

## TRACES OF $C^k$ FUNCTIONS ON WEAK MARKOV SUBSETS OF $\mathbb{R}^n$

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ABSTRACT. A wide class of closed subsets of  $\mathbb{R}^n$  is introduced; these subsets admit constructive  $C^{k,\omega}$  extensions with good bounds for the corresponding extension constants.

### §1. INTRODUCTION

The problem whose special case is studied in this paper goes back to the classical 1934 Whitney papers [W1] and [W2] but has been intensively investigated only since the beginning of the 1980s, see Fefferman’s survey [F1] and Chapter 10 of the forthcoming book [BB1]. The problem deals with various spaces of  $C^k$  functions defined on  $\mathbb{R}^n$  whose higher derivatives satisfy certain conditions. The most important classes are the spaces  $C_b^k(\mathbb{R}^n)$  and  $C^{k,\omega}(\mathbb{R}^n)$  equipped with the norms

$$(1.1) \quad \|f\|_{C_b^k(\mathbb{R}^n)} := \sup_{\mathbb{R}^n} |f| + \max_{|\alpha|=k} \sup_{\mathbb{R}^n} |D^\alpha f|,$$

$$(1.2) \quad \|f\|_{C^{k,\omega}(\mathbb{R}^n)} := \sup_{\mathbb{R}^n} |f| + \max_{|\alpha|=k} \sup_{x \neq y} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{\omega(\|x - y\|)},$$

and the space  $C^k \Lambda^\omega(\mathbb{R}^n)$  whose norm differs from that of (1.2) by replacing the first difference by the second one, i.e.,  $D^\alpha f(x) - 2D^\alpha f(\frac{x+y}{2}) + D^\alpha f(y)$ .

We shall study only the first two spaces; in this case  $\omega : (0, +\infty) \rightarrow \mathbb{R}$  is a monotone nondecreasing concave function with  $\omega(0+) = 0$ .

Let  $X$  be one of these spaces and  $S$  a closed subset of  $\mathbb{R}^n$ .

**Trace problem.** *Given a function  $f : S \rightarrow \mathbb{R}$ , does there exist a function  $F \in X$  such that the trace  $F|_S$  coincides with  $f$ ?*

*If such an  $F$  exists, how to estimate the trace norm of  $f$ ?*

Here the trace norm of  $f$  is defined by the formula

$$(1.3) \quad \|f\|_{X[S]} := \inf\{\|g\|_X ; g|_S = f\},$$

where  $X[S]$  denotes the corresponding trace space.

**Linearity problem.** *Does there exist a linear bounded extension operator  $E$  from  $X[S]$  into  $X$ , i.e., an operator such that*

$$Ef|_S = f$$

*for every  $f \in X[S]$ ?*

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Whitney [W2] solved these problems for the space  $C_b^k(\mathbb{R})$  and an arbitrary closed subset  $S \subset \mathbb{R}$ . His elegant results are as follows.

- (a) A function  $f : S \rightarrow \mathbb{R}$  belongs to the trace space  $C_b^k[S]$  if and only if
- $$(1.4) \quad \lim_{\Sigma \rightarrow \{x\}} f[\Sigma] \operatorname{diam} \Sigma = 0 \quad \text{for every } x \in S;$$

here  $\Sigma$  belongs to the set of all  $k + 2$  point subsets of  $S$  and  $f[\Sigma]$  stands for the divided difference of  $f$  with node set  $\Sigma$ .

- (b) The trace norm of  $f : S \rightarrow \mathbb{R}$  is equivalent to

$$\sup_S |f| + \sup_{\Sigma} |f[\Sigma]| \operatorname{diam} \Sigma,$$

where  $\Sigma$  runs over all  $k + 2$  point subsets of  $S$ . The constants of equivalence depend only on  $k$ .

- (c) There exists a linear extension operator  $E : C_b^k[S] \rightarrow C_b^k(\mathbb{R})$  whose norm is bounded by a constant depending only on  $k$ .

Yet the multivariate problems turned out to be extremely difficult. To tackle the trace problem, Yu. BrudnyĀ proposed to begin with a seemingly simpler problem concerning the *finiteness property*. To formulate this problem for the  $C^k$  spaces under consideration, we need the following definition.

**Definition 1.1.** Let  $\mathcal{C}$  be a class of nonempty closed subsets of  $\mathbb{R}^n$ . A Banach space  $X$  of  $C^k$  functions on  $\mathbb{R}^n$  has the *finiteness property* (briefly, FP) with respect to  $\mathcal{C}$  if there exists an integer  $N > 1$  such that

$$(1.5) \quad \|f\|_{X[S]} \leq \gamma \sup\{\|f\|_{X[\Sigma]}; \Sigma \subset S \text{ and } \operatorname{card} \Sigma \leq N\}$$

for every  $S \in \mathcal{C}$ , where  $\gamma$  is independent of  $f$  and  $S$ .

The optimal constants in (1.5) are denoted by  $N_{\mathcal{C}}(X)$  and  $\gamma_{\mathcal{C}}(X)$  and called the *finiteness constants*. We omit the indices if  $\mathcal{C}$  consists of all nonempty closed subsets of  $\mathbb{R}^n$ .

In particular, this means that  $f : S \rightarrow \mathbb{R}$ ,  $S \in \mathcal{C}$ , belongs to the trace space  $X[S]$  if every its trace to a subset  $\Sigma \subset S$  such that  $\operatorname{card} \Sigma \leq N$  extends to a function belonging to a ball of  $X$  centered at 0 and of radius depending only on  $k$  and  $n$ .

If, in addition, these extensions can be chosen to form a precompact set in the ball, we shall say that  $X$  has the *strong finiteness property* SFP with respect to  $\mathcal{C}$ .

At the beginning of the 1980s, Yu. BrudnyĀ conjectured that the generalized Zygmund spaces  $\Lambda^{k,\omega}(\mathbb{R}^n)$  possess FP and the linearity problem has a positive solution. We recall that

$$\|f\|_{\Lambda^{k,\omega}(\mathbb{R}^n)} := \sup_{\mathbb{R}^n} |f| + \sup_{x,h \in \mathbb{R}^n} \frac{|\Delta_h^k f(x)|}{\omega(\|h\|)}$$

and this family includes  $C^{k,\omega}(\mathbb{R}^n)$  and  $C^k \Lambda^{\omega}(\mathbb{R}^n)$  for the quasipower majorants  $\omega$ .

In view of the recent development, it seems to be natural to strengthen the above claim as follows.

**Conjecture.** *The space  $C^s \Lambda^{k,\omega}(\mathbb{R}^n)$  has FP, and the linearity problem has a positive solution for this space.*

For the time being, the case where  $s \geq 1$  and  $k \geq 2$  is far from understanding, but for  $s \geq 1$  and  $k = 1$  or  $s = 0$  and  $k = 2$  the situation is much better as the following facts show.

- (1) The Whitney method [W2] applied to the space  $C^{k,\omega}(\mathbb{R})$  bounds its finiteness constant by  $k + 2$ . Counting the number of derivatives that should be restored and

taking into account the Lipschitz condition for the upper derivative, we easily conclude that this constant is at least  $k + 2$ . Hence,

$$N(C^{k,\omega}(\mathbb{R})) = k + 2.$$

From this one can derive the same relation for the finiteness constant for  $C_b^k(\mathbb{R})$  and prove that this space has SFP.

Also, using these facts one can easily derive the Whitney theorem, see (1.4), and prove that a function  $f : S \rightarrow \mathbb{R}$ ,  $S \subset \mathbb{R}$ , belongs to the trace space  $C_b^k[S]$  if and only if

$$\sup \left\{ \frac{|f[\Sigma]| \operatorname{diam} \Sigma}{\omega(\operatorname{diam} \Sigma)}; \Sigma \subset S \text{ and } \operatorname{card} \Sigma \leq k + 2 \right\}$$

is finite. Moreover, this quantity is equivalent to  $\|f\|_{C^{k,\omega}[S]}$  with constants of equivalence depending only on  $k$  and growing polynomially in  $k$ . In particular, this bounds the second finiteness constant  $\gamma(C^{k,\omega}(\mathbb{R}))$  by  $c_0 k^{c_1}$ , where  $c_0, c_1 > 0$  are numerical constants.

Since the final part of Whitney's proof exploits the linear extension operator presented in [W1], the linearity problem has also a positive solution for these two spaces.

Moreover, the Whitney extension method is universal in the sense that it does not depend on  $\omega$ . This leads to a positive solution of the linearity problem also for  $C_u^k(\mathbb{R})$ , the closed subspace of  $C_b^k(\mathbb{R})$  consisting of functions with uniformly continuous higher derivatives.

(2) The next results in this direction were proved in the 1984 PhD thesis by Shvartsman [Sh1] and in the paper [BSh1]; they concern the Zygmund space  $\Lambda^\omega(\mathbb{R}^n)$ . In the former paper it was proved that

$$(1.6) \quad N(\Lambda^\omega(\mathbb{R}^n)) = 3 \cdot 2^{n-1}$$

while in the latter the linearity problem was solved positively for this space. It should be noted that the construction of a linear extension operator for this case exploits some facts of hyperbolic geometry, see [BSh2] for the details.

Adapting the methods of the papers cited above to the case of  $C^{1,\omega}(\mathbb{R}^n)$ , in [BSh3] it was proved that  $N(C^{1,\omega}(\mathbb{R}^n)) = 3 \cdot 2^{n-1}$  and the linearity problem has a positive solution for this space. It was also shown that the linearity problem for the space  $C_u^1(\mathbb{R}^n)$  has a negative solution.

(3) The finiteness and linearity problems for the spaces  $C_b^k(\mathbb{R}^n)$  and  $C^{k,\omega}(\mathbb{R}^n)$  were solved not quite long ago by Ch. Fefferman in the series of papers [F1, F2, F3, F4, F5, F6]. In particular, he proved that

$$(1.7) \quad N(C^{k,\omega}(\mathbb{R}^n)) \leq N^* := (d + 1)^{3 \cdot 2^d},$$

where  $d := \binom{n+k}{k}$  is the dimension of the space  $\mathcal{P}_k(\mathbb{R}^n)$  of polynomials of degree  $k$  on  $\mathbb{R}^n$ . Also, Ch. Fefferman proves the finiteness property for  $C_b^k(\mathbb{R}^n)$  with the same estimate of the finiteness constant. By using this result, in [BB1, Th.10.92] it was established that, in fact,  $C_b^k(\mathbb{R}^n)$  has the strong finiteness property (with the same estimate for the finiteness constant).

In Ch. Fefferman's result, the second finiteness constant  $\gamma(C^{k,\omega}(\mathbb{R}^n))$  is bounded by some very large constant, say  $N^{**}$ , of the same growth in  $k$  and  $n$  as  $N^*$ . Using (1.7), Bierstone and P. Milman [BM] and, independently and by a different method, Shvartsman [Sh2] essentially improved the upper bound for  $N(C^{k,\omega}(\mathbb{R}^n))$  replacing  $N^*$  by  $2^d$ . Unfortunately, their proofs enlarge the upper bound for  $\gamma(C^{k,\omega}(\mathbb{R}^n))$  substantially, replacing  $N^{**}$  by a constant of order of  $c^{N^*}$ , where  $c > 1$  is a numerical constant.

The upper bounds for the linear extension operators constructed in the cited papers of Ch. Fefferman are also very large.

The enormous growth in  $n$  and  $k$  of the multivariate extension constants reflects the high complexity of the corresponding extension algorithms. In particular, unlike the Whitney extension method [W2], no linear extension operator for  $C_b^k(\mathbb{R}^n)$  can be well localized. More precisely, Fefferman's example [F7] presents the trace space of  $C_b^1(\mathbb{R}^2)$  where any linear bounded extension operator is of infinite *depth* in the sense of the next definition.

**Definition 1.2.** A linear extension operator  $E : X[S] \rightarrow X$  has depth  $d$  if for every  $x \in \mathbb{R}^n \setminus S$  there exist numbers  $\lambda_i(x)$  and points  $x^i \in S$ ,  $1 \leq i \leq d$ , such that

$$(1.8) \quad (Ef)(x) = \sum_{i=1}^d \lambda_i(x) f(x).$$

This operator is of depth of order  $k$  if  $X \subset C^k(\mathbb{R}^n)$  and all  $\lambda_i$  belong to  $C^k(\mathbb{R}^n)$ .

*Remark 1.3.* Recently, Luli [L] proved that a modified Fefferman extension operator for  $C^{k,\omega}(\mathbb{R}^n)$  is of depth of order  $k$  bounded by some constant  $c(k, n)$ .

In this paper, we present a wide class  $\mathcal{C}$  of closed subsets of  $\mathbb{R}^n$  for which the two problems formulated above have positive solutions with good estimates of the corresponding extension constants. For instance, an upper bound for  $N_{\mathcal{C}}(C^{k,\omega}(\mathbb{R}^n))$  with respect to this class equals  $2\binom{n+k}{k}$  (compare this with the number  $\binom{n+k}{k} + \binom{n+k-1}{k}$  of derivatives to be restored plus the number of Lipschitz conditions for the higher derivatives). The class of subsets in question, denoted by  $\text{Mar}_k^*(\mathbb{R}^n)$ , contains, in particular, the closure of any open set, the Ahlfors  $p$ -regular compact subsets of  $\mathbb{R}^n$  with  $p > n - 1$ , a wide class of fractals of arbitrary positive Hausdorff measure and the unions of any combination of such sets. The class  $\text{Mar}_0^*(\mathbb{R}^n)$  admits a filtration

$$\text{Mar}_0^*(\mathbb{R}^n) \supset \text{Mar}_1^*(\mathbb{R}^n) \supset \text{Mar}_2^*(\mathbb{R}^n) \supset \cdots \supset \text{Mar}_k^*(\mathbb{R}^n) \supset \cdots$$

The elements of the class  $\text{Mar}_k^*(\mathbb{R}^n)$  will be called *weak  $k$ -Markov sets*.

The definition and a detailed description of properties of this class and its relationship with the class of Markov sets introduced by Jonsson and Wallin [JW, Chapter 2] will be presented in the next section. The final section is devoted to the formulation and proof of the main result of the paper, the solution to the finiteness and linearity problems mentioned above, and estimates of the corresponding extension constants.

## §2. WEAK MARKOV SETS

In what follows,  $\mathcal{P}_k(\mathbb{R}^n)$  denotes the space of real polynomials of degree  $k$  in  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ , and  $Q_r(x)$  stands for the open cube of side length  $2r$  and centered at  $x$ .

**Definition 2.1.** A closed set  $S \subset \mathbb{R}^n$  is said to be *weak  $k$ -Markov* if there exists a dense subset  $S_0 \subset S$  such that for every  $x \in S_0$  we have

$$(2.1) \quad \liminf_{r \rightarrow 0} \left( \sup_{p \in \mathcal{P}_k(\mathbb{R}^n) \setminus \{0\}} \left\{ \frac{\sup_{Q_r(x)} |p|}{\sup_{S \cap Q_r(x)} |p|} \right\} \right) < \infty.$$

The class of sets with this property is denoted by  $\text{Mar}_k^*(\mathbb{R}^n)$ .

*Remark 2.2.* We recall, see [JW], that a closed set  $S \subset \mathbb{R}^n$  belongs to the class of Markov sets denoted by  $\text{Mar}(\mathbb{R}^n)$  if for some constant  $c > 0$  and every  $x \in S$  and  $p \in \mathcal{P}_1(\mathbb{R}^n)$  the ratio in (2.1) is bounded by  $c$ .

It was proved in [JW, Chapter 2] that if the above condition is fulfilled for polynomials of degree 1, then it does for polynomials of every degree  $k$ . Hence,

$$(2.2) \quad \text{Mar}(\mathbb{R}^n) \subset \text{Mar}_k^*(\mathbb{R}^n),$$

for every  $k \geq 1$ ; however, we shall see that  $\text{Mar}(\mathbb{R}^n)$  is a small part of  $\text{Mar}_k^*(\mathbb{R}^n)$ .

Now, we discuss the basic properties of weak Markov sets.

**Proposition 2.3.** (a) *If  $\{S_i\}_{i \in I} \subset \text{Mar}_k^*(\mathbb{R}^n)$ , then  $\overline{\cup_{i \in I} S_i} \in \text{Mar}_k^*(\mathbb{R}^n)$ .*

(b) *If  $S_i \in \text{Mar}_k^*(\mathbb{R}^{n_i})$ ,  $i = 1, 2$ , then  $S_1 \times S_2 \in \text{Mar}_k^*(\mathbb{R}^{n_1+n_2})$ .*

(c) *If  $L : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a linear automorphism or a linear projection onto an affine subspace of  $\mathbb{R}^n$ , then  $\overline{L(S)}$  belongs to  $\text{Mar}_k^*$  whenever  $S$  does.*

(d) *Let  $S \subset \text{Mar}_k^*(\mathbb{R}^n)$ , and let  $\varphi : S_\varepsilon \rightarrow \mathbb{R}^n$  be a homeomorphism of an  $\varepsilon$ -neighborhood of  $S$ . Assume that the set of points in  $S$  where the derivative  $D\varphi$  exists and is invertible is open and dense. Then  $\varphi(S) \in \text{Mar}_k^*(\mathbb{R}^n)$ .*

*Proof.* Assertions (a)–(c) follow directly from the definition.

(d) Let  $S_0$  denote the set of points in  $S$  where (2.1) is fulfilled, and  $S_1 \subset S$  the set of points where  $D\varphi$  exists and is invertible. Since  $S_0$  is dense and  $S_1$  is open and dense in  $S$ , the set  $\varphi(S_0 \cap S_1)$  is dense in  $\varphi(S)$ . Therefore, it suffices to prove that

$$(2.3) \quad \text{if } \hat{x} := \varphi(x) \in \varphi(S_0 \cap S_1), \quad \text{then (2.1) holds true for } \hat{x} \text{ and } \varphi(S).$$

To prove this, first we note that, due to the invariance of  $\text{Mar}_k^*(\mathbb{R}^n)$  with respect to linear automorphisms of  $\mathbb{R}^n$ , we may (and shall) assume that  $\varphi(x) = x$  and  $D\varphi$  is the identity map at  $x \in S_1$ . Hence, we get

$$(2.4) \quad \varphi(y) = y + R(x, y), \quad \text{where } \|R(x, y)\| = o(\|x - y\|) \text{ as } y \rightarrow x.$$

In particular, for some  $\rho := \rho(r) \geq r$  such that  $\lim_{r \rightarrow 0} \frac{\rho(r)}{r} = 1$  we have

$$(2.5) \quad \varphi^{-1}(Q_r(\hat{x})) \subset Q_\rho(x) \quad \text{and} \quad \varphi(Q_{\rho/4}(x)) \subset Q_{r/2}(\hat{x}).$$

These relations imply that, for  $p \in \mathcal{P}_k(\mathbb{R}^n)$ ,

$$\max_{Q_r(\hat{x})} |p| = \max_{\varphi^{-1}(Q_r(\hat{x}))} |p \circ \varphi| \leq \max_{Q_\rho(x)} |p \circ \varphi|.$$

Further, for sufficiently small  $r > 0$  the point  $\varphi(y)$  belongs to  $Q_{2\rho}(x)$  whenever  $y \in Q_\rho(x)$ , see (2.4). Hence, the maximum on the right-hand side does not exceed  $\max_{Q_{2\rho}(x)} |p|$ , which, in its turn, by the Remez type inequality [BG] is bounded by<sup>1</sup> the quantity  $c(k, n) \left(\frac{2\rho}{\rho/4}\right)^{nk} \max_{Q_{\rho/4}(x)} |p|$ . Therefore, we conclude that

$$(2.6) \quad \max_{Q_r(\hat{x})} |p| \leq c_1(n, k) \max_{Q_{\rho/4}(x)} |p|.$$

To proceed, now we use condition (2.1) for the point  $x$  to show that there exists a monotone decreasing sequence  $\{r_i\}_{i \geq 1}$  and a constant  $\gamma = \gamma(x, S, k) > 0$  such that for every  $p \in \mathcal{P}_k(\mathbb{R}^n)$  and  $i \in \mathbb{N}$  we have

$$\max_{Q_{r_i}(x)} |p| \leq \gamma \max_{S \cap Q_{r_i}(x)} |p|.$$

Now, let  $\hat{r}_i$  satisfy  $r_i = \frac{1}{4}\rho(\hat{r}_i)$ . Changing variables by the rule  $y \mapsto \varphi^{-1}(y)$  and applying (2.4), we obtain

$$\max_{S \cap Q_{r_i}(x)} |p| = \max_{\varphi(S) \cap \varphi(Q_{r_i}(x))} |p \circ \varphi^{-1}| \leq \max_{\varphi(S) \cap Q_{\hat{r}_i/2}(\hat{x})} |p \circ \varphi^{-1}|.$$

For small  $\hat{r}_i$ , the points  $\varphi^{-1}(y)$  belong to  $Q_{\hat{r}_i}(\hat{x})$  whenever  $y \in Q_{\hat{r}_i/2}(\hat{x})$ ; therefore, using (2.4), Definition 2.1, and the Remez and Markov polynomial inequalities for sufficiently large  $i$ , we conclude that the right-hand side of the above inequality is bounded by  $2 \max_{\varphi(S) \cap Q_{\hat{r}_i}(\hat{x})} |p|$ , see [BB1, Chapter 10] for the details.

<sup>1</sup>Hereafter,  $c(\alpha, \beta, \dots)$  denotes a positive constant depending only on the parameters in the parentheses.

Combining the above estimates, for sufficiently large  $i$ , from (2.6) we derive the inequality

$$\max_{Q_{\hat{r}_i}(\hat{x})} |p| \leq \gamma_1 \max_{\varphi(S) \cap Q_{\hat{r}_i}(\hat{x})} |p|,$$

where  $\gamma_1 := 2c_1(n, k)\gamma$ .

Thus, for every point  $\hat{x}$  of the dense subset  $\varphi(S_0 \cap S_1)$  of  $\varphi(S)$  condition (2.1) is true, i.e.,  $\varphi(S) \in \text{Mar}_k^*(\mathbb{R}^n)$ .  $\square$

The next result requires a more involved proof.

**Theorem 2.4.** *Every closed subset  $S \subset \mathbb{R}^n$  can be presented as disjoint union of a weak  $k$ -Markov (possibly empty) subset and a set of Hausdorff dimension at most  $n - 1$ .*

*Proof.* By Proposition 2.3 (a), there exists a (unique) maximal weak  $k$ -Markov subset of  $S$ . Denoting it by  $S_{\max}$ , we show that

$$\dim_{\mathcal{H}}(S \setminus S_{\max}) \leq n - 1,$$

proving the result.

Assume, to the contrary, that the Hausdorff dimension of this set is greater than  $n - 1$ . Since by definition

$$\dim_{\mathcal{H}}(S \setminus S_{\max}) := \sup\{d > 0; \mathcal{H}_d(S \setminus S_{\max}) = \infty\},$$

where  $\mathcal{H}_d$  is the Hausdorff  $d$ -measure, there exists a subset  $\Omega \subset S \setminus S_{\max}$  such that  $\mathcal{H}_\delta(\Omega) = \infty$  for some  $\delta > n - 1$ . By the classical Besicovitch theorem [B], there exists a compact subset  $C \subset \Omega$  such that

$$0 < \mathcal{H}_\delta(C) < \infty.$$

By another Besicovitch theorem (see, e.g., [Fed]), for some  $C_0 \subset C$  such that  $\mathcal{H}_\delta(C_0) = 0$ , the upper density of every point  $x \in C \setminus C_0$  satisfies

$$2^{-\delta} \leq \limsup_{r \rightarrow 0} \frac{\mathcal{H}_\delta(S \cap Q_r(x))}{(2r)^\delta} \leq 1.$$

Since  $\mathcal{H}_\delta(C_0) = 0$ , there exists a compact subset  $C_1 \subset C \setminus C_0$  such that  $0 < \mathcal{H}_\delta(C_1) < \infty$ . Applying to  $C_1$  the latter Besicovitch theorem and using the compactness of this set, we see that there exists a subset  $C_2 \subset C_1$  and a constant  $a > 0$  such that

$$\begin{aligned} (a) \quad & \mathcal{H}_\delta(C_2) = 0, \\ (2.7) \quad (b) \quad & 1 \leq \limsup_{r \rightarrow 0} \frac{\mathcal{H}_\delta(C_1 \cap Q_r(x))}{r^\delta} \quad \text{for every } x \in C_1 \setminus C_2, \\ (c) \quad & \mathcal{H}_\delta(C_1 \cap Q_r(x)) \leq ar^\delta \quad \text{for every } x \in C_1, r > 0. \end{aligned}$$

Hence, for every  $x \in C_1 \setminus C_2$  there exists a sequence  $R(x)$  of positive numbers decreasing to zero and such that

$$(2.8) \quad \frac{1}{2} < \frac{\mathcal{H}_\delta(C_1 \cap Q_r(x))}{r^\delta} \quad \text{for all } r \in R(x).$$

Now let  $r \in R(x)$ . By Theorem 2.6 of the authors' paper [BB2] (which employs inequalities (2.8) and (2.7) (c)), for  $Q := Q_r(x)$ ,  $\tilde{C} := C_1 \setminus C_2$ , and every polynomial  $p \in \mathcal{P}_k(\mathbb{R}^n)$  we have

$$(2.9) \quad \sup_Q |p| \leq \gamma \sup_{Q \cap \tilde{C}} |p|,$$

where  $\gamma$  depends only on  $k, n, a, \delta$ .

We conclude that  $\tilde{C}$  is subject to Definition 2.1 and, therefore,  $S_{\max} \cup \tilde{C}$  is a weak  $k$ -Markov set contained in  $S$ , which contradicts the maximality of  $S_{\max}$ .  $\square$

**Example 2.5.** (a) We recall that a closed set  $S \subset \mathbb{R}^n$  is (Ahlfors)  $d$ -regular ( $0 \leq d \leq n$ ) if for every cube  $Q_r(x)$  with  $x \in S$  and  $0 < r < \text{diam } S$  we have

$$c_0 r^d \leq \mathcal{H}_d(S \cap Q_r(x)) \leq c_1 r^d,$$

where  $c_0, c_1 > 0$  are constants independent of  $x$  and  $r$ .

In [JW, Chapter 2] it was proved that a  $d$ -regular compact set with  $d > n - 1$  is Markov. This, Proposition 2.3 (a) and relation (2.2) imply that the closure of the union of  $d_i$ -regular compact sets  $S_i \subset \mathbb{R}^n$  with  $d_i > n - 1$ ,  $i \in I$ , is weak  $k$ -Markov for all  $k \geq 1$ .

(b) The previous example might suggest that weak  $k$ -Markov sets with  $k \geq 1$  should be sufficiently massive. However, they may have arbitrarily small Hausdorff dimension. For instance, consider a Cantor set  $C_n$  in  $\mathbb{R}^n$  that is  $d$ -regular with an arbitrarily small  $d > 0$  and is a direct product of  $\frac{d}{n}$ -regular one-dimensional Cantor sets. These one-dimensional sets are Markov by the result formulated in (a) (for  $n = 1$ ). Hence,  $C_n$  is weak  $k$ -Markov by Proposition 2.3 (b).

In fact, most commonly used classical fractals including the Sierpinski gasket, the von Koch curve and the Antoine necklace are Markov, hence, belong to  $\text{Mar}_k^*(\mathbb{R}^n)$ ,  $k \geq 1$ . More generally, let  $S \subset \mathbb{R}^n$  be a self-similar compact fractal, see, e.g., [Fal, 9.2], satisfying

$$\dim(\text{conv } S) = n.$$

Then  $S \in \text{Mar}(\mathbb{R}^n) \subset \text{Mar}_k^*(\mathbb{R}^n)$ ,  $k \geq 1$ , see [BB1, Example 9.25(a)].

(c) If a closed set  $S \subset \mathbb{R}^n$  has an isolated point, say  $x_0$ , then it does not belong to  $\text{Mar}_k^*(\mathbb{R}^n)$  for any  $k \geq 1$  (but, clearly, belongs to  $\text{Mar}_0^*(\mathbb{R}^n)$  as every closed set does). In fact, for  $p(x) := (x - x_0)^\alpha$ ,  $x \in \mathbb{R}^n$ , where  $|\alpha| = k \geq 1$ , and all sufficiently small  $r > 0$  we have

$$\sup_{Q_r(x_0)} |p| > 0 \quad \text{while} \quad \sup_{S \cap Q_r(x_0)} |p| = 0.$$

(d) Clearly,  $\text{Mar}_0^*(\mathbb{R}^n)$  consists of all nonempty closed sets in  $\mathbb{R}^n$ . It should be noted that  $\text{Mar}_1^*(\mathbb{R})$  consists of all closed subsets of the real line without isolated points (including those of Hausdorff dimension 0).

*Remark 2.6.* Unlike the case of Markov sets, the classes  $\text{Mar}_k^*(\mathbb{R}^n)$  with different  $k \geq 1$  do not coincide. We briefly describe the corresponding example for  $k = 1$  and 2 (Proposition 2.3 (b) shows that it suffices to consider the case of  $n = 1$ ).

Set  $c_k := 2^{-k^2}$ ,  $k \in \mathbb{N}$ , and, for an ordered (finite or infinite) set  $K := \{k_1, k_2, \dots\} \subset \mathbb{N}$  (i.e.,  $k_1 < k_2 < \dots$ ), put

$$(2.10) \quad c_K := c_{k_1} + c_{k_1} c_{k_2} + \dots + c_{k_1} \dots c_{k_j} + \dots$$

It can be checked that  $S := \{c_K; K \subset \mathbb{N}\}$  is a compact subset of  $\mathbb{R}$  without isolated points, and that every  $x \in S$  admits a unique presentation in the form (2.10). Moreover,  $c_K < c_{\tilde{K}}$  if and only if  $K > \tilde{K}$  in the lexicographic order (i.e., there exists  $\ell \geq 1$  such that  $k_i = \tilde{k}_j$  for  $1 \leq i \leq \ell$  and  $k_{\ell+1} > \tilde{k}_{\ell+1}$ ).

Using these properties, formula (2.10), and a compactness argument, from the inequalities

$$\sup_{Q_{r_i}(x)} |p| \leq \gamma(x) \sup_{S \cap Q_{r_i}(x)} |p|, \quad x \in S,$$

which are fulfilled for some sequence  $\{r_i\} \subset \mathbb{R}_+$  tending to 0 and all  $p \in \mathcal{P}_2(\mathbb{R})$ , we can deduce that

$$\max_{[0,1]} |p| \leq \gamma(x) \max\{|p(0)|, |p(1)|\},$$

which is clearly not true.

Hence,  $S \notin \text{Mar}_2^*(\mathbb{R})$ , but  $S$  belongs to  $\text{Mar}_1^*(\mathbb{R})$  by Example 2.5 (d).

## §3. FINITENESS PROBLEM FOR WEAK MARKOV SETS

Then main result of this section is the following.

**Theorem 3.1.** (a) *We have*

$$N_{\text{Mar}_k^*(\mathbb{R}^n)}(C^{k,\omega}(\mathbb{R}^n)) \leq 2 \binom{n+k}{n}.$$

(b) *If  $S \in \text{Mar}_k^*(\mathbb{R}^n)$ , then there exists a linear bounded extension operator from  $C^{k,\omega}[S]$  to  $C^{k,\omega}(\mathbb{R}^n)$  whose norm and depth of order  $k$  are bounded by constants depending only on  $n$  and  $k$ .*

*Proof.* (a) We begin with three basic facts exploited in the proof. The first is a simple reformulation of the Whitney–Glaeser theorem.

**Proposition 3.2.** *Given a set  $S \subset \mathbb{R}^n$  and a family of polynomials  $\{p_x\}_{x \in S} \subset \mathcal{P}_k(\mathbb{R}^n)$ , there exists a function  $F \in C^{k,\omega}(\mathbb{R}^n)$  whose Taylor polynomials of order  $k$  satisfy*

$$(3.1) \quad T_x^k(F) = p_x \quad \text{for all } x \in S$$

*if and only if for some constant  $\lambda > 0$  and all  $x, y \in S$  and  $z \in \{x, y\}$  we have*

$$(3.2) \quad \max_{|\alpha| \leq k} |D^\alpha p_x(x)| \leq \lambda \quad \text{and} \quad \max_{|\alpha| \leq k} \frac{|D^\alpha(p_x - p_y)(z)|}{\|x - y\|^{k-|\alpha|}} \leq \lambda \omega(\|x - y\|).$$

*Moreover, we have the equivalence*

$$\inf \{ \|F\|_{C^{k,\omega}[S]} ; T_x^k(F) = p_x, x \in S \} \approx \inf \lambda$$

*with constants depending only on  $k$  and  $n$ .*

It suffices to write

$$p_x(y) := \sum_{|\alpha| \leq k} f_\alpha(x) \frac{(y-x)^\alpha}{\alpha!}$$

and observe that condition (3.2) is equivalent to the Glaeser condition [G] for the  $k$ -jet  $\{f_\alpha\}_{|\alpha| \leq k}$ .

*Remark 3.3.* Whitney’s extension construction exploits only a countable subfamily of the family  $\{p_x\}_{x \in S}$ . In particular, it suffices to take a subfamily indexed by a dense subset of  $S$ , see, e.g., Lemma 3.7 below.

The second fact is a compactness result for the space  $C^{k,\omega}(\mathbb{R}^n)$ .

**Proposition 3.4.** *Let a sequence  $\{f_j\}_{j \in \mathbb{N}} \subset C^{k,\omega}(\mathbb{R}^n)$  be bounded. Then there exists a subsequence  $\{f_j\}_{j \in J}$  and a function  $f \in C^{k,\omega}(\mathbb{R}^n)$  such that*

$$\lim_{j \in J} \|f - f_j\|_{C^k(\bar{Q})} = 0$$

*for every closed cube  $\bar{Q} \subset \mathbb{R}^n$ , and moreover, the moduli of continuity of the higher derivatives of  $f$  satisfy*

$$\max_{|\alpha|=k} \sup_{t>0} \frac{\omega(t; D^\alpha f)}{\omega(t)} \leq \sup_{j \in \mathbb{N}} \|f_j\|_{C^{k,\omega}(\mathbb{R}^n)}.$$

*Proof.* Without loss of generality, we assume that the sequence lies in the unit ball of  $C^{k,\omega}(\mathbb{R}^n)$ . Then for every  $j \geq 1$  and  $|\alpha| = k$ , for a fixed closed cube  $\bar{Q} \subset \mathbb{R}^n$  we have

$$(3.3) \quad \max_{\bar{Q}} |D^\alpha f_j| \leq 1 \quad \text{and} \quad \sup_{t>0} \frac{\omega(t; D^\alpha f_j)}{\omega(t)} \leq 1.$$

Putting the set of multiindices  $\{\alpha; |\alpha| = k\}$  in some order  $\alpha^1, \alpha^2, \dots$ , we apply the Arcela–Ascoli compactness theorem to the sequence  $\{D^\alpha f_j\}_{j \in \mathbb{N}}$  with  $\alpha = \alpha^1$ , obtaining a subsequence  $\{D^\alpha f_j\}_{j \in J_1}$  and a function  $f_\alpha \in C(\bar{Q})$  with this  $\alpha$  such that

$$(3.4) \quad \lim_{j \in J_1} \|f_\alpha - D^\alpha f_j\|_{C(\bar{Q})} = 0.$$

Then we apply the Arcela–Ascoli criterion to the sequence  $\{D^\alpha f_j\}_{j \in J_1}$  with  $\alpha = \alpha^2$  to find a subsequence  $\{D^\alpha f_j\}_{j \in J_2}$ , where  $J_2 \subset J_1$ , and the function  $f_\alpha \in C(\bar{Q})$  so that (3.4) holds true for  $j \in J_2$  and  $\alpha = \alpha^2$ .

Continuing, we obtain a subsequence  $\{f_j\}_{j \in J_d}$ , where  $d := \text{card}\{\alpha; |\alpha| = k\}$ , and a family of functions  $\{f_\alpha\}_{|\alpha|=k} \subset C(\bar{Q})$  such that  $\lim_{j \in J_d} \|D^\alpha f_j - f_\alpha\|_{C(\bar{Q})} = 0$  for all  $\alpha$ .

We rename  $J_d$  by  $I(k)$  and proceed with the derivatives of order  $\alpha$  for  $|\alpha| = k - 1$ . Using the norm

$$\|f\|^* := \max_{|\alpha| \leq k} \sup_{\mathbb{R}^n} |D^\alpha f| + |f|_{C^{k,\omega}(\mathbb{R}^n)}$$

equivalent to the original norm (1.2), from the boundedness of  $\{\|f_j\|^*\}_{j \in I(k)}$  we derive an inequality similar to (3.3), but now, for  $j \in I(k)$  and  $|\alpha| = k - 1$ ,

$$\max_{\bar{Q}} |D^\alpha f_j| \leq c \quad \text{and} \quad \sup_{t>0} \frac{\omega(t; D^\alpha f_j)}{t} \leq c,$$

where  $c$  depends only on  $k$  and  $n$ .

Then we apply the argument already used to find a subsequence  $I(k-1) \subset I(k)$  and a family of functions  $\{f_\alpha\}_{|\alpha|=k-1} \subset C(\bar{Q})$  such that

$$\lim_{j \in I(k-1)} \|D^\alpha f_j - f_\alpha\|_{C(\bar{Q})} = 0$$

for every  $\alpha$  with  $|\alpha| = k - 1$ .

Proceeding in this way for  $k - 2, \dots, 0$ , finally we obtain a subsequence  $\{f_j\}_{j \in I(0)}$  and a family  $\{f_\alpha\}_{|\alpha| \leq k} \subset C(\bar{Q})$  such that

$$\lim_{j \in I(0)} \|D^\alpha f_j - f_\alpha\|_{C(\bar{Q})} = 0$$

for every  $|\alpha| \leq k$ .

We show that  $\{f_j\}_{j \in I(0)}$  is the desired subsequence. For this, we write the Taylor formula for  $f_j$  and estimate the remainder by using the second inequality in (3.3). Then, for  $x, y \in \bar{Q}$ , we have

$$f_j(x) = \sum_{|\alpha| \leq k} \frac{D^\alpha f_j(x)}{\alpha!} (x - y)^\alpha + R_k(f_j; x, y)$$

where

$$|R_k(f_j; x, y)| \leq c(k, n) \|x - y\|^k \omega(\|x - y\|).$$

Passing to the limit along the subsequence  $I(0)$ , we see that

$$\left| f_0(x) - \sum_{|\alpha| \leq k} \frac{f_\alpha(x)}{\alpha!} (x - y)^\alpha \right| \leq c(k, n) \|x - y\|^k \omega(\|x - y\|)$$

for arbitrary  $x, y \in \bar{Q}$ . By the multivariate analog of the Whitney theorem, as given in [BB1, Theorem 2.34], from this inequality we see that  $f_0 \in C^k(\bar{Q})$  and, moreover,  $D^\alpha f_0 = f_\alpha$ .

Finally, passing to the limit in the second inequality (3.3) and then letting  $\bar{Q}$  tend to  $\mathbb{R}^n$ , we obtain the remaining required inequality

$$\max_{|\alpha|=k} \sup_{t>0} \frac{\omega(t; D^\alpha f_0)}{\omega(t)} \leq 1 = \sup_{j \in \mathbb{N}} \|f_j\|_{C^{k,\omega}(\mathbb{R}^n)}.$$

The result is proved.  $\square$

The third fact concerns interpolating properties of subsets satisfying Markov type inequalities.

**Proposition 3.5.** *Suppose a subset  $S$  of the closed unit cube  $\bar{Q} := \bar{Q}_1(0)$  in  $\mathbb{R}^n$  is such that for some constant  $\gamma > 0$  and all polynomials  $p \in \mathcal{P}_k(\mathbb{R}^n)$  we have*

$$(3.5) \quad \max_{\bar{Q}} |p| \leq \gamma \sup_S |p|.$$

*Then there exists a finite subset  $F \subset S$  and a constant  $c = c(k, n, \gamma) > 0$  such that*

- (a)  $\text{card } F = \dim \mathcal{P}_k(\mathbb{R}^n)$ ;
- (b) *for every  $p \in \mathcal{P}_k(\mathbb{R}^n)$ ,*

$$\sup_{\bar{Q}} |p| \leq c \sup_F |p|.$$

*In particular,  $F$  is a minimal interpolating set for the polynomials of degree  $k$  on  $\mathbb{R}^n$ .*

*Proof.* For brevity, we put  $d := \dim \mathcal{P}_k(\mathbb{R}^n)$  and denote by  $\{g_j\}_{1 \leq j \leq d}$  the set of monomials  $\{x^\alpha\}_{|\alpha| \leq k}$  ordered lexicographically. Given a finite subset  $\{x^j\}_{1 \leq j \leq d} \subset S \subset \bar{Q}$  and a polynomial  $p \in \mathcal{P}_k(\mathbb{R}^n)$ , we consider the linear system

$$\sum_{i=1}^d a_i g_i(x^j) = p(x^j), \quad 1 \leq j \leq d.$$

Assume for a while that the matrix  $G := (g_i(x^j))_{1 \leq i, j \leq d}$  is invertible. Then, for  $1 \leq i \leq d$ , the Kramer rule yields

$$a_i = \frac{\sum_{j=1}^d \varphi_{ij} p(x^j)}{\det G},$$

where the  $\varphi_{ij}$  are polynomials in the variables  $g_i(x^j)$  of degree  $d - 1$  whose coefficients equal  $\pm 1$ . Since all  $x^j$  belong to  $\bar{Q}$  and every monomial  $g_i$  is bounded in absolute value by 1 on  $\bar{Q} := \bar{Q}_1(0)$ , we see that

$$|a_i| \leq c \frac{\max_{1 \leq j \leq d} |p(x^j)|}{|\det G|}$$

for some  $c = c(k, n)$ . If we show that in the set  $S$  there exists a subset  $\{x^j\}_{1 \leq j \leq d}$  such that

$$|\det G| \geq c_1 > 0,$$

where  $c_1$  depends only on  $k, n$ , and  $\gamma$ , then this  $d$ -tuple will satisfy the claim of the proposition. Indeed, if  $p \in \mathcal{P}_k(\mathbb{R}^n)$ , then  $p = \sum_{i=1}^d a_i g_i$  and, therefore,

$$\max_{\bar{Q}} |p| \leq d \max_i |a_i| \leq \frac{dc}{c_1} \max_{\{x^j\}} |p|,$$

as required.

Hence, it remains to prove the following.

**Lemma 3.6.** *There exists a subset  $F := \{x^j\}_{1 \leq j \leq d} \subset S$  and a constant  $c = c(k, n, \gamma) > 0$  such that*

$$|\det(g_i(x^j))| \geq c.$$

*Proof.* We exploit a (Veronese) map  $V : \mathbb{R}^n \rightarrow \mathbb{R}^d$  given by

$$V(x) := (g_i(x))_{1 \leq i \leq d}.$$

Let  $\hat{Q}$  denote the symmetric convex hull of  $V(\bar{Q})$ , i.e.,  $\hat{Q} := \text{conv}(V(\bar{Q}) \cup (-V(\bar{Q})))$ , and let  $\hat{S}$  be the symmetric convex hull of  $V(S)$ . We verify that

$$(3.6) \quad \dim \hat{Q} = d.$$

It suffices to show that  $V(\bar{Q})$  is not contained in a hyperspace of  $\mathbb{R}^d$ , say  $H$ . But if  $V(\bar{Q}) \subset H$ , then for some nonzero vector  $\ell \in \mathbb{R}^d$  and all  $x \in \bar{Q}$ , we have

$$\langle \ell, V(x) \rangle := \sum_{i=1}^d \ell_i g_i(x) = 0.$$

This means that the  $d$ -cube  $\bar{Q}$  is contained in a  $(d-1)$ -dimensional polynomial surface, a contradiction.

The dual space  $(\mathbb{R}^d)^*$  can be identified with the space of polynomials  $\mathcal{P}_k(\mathbb{R}^n)$ . Under this identification, inequality (3.5) implies that

$$\sup_{V(\bar{Q})} |\varphi| \leq \gamma \sup_{V(S)} |\varphi|$$

for every  $\varphi \in (\mathbb{R}^d)^*$ . Since the supremum of the absolute value of a linear functional on a subset of  $\mathbb{R}^d$  equals its supremum on the symmetric convex hull of that subset, we conclude that

$$(3.7) \quad \sup_{\hat{Q}} |\varphi| \leq \gamma \sup_{\hat{S}} |\varphi|$$

for every  $\varphi \in (\mathbb{R}^d)^*$ .

From this we deduce that

$$(3.8) \quad \hat{Q} \subset \gamma \hat{S} := \{\gamma x; x \in \hat{S}\}.$$

If, to the contrary, there exists a point  $x \in \hat{Q} \setminus \hat{S}$ , then the Hahn–Banach theorem implies the existence of a functional  $\varphi \in (\mathbb{R}^d)^*$  separating  $x$  and  $\gamma \hat{S}$ , i.e.,

$$\sup_{\gamma \hat{S}} |\varphi| < |\varphi(x)| \left( \leq \sup_{\hat{Q}} |\varphi| \right).$$

But the left-hand side equals  $\gamma \sup_{\hat{S}} |\varphi|$ , and this inequality contradicts (3.7).

Next, up to the factor  $\pm \frac{1}{d!}$ , the quantity  $\det(g_i(x^j))$  to be estimated is equal to the  $d$ -volume of the simplex  $\text{conv}(V(x^1), \dots, V(x^d))$ . Hence, it suffices to find a  $d$ -tuple  $\{x^1, \dots, x^d\} \subset S$  such that

$$(3.9) \quad |\text{conv}\{V(x^1), \dots, V(x^d)\}| \geq c(k, n, \gamma) > 0,$$

where  $|\cdot|$  stands for the Lebesgue  $d$ -measure.

For this, we use the John theorem [J] to find an ellipsoid  $\mathcal{E} \subset \mathbb{R}^d$  centered at 0 and such that

$$(3.10) \quad \mathcal{E} \subset \hat{S} \subset \sqrt{d}\mathcal{E}.$$

Due to (3.8) and (3.6), the set  $\hat{S}$  is of dimension  $d$ , whence so is  $\mathcal{E}$ .

Let  $\Delta := \text{conv}\{s^j\}_{1 \leq j \leq d}$  be the simplex of maximal volume inscribed in  $\mathcal{E}$ . To evaluate its volume, we use the map  $(s_1, \dots, s_d) \mapsto (a_1 s_1, \dots, a_d s_d)$ , where the numbers  $a_i$  are the half-lengths of the principal axes of  $\mathcal{E}$ . Then the maximal simplex  $\Delta$  is the image of the regular  $d$ -simplex inscribed in the unit Euclidean  $(d-1)$ -sphere. Denoting the volume of the latter by  $\sigma_d$ , we then have

$$|\Delta| = \sigma_d \prod_{i=1}^d a_i = \frac{\sigma_d}{\beta_d} |\mathcal{E}|,$$

where  $\beta_d$  is the volume of the unit Euclidean  $d$ -ball. This, (3.10), and (3.7) yield

$$|\Delta| \geq d^{-\frac{d}{2}} \frac{\sigma_d}{\beta_d} |\hat{S}| \geq c(k, n) \gamma^{-d} |\hat{Q}|.$$

On the other hand,  $\dim \widehat{Q} = d$ , so that its  $d$ -volume is positive and obviously depends only on  $d = d(k, n)$  and  $n$ . Hence, finally we have

$$(3.11) \quad |\det(s^1, \dots, s^d)| \geq \tilde{c}(k, n)\gamma^{-d} > 0.$$

We use this to find the desired  $d$ -tuple satisfying (3.9) and in this way complete the proof. By the Carathéodory theorem (see, e.g., [DGK]), every point  $s^j$  of the convex hull  $\widehat{S} := \text{conv}(V(S) \cup (-V(S)))$  is a convex combination of  $d + 1$  points of the initial set. Hence, there exists a  $(d + 1)$ -tuple of points  $\{x^{j_i}\}_{1 \leq i \leq d+1} \subset S$  and a sequence  $\{\lambda_{j_i}\}_{1 \leq i \leq d} \subset \mathbb{R}$  such that

$$s^j = \sum_{i=1}^{d+1} \lambda_{j_i} V(x^{j_i}) \quad \text{and} \quad \sum_{i=1}^{d+1} |\lambda_{j_i}| = 1, \quad 1 \leq j \leq d + 1.$$

Since the determinant is a multilinear function of its entries, we have

$$|\det(s^1, \dots, s^d)| \leq \max |\det(y^1, \dots, y^d)|,$$

where the maximum is taken over all tuples  $\{y^j\}_{1 \leq j \leq d}$  with  $y^j \in \{V(x^{j_1}), \dots, V(x^{j_{d+1}})\}$ . If the maximum is attained at  $\{\bar{y}^1, \dots, \bar{y}^d\}$ , then the above inequality and (3.11) give

$$|\det(\bar{y}^1, \dots, \bar{y}^d)| \geq \tilde{c}(k, n)\gamma^{-d} > 0.$$

The left-hand side divided by  $d!$  equals  $|\text{conv}(\bar{y}^1, \dots, \bar{y}^d)|$ . Hence, the desired inequality (3.9) holds true for  $V(x^i) := \bar{y}^i$  and the constant  $\frac{\tilde{c}(n, k)\gamma^{-d}}{d!}$ .  $\square$

The proof of the proposition is complete.  $\square$

Now we are in a position to prove the required result.

Let a set  $S$  belong to  $\text{Mar}_k^*(\mathbb{R}^n)$ . Suppose the trace of a function  $f : S \rightarrow \mathbb{R}$  on every subset of  $S$  of cardinality at most  $2 \binom{n+k}{n}$  admits an extension to a function belonging to the unit ball of the space  $C^{k, \omega}(\mathbb{R}^n)$ . We must prove that the function  $f$  belongs to the trace space  $C^{k, \omega}[S]$  and its trace norm is bounded by a constant  $c = c(n, k)$ .

For this, we construct a family of polynomials  $\{T_x^k\}_{x \in S}$  of degree  $k$  satisfying the assumptions of the Whitney–Glaeser Proposition 3.2 with  $\lambda = c(k, n) > 0$  such that  $T_x^k(x) = f(x)$  for all  $x \in S$ . As was noted in Remark 3.3, we may work with a dense subset  $S_0$  of  $S$ . In particular,  $S_0$  may be taken as the subset of  $S$  ( $\in \text{Mar}_k^*(\mathbb{R}^n)$ ) subject to Definition 2.1. Since it suffices to obtain the required extension for  $f|_{S_0}$ , below we may and shall assume that  $S_0 = S$ , i.e., every point of  $S$  satisfies condition (2.1) of this definition. Should the family  $\{T_x^k; x \in S\}$  be constructed, the proposition cited would imply that there exists a function  $F \in C^{k, \omega}(\mathbb{R}^n)$  with norm bounded by a constant depending only on  $k, n$  such that  $F(x) = T_x^k(x)$  for all  $x \in S$ . Clearly, this means that  $F$  is an extension of  $f$  and

$$\|f\|_{C^{k, \omega}[S]} \leq \|F\|_{C^{k, \omega}(\mathbb{R}^n)} \leq c(k, n),$$

as required.

To define the desired polynomial  $T_x^k$  for  $x \in S$ , we use a compactness argument. By Definition 2.1, there exists a sequence of positive numbers  $\{r_j\}$  tending to 0 and a constant  $\gamma(x) > 0$  such that for every polynomial  $p \in \mathcal{P}_k(\mathbb{R}^n)$  and  $r = r_j$  we have

$$\max_{\widehat{Q}_r(x)} |p| \leq \gamma(x) \sup_{Q_r(x) \cap S} |p|.$$

Applying Proposition 3.5 to the polynomial  $x \mapsto p(rx)$ , where  $p \in \mathcal{P}_k(\mathbb{R}^n)$ , from the previous inequality we deduce the existence of a subset  $S_r(x) \subset Q_r(x) \cap S$  such that

$$(3.12) \quad \text{card } S_r(x) = \binom{n+k}{n}$$

and

$$(3.13) \quad \max_{\bar{Q}_r(x)} |p| \leq c(k, n, \gamma(x)) \sup_{S_r(x)} |p|$$

for every  $p \in \mathcal{P}_k(\mathbb{R}^n)$  and  $r = r_j$ ,  $j \in \mathbb{N}$ .

By the assumption of the theorem, there exists a function  $F_j \in C^{k,\omega}(\mathbb{R}^n)$ ,  $j \in \mathbb{N}$ , such that

$$(3.14) \quad F_j = f \quad \text{on} \quad S_{r_j}(x)$$

and

$$(3.15) \quad \|F_j\|_{C^{k,\omega}(\mathbb{R}^n)} \leq 1.$$

This inequality shows that the family  $\{F_j\}_{j \in \mathbb{N}}$  satisfies the compactness criterion of Proposition 3.4. Hence, a subsequence of  $\{F_j\}_{j \in \mathbb{N}}$  is convergent in  $C^k$  on every closed cube of  $\mathbb{R}^n$  to a function  $F \in C^{k,\omega}(\mathbb{R}^n)$  such that  $\|F\|_{C^{k,\omega}(\mathbb{R}^n)} \leq 1$ .

Now the desired polynomial  $T_x^k$  is defined as the Taylor polynomial of  $F$  at  $x$  of degree  $k$ . To prove that  $T_x^k$  is well defined, we need to show that it is independent of the choice of a sequence satisfying (3.14) and (3.15).

Let  $\{F'_j\}_{j \in \mathbb{N}}$  be such a sequence distinct from  $\{F_j\}_{j \in \mathbb{N}}$ . Then

$$(3.16) \quad \|F_j - F'_j\|_{C^{k,\omega}(\mathbb{R}^n)} \leq 2$$

and, moreover,

$$(3.17) \quad F_j - F'_j = 0 \quad \text{on} \quad S_{r_j}(x).$$

Let  $T_j$  and  $T'_j$  be the Taylor polynomials at  $x$  of degree  $k$  for  $F_j$  and  $F'_j$ , respectively. In accordance with the Taylor formula (the necessary condition of Proposition 3.2) and (3.16), for every  $z \in S_{r_j}(x) \subset \bar{Q}_{r_j}(x)$  we have

$$(3.18) \quad |(T_j - T'_j)(z)| \leq c(k, n) \|x - z\|^k \omega(\|x - z\|) \leq c(k, n) r_j^k \omega(r_j).$$

Combining this and (3.13), we get

$$\max_{\bar{Q}_{r_j}(x)} |T_j - T'_j| \leq c(k, n, \gamma(x)) r_j^k \omega(r_j).$$

Now we apply the Remez type inequality [BG] with  $n = 1$  for the restriction of  $T_j - T'_j$  to a straight line passing through  $x$ . Along with the preceding inequality, this yields

$$(3.19) \quad \max_{\bar{Q}_1(x)} |T_j - T'_j| \leq c(n, k) \left(\frac{1}{r_j}\right)^k \max_{\bar{Q}_{r_j}(x)} |T_j - T'_j| \leq c(n, k) c(n, k, \gamma(x)) \omega(r_j).$$

Finally, let  $J \subset \mathbb{N}$  be a subsequence such that, for some functions  $F, F' \in C^k(\mathbb{R}^n)$ ,

$$\lim_j F_j = F, \quad \lim_j F'_j = F' \quad (\text{convergence in } C^k(\bar{Q}_1(x))).$$

Then  $T_j$  and  $T'_j$  tend to the Taylor polynomials at  $x$  of order  $k$  for  $F$  and  $F'$ , respectively. But (3.19) shows that these polynomials coincide.

Hence, the polynomial  $T_x^k$ ,  $x \in S$ , does not depend on the choice of  $\{F_j\}$ , as required.

Now we check that the family  $\{T_x^k\}_{x \in S}$  satisfies the assumptions of Proposition 3.2. For this, given two points  $x \neq y$  in  $S$  and  $r_j, \tilde{r}_j > 0$ , we find the corresponding finite subsets  $S_{r_j}(x)$  and  $S_{\tilde{r}_j}(y)$ ,  $j \in \mathbb{N}$ . Since

$$\text{card}(S_{r_j}(x) \cup S_{\tilde{r}_j}(y)) \leq 2 \binom{n+k}{n},$$

the assumption of the theorem implies the existence of a function  $F_j$  such that

$$F_j = f \quad \text{on} \quad S_{r_j}(x) \cup S_{\tilde{r}_j}(y)$$

and

$$\|F_j\|_{C^{k,\omega}(\mathbb{R}^n)} \leq 1 \quad \text{for all } j \in \mathbb{N}.$$

Employing the above compactness argument again, we find a subsequence  $\{F_j\}_{j \in J}$  and a function  $F \in C^k(\mathbb{R}^n)$  such that the limit of  $\{F_j\}_{j \in J}$  in  $C^k(\bar{Q})$  equals  $F|_{\bar{Q}}$  for any cube  $\bar{Q}$  (in particular, for a cube containing  $x$  and  $y$ ).

Passing then to the limit as  $r_j, \tilde{r}_j \rightarrow 0$  in the previous two relations, we get

$$(3.20) \quad F(x) = f(x), \quad F(y) = f(y) \quad \text{and} \quad \|F\|_{C^{k,\omega}(\mathbb{R}^n)} \leq 1.$$

Now let  $T_x^k(F)$  and  $T_y^k(F)$  be the Taylor polynomials of degree  $k$  at the points  $x$  and  $y$  in  $S$ , respectively. Both of them are determined by the sequence  $\{F_j\}_{j \in \mathbb{N}}$  and, therefore, belong to the family  $\{T_z^k\}_{z \in S}$  introduced above, i.e.,  $T_z^k(F) = T_z^k$  for  $z \in \{x, y\}$ .

Since  $\|F\|_{C^{k,\omega}(\mathbb{R}^n)} \leq 1$ , the Taylor formula for  $F$  yields

$$|D^\alpha(T_x^k - T_y^k)(z)| \leq c(k, n)\|x - y\|^{k-|\alpha|}\omega(\|x - y\|)$$

for  $z \in \{x, y\}$  and  $|\alpha| \leq k$ . This and the first relation in (3.20) show that the sufficiency condition of Proposition 3.2 is fulfilled. Hence, the required function  $g \in C^{k,\omega}(\mathbb{R}^n)$  that extends  $f$  and satisfies  $\|g\|_{C^{k,\omega}(\mathbb{R}^n)} \leq c(n, k)$  does exist.

(b) The desired linear extension operator  $E : C^{k,\omega}[S] \rightarrow C^{k,\omega}(\mathbb{R}^n)$  with  $S \in \text{Mar}_k^*(\mathbb{R}^n)$  is a modification of the classical Whitney extension construction [W1]. The latter was used, e.g., in the proof of Proposition 3.2 to recover the function  $F \in C^{k,\omega}(\mathbb{R}^n)$  by the formula

$$(3.21) \quad F(x) := \begin{cases} p_x(x) & \text{if } x \in S, \\ \sum_{Q \in \mathcal{W}_S} p_x(Q)(x)\varphi_Q(x) & \text{if } x \in S^c := \mathbb{R}^n \setminus S. \end{cases}$$

We recall some properties of the ingredients involved here in a form used in the sequel. Below  $\lambda Q$ ,  $\lambda > 0$ , stands for  $Q_{\lambda r}(x)$  provided  $Q := Q_r(x)$ , and all distances are measured in the  $\ell_\infty$ -norm of  $\mathbb{R}^n$ , i.e.,  $\|x\| := \max_{1 \leq i \leq n} |x_i|$ .

**Lemma 3.7.** *There exists a family  $\mathcal{W}_S$  of cubes with the following properties.*

- (a)  $\mathcal{W}_S$  is a family of closed dyadic cubes in  $S^c$  with pairwise disjoint interiors covering  $S^c$ .
- (b)  $\mathcal{W}_S^* := \{Q^* := \frac{9}{8}Q; Q \in \mathcal{W}_S\}$  is a family of cubes in  $S^c$  covering this set with multiplicity <sup>2</sup>

$$(3.22) \quad \text{mult}(\mathcal{W}_S^*) \leq c(k, n).$$

- (c) For every  $Q := Q_r(x) \in \mathcal{W}_S$  and  $y \in Q^*$ , the point  $x(Q)$  belongs to  $S$ , and

$$(3.23) \quad \|x(Q) - y\| \approx d(Q, S) \approx r$$

with equivalence constants depending only on  $n$ .

- (d)  $\{\varphi_Q\}_{Q \in \mathcal{W}_S}$  is a  $C^\infty$ -partition of unity subordinate to  $\mathcal{W}_S^*$  and such that for every  $\alpha \in \mathbb{Z}_+^n$  and  $Q := Q_r(x) \in \mathcal{W}_S$  we have

$$(3.24) \quad \max_{\mathbb{R}^n} |D^\alpha \varphi_Q| \leq c(\alpha, n)r^{-|\alpha|}.$$

*Remark 3.8.*  $x(Q)$  is a point of  $S$  almost closest to  $Q$ , e.g., we may take any  $x(Q) \in S$  satisfying

$$(3.25) \quad d(x(Q), Q) \leq 2d(S, Q).$$

<sup>2</sup>That is every point of  $x \in S^c$  is covered by at most  $c(k, n)$  cubes from  $\mathcal{W}_S^*$ .

Now let  $f \in C^{k,\omega}[S]$ , where

$$(3.26) \quad \|f\|_{C^{k,\omega}[S]} \leq 1,$$

and let  $S_0$  be a dense subset of  $S \in \text{Mar}_k^*(\mathbb{R}^n)$  subject to Definition 2.1. If we select points  $x(Q)$  in  $S_0$ , then for every  $Q \in \mathcal{W}_S$  there exists a sequence  $\mathcal{R}_Q \subset \mathbb{R}_+$  decaying to zero, and a family of finite subsets  $\{S_r(x(Q))\}_{r \in \mathcal{R}_Q}$  of  $S$  such that

$$(3.27) \quad S_r(x(Q)) \subset S \cap Q_r(x(Q)) \quad \text{and} \quad \text{card } S_r(x(Q)) = \binom{n+k}{k},$$

and

$$(3.28) \quad \max_{Q_r(x(Q))} |p| \leq c(k, n, x(Q)) \sup_{S_r(x(Q))} |p|$$

for every  $p \in \mathcal{P}_k(\mathbb{R}^n)$ ; see Proposition 3.5.

In what follows,  $c(\cdot)$  denotes a positive constant depending only on parameters in the parentheses. It may change from line to line or within the same line.

Now for  $r \in \mathcal{R}_Q$ , we denote by  $p_{r,Q}$  a (unique) polynomial interpolating  $f$  at points of  $S_r(x(Q))$  and compare this with the Taylor polynomial  $T_{x(Q)}^k$  constructed for  $f$  in the preceding part of the proof (since  $f \in C^{k,\omega}[S]$ , it clearly satisfies the finiteness condition of this part). For  $z \in S_r(x(Q))$  we have

$$\begin{aligned} |(T_{x(Q)}^k - p_{r,Q})(z)| &= |(T_{x(Q)}^k - f)(z)| \\ &= |(T_{x(Q)}^k - T_z^k)(z)| \leq c(k, n) \|x(Q) - z\|^k \omega(\|x(Q) - z\|), \end{aligned}$$

where, as in (3.18), the inequality follows from the Taylor formula and (3.26).

Estimating the right-hand side by (3.23) and (3.28), we obtain

$$(3.29) \quad \max_{Q_r(x(Q))} |T_{x(Q)}^k - p_{r,Q}| \leq c(k, n, x(Q)) r^k \omega(r).$$

Now, let  $r_Q$  stand for the  $\ell_\infty$ -radius (length of the half-edge) of  $Q$ , and let  $\rho(Q)$  be the smallest number  $\rho$  such that  $Q \subset Q_\rho(x(Q))$ . By (3.23),

$$(3.30) \quad \rho(Q) \approx r_Q \approx d(Q, S)$$

with equivalence constants depending only on  $n$ . Using then (3.29), (3.30), and the univariate Remez inequality, cf. (3.19), we get

$$\begin{aligned} \max_Q |T_{x(Q)}^k - p_{r,Q}| &\leq c(k, n) \left( \frac{\rho(Q)}{r} \right)^k \cdot \max_{Q_r(x(Q))} |T_{x(Q)}^k - p_{r,Q}| \\ &\leq c(k, n, x(Q)) \left( \frac{r_Q}{r} \right)^k \cdot r^k \omega(r) = c(k, n, x(Q)) r_Q^k \omega(r). \end{aligned}$$

Since  $\omega(r) \rightarrow 0$  as  $r \rightarrow 0$ , we can choose  $r \in \mathcal{R}_Q$  to be a number such that the right-hand side is bounded by  $\hat{r}^{k+1} \min\{1, \omega(\hat{r})\}$ , where  $\hat{r}_Q := \min\{1, r_Q\}$ . Denoting  $p_{r,Q}$  with this  $r$  by  $p_Q$ , we see that the following claim is proved.

**Lemma 3.9.** *For every  $Q \in \mathcal{W}_S$ , there exists a polynomial  $p_Q$  interpolating  $f$  at some points of  $S$  and satisfying*

$$(3.31) \quad \max_Q |T_{x(Q)}^k - p_Q| \leq \hat{r}^{k+1} \min\{1, \omega(\hat{r})\} \quad \text{with} \quad \hat{r}_Q := \min\{1, r_Q\}.$$

Now we define the desired extension of the function  $f$  by setting

$$(3.32) \quad Ef := \begin{cases} f & \text{on } S, \\ \sum_{Q \in \mathcal{W}_S} p_Q \varphi_Q & \text{on } S^c. \end{cases}$$

Clearly, this gives rise to a linear extension operator acting from  $C^{k,\omega}[S]$ , and first we show that it acts into  $C^{k,\omega}(\mathbb{R}^n)$ . For this, we choose an extension of  $f$ , say  $F$ , of class  $C^{k,\omega}(\mathbb{R}^n)$  such that

$$(3.33) \quad \|F\|_{C^{k,\omega}(\mathbb{R}^n)} \leq 2\|f\|_{C^{k,\omega}[S]} (\leq 2)$$

and then use the Taylor polynomials of  $F$  to write

$$f_1 := \begin{cases} f & \text{on } S, \\ \sum_{Q \in \mathcal{W}_S} (T_{x(Q)}^k F) \varphi_Q & \text{on } S^c. \end{cases}$$

By the proof of part (a), the Taylor polynomials at points of  $S$  of any  $C^{k,\omega}(\mathbb{R}^n)$ -extension of  $f$  are uniquely determined by  $f$ . Hence, we have the identity

$$T_{x(Q)}^k = T_{x(Q)}^k F,$$

which, in its turn, implies

$$Ef = f_1 + f_2,$$

where

$$(3.34) \quad f_2 := \begin{cases} 0 & \text{on } S, \\ \sum_{Q \in \mathcal{W}_S} (p_Q - T_{x(Q)}^k) \varphi_Q & \text{on } S^c. \end{cases}$$

Since, clearly, the family  $\{T_x^k F\}_{x \in S}$  satisfies the assumptions of Proposition 3.2 with  $\lambda := \|F\|_{C^{k,\omega}(\mathbb{R}^n)}$ , that proposition and (3.33) show that  $f_1 \in C^{k,\omega}(\mathbb{R}^n)$  and

$$(3.35) \quad \|f_1\|_{C^{k,\omega}(\mathbb{R}^n)} \leq c(k, n) \|F\|_{C^{k,\omega}(\mathbb{R}^n)} \leq c(k, n).$$

It remains to prove a similar inequality for  $f_2$ . Differentiating  $f_2$  at a point  $x \in S^c$  and applying (3.31), the Markov polynomial inequality, and (3.24), for  $|\alpha| \leq k+1$  we get

$$\begin{aligned} |D^\alpha f_2|(x) &\leq \sum_{\mathcal{W}_S^* \ni Q \ni x} \left( \sum_{\gamma \leq \alpha} |D^\gamma (p_Q - T_{x(Q)}^k)|(x) \cdot |D^{\alpha-\gamma} \varphi_Q|(x) \right) \\ &\leq c(k, n) \sum_{\mathcal{W}_S^* \ni Q \ni x} \frac{\hat{r}_Q^{k+1-|\gamma|} \min\{1, \omega(\hat{r}_Q)\}}{r_Q^{|\alpha|-|\gamma|}} \leq c(k, n) \sum_{\mathcal{W}_S^* \ni Q \ni x} \hat{r}_Q^{k+1-|\alpha|} \min\{1, \omega(\hat{r}_Q)\}. \end{aligned}$$

Setting  $d(x) := \min\{d(x, S), 1\}$  for brevity and applying (3.22) and (3.23), we conclude that

$$(3.36) \quad |D^\alpha f_2|(x) \leq c(k, n) d(x)^{k+1-|\alpha|} \min\{1, \omega(d(x))\}$$

provided  $x \in S^c$  and  $|\alpha| \leq k+1$ .

This implies that  $f_2$  is  $(k+1)$ -times differentiable and all its derivatives equal zero on  $S$ . Indeed, from (3.36) with  $|\alpha| = 1$  and that fact that  $f_2|_S = 0$  it follows that every first order derivative of  $f_2$  exists and equals zero at the points of  $S$ . Using this and (3.36) with  $|\alpha| = 2$ , we prove the same for the second order derivatives and so forth.

Now we estimate the  $C^{k,\omega}$ -norm of  $f_2$ . Inequality (3.36) immediately implies the formula

$$\max_{|\alpha| \leq k} \sup_{\mathbb{R}^n} |D^\alpha f_2| \leq c(k, n) \min\{1, \omega(1)\} \leq c(k, n),$$

and it remains to estimate the moduli of continuity of the  $k$ th derivatives. For  $|\alpha| = k$ , from (3.36) with  $|\alpha| = k+1$  we obtain

$$|D^\alpha f_2(x) - D^\alpha f_2(y)| \leq n \max_{|\beta|=k+1} |D^\beta f_2| \cdot \|x - y\| \leq c(k, n) \omega(1) \|x - y\|.$$

Hence, since the function  $t \mapsto \frac{\omega(t)}{t}$  decreases monotonically as  $t$  increases, for  $t \leq 1$  we get

$$\frac{\omega(t; D^\alpha f_2)}{\omega(t)} \leq c(k, n)\omega(1) \frac{t}{\omega(t)} \leq c(k, n).$$

Also, if  $\|x - y\| \geq 1$  and  $|\alpha| = k + 1$ , by (3.36) we have

$$|D^\alpha f_2(x) - D^\alpha f_2(y)| \leq c(k, n)\omega(1).$$

Since  $t \mapsto \omega(t)$  is a monotone nondecreasing function, the above two inequalities imply

$$\sup_{0 < t < \infty} \frac{\omega(t; D^\alpha f_2)}{\omega(t)} \leq c(k, n).$$

Combining the inequalities obtained, we then get

$$\max_{|\alpha| \leq k} \left( \sup_{\mathbb{R}^n} |D^\alpha f_2| + \sup_{0 < t \leq \infty} \frac{\omega(t; D^\alpha f_2)}{\omega(t)} \right) \leq c(k, n).$$

But the left-hand side is the  $C^{k, \omega}(\mathbb{R}^n)$ -norm of  $f_2$ . Therefore, the last inequality, (3.35), and (3.26) imply the required estimate

$$\|Ef\|_{C^{k, \omega}(\mathbb{R}^n)} := \|f_1 + f_2\|_{C^{k, \omega}(\mathbb{R}^n)} \leq c(k, n)\|f\|_{C^{k, \omega}[S]}.$$

It remains to show that  $E$  is of depth of order  $k$  bounded by some  $c(k, n)$ . By definition (see (3.32)), for  $x \in S^c$  we have

$$(Ef)(x) = \sum_{\mathcal{W}_S^* \ni Q \ni x} p_Q(x) \varphi_Q(x).$$

Moreover,  $p_Q = f(x_Q^i) \ell_i^Q$ , where  $\{x_Q^i\}$  is a subset of  $S$  with  $\binom{n+k}{k}$  points and the  $\ell_i^Q$  are the fundamental polynomials of Lagrange interpolation. Hence, the above identity can be rewritten as

$$(Ef)(x) = \sum_{\mathcal{W}_S^* \ni Q \ni x} \sum_i f(x_Q^i) \lambda_i^Q(x),$$

where all  $\lambda_i^Q$  are  $C_0^\infty$ -functions, and the number of points in the family  $\{x_Q^i\} \subset S$  is at most  $\binom{n+k}{k} \text{mult } \mathcal{W}_S^* \leq c(n) \binom{n+k}{k}$ . This means that the depth  $E$  of order  $k$  is bounded by some  $c(k, n)$ .

The theorem is proved.  $\square$

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