$ and α are related to the Weyl–Titchmarsh function $v(z)$ by the formula (see [17, Chapter 10])

$$v(z) = \alpha + \int_{-\infty}^{\infty} \Big(\frac{1}{u-z} - \frac{u}{1+u^2}\Big)\, d\sigma(u). \tag{1.3}$$

The corresponding Weyl–Titchmarsh function $v(z)$ is defined by (see [17, 18])

$$v(z) = (-i) \lim_{\ell\to\infty} P_2^{-1}(\ell, z)\widehat{P}_2(\ell, z), \quad \operatorname{Im} z > 0. \tag{1.4}$$

The well-known asymptotic theorems (see [1, 3]) yield

$$\operatorname{col}[P_1(r,z), P_2(r,z)] \sim m_1(z)\exp(r\lambda_1(z))e_1 + m_2(z)\exp(r\lambda_2(z))e_2, \tag{1.5}$$

$$\operatorname{col}[\widehat{P}_1(r,z), \widehat{P}_2(r,z)] \sim \widehat{m}_1(z)\exp(r\lambda_1(z))e_1 + \widehat{m}_2(z)\exp(r\lambda_2(z))e_2, \tag{1.6}$$

where $r \to \infty$ and $\operatorname{Im} z > 0$. Here $\lambda_1(z)$ and $\lambda_2(z)$ are the eigenvalues of the matrix

$$D(z) = \begin{bmatrix} iz & -\bar{C} \\ -C & 0 \end{bmatrix}, \tag{1.7}$$

namely,

$$\lambda_1(z) = \frac{iz - \sqrt{4|C^2| - z^2}}{2}, \quad \lambda_2(z) = \frac{iz + \sqrt{4|C^2| - z^2}}{2}, \quad \operatorname{Im} z > 0, \tag{1.8}$$

where

$$\sqrt{4|C^2| - z^2} > 0 \quad \text{for} \quad z = \bar{z}, \quad |z| < 2|C|. \tag{1.9}$$

The corresponding eigenvectors look like this:

$$e_1(z) = \operatorname{col}[1, \lambda_2(z)/\bar{C}], \quad e_2(z) = \operatorname{col}[-\lambda_2(z)/C, 1]. \tag{1.10}$$

We introduce the matrix

$$V(r,z) = \begin{bmatrix} P_1(r,z) & \widehat{P}_1(r,z) \\ P_2(r,z) & \widehat{P}_2(r,z) \end{bmatrix}. \tag{1.11}$$

Relations (1.5), (1.6), and (1.11) show that

$$V(r,z) \sim V_1(r,z)M(z), \quad r \to \infty. \tag{1.12}$$

The matrices $V_1(r,z)$ and $M(z)$ are defined by the relations

$$V_1(r,z) = \begin{bmatrix} \exp(r\lambda_1(z)) & (-\frac{\lambda_2(z)}{C})\exp(r\lambda_2(z)) \\ (\frac{\lambda_2(z)}{\bar{C}})\exp(r\lambda_1(z)) & \exp(r\lambda_2(z)) \end{bmatrix}, \tag{1.13}$$

$$M(z) = \begin{bmatrix} m_1(z) & \widehat{m}_1(z) \\ m_2(z) & \widehat{m}_2(z) \end{bmatrix}. \tag{1.14}$$

We write system (0.6) in the matrix form:

$$\frac{dV(r,z)}{dr} = [izP - Q(r)]V(r,z), \tag{1.15}$$

where

$$Q(r) = \begin{bmatrix} 0 & \overline{A(r)} \\ A(r) & 0 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{1.16}$$

We put

$$j = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{1.17}$$

It is easily seen that

$$j[izP - Q(r)]j = -[izP - Q(r)]^*, \quad z = \bar{z}. \tag{1.18}$$

Proposition 1.1. *If $z = \bar{z}$, then*

$$V^*(r,z)jV(r,z) = J. \tag{1.19}$$

Proof. By (1.1), (1.2), and (1.11), we have

$$V^*(0,z)jV(0,z) = J, \quad z = \bar{z}. \tag{1.20}$$

Now, (1.19) follows from (1.15)–(1.18). □

We introduce the matrix-valued functions

$$V_1(r,u) = \lim_{y\to+0} V_1(r,z), \quad M(u) = \lim_{y\to+0} M(z), \quad z = u + iy, \tag{1.21}$$

and the functions

$$\lambda_k(u) = \lim_{y\to+0} \lambda_k(z), \quad z = u + iy, \ \ k = 1,2. \tag{1.22}$$

Using (1.13) and the identity

$$\lambda_2(u) = -\overline{\lambda_2(u)}, \quad |u| > 2|C|, \tag{1.23}$$

we deduce the formula

$$\lim_{r\to\infty}[V_1^*(r,u)jV_1(r,u)] = G(u), \tag{1.24}$$

where

$$G(u) = (1 - |\lambda_2{}^2|/|C^2|)j. \tag{1.25}$$

Relations (1.12), (1.19), and (1.25) imply that

$$M^*(u)G(u)M(u) = J, \quad |u| > 2C. \tag{1.26}$$

Hence,

$$\overline{m_1}\widehat{m_1} - \overline{m_2}\widehat{m_2} = 1/(1 - |\lambda_2/C|^2). \tag{1.27}$$

Now we use the following relations:

$$e^{-iru/2}P_1(r,u) = \overline{e^{-iru/2}P_2(r,u)}, \quad e^{-iru/2}\widehat{P}_1(r,u) = -\overline{e^{-iru/2}\widehat{P}_2(r,u)}. \tag{1.28}$$

Formulas (1.5), (1.6), and (1.28) show that

$$m_1(u) = \overline{m_2(u)}, \quad \widehat{m}_1(u) = -\overline{\widehat{m}_2(u)}, \quad |u| > 2|C|. \tag{1.29}$$

Using (1.5) and (1.29), we see that

$$m_2(u) \neq 0, \quad u = \bar{u}, \quad |u| > 2|C|. \tag{1.30}$$

Consider the case where $u = \bar{u}$, $|u| < 2|C|$. Since

$$\sqrt{4|C^2| - u^2} > 0 \quad \text{for} \quad u = \bar{u}, \quad |u| < 2|C|, \tag{1.31}$$

formulas (1.5), (1.6) yield

$$m_2(u) = -\overline{m_2(u)}(\overline{\lambda_2(u)}/\bar{C}), \quad \widehat{m}_2(u) = \overline{\widehat{m}_2(u)}(\overline{\lambda_2(u)}/\bar{C}), \tag{1.32}$$

where $|u| < 2|C|$. We consider

$$v(u) = \lim_{y\to+0} v(z), \quad z = u + iy.$$

By (1.4),

$$v(u) = -i\frac{\widehat{m}_2(u)}{m_2(u)}, \quad u = \bar{u}. \tag{1.33}$$

Relations (1.27), (1.29) and (1.32), (1.33) imply that

$$\operatorname{Im} v(u) = \frac{1}{2|m_2(u)|^2(1-|\lambda_2(u)/C|^2)}, \quad u=\bar{u}, \quad |u|>2|C|, \tag{1.34}$$

$$\operatorname{Im} v(u) = 0, \quad u=\bar{u}, \quad |u|<2|C|. \tag{1.35}$$

Using relations (1.35) and (1.34) we deduce the following assertion.

Theorem 1.2. *Under conditions* (0.12)*, the corresponding spectral function* $\sigma'(u)$ *is defined by the formula*

$$\sigma'(u,C) = \begin{cases} \frac{1}{2\pi|m_2(u)|^2(1-|\lambda_2(u)/C|^2)}, & u=\bar{u}, \quad |u|>2|C|, \\ 0, & u=\bar{u}, \quad |u|<2|C|. \end{cases} \tag{1.36}$$

Remark 1.3. We think that formula (1.36) is new.

If $C=0$ then the corresponding result has the form

$$\sigma'(u,0) = \frac{1}{2\pi|m_2(u)|^2}. \tag{1.37}$$

We use the relation

$$\lim_{C\to 0} \frac{\lambda_2(u)}{C} = 0. \tag{1.38}$$

Corollary 1.4. *Under conditions* (0.12)*, the absolutely continuous spectrum of system* (0.6) *coincides with* $(-\infty,-2|C|)\cup(2|C|,\infty)$*.*

Remark 1.5. Corollary 1.4 was known earlier under some other conditions.

From (1.11)–(1.13) it follows that there exists a solution $Y(r,u)$ ($u=\bar{u}$, $|u|<2|C|$) of system (0.6) that has the form

$$Y(r,u) \sim \exp(r\lambda_1(u))e_1 + s(u)\exp(r\lambda_2(u))e_2, \quad r\to\infty.$$

Definition 1.6. We call the function $s(u)$ the *scattering function* of system (0.6).

Using (1.5) and (1.10), we obtain the formula

$$s(u) = m_2(u)/\overline{m_2(u)} \qquad (u=\bar{u},\ |u|>2|C|). \tag{1.39}$$

§2. $A(x)$ is constant

We consider the case of system (0.6) with $A(x)=C$. Then

$$V(r,z) = L(z)D(r,z)L^{-1}(z)V(0,z), \tag{2.1}$$

where

$$L(z) = \begin{bmatrix} 1 & -\lambda_2(z)/C \\ \lambda_2(z)/\bar{C} & 1 \end{bmatrix}, \quad D(r,z) = \begin{bmatrix} e^{r\lambda_1(z)} & 0 \\ 0 & e^{r\lambda_2(z)} \end{bmatrix}. \tag{2.2}$$

Using (2.2) and the relation

$$V(0,z) = \begin{bmatrix} 1 & 1/2 \\ 1 & -1/2 \end{bmatrix}, \tag{2.3}$$

we deduce the formula

$$M(u) = L^{-1}(u)V(0,u) = \frac{1}{1+\lambda_2^2(u)/|C|^2} \begin{bmatrix} 1+\lambda_2(u)/C & [1-\lambda_2(u)/C]/2 \\ 1-\lambda_2(u)/\bar{C} & -[1+\lambda_2(u)/\bar{C}]/2 \end{bmatrix}, \tag{2.4}$$

where $u=\bar{u}, \quad |u|>2|C|$.

Now, assuming (0.13), we can write $m_k(u)$ and $\widehat{m}_k(u)$ explicitly (see (1.14) and (2.4)):

$$m_1(u) = \overline{m_2(u)} = (1 + \lambda_2(u)/C)/(1 + \lambda_2^2(u)/|C|^2), \tag{2.5}$$

$$\widehat{m_1}(u) = -\overline{\widehat{m_2}(u)} = [(1 - \lambda_2(u)/C)/(1 + \lambda_2^2(u)/|C|^2)]/2. \tag{2.6}$$

By (1.33) and (1.39), we have

$$v(u) = i[(1 + \lambda_2(u)/\bar{C})/(1 - \lambda_2(u)/\bar{C})]/2, \qquad u = \bar{u}, \quad |u| > 2|C|, \tag{2.7}$$

and

$$s(u) = (1 - \lambda_2(u)/C)/(1 + \lambda_2(u)/\bar{C}), \qquad u = \bar{u}, \quad |u| > 2|C|. \tag{2.8}$$

In the case where $C = \bar{C}$, formulas (2.7) and (2.8) simplify to

$$v(u) = i\frac{\sqrt{u^2 - 4C^2} + 2iC \operatorname{sign} u}{2|u|}, \qquad u = \bar{u}, \quad |u| > 2|C|, \tag{2.9}$$

$$s(u) = \frac{|u|}{\sqrt{u^2 - 4C^2} + 2iC \operatorname{sign} u}, \qquad u = \bar{u}, \quad |u| > 2|C|. \tag{2.10}$$

§3. The sine kernel

Consider the operator

$$S_\xi f = f(x) + \int_0^\xi k(x-u) f(u)\, du, \quad f(u) \in L^2(0, \xi), \tag{3.1}$$

where

$$k(x) = -\frac{\sin x\pi}{x\pi}. \tag{3.2}$$

The operator S_ξ is invertible (see [4, p. 167]). Hence,

$$S_\xi^{-1} f = f(x) + \int_0^\xi R_\xi(x, u) f(u)\, du, \quad f(u) \in L^2(0, \xi), \tag{3.3}$$

where the kernel $R_\xi(x, u)$ is continuous with respect to the variables ξ, x, u. The operator S_ξ plays an important role in a number of theoretical and applied problems (random matrix theory [4, 20], optical problems [9]).

In the case of (3.1) and (3.2), the first condition (0.9) fails. We use the relation

$$\frac{\sin x\pi}{x\pi} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ix\lambda}\, d\sigma_0(\lambda), \tag{3.4}$$

where

$$\sigma_0(u) = \frac{1}{2\pi} \begin{cases} \pi & \text{if } u > \pi, \\ u & \text{if } u \in [-\pi, \pi], \\ -\pi & \text{if } u < -\pi. \end{cases}$$

We deduce that the spectral function $\sigma(u)$ of system (0.6) has the form

$$\sigma(u) = \frac{1}{2\pi} u - \sigma_0(u), \tag{3.5}$$

showing that

$$\sigma(u) = 0, \quad u \in [-\pi, \pi]. \tag{3.6}$$

Hence, in the case of (3.1) and (3.2) the second condition (0.10) also fails. To study the operator (3.1), (3.2) in greater detail we shall need triangular factorization.

§4. Triangular factorization

We introduce the operator

$$(P_\xi f)(x) = \begin{cases} 0 & \text{if } 0 < x < \xi, \\ f(x) & \text{if } \xi < x < b, \end{cases} \quad f \in L^2(0, \xi).$$

Definition 4.1 ([11]). We say that the positive operator S_b, $0 < b < \infty$, admits triangular factorization if it can be represented in the form

$$S_b = S_- S_-^\star, \tag{4.1}$$

where the operators $S_-^{\pm 1}$ are bounded and

$$S_-^{\pm 1} P_\xi = P_\xi S_-^{\pm 1} P_\xi. \tag{4.2}$$

Using Krein's result (see [6, Chapter IV]), we deduce the following assertion.

Proposition 4.2. *The operator S_b defined by* (3.1), (3.2) *admits triangular factorization* (4.1)*, and*

$$(S_-^{-1} f)(x) = f(x) + \int_0^x R_x(x, u) f(u)\, du. \tag{4.3}$$

We introduce the functions

$$q(x) = S_-^{-1} e^{ix\pi}, \tag{4.4}$$

$$q_1(x) = S_-^{-1} 1, \quad M(x) = \frac{1}{2} - \mu \int_0^x \frac{\sin \pi t}{\pi t}\, dt, \quad q_2(x) = S_-^{-1} M(x). \tag{4.5}$$

The kernel $k(x)$ (see (3.2)) satisfies

$$k(x) = k(-x) = \overline{k(x)}. \tag{4.6}$$

In the paper [16] it was shown that relations (4.6) imply the identity

$$q_1(x) q_2(x) = 1/2. \tag{4.7}$$

§5. Two operator identities and two canonical differential systems

The operator S_ξ satisfies simultaneously the two operator identities of [15, 17]. The first operator identity has the form

$$(QS_\xi - S_\xi Q) f = -\frac{1}{2i\pi} \int_0^\xi [e^{i(x-u)\pi} - e^{-i(x-u)\pi}] f(u)\, du, \tag{5.1}$$

where

$$Qf = x f(x). \tag{5.2}$$

The second operator identity has the form

$$(AS_\xi - S_\xi A^\star) f = i \int_0^\xi [M(x) + M(u)] f(u)\, du, \tag{5.3}$$

where

$$Af = i \int_0^x f(u)\, du. \tag{5.4}$$

The operator S_ξ and the operator identities (5.1), (5.3) generate two canonical differential systems. The first system is related to (5.1) and looks like this [15]:

$$\frac{d}{dx} W_1(x, z) = -i \frac{j H_1(x)}{z - x} W_1(x, z), \quad W_1(0, z) = I_2, \tag{5.5}$$

where

$$H_1(x) = \frac{1}{2\pi}\begin{bmatrix} |q(x)|^2 & q^2(x) \\ \overline{q(x)}^2 & |q(x)|^2 \end{bmatrix}. \tag{5.6}$$

Note that j and $q(x)$ are defined by (1.17) and (4.4). The second system is related to (5.3) and has the form [15]

$$\frac{d}{dx}W_2(x,z) = izJH_2(x)W_2(x,z), \quad W_2(0,z) = I_2, \tag{5.7}$$

where

$$H_2(x) = \begin{bmatrix} q_1^2(x) & 1/2 \\ 1/2 & q_2^2(x) \end{bmatrix}. \tag{5.8}$$

Now, J and the $q_k(x)$, $k = 1, 2$, are defined by (1.17) and (4.5).

Remark 5.1. System (5.5) plays an important role in random matrix theory (see [4, 19, 20]). Below, we shall show that system (5.7) can be reduced to the Krein system.

§6. Asymptotic behavior of the Hamiltonians $H_1(x)$ and $H_2(x)$ as $x \to \infty$

Along with the operator S_ξ, we consider the operator

$$C_\xi f = f(x) - \int_{-\xi}^{\xi} k(x-v)f(v)\,dv, \quad f(v) \in L^2(-\xi,\xi). \tag{6.1}$$

The operator

$$U_\xi f(x) = f(u+\xi) \tag{6.2}$$

maps the space $L^2(0, 2\xi)$ unitarily onto $L^2(-\xi,\xi)$. It is easily seen that

$$U_\xi^{-1}C_\xi U_\xi f = S_{2\xi} f. \tag{6.3}$$

By (4.3) and (6.3), we have

$$C_\xi^{-1} f = f(x) + \int_{-\xi}^{\xi} Q_\xi(x,u)f(u)\,du, \quad f(u) \in L^2(-\xi,\xi), \tag{6.4}$$

where

$$R_{2\xi}(x,y) = Q_\xi(x-\xi, y-\xi), \tag{6.5}$$

which implies that

$$R_{2\xi}(2\xi, 2\xi) = Q_\xi(\xi,\xi), \quad R_{2\xi}(2\xi, 0) = Q_\xi(\xi, -\xi). \tag{6.6}$$

Following Tracy and Widom [20], we introduce the function

$$s(t) = e^{it\pi} + \int_{-t}^{t} e^{iu\pi}Q_t(t,u)\,du. \tag{6.7}$$

Relations (4.4) and (6.3), (6.7) show that

$$q(2t) = s(t)e^{it\pi}. \tag{6.8}$$

We use the well-known system (see [20])

$$\frac{d}{dt}[tQ_t(t,t)] = |s(t)|^2, \quad \frac{d}{dt}[tQ_t(-t,t)] = \mathrm{Re}[s(t)^2], \tag{6.9}$$

$$\frac{d}{dt}[Q_t(t,t)] = 2Q_t^2(-t,t), \quad 2\pi tQ_t(-t,t)] = \mathrm{Im}[s(t)^2], \tag{6.10}$$

and the asymptotic representation (see [20])

$$Q_t(t,t) \sim \frac{t\pi^2}{2} + \frac{1}{8t} - \sum_{n=1} \frac{c_{2n}}{2t^{2n+1}(2\pi)^{2n}}, \tag{6.11}$$

where $t \to \infty$, $c_2 = -\frac{1}{4}$, $c_4 = -\frac{5}{2}$.

Formulas (6.9) and (6.11) give us the asymptotic formula

$$|s^2(t)| \sim \pi^2 t + \sum_{n=1}^{\infty} \frac{nc_{2n}}{(2\pi)^{2n}t^{2n+1}}, \quad t \to \infty. \tag{6.12}$$

By (6.10) and (6.11),

$$Q_t^2(-t,t) \sim \sum_{n=0}^{\infty} \frac{b_{2n}}{t^{2n}}, \quad t \to \infty, \tag{6.13}$$

where

$$b_0 = \frac{\pi^2}{4}, \quad b_2 = -\frac{1}{16}, \quad b_{2(n+1)} = \frac{c_{2n}(2n+1)}{4(2\pi)^{2n}}, \quad n \geq 1. \tag{6.14}$$

Consequently,

$$Q_t(-t,t) \sim \sum_{n=0}^{\infty} \frac{a_{2n}}{t^{2n}}, \quad t \to \infty, \tag{6.15}$$

where

$$b_{2n} = \sum_{i+j=n} a_{2i}a_{2j}, \quad i,j \geq 0. \tag{6.16}$$

Comparing (6.14) and (6.16), we see that $a_0^2 = \frac{\pi^2}{4}$, i.e., $a_0 = \pm\frac{\pi}{2}$. Using [20, formula (90)], we obtain

$$a_0 = \frac{\pi}{2}, \quad a_2 = -\frac{1}{16\pi}. \tag{6.17}$$

By (6.9), (6.11), and (6.15), the relation

$$s^2(t) \sim \pi^2 it + \frac{\pi}{2} + 2i\pi \sum_{n=1}^{\infty} \frac{a_{2n}}{t^{2n-1}} - \sum_{n=1}^{\infty} \frac{(2n-1)a_{2n}}{t^{2n}}, \quad t \to \infty, \tag{6.18}$$

is valid.

Formulas (6.8) and (6.18) imply that

$$q^2(t) \sim e^{it\pi}\left[\pi^2 it/2 + \frac{\pi}{2} + 2i\pi \sum_{n=1}^{\infty} \frac{a_{2n}}{(t/2)^{2n-1}} - \sum_{n=1}^{\infty} \frac{(2n-1)a_{2n}}{(t/2)^{2n}}\right] \tag{6.19}$$

as $t \to \infty$. The asymptotic behavior of $H_1(t)$ as $t \to \infty$ is a consequence of (5.6) and (6.19):

$$H_1(t) \sim \frac{t\pi}{4}\begin{bmatrix} 1 & e^{it\pi}i \\ -e^{-it\pi}i & 1 \end{bmatrix} + \begin{bmatrix} 0 & e^{it\pi}\pi/2 \\ e^{-it\pi}\pi/2 & 0 \end{bmatrix} + \cdots . \tag{6.20}$$

In order to find the asymptotic behavior of $H_2(x)$ as $x \to \infty$ we use the well-known Krein's formula ([6, Chapter IV])

$$q_1(x) = \exp\left[\int_0^x R_t(t,0)\, du\right]. \tag{6.21}$$

From (6.6), (6.13), and (6.21) we deduce that

$$\log q_1(x) = (\pi/2)x + \beta + \sum_{j=0}^{\infty} c_j x^{-j-1}, \quad x \to \infty, \tag{6.22}$$

where

$$\beta = \int_0^{\infty} R_t(t,0)\,dt. \tag{6.23}$$

By (4.7) and (6.22), we have

$$\log q_2(x) = -\log 2 - (\pi/2)x - \beta - \sum_{j=0}^{\infty} c_j x^{-j-1}, \quad x \to \infty. \tag{6.24}$$

§7. The Krein differential system

We consider the canonical system (5.7). We use the identity

$$JH_2(x) = T(x)PT^{-1}(x), \tag{7.1}$$

where

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad T(x) = \begin{bmatrix} q_2(x) & -q_2(x) \\ q_1(x) & q_1(x) \end{bmatrix}. \tag{7.2}$$

Then

$$T(x)jT^{\star}(x) = J. \tag{7.3}$$

Using (5.7) and (7.1), we obtain

$$\frac{dV}{dx} = izPV - Q(x)V, \tag{7.4}$$

where

$$V(x,z) = T^{-1}(x)W_2(x,z)T(0), \quad Q(x) = T^{-1}(x)T'(x). \tag{7.5}$$

By (7.2) and (7.5),

$$Q(x) = \begin{bmatrix} 0 & A(x) \\ A(x) & 0 \end{bmatrix}, \quad A(x) = -q_1^{-1}(x)q_1'(x). \tag{7.6}$$

So, we have reduced the canonical system (5.7) to the Krein system (1.15), (1.16).

Relations (6.22) and (7.6) imply that

$$A(x) = -\pi/2 + O(1/x^2), \quad x \to \infty. \tag{7.7}$$

This means that in the case of Example 0.2 condition (0.12) is fulfilled. Hence, the results of §1 are true for this example.

§8. The sine kernel, $0 < \mu < 1$

Consider the operator

$$S_{\xi,\mu} f = f(x) - \mu \int_0^{\xi} k(x-u)f(u)\,du, \quad f(u) \in L^2(0,\xi), \;\; 0 < \mu < 1, \tag{8.1}$$

where

$$k(x) = \frac{\sin x\pi}{x\pi}. \tag{8.2}$$

In §3 we considered the case (8.2) with $\mu = 1$.

The operator $S_{\xi,\mu}$ is invertible (see [4, p. 167]). We see that in the case of Example 0.3 the first condition (0.9) fails. Using (3.4), we deduce that, in the case of Example 0.3, the spectral function $\sigma(u,\mu)$ of system (0.6) has the form

$$\sigma(u,\mu) = \frac{1}{2\pi}u - \mu\sigma_0(u). \tag{8.3}$$

Hence, the second condition (0.10) is fulfilled. It follows that the fundamental Krein theorem [7] (see also [18]) can be applied to Example 0.3.

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