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GEOMETRY OF ROOT ELEMENTS IN GROUPS OF TYPE E_6

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Abstract. Root elements in the 27-dimensional representation of the simply connected Chevalley group of type E_7 over a field are studied. With every root subgroup, a certain six-dimensional singular subspace is associated; this correspondence is shown to be a natural bijection. Also, the notion of the angle between root subgroups is described in terms of singular subspaces.

Introduction

Let $G = G_{\mathrm{sc}}(\mathrm{E}_6, K)$ be a simply connected Chevalley group of type E_6 over a field K, and let G act on a minimal 27-dimensional module V. We aim to study the relationship between the geometry of V and the root elements of G; namely, we describe a natural correspondence between root subgroups and six-dimensional singular subspaces. It is clear that for any nontrivial root element $g \in G$ the subspace $V^g = \mathrm{Im}(g - E)$ is six-dimensional and singular. Moreover, the same singular subspace corresponds to elements of the same root subgroup, while different subspaces correspond to elements of different root subgroups. This allows us to construct an injective map from the set of all root subgroups to the set of six-dimensional singular subspaces. This map is compatible with the action of G, where G acts on the set of root subgroups by conjugation. Our aim in the present paper is to prove that the image of the set of root elements under the injective map described above coincides with the set of all six-dimensional singular subspaces. More precisely, we prove the following theorem.

Theorem 2. *Let $n = 1, 2, 3, 4,$ or 6. Let $\{u^i\}_{i=1}^n$ and $\{v^i\}_{i=1}^n$ be two sets of singular vectors that generate two n-dimensional singular subspaces. Then there exists a matrix $g \in G_{\mathrm{sc}}(\mathrm{E}_6, K)$ such that $u^i = gv^i$ for $i \le \min(n, 5)$. If $n = 6$, then $u^6 = agv^6$ for some $a \in K^*$.*

From this theorem it follows easily that the map from the set of root subgroups to the set of six-dimensional singular subspaces, as described above, is a bijection.

From its beginnings, the theory of Lie groups was intimately related to geometry, because the geometry of a space is largely determined by the group of its automorphisms. The theory of buildings, which emerged in the work of Jacques Tits, lifted this link to a new level: it became possible to study the geometry of a group itself. The root subgroups and their relative position play a key role in this study.

The root subgroups (especially the long ones) are among the most important and, at the same time, the simplest objects in Chevalley groups. However, the study of 'relative position' (in some sense) of several long root subgroups is a very complicated and interesting problem, which has been considered by many authors. These problems and the

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intimately related problems of description of subgroups generated by root elements were treated by Jack McLaughlin, Ascher Wagner, Michael Aschbacher, Gary Seitz, William Kantor, Bruce Cooperstein, Alexander Zalesskiĭ, and many others (see, in particular, [9, 28, 50, 54, 55], [60]–[64], [73, 75, 102], and further references in [12, 26, 27, 29, 74]). In the mid-1990s, Liebeck and Seitz found astonishingly simple proofs and generalizations of the results concerning generation by root elements in the context of algebraic group theory [76]. On the other hand, an extremely deep theory of *abstract root subgroups* was developed by F. Timmesfeld and his students A. Steinbach, H. Cuypers, and others (see [65]–[68] and [83]–[94]). E. L. Bashkirov's remarkable papers [1]–[5] and [51]–[53] were concerned with similar problems over division algebras.

A relative position of two long root subgroups was described many times by many authors (see, for example, [9] and the further references there). It is well known that any pair of long root subgroups is conjugated to a pair of elementary long root subgroups. Therefore, their relative position is uniquely determined by the angle between the subgroups. In the present paper we give another (geometric) proof of this fact and describe how the angle between root subgroups can be recovered from the corresponding two six-dimensional singular subspaces. More precisely, we prove the following proposition.

Proposition 12. *For any root elements g and h, exactly one of the following statements is true*:

(1) *if* $V^g = V^h$, *then* $\angle(g,h) = 0$;
(2) *if* $\dim(V^g \cap V^h) = 3$, *then* $\angle(g,h) = \pi/3$;
(3) *if* $\dim(V^g \cap V^h) = 1$, *then* $\angle(g,h) = \pi/2$;
(4) *if* $V^g \cap V^h = 0$ *and there exists a six-dimensional singular subspace* W *such that* $\dim(V^g \cap W) = \dim(V^h \cap W) = 3$, *then* $\angle(g,h) = 2\pi/3$;
(5) *if* $V^g \cap V^h = 0$ *and for any vector* $v \in V^g$ *there exists a unique* (*up to scalar multiplication*) *vector* $u \in V^h$ *such that* $v + u$ *is singular, then* $\angle(g,h) = \pi$.

To attack the problem of generation, studied in the subsequent papers of the author, we need a direct answer to the following question: what is the subgroup generated by three given root subgroups in $G_{\mathrm{sc}}(\mathrm{E}_6, K)$? It is not easy to extract an answer to this question from the literature. On the one hand, the answer for *finite* fields was stated in [62] and, most likely, follows from the work of Cooperstein [60]–[64]. However, first, the answer is scattered over dozens of pages; second, the techniques applied in these papers cannot be generalized to an arbitrary field. On the other hand, undoubtedly, the answer we need is contained in the work of Timmesfeld, but it is not stated explicitly. It is probable that the description we need can be found in the work of Bashkirov, who studied this question in a much more general setting. A similar result with an elementary proof can be found in the paper by Di Martino and Vavilov [71], where it was discussed how the subgroup generated by three root subgroups in $\mathrm{SL}(n, K)$ two of which are opposite may look like. The recent paper [21] by Vavilov and the author was devoted to a similar question for any simply connected group $G(\Phi, K)$, where $\operatorname{char} K \neq 2$. In the present paper we show how to determine (using an invariant geometric language) the group generated by three root subgroups in $\mathrm{SO}(2n, K)$ and $G_{\mathrm{sc}}(\mathrm{E}_6, K)$ two of which are opposite.

In the present paper we study only the geometry of long root subgroups. Nevertheless, we must say a few words about the geometry of short root subgroups and tori. These geometries and related problems are much harder and much less studied. However, the recent papers of Nesterov and Vavilov [18, 19] and [34]–[36] were devoted to these extremely interesting objects. In particular, the orbits of *pairs* of microweight tori were listed (in [18] and [19]), as well as those of *pairs* of short root subgroups (in [34]–[36]). The former list has about the same complexity as the list of *triples* of long root subgroups in [21], while the latter list is even more complicated.

The paper is organized as follows. The basic notions are described in §1. Simple facts concerning roots, weights, and singular vectors are collected in §2. In §3 we study root elements in $G_{\mathrm{sc}}(\mathrm{E}_6, K)$. In particular, we prove Theorem 1, which describes 22 independent variables defining a root element (under the condition that a certain matrix coefficient is nonzero). Moreover, we find some equations on the matrix coefficients of root elements. Theorem 2 in §4 claims that the natural correspondence between root subgroups and six-dimensional singular subspaces is a bijection; we also consider the relationship between the relative position of two six-dimensional singular subspaces and the angle between the corresponding root subgroups. Finally, in §5 we describe how to determine the group generated by the corresponding root subgroups by the relative position of three six-dimensional singular subspaces two of which are opposite.

Note that most of the claims of the first sections are either well known to specialists, or can easily be deduced from classical and/or well-known facts, or are perfectly natural and even intuitively obvious, though the proof can take some time. Many of them can be found in the literature. However, it is pretty hard to pinpoint a publication where a given trivial fact was stated explicitly, and it is almost impossible to find the one where this was done for the first time. Moreover, it is sometimes harder to show that several different statements are in fact equivalent than to reprove it once again. Because of that, we took the liberty of omitting the authors of simple facts we prove.

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§1. Basic notation

1. Chevalley groups. All our notation pertaining to roots, weights, Lie algebras, algebraic groups and representations is fairly standard and follows [6]–[8] and [41, 43] (see also [14, 97, 98, 79]), where one can find many further references. We do not reiterate the definitions of Chevalley groups and their basic subgroups. They can be found, for example, in [6, 41, 77, 97, 98, 101]. Here we simply fix the basic notation for what follows.

First, let Φ be a reduced irreducible root system of rank l, $\Pi = \{\alpha_1, \dots, \alpha_l\}$ a simple root system in Φ, and Φ^+ and Φ^- the sets of positive and negative roots, respectively. The elements of Π are called *simple* roots, and our numbering always follows [7]. Since our work is devoted mainly to the system $\Phi = \mathrm{E}_6$ (and in some respect to its subsystems), we are especially interested in the case where all roots in Φ have the same length; these root systems are called *simply laced*, as opposed to *multiply laced*. As usual, $W = W(\Phi)$ denotes the Weyl group of a root system Φ; w_α denotes the reflection with respect to a root $\alpha \in \Phi$, while $w_i = w_{\alpha_i}$, $1 \le i \le l$, are the fundamental reflections. The fundamental root system Π induces some order on Φ. Let δ be the maximal root of Φ with respect to this order. In the case where $\Phi = \mathrm{E}_6$, we have $\delta = \begin{smallmatrix}12321\\2\end{smallmatrix}$.

The construction of Chevalley groups is based on a choice of a Chevalley basis in a simple complex Lie algebra L of type Φ. Recall that the choice of a Cartan subalgebra H in L gives a root decomposition $L = H \bigoplus \sum L_\alpha$, where the L_α are one-dimensional H-invariant root subspaces. For every root $\alpha \in \Phi^+$, we choose some nonzero root vector $e_\alpha \in L_\alpha$ and identify α with a linear functional on H such that $[h, e_\alpha] = \alpha(h)e_\alpha$. The restriction of the Killing form of L to H is nonsingular; therefore, it induces the canonical isomorphism $H \cong H^*$, and we can even suppose that $\alpha \in H$. However, it is usually more convenient to introduce the coroots $h_\alpha = 2\alpha/(\alpha, \alpha)$. Hence, any choice of nonzero $e_\alpha \in L_\alpha$, $\alpha \in \Phi^+$, uniquely determines $e_{-\alpha} \in L_{-\alpha}$, $\alpha \in \Phi^+$, such that $[e_\alpha, e_{-\alpha}] = h_\alpha$. The set $\{e_\alpha, \alpha \in \Phi; h_\alpha, \alpha \in \Pi\}$ is a basis of the Lie algebra L; it is called a Weyl basis. Moreover, $[h_\alpha, e_\beta] = A_{\alpha\beta} e_\beta$, where the $A_{\alpha\beta} = 2(\alpha, \beta)/(\alpha, \alpha) \in \mathbb{Z}$ are the Cartan

numbers. The structure constants $N_{\alpha\beta}$ are defined by the identities $[e_\alpha, e_\beta] = N_{\alpha\beta} e_{\alpha+\beta}$. A Weyl basis can be normalized so that the $N_{\alpha\beta}$ are integers; then it is called a Chevalley basis, and the set $\{e_\alpha, \alpha \in \Phi\}$ is called a Chevalley system.

For simply laced systems we always have $N_{\alpha\beta} = 0, \pm 1$, so that we only need to fix the *signs* of the structure constants. We fix a positive Chevalley basis, defined by the conditions $N_{\alpha\beta} > 0$ for all extra-special pairs (see [95, 56, 72, 99]). For simply laced systems, this is equivalent to $N_{\alpha_i\beta} = +1$ for all $\alpha_i + \beta \in \Phi$ such that if $\alpha_j + \gamma = \alpha_i + \beta$ for some simple root α_j and some positive root γ, then $j > i$.

By $Q(\Phi)$ we denote the root lattice of Φ, and by $P(\Phi)$ we denote its weight lattice. Recall that $P(\Phi)$ consists of integer linear combinations of the fundamental weights $\varpi_1, \dots, \varpi_l$, which form the basis dual to $h_{\alpha_1}, \dots, h_{\alpha_l}$, where $h_\alpha = 2\alpha/(\alpha, \alpha)$. In particular, $Q(\Phi) \subseteq P(\Phi)$. Let P be some lattice between $Q(\Phi)$ and $P(\Phi)$. As usual, $P_{++}(\Phi)$ denotes a cone of *dominant* integer weights, which are nonnegative integer linear combinations of the fundamental weights $\varpi_1, \dots, \varpi_l$.

Now, let R be a commutative ring with 1. It is well known that we can construct a Chevalley group $G = G_P(\Phi, R)$ as a group of points over R of a certain affine group scheme $G = G_P(\Phi, -)$, namely, the Chevalley–Demazure scheme. In our case where $\Phi = \mathrm{E}_6$, it is known that $|P(\Phi) : Q(\Phi)| = 3$, so P equals either $P(\Phi)$ or $Q(\Phi)$. Hence, when $\Phi = \mathrm{E}_6$, there are two possibilities for the group $G = G_P(\mathrm{E}_6, R)$, namely, the adjoint group $G_{\mathrm{ad}}(\mathrm{E}_6, R) = G_{P(\Phi)}(\mathrm{E}_6, R)$ and the simply connected group $G_{\mathrm{sc}}(\mathrm{E}_6, R) = G_{Q(\Phi)}(\mathrm{E}_6, R)$.

Let $G = G(\Phi, R)$ be a Chevalley group of type Φ over R. The choice of a Chevalley basis fixes, in particular, a split maximal torus $T = T(\Phi, R)$ in G and a parametrization of the root unipotent subgroups X_α, $\alpha \in \Phi$, with respect to T. For this fixed parametrization, let $x_\alpha(a)$ be the elementary root unipotent corresponding to $\alpha \in \Phi$, $a \in R$. Then

$$X_\alpha = \{x_\alpha(a) \mid a \in R\}.$$

For x, y in G, we denote by $[x, y]$ their left-normed commutator $xyx^{-1}y^{-1}$. The Chevalley commutator formula claims that

$$[x_\alpha(a), x_\beta(b)] = \prod x_{i\alpha+j\beta}(N_{\alpha\beta ij} a^i b^j)$$

for any $\alpha, \beta \in \Phi$ such that $\alpha+\beta \neq 0$, and for any $a, b \in R$. The product on the right-hand side of this formula is taken over all roots $i\alpha+j\beta \in \Phi$, $i, j \in \mathbb{N}$, in some fixed order. Here the structure constants $N_{\alpha\beta ij}$ do not depend on a and b. Moreover, the $N_{\alpha\beta 11} = N_{\alpha\beta}$ are exactly the structure constants of the Lie algebra L in the corresponding Chevalley basis. For a simply laced system, the only positive linear combination of α and β that can be a root is their sum $\alpha + \beta$. Therefore, in the simply laced case, the Chevalley commutator formula can be written as $[x_\alpha(a), x_\beta(b)] = e$ if $\alpha + \beta$ is not a root, and as

$$[x_\alpha(a), x_\beta(b)] = x_{\alpha+\beta}(N_{\alpha\beta} ab)$$

if $\alpha + \beta$ is a root. The group $E_P(\Phi, R) = \langle X_\alpha, \alpha \in \Phi\rangle$ generated by all elementary root subgroups is called the elementary subgroup of the Chevalley group $G_P(\Phi, R)$. In the present paper we are only interested in the case where $R = K$ is a field. Hence, if the root system Φ has rank greater than 0, then $E_{\mathrm{sc}}(\Phi, K) = G_{\mathrm{sc}}(\Phi, K)$.

2. Weyl modules. Usually, we consider a Chevalley group $G = G_P(\Phi, R)$ together with its action on the Weyl module $V = V(\omega)$ for a dominant weight ω. We fix a weight $\omega \in P_{++}(\Phi)$, and let $V = V(\omega)$ be the Weyl module of G with a highest weight ω. The corresponding representation $G \longrightarrow \mathrm{GL}(V)$ will be denoted by $\pi = \pi(\omega)$. Let $\Lambda = \Lambda(\omega)$ be a *multiset* of weights of $V = V(\omega)$ *with multiplicities*. We write $\bar{\Lambda}(\omega)$ for a *set* of weights *without* multiplicities. In this paper we consider only microweight modules

(see [8], [77]–[79], [97, 98] and the references therein). Since all weights of a microweight representation are extremal, they have multiplicity 1, and in this case $\Lambda = \bar{\Lambda}(\omega)$ coincides with the Weyl orbit of the highest weight: $\Lambda = W\omega$.

In what follows, we fix an admissible basis v^ρ, $\rho \in \Lambda$, of a module V. Recall that a basis is said to be admissible when the following conditions are satisfied.

• Every vector v^ρ is indeed a vector of weight ρ, provided we view ρ as a weight *without multiplicity*.

• The action of the root unipotents $x_\alpha(a)$, $\alpha \in \Phi$, $a \in R$, in the basis v^ρ, $\rho \in \Lambda(\omega)$, is given by matrices of polynomials in a with integer coefficients.

By the Matsumoto lemma (see [77]) we can normalize an admissible basis for a microweight representation in such a way that

$$x_\alpha(a)v^\rho = v^\rho + c_{\rho+\alpha,\rho} a v^{\rho+\alpha},$$

where all action structure constants $c_{\rho+\alpha,\rho}$ are equal to ± 1. These constants are usually denoted by $c_{\rho\alpha}$, but it is more convenient for us to write $c_{\rho+\alpha,\rho}$. In what follows, we always choose a crystal basis where all the structure constants $c_{\rho+\alpha,\rho}$ are equal to $+1$ for the fundamental and the negative fundamental roots, i.e., $c_{\rho+\alpha,\rho} = +1$ for $\alpha \in \pm\Pi$. Hence, $c_{\rho+\delta,\rho}$ is equal to $+1$ for all $\rho, \rho+\delta \in \Lambda$. The existence of such a basis follows from the general results of Lusztig and Kashiwara; an elementary proof can be found in [98] and [13].

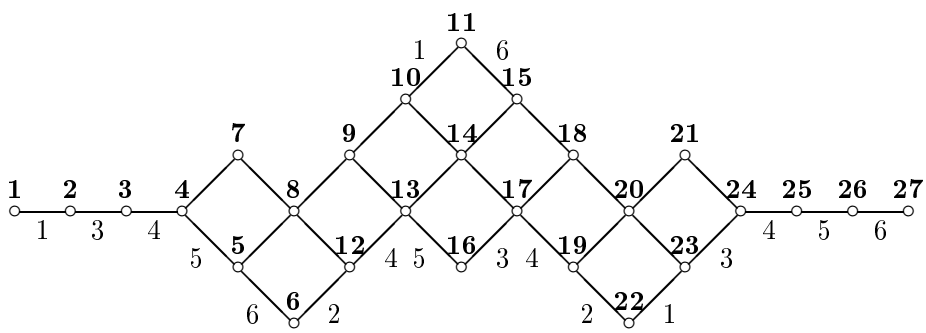
We identify a vector $a \in V$, $a = \sum v^\rho a_\rho$, with a coordinate *column* $a = (a_\rho)$, $\rho \in \Lambda$. Then an element b of the contragradient module V^* can be viewed naturally as a *row* $b = (b_\rho)$, $\rho \in \Lambda$. Of course, with respect to the weights Λ^* of the contragradient module V^*, the picture is reversed: the elements of V^* are represented by the *columns* $b = (b_\rho)$, $\rho \in \Lambda^*$, while the elements of V by the *rows* $a = (a_\rho)$, $\rho \in \Lambda^*$. It should be emphasized that we index both columns and rows by the weights of V: the indices $\rho, \sigma, \lambda, \dots$ are in Λ. In other words, we find it convenient to index the coordinates of vectors in V^* by the weights of V and visualize them as rows, while the usual way is to index them by the weights of V^* and visualize them as columns.

A crucial technical issue is that the components of these rows are not linearly ordered, but rather partially ordered, in accordance with the order on Λ induced by the choice of the fundamental root system Π. Namely, we set $\rho \geq \sigma$ when $\rho - \sigma = \sum m_i\alpha_i$, $m_i \geq 0$. Interpreting the elements of the module V as above, it is natural to identify the elements of our Chevalley group with matrices $g = (g_{\rho\sigma})$, $\rho, \sigma \in \Lambda$, with respect to the basis v^ρ. As usual, the columns of this matrix are the coordinate columns of the vectors gv^σ, $\sigma \in \Lambda$, with respect to the basis v^ρ, $\rho \in \Lambda$. We often use the following notation: the σth column of the matrix g is denoted by $g_{*\sigma}$, while the ρth row of it is denoted by $g_{\rho*}$.

In the present paper we consider the group $G_{\mathrm{sc}}(\mathrm{E}_6, R)$ together with its action on the 27-dimensional module $V = V(\varpi_1)$. The corresponding representation is well known to be microweight. We need some properties of weight systems that are not apparent from the description of Λ above, so now we want to present a slightly different construction of this set. Consider the root system $\Delta = \mathrm{E}_7$. Its subsystem formed by the roots having coefficient 0 at α_7 in the expansion into simple roots is canonically isomorphic to $\Phi = \mathrm{E}_6$. Let Λ_1 be the set of roots having coefficient 1 at α_7 in the expansion into simple roots. The group $W_\Phi = \langle w_{\alpha_i}; 1 \leq i \leq 6\rangle$ acts on these roots in a standard way. It is easily seen that this action is transitive. Let $\overline{}$ denote the orthogonal projection onto the hyperplane spanned by the roots α_i, $1 \leq i \leq 6$. Consider the root $\alpha = \overset{234321}{2}$. Note that α is orthogonal to the roots α_i, $2 \leq i \leq 6$, and is at an angle of $\pi/3$ with α_1. In other words, $(\alpha, \alpha_1) = 1/2$ and $(\alpha, \alpha_i) = 0$ for $2 \leq i \leq 6$. Since $\alpha - \overline{\alpha} \perp \alpha_i$ for $1 \leq i \leq 6$, it follows that $\overline{\alpha} = \varpi_1$ is the first fundamental weight. Therefore, $\Lambda = W\varpi_1 = W\overline{\alpha} = \overline{W\alpha} = \overline{\Lambda_1}$.

It is easy to show that the restriction of the projection $\overline{}$ to Λ_1 is bijective. Moreover, it is clear that this projection agrees with the action of W, and that $\overline{\rho}+\beta=\overline{\rho+\beta}$ for all $\beta\in\Phi$ and $\rho,\rho+\beta\in\Lambda_1$. Hence, we can identify Λ with Λ_1. This construction, in much more detail and in a slightly different language, was given in [17], where one can also find many further references.

In order to work with our 27-dimensional representation of $G_{\mathrm{sc}}(\mathrm{E}_6,R)$, it is convenient to fix its weights. As we noticed above, the set of weights can be given a natural partial order; it is usual to consider only linear orders that are compatible with this partial order. Still, there are a lot of such orders; to avoid the mix-up, we try not to use a specific order heavily. Anywhere except in §5 we need only the fact that the linear order in question is compatible with the α_5-branching (see, for example, [17]). This means that $i+21=i-\delta$ for all $1\le i\le 6$ (here we are imprecise slightly: we mean that if we add δ to a weight with a number $i+21$, we get a weight with the number i). This condition uniquely determines (together with a given partial order) the first eight and the last eight weights. Unfortunately, in §5 we are forced to use the entire linear order. The order we use is different from those in [17]; it is described on the weight diagram below.



As usual, the vertices of the diagram correspond to weights. We connect two vertices with an edge marked by i if the difference between the corresponding weights is a simple root α_i; parallel edges are marked by the same simple root. A more detailed description of weight diagrams, with historical perspective and further references, can be found in [17]. We denote the basic vectors of V by e^ρ, $\rho\in\Lambda$; the basic covectors are denoted by e_ρ, $\rho\in\Lambda$.

3. Trilinear form and the 3-form. Let $V=V(\varpi_1)$ be the 27-dimensional module for the Chevalley group $G=G_{\mathrm{sc}}(\mathrm{E}_6,R)$. Then there exists a trilinear form $F:V\times V\times V\to R$ such that G is the group of isometries of F; in other words, G is the group of all $g\in\mathrm{GL}(V,R)$ such that $F(gu,gv,gw)=F(u,v,w)$ for all $u,v,w\in V$.

The form F was introduced by Dickson in 1901. It was then actively studied and used by Chevalley, Freudenthal, Springer, Tits, Seligman, Jacobson, Veldkamp, Cohen, Cooperstein and others (see [17, 16] for the further references). Initially, the form F was studied over fields of characteristic zero, but then it was generalized to arbitrary fields of characteristic not equal to 2 or 3. In [44]–[49] Aschbacher showed that the condition of invertibility of 2 and 3 can be lifted. In [97, 16] it was proved that the group G can be described as the group of isometries of this form over any commutative ring, though we are only interested in the case of a field.

In fact, Aschbacher used a 3-form $\mathfrak{F} = (T, Q, F)$, where T is a cubic form, Q is its partial polarization, and F is its full polarization. More precisely, a 3-form $\mathfrak{F}$ is a triple (T, Q, F) such that

(1) F is a trilinear form;
(2) $Q : V \times V \to K$ is linear in the first variable, and $Q(x, ay) = a^2 Q(x, y)$, $Q(x, y + z) = Q(x, y) + Q(x, z) + F(x, y, z)$ for all $a \in K$ and $x, y, z \in V$;
(3) $T : V \to K$ satisfies $T(ax) = a^3 T(x)$ and $T(x + y) = T(x) + T(y) + Q(x, y) + Q(y, x)$ for all $a \in K$ and $x, y \in V$.

In particular, Aschbacher proved that over an arbitrary field the group $G_{\mathrm{sc}}(\mathrm{E}_6, K)$ coincides with the group of isometries of the 3-form $\mathfrak{F}$, and moreover, with the groups of isometries of the forms F and Q. In the present paper, besides the trilinear form F, we use the form Q in the definition of singular vectors. We do this in order to treat fields of arbitrary characteristics uniformly.

The explicit expression for T (it is easy to deduce Q and F from T) was calculated in [17]. In our chosen numeration of weights, it looks like this:

$$\begin{aligned}
T(x) = {} & x_1x_{11}x_{27} - x_1x_{15}x_{26} + x_1x_{18}x_{25} - x_1x_{20}x_{24} + x_1x_{21}x_{23} \\
& - x_2x_{10}x_{27} + x_2x_{14}x_{26} - x_2x_{17}x_{25} + x_2x_{19}x_{24} - x_2x_{21}x_{22} \\
& + x_3x_9x_{27} - x_3x_{13}x_{26} + x_3x_{16}x_{25} - x_3x_{19}x_{23} + x_3x_{20}x_{22} \\
& - x_4x_8x_{27} + x_4x_{12}x_{26} - x_4x_{16}x_{24} + x_4x_{17}x_{23} - x_4x_{18}x_{22} \\
& + x_5x_7x_{27} - x_5x_{12}x_{25} + x_5x_{13}x_{24} - x_5x_{14}x_{23} + x_5x_{15}x_{22} \\
& - x_6x_7x_{26} + x_6x_8x_{25} - x_6x_9x_{24} + x_6x_{10}x_{23} - x_6x_{11}x_{22} \\
& + x_7x_{16}x_{21} - x_7x_{17}x_{20} + x_7x_{18}x_{19} - x_8x_{13}x_{21} + x_8x_{14}x_{20} \\
& - x_8x_{15}x_{19} + x_9x_{12}x_{21} - x_9x_{14}x_{18} + x_9x_{15}x_{17} - x_{10}x_{12}x_{20} \\
& + x_{10}x_{13}x_{18} - x_{10}x_{15}x_{16} + x_{11}x_{12}x_{19} - x_{11}x_{13}x_{17} + x_{11}x_{14}x_{16}.
\end{aligned}$$

In order to answer most of our questions, it suffices to know that $T(x) = \sum \pm x_\rho x_\sigma x_\tau$, where the sum runs over all unordered *triads* $\{\rho, \sigma, \tau\}$, i.e., over the triples of pairwise nonadjacent weights (see §2). Furthermore, $F(x, y, z) = \sum \pm x_\rho y_\sigma z_\tau$, where the sum runs over all ordered triads (ρ, σ, τ); while $Q(x, y) = \sum \pm x_\rho y_\sigma y_\tau$ (the sum runs over all triads $(\rho, \{\sigma, \tau\})$, where the pair consisting of the second and third weight is unordered). These definitions were discussed more thoroughly in [17]. Note that the same form is defined on the contragradient module V^*, and we use rows to picture its elements. This form is also denoted by $\mathfrak{F} = (T, Q, F)$.

§2. Roots, weights and singular vectors

1. Roots and weights. In this subsection we prove some basic facts concerning the sets Λ and Φ. We shall make frequent use of the weight diagram. All propositions in this subsection are of a combinatorial nature, because there are finitely many roots and weights. Nevertheless, a brute force proof is usually cumbersome, so that we employ several tricks. The main trick is using the Weyl group. It is known that the Weyl group W acts on Λ and on Φ. Moreover, if $\rho, \rho + \alpha \in \Lambda$ for some $\alpha \in \Phi$, then $w(\rho) + w(\alpha) = w(\rho + \alpha) \in \Lambda$. In other words, the actions of the Weyl group on the set of weights and on the set of roots are compatible. We tried to reduce the number of the properties of the Weyl group we use; in fact, we need only three properties. Namely,

(1) *If $\alpha \in \Phi$ and $\rho \in \Lambda$ is a weight such that $\rho + \alpha, \rho - \alpha \notin \Lambda$, then $w_\alpha(\rho) = \rho$. If, however, $\rho, \rho + \alpha \in \Lambda$, then $w_\alpha(\rho) = \rho + \alpha$ and $w_\alpha(\rho + \alpha) = \rho$.* This is a standard property of microweight representations. In our case of the 27-dimensional

representation $\pi(\varpi_1)$, this property becomes obvious if we use the isomorphism $\Lambda \cong \Lambda_1$ described in §1.

(2) *Any pair of roots can be mapped to any other pair with the same angle between them by the action of the Weyl group; in particular, any root can be mapped to any other.* This is a simple property of root systems.

(3) *Any weight can be mapped to any other by the action of the Weyl group.* In the previous section we mentioned that Λ coincides with the Weyl orbit $W\omega$ of the highest weight; this immediately implies the property we need. Also, this property can easily be deduced from the first property and the fact that the weight diagram is connected.

In §4 we also need to consider the extended Weyl group. Namely, let

$$w_\alpha(t) = x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t) \in G_{\mathrm{sc}}(\mathrm{E}_6, K),$$

and let $\widetilde{W} = \langle w_\alpha(1); \alpha \in \Phi\rangle$. It is easily seen that $w_\alpha(1)$ maps the set $\Lambda^\pm = \{\pm e^\rho; \rho \in \Lambda\}$ to itself. Meanwhile, if $\rho, \rho+\alpha \in \Lambda$, then $(w_\alpha(1))^2 e^\rho = -e^\rho$ and $(w_\alpha(1))^2 e^{\rho+\alpha} = -e^{\rho+\alpha}$. The elements of $\Lambda^\pm$ are called plus/minus weights. Note that there exists a natural projection of $\Lambda^\pm$ to Λ that maps $\pm e^\rho$ to ρ. This projection is compatible with the action of $\widetilde{W}$. Moreover, it is clear that, under this projection, the action of $w_\alpha(1) \in \widetilde{W}$ on $\Lambda^\pm$ goes to the action of $w_\alpha \in W$ on Λ; similarly, $\widetilde{W}$ maps to W. It is easy to show that $w_\alpha(1)x_\beta(a)w_\alpha(1)^{-1} = x_{w_\alpha\beta}(\pm a)$. All these and other similar facts are proved in [41].

Proposition 1. *For any root $\alpha \in \Phi$, there exist exactly six weights $\rho_i \in \Lambda$ such that $\rho_i + \alpha \in \Lambda$. The difference between any two such ρ_i is a root.*

Proof. For the root $\alpha = \alpha_1$ the desired property can be checked directly, while for an arbitrary root it follows from the fact that any root can be mapped to any other by the Weyl group action. □

Proposition 2. *Let $\alpha, \beta \in \Phi$ be two roots such that $\beta \neq -\alpha$. Then:*

(1) *there is no weight ρ such that $\rho + \alpha$ and $\rho + 2\alpha$ are weights;*

(2) *there is no weight ρ such that $\rho + \alpha$, $\rho + \alpha + \beta$, and $\rho + 2\alpha + \beta$ are weights.*

Proof. The conclusion follows from the fact that any pair of roots can be mapped to any other pair with the same angle between them by the Weyl group action. □

Let

$$I_1^\alpha = \{\rho; \rho, \rho - \alpha \in \Lambda\}, \quad I_2^\alpha = \{\rho; \rho \in \Lambda, \rho \pm \alpha \notin \Lambda\}, \quad I_3^\alpha = \{\rho; \rho, \rho + \alpha \in \Lambda\}.$$

Clearly, Λ is the union of I_1^α, I_2^α, and I_3^α, while I_2^α does not intersect the other two sets. Part (1) of Proposition 1 shows that I_1^α does not intersect I_3^α. By Proposition 1, the sets I_1^α and I_3^α have 6 elements each, whence I_2^α has exactly 15 elements. Finally, it is easy to check that conjugating the sets $I_1^\alpha, I_2^\alpha, I_3^\alpha$ by w_β, we get $I_1^{w_\beta(\alpha)}, I_2^{w_\beta(\alpha)}, I_3^{w_\beta(\alpha)}$, respectively. We are mainly interested in the case where $\alpha = \delta$; for brevity, we put $I_1 = I_1^\delta, I_2 = I_2^\delta$, and $I_3 = I_3^\delta$. In our numeration of weights, we have

$$I_1 = \{i; 1 \le i \le 6\}, \quad I_2 = \{i; 6 < i < 22\}, \quad I_3 = \{i; 22 \le i \le 27\}.$$

Recall that, by definition, Λ is a subset of the weight lattice $P(\Phi)$. Since $\Lambda = W\varpi$, the difference of any two weights in Λ lies in the root lattice $Q(\Phi)$. In other words, the difference of any two weights in Λ is a sum of several roots (this also follows from the connectedness of the weight diagram).

Definition. The distance between two different weights ρ and σ (notation: $d(\rho, \sigma)$) is the minimal number of roots such that their sum equals $\rho - \sigma$. If the weights coincide, we define the distance between them to be 0.

Now we can reformulate part (2) of Proposition 2.

Corollary. *Let $\rho \in I_1^\alpha$ and $\sigma \in I_3^\alpha$. If $d(\rho, \sigma) = 1$, then $\rho = \sigma + \alpha$.*

Proof. This follows directly from part (2) of Proposition 2. □

Proposition 3. *Let $\alpha \in \Phi$ be an arbitrary root. Then*:

(1) *$\alpha = \delta$ if and only if in the expansion of α into simple roots the root α_2 has coefficient* 2;
(2) *$\angle(\alpha, \delta) = \pi/3$ if and only if in the expansion of α into simple roots the root α_2 has coefficient* 1;
(3) *$\angle(\alpha, \delta) = \pi/2$ if and only if in the expansion of α into simple roots there is no α_2*;
(4) *$\angle(\alpha, \delta) = 2\pi/3$ if and only if in the expansion of α into simple roots the root α_2 has coefficient* -1;
(5) *$\alpha = -\delta$ if and only if in the expansion of α into simple roots the root α_2 has coefficient* -2.

Proof. Parts (1) and (5) are trivial. Note that the maximal root is orthogonal to every simple root except α_2, and has angle $\pi/3$ with α_2. Hence, if there is no root α_2 in the expansion of α into simple roots, then $\alpha \perp \delta$. Next, if α_2 has coefficient 1 in the expansion of α, then the scalar product of α and δ equals $\frac{1}{2}$, so that $\angle(\alpha, \delta) = \pi/3$. The case of the coefficient -1 is similar. The reverse statements directly follow from the fact that all possible coefficients of α_2 in the expansion of α are 2, 1, 0, -1, and -2. □

Proposition 4. *Let $\alpha, \beta \in \Phi$ be fixed roots, and let $\rho, \sigma \in \Lambda$ be weights such that $\rho + \beta = \sigma$.*

(1) *If $\alpha = \beta$, then $\rho \in I_3^\alpha$ and $\sigma \in I_1^\alpha$.*
(2) *If $\angle(\alpha, \beta) = \pi/3$, then either $\rho \in I_3^\alpha$ and $\sigma \in I_2^\alpha$, or $\rho \in I_2^\alpha$ and $\sigma \in I_1^\alpha$. These cases are realized three times each.*
(3) *If $\angle(\alpha, \beta) = \pi/2$, then either $\rho, \sigma \in I_1^\alpha$, or $\rho, \sigma \in I_2^\alpha$, or $\rho, \sigma \in I_3^\alpha$. The first and third cases are realized once each, while the second case is realized four times.*
(4) *If $\angle(\alpha, \beta) = 2\pi/3$, then either $\rho \in I_2^\alpha$ and $\sigma \in I_3^\alpha$, or $\rho \in I_1^\alpha$ and $\sigma \in I_2^\alpha$. These cases are realized three times each.*
(5) *If $\alpha = -\beta$, then $\rho \in I_1^\alpha$ and $\sigma \in I_3^\alpha$.*

Proof. This follows directly from the fact that any pair of roots can be mapped to any other pair with the same angle by the Weyl group action. □

Proposition 5.

(1) *In our* 27*-dimensional representation, the distance between weights can be equal to* 0, 1, *or* 2. *For any weight there are exactly* 16 *weights at a distance of* 1 *from it, and exactly* 10 *weights at a distance of* 2 *from it.*
(2) *For any two weights at a distance of* 2 *there is exactly one weight at a distance of* 2 *from each of them.*
(3) *Let $\alpha \in \Phi$ be an arbitrary root. Let $\rho_1, \rho_2, \rho_3 \in \Lambda$ be an arbitrary triple of weights such that $d(\rho_1, \rho_2) = d(\rho_1, \rho_3) = d(\rho_2, \rho_3) = 2$. Then either every ρ_i lies in I_2^α, or one of them lies in I_1^α, the other in I_2^α, and the third in I_3^α.*

Remark. The weights with distance 1 between them are said to be *adjacent*; the weights with distance 2 between them are *nonadjacent*. Triples of pairwise nonadjacent weights are called *triads*.

Proof. The first two parts are obvious, because the action of W can map any weight to any other. We prove the third part. By Proposition 1, it suffices to prove that there is no triple of pairwise nonadjacent weights such that two of these three weights lie in I_2^α while the third lies in I_1^α (the case where two of them lie in I_2^α while the third lies in I_3^α can be obtained by substituting $-\alpha$ for α). Suppose such a triple does exist. Apply w_α to its weights. The weights in I_2^α are not affected, while the weights in I_1^α are taken to the weights in I_3^α. But part (2) shows that any triad is uniquely determined by two of its weights. This contradiction finishes the proof. □

Proposition 6. *Let $\alpha \in \Phi$ be an arbitrary root.*

(1) *For any root β orthogonal to α, w_β transposes some two weights in I_1^α and fixes all other weights in I_1^α.*
(2) *For any weight $\rho \in I_2^\alpha$, there exist exactly two weights $\rho_1, \rho_2 \in I_1^\alpha$ that are nonadjacent to ρ.*
(3) *If two weights $\rho \in I_2^\alpha$ and $\rho_1 \in I_1^\alpha$ are nonadjacent, then the weights ρ and $\rho_1 - \alpha$ are also nonadjacent.*
(4) *For any pair of weights $\rho_1, \rho_2 \in I_1^\alpha$, there exists exactly one weight $\rho \in I_2^\alpha$ that is nonadjacent to each of ρ_1, ρ_2.*
(5) *Let $\rho, \sigma \in I_2^\alpha$ be arbitrary weights, and let $\{\rho_1, \rho_2\}, \{\sigma_1, \sigma_2\}$ be the corresponding pairs of weights in I_1^α (see (2)). Then $d(\rho, \sigma) = 2$ if and only if $\{\rho_1, \rho_2\} \cap \{\sigma_1, \sigma_2\} = \varnothing$.*

Proof. The first part follows from Proposition 4, part (3). We have already mentioned that any weight can be mapped to any other weight by an element of W. Together with Proposition 4, this proves part (2). We prove the third part. Suppose that the weights ρ and $\rho_1 - \alpha$ are adjacent. Then $\rho - (\rho_1 - \alpha) = \beta \in \Phi$. By Proposition 4, part (2), we have $\angle(\alpha, \beta) = \pi/3$, whence $\alpha - \beta \in \Phi$, though $\alpha - \beta = \rho_1 - \rho$, a contradiction.

Now we prove part (4). It suffices to show that ρ is unique; its existence will follow from part (2), because the number of pairs $\{\rho_1, \rho_2\}$ with $\rho_1, \rho_2 \in I_1^\alpha$ coincides with the number of elements in I_2^α. Note that by part (3) the weight ρ is nonadjacent both to ρ_1 and to $\rho_2 - \alpha$. By Corollary of Proposition 2, the weights ρ_1 and $\rho_2 - \alpha$ are nonadjacent, so that ρ, ρ_1, and $\rho_2 - \alpha$ form a triad. By Proposition 5, part (2), ρ is uniquely determined by ρ_1 and ρ_2, which is exactly what we needed.

We prove that if $d(\rho, \sigma) = 2$, then $\{\rho_1, \rho_2\} \cap \{\sigma_1, \sigma_2\} = \varnothing$. Suppose this is not the case. Then the weight $\chi \in \{\rho_1, \rho_2\} \cap \{\sigma_1, \sigma_2\}$ is nonadjacent both to ρ and to σ; hence, the weights ρ, σ, and χ form a triad. Moreover, this triad has two weights in I_2^α and one weight in I_1^α, which contradicts Proposition 5, part (3). It remains to show that if $\{\rho_1, \rho_2\} \cap \{\sigma_1, \sigma_2\} = \varnothing$, then $d(\rho, \sigma) = 2$. By part (2), ρ is not equal to σ. Suppose $\sigma - \rho = \beta \in \Phi$. Then w_β maps ρ to σ; therefore, it maps $\{\rho_1, \rho_2\}$ to $\{\sigma_1, \sigma_2\}$, which by (1) contradicts our assumption. □

Lemma 2.1.

(1) *Suppose $\alpha, \beta \in \Phi$ and $\alpha \perp \beta$. Let ρ be a weight such that $\rho + \alpha$ and $\rho + \beta$ are weights. Then*

$$c_{\rho,\rho+\alpha} c_{\rho+\alpha,\rho+\alpha+\beta} = c_{\rho,\rho+\beta} c_{\rho+\beta,\rho+\alpha+\beta}.$$

In particular, $\rho + \alpha + \beta$ is also a weight.

(2) *Suppose $\alpha, \beta, \gamma \in \Phi$, $\alpha \perp \beta, \gamma$, and $\angle(\beta, \gamma) = 2\pi/3$. Also, suppose there exist a weight σ such that $\sigma + \alpha, \sigma + \beta, \sigma + \beta + \gamma \in \Lambda$. Then*

$$c_{\sigma,\sigma+\beta} c_{\sigma,\sigma+\beta+\gamma} c_{\sigma+\beta,\sigma+\beta+\gamma} = c_{\sigma+\alpha,\sigma+\alpha+\beta} c_{\sigma+\alpha,\sigma+\alpha+\beta+\gamma} c_{\sigma+\alpha+\beta,\sigma+\alpha+\beta+\gamma}. \tag{2.1}$$

In particular, $\sigma + \alpha + \beta$ and $\sigma + \alpha + \beta + \gamma$ are weights.

Proof. We prove the first part. Proposition 4 implies that $\rho+\alpha+\beta$ is a weight. As in the previous section, for $\gamma \in \Phi$ and $\tau \in \Lambda$ we have $x_\gamma(a)e^\tau = e^\tau + c_{\tau+\gamma,\tau}ae^{\tau+\gamma}$ if $\tau+\gamma \in \Lambda$, and $x_\gamma(a)e^\tau = e^\tau$ otherwise. By the Chevalley commutator formula (see §1) we have $x_\alpha(1)x_\beta(1) = x_\beta(1)x_\alpha(1)$. Applying the two parts of this relation to e^ρ, we get the required statement. By part (3) of Proposition 4, ρ is unique. Now we prove the second claim. Again, from Proposition 4 it follows that $\sigma+\alpha+\beta$ and $\sigma+\alpha+\beta+\gamma$ are weights. In order to prove (2.1), we multiply the equations in part (1) for the pairs of roots (α,β) (here we put $\rho=\sigma$), (α,γ) (put $\rho=\sigma+\beta$), and $(\alpha,\beta+\gamma)$ (put $\rho=\sigma$). $\square$

Remark. Note that in part (1) the roots α and β form a "square" with the vertices $\rho, \rho+\alpha, \rho+\beta,\ \rho+\alpha+\beta$, and we can formulate our lemma as follows: the product of signs along a pair of edges is equal to the product along the other pair. Similarly, in part (2), the roots $\alpha,\beta,\gamma,\beta+\gamma$ form a triangular prism with the vertices $\sigma, \sigma+\beta, \sigma+\beta+\gamma, \sigma+\alpha, \sigma+\alpha+\beta, \sigma+\alpha+\beta+\gamma$, and we can formulate the statement of the lemma as follows: the product of the signs on one triangular base is equal to the product of the signs along the other base. The proof can therefore be visualized as multiplying the equations for lateral faces.

2. Singular vectors: elementary properties.

Definition. A vector v is said to be *singular* (with respect to a 3-form $\mathfrak{F}$) if $Q(x,v)=0$ for any vector x. A subspace consisting of singular vectors is said to be *singular*.

Remark. The subspaces consisting of singular vectors are sometimes called totally singular.

From the definition of a 3-form we can deduce the following properties.

(1) If $\operatorname{char} K \neq 2$, we can write $F(v,v,x)=0$ instead of $Q(x,v)=0$ in the definition of a singular vector. If $\operatorname{char} K = 2$, we have $F(v,v,x)=0$ for any two vectors v, x.
(2) Let W be a subspace with a basis consisting of singular vectors w^i, $1 \le i \le n$, for $n>1$. Then W is singular if and only if $F(w^i,w^j,x)=0$ for any vector x and for any i,j, $1 \le i,j \le n$.
(3) Suppose that vectors $u, v, u+v$ are singular. Then the two-dimensional subspace they span is singular.

Definition. We say that the distance between two distinct singular vectors u, v, denoted by $d(u,v)$, is equal to 1 if $u-v$ is singular, and $d(u,v)=2$ otherwise. The vectors u,v are said to be adjacent in the former case, and nonadjacent in the latter case. If $u=v$, we put $d(u,v)=0$.

Recall that $F(u,v,w)=\sum \pm u_\rho v_\sigma w_\tau$, where the sum is taken over all ordered triples of weights ρ,σ,τ such that $d(\rho,\sigma)=d(\rho,\tau)=d(\sigma,\tau)=2$. It follows that $d(\rho,\sigma)=n$ if and only if $d(e^\rho,e^\sigma)=n$ for all weights $\rho,\sigma \in \Lambda$. In what follows, for simplicity, sometimes we shall replace the distance between the basis weights with the distance between the corresponding basis vectors.

Since any matrix $A \in G_{\mathrm{sc}}(\mathrm{E}_6,K)$ preserves the form F, it maps singular vectors to singular vectors, and nonsingular to nonsingular. Hence, A preserves the distance between vectors.

Proposition 7.

(1) *Let u be a singular vector, and let ρ be any weight. Suppose that for any weight $\sigma \in \Lambda$ nonadjacent to ρ the coefficient u_σ is equal to* 0. *Then u is adjacent to e^ρ.*
(2) *Let u be a singular vector adjacent to a base vector e^ρ, and let $\sigma \in \Lambda$ be any weight nonadjacent to ρ. Then $u_\sigma = 0$.*

Proof. The adjacency of u to e^ρ is equivalent to the relation $F(u, e^\rho, x) = 0$ for all vectors x. Together with the explicit expression for F, this yields the first part of the proposition. In order to prove the second part, suppose that $u_\sigma \neq 0$. Let τ be a weight adjacent to neither ρ nor σ. From the formula for F we get $F(u, e^\rho, e^\tau) = \pm u_\sigma \neq 0$, whence u is nonadjacent to e^ρ, which gives a contradiction. □

Let $\alpha \in \Phi$ be any root. Let $D_\alpha = \langle X_\beta; \beta \perp \alpha\rangle$. From part (3) of Proposition 5 it follows that our space is a sum of three D_α-invariant subspaces: $V_k^\alpha = \langle e^\rho; \rho \in I_k^\alpha\rangle$, $1 \leq k \leq 3$. Consider the restriction of D_α to V_k^α. Again by Proposition 4, the restriction of any X_β to V_1^α is a transvection; hence, the restriction of D_α to V_1^α coincides with $\mathrm{SL}(V_1^\alpha, K)$. Similar facts are valid for the restriction to V_3^α.

For brevity, we write D instead of D_δ. We denote the subspaces V_k^δ by V_k. It is easy to show that for a matrix $A \in D$ the blocks $A|_{V_1}$ and $A|_{V_3}$ coincide (more precisely, we have $A_{\rho-\delta,\sigma-\delta} = A_{\rho\sigma}$ for any $\rho, \sigma \in I_1$).

Lemma 2.2. *Let u, v be two adjacent singular vectors, and let $u \in V_1$. Then $u_\rho v_{\sigma-\delta} = u_\sigma v_{\rho-\delta}$ for all $\rho, \sigma \in I_1$.*

Proof. Let A_1 be a matrix in $\mathrm{SL}(V_1, K)$ that maps u to e^1. Then, as above, there exists a matrix $A \in D$ such that $A|_{V_1} = A_1$. Arguing as above, we see that $Au = e^1$ is adjacent to $Av = v'$, and by Proposition 7 we have $v'_\rho = 0$ for $22 < \rho \leq 27$. Now the lemma follows directly from the fact that $A|_{V_1} = A|_{V_3}$. □

§3. Root elements

Let g be a root element. Since our representation is microweight, we have $g = e + x$, where x is a root element of the Chevalley algebra L_K, i.e., an element conjugate to ae_α. Then x can be expressed as a linear combination of e_α, $\alpha \in \Phi = \mathrm{E}_6$, and h_i, $1 \leq i \leq 6$. Since we shall make heavy use of the coefficients of this linear combination, we introduce the following notation.

Definition. The coefficient of e_α in the expansion of $g - e$ in the Chevalley basis is denoted by $(\alpha)_g$ and called a *coordinate of a root element* g (or of a matrix g) *at the root* $\alpha \in \Phi = \mathrm{E}_6$.

Therefore, for a root element g we have an expansion $g = e + \sum(\alpha)_g e_\alpha + \sum a_i h_i$, where the first sum is over all $\alpha \in \Phi = \mathrm{E}_6$, and the second sum is over $1 \leq i \leq 6$. The diagonal coefficients of the first sum, $\sum(\alpha)_g e_\alpha$, are equal to 0, while the second sum is, in fact, diagonal.

Lemma 3.1. *Let g be a root element. Then*:

(1) $g_{\rho\sigma} = 0$ *for all* $\rho, \sigma \in \Lambda$ *such that* $d(\rho, \sigma) = 2$;
(2) $g_{\rho\sigma} = c_{\rho\sigma}(\rho - \sigma)_g$ *for all* $\rho, \sigma \in \Lambda$ *such that* $d(\rho, \sigma) = 1$.

Proof. As was noted before, $g = e + \sum(\alpha)_g e_\alpha + \sum a_i h_i$, the first sum being over all $\alpha \in \Phi = \mathrm{E}_6$, and the second being over $i = 1, \ldots, 6$. Moreover, since the second sum is represented by a diagonal matrix, we only need to consider the first sum. It is clear that if $d(\rho, \sigma) = 2$, then the matrix coefficients of all e_α in the position (ρ, σ) are equal to 0; this proves (1). On the other hand, if $d(\rho, \sigma) = 1$, then the only elementary root element e_α with nonzero matrix coefficient in the position (ρ, σ) is $e_{\rho-\sigma}$. Now (2) follows from the definition of $(\rho - \sigma)_g$ and $c_{\rho\sigma}$. □

Lemma 3.2. *Let g be a root element and $x_\alpha(a)$ an elementary root element. Let $g' = x_\alpha(a) g x_\alpha(-a)$. If $\angle(\alpha, \beta) = \pi/3$, the $(\beta)_{g'} = (\beta)_g + N_{\alpha\beta} a(\beta - \alpha)_g$. On the other hand, if $\angle(\alpha, \beta) > \pi/3$, then $(\beta)_{g'} = (\beta)_g$.*

Proof. As was noted before, $g = e + \sum(\beta)_g e_\beta + \sum a_i h_i$, where the first sum is over all $\beta \in \Phi$, and the second is over $i = 1, \dots, 6$. Hence,

$$g' = x_\alpha(a) g x_\alpha(-a) = e + \sum(\beta)_g x_\alpha(a) e_\beta x_\alpha(-a) + \sum a_i x_\alpha(a) h_i x_\alpha(-a).$$

Clearly, the second sum is equal to $\sum a_i h_i + b e_\alpha$ for some $b \in K$, so that it is irrelevant for our purposes. Consider the first sum. If $\angle(\alpha, \beta) < 2\pi/3$, then $(\beta)_g x_\alpha(a) e_\beta x_\alpha(-a) = (\beta)_g e_\beta$; if $\angle(\alpha, \beta) = 2\pi/3$, then $(\beta)_g x_\alpha(a) e_\beta x_\alpha(-a) = (\beta)_g (e_\beta + N_{\alpha\beta} a e_{\beta+\alpha})$; finally, if $\beta = -\alpha$, then $(\beta)_g x_\alpha(a) e_\beta x_\alpha(-a) = (\beta)_g (e_\beta + h_\alpha(ab) + c e_\alpha)$ for some $b, c \in K$. Now our claim follows easily. □

Proposition 8.

(1) *Suppose $g \in \mathrm{M}(27, K)$ equals h_α for some $\alpha \in \Phi = \mathrm{E}_6$. Then for any triple of pairwise nonadjacent weights ρ, σ, τ, we have $g_{\rho\rho} + g_{\sigma\sigma} + g_{\tau\tau} = 0$.*

(2) *For any diagonal matrix $g = e + \sum_{i=1}^{6} a_i h_i \in \mathrm{M}(27, K)$ and any triple of pairwise nonadjacent weights ρ, σ, τ, we have $g_{\rho\rho} + g_{\sigma\sigma} + g_{\tau\tau} = 3$.*

(3) *For any root element g and any triple of pairwise nonadjacent weights ρ, σ, τ, we have $g_{\rho\rho} + g_{\sigma\sigma} + g_{\tau\tau} = 3$.*

Proof. It is clear that if $\phi \in I_1^\alpha$, then $g_{\phi\phi} = 1$; if $\phi \in I_2^\alpha$, then $g_{\phi\phi} = 0$; and finally, if $\phi \in I_3^\alpha$, then $g_{\phi\phi} = -1$. Part (3) of Proposition 5 shows that either $\rho, \sigma, \tau \in I_2^\alpha$, or each of $I_1^\alpha, I_2^\alpha, I_3^\alpha$ contains exactly one of ρ, σ, τ. Hence, we have proved part (1). Part (2) follows from part (1) immediately. In order to prove (3), recall that whenever g is a root element, we have $g = e + \sum(\alpha)_g e_\alpha + \sum a_i h_i$, and the first sum contains no diagonal coefficients. □

Lemma 3.3. *Let α be a root. A diagonal matrix $g = e + \sum a_i h_i \in \mathrm{M}(27, K)$ is determined by the 15 coefficients g_{ν_k, ν_k}, where $\nu_k \in I_2^\alpha$, uniquely up to addition of $a h_\alpha$.*

Proof. By linearity, it remains to show that if $g_{\nu_k, \nu_k} = 1$ for all $\nu_k \in I_2^\alpha$, then $g = e + a h_\alpha$ for some a. Suppose $g_{\nu_k, \nu_k} = 1$ for all $\nu_k \in I_2^\alpha$. If $\lambda \in I_1^\alpha$ and $\mu \in I_3^\alpha$ are any two weights with $\mu + \delta \neq \lambda$, then, by the previous lemma, $g_{\lambda\lambda} + g_{\mu\mu} = 2$ (by part (3) of Proposition 5 any weight adjacent to neither λ nor μ lies in I_2^α). Since this is true for any λ and μ, the lemma is proved. □

Before proceeding to Theorem 1, we recall a definition.

Definition. Let A be any matrix. The rank of the matrix $A - E$ is called a *residue* of A. It is denoted by $\operatorname{res} A$.

We shall need two properties of the residue: 1) it is independent of the basis change; 2) $\operatorname{res} AB \leq \operatorname{res} A + \operatorname{res} B$.

Theorem 1. *Let g be a root element of $G_{\mathrm{sc}}(\mathrm{E}_6, K)$, let $\alpha \in \Phi = \mathrm{E}_6$ be a root such that $(\alpha)_g \neq 0$, and let λ be a weight such that $\lambda, \lambda + \alpha \in \Lambda$. Then the variables $(\alpha)_g$, all $(\beta)_g$ such that $\angle(\alpha, \beta) = \pi/3$, and the $g_{\lambda\lambda}$ are independent and determine g uniquely.*

Remark. There are exactly 20 roots β that form an angle of $\pi/3$ with α. Hence, Theorem 1 states that a root element g is uniquely determined by 22 independent variables.

Proof. 1. Note that the residue of any root element equals 6. Suppose $\lambda_i \in I_1^\alpha$, $\nu_k \in I_2^\alpha$ and $\mu_i \in I_3^\alpha$ for $1 \leq i \leq 6$, $1 \leq k \leq 15$, while $\lambda_i = \mu_i + \alpha$. Now the corollary to Proposition 2 together with the relation $(\alpha)_g \neq 0$ yields $d(\lambda_i, \mu_j) = 2$ for $i \neq j$. Hence, by Lemma 3.1 the matrix formed by intersection of the rows labeled by λ_i with the columns labeled by μ_j is an invertible diagonal matrix. The rank of the matrix $h = g - E$ equals 6, so that any of its columns can be expressed as a linear combination of columns labeled by μ_j.

Therefore, for $s \in I_2^\alpha \cup I_3^\alpha$ and $t \in I_1^\alpha \cup I_2^\alpha$, we have

$$h_{st} \prod_{i=1}^{6} h_{\lambda_i,\mu_i} = \sum_{i=1}^{6} \Big(h_{\lambda_i,t} h_{s,\mu_i} \prod_{j \neq i} h_{\lambda_j,\mu_j} \Big), \tag{3.1}$$

or, by the definition of the matrices h and c,

$$(g_{st} - \delta_{s,t})(\alpha)_g = \sum_{i=1}^{6} ((g_{\lambda_i,t} - \delta_{\lambda_i,t})(g_{s,\mu_i} - \delta_{s,\mu_i}) c_{\lambda_i,\mu_i}). \tag{3.2}$$

From Lemma 3.1 it follows easily that this formula holds true for all s and t; moreover, we shall show that it is valid even if $(\alpha)_g = 0$.

2. Consider any root γ orthogonal to α. By Proposition 4, there are weights $\nu_k, \nu_l \in I_2^\alpha$ such that $\nu_k = \nu_l + \gamma$, whence $(\gamma)_g = c_{\nu_k,\nu_l} g_{\nu_k,\nu_l}$. Then, by the first part, g_{ν_k,ν_l} can be expressed via g_{λ_i,ν_l}, g_{ν_k,μ_i} and $(\alpha)_g$. The coefficient g_{λ_i,ν_l} is equal to $c_{\lambda_i,\nu_l}(\beta)_g$ when $\beta = \lambda_i - \nu_l$, and to 0 when $\lambda_i - \nu_l \notin \Phi$. By the same argument, g_{ν_k,μ_i} is equal to $c_{\nu_k,\mu_i}(\beta)_g$ when $\beta = \nu_k - \mu_i$, and to 0 when $\nu_k - \mu_i \notin \Phi$. By Proposition 4, $\angle(\alpha,\beta) = \pi/3$. Therefore, $(\gamma)_g$ can be expressed polynomially in terms of $(\beta)_g$ for a fixed $(\alpha)_g$, where $\angle(\alpha,\gamma) = \pi/2$, $\angle(\alpha,\beta) \le \pi/3$. Hence, all coefficients g_{μ_i,μ_j} and g_{λ_i,λ_j} for $i \neq j$ can be expressed polynomially in terms of those elements.

Now we prove that all diagonal coefficients can be expressed in terms of those 22 variables. As before, g_{ν_k,ν_k} can be expressed in terms of $(\beta)_g$ for $\angle(\alpha,\beta) \le \pi/3$. By Lemma 3.3, the remaining diagonal coefficients can be uniquely determined up to adding some $a \in K$ to g_{λ_i,λ_i} and subtracting it from g_{μ_i,μ_i}. Therefore, all diagonal coefficients can be expressed via $(\beta)_g$ for $\angle(\alpha,\beta) \le \pi/3$, and $g_{\lambda\lambda}$. It remains to show that if we fix $(\alpha)_g$, this expression is polynomial. This is clear for g_{ν_k,ν_k}. Next, for any $\rho \in \Lambda$ the coefficient $g_{\rho\rho}$ can be expressed linearly via h_{α_i} for $1 \le i \le 6$. Thus, h_{α_i} can be expressed linearly via g_{ν_k,ν_k} and $g_{\lambda\lambda}$; this means that they can be expressed polynomially via $(\beta)_g$ for a fixed $(\alpha)_g$, where $\angle(\alpha,\beta) = \pi/3$, and $g_{\lambda\lambda}$. Hence, $g_{\rho\rho}$ can be expressed polynomially via the same variables for a fixed $(\alpha)_g$.

Therefore, the coefficients $g_{\lambda_i,*}$ and g_{*,μ_i} of the matrix g can be expressed polynomially via $(\beta)_g$ and $g_{\lambda\lambda}$ for a fixed $(\alpha)_g$, where $\angle(\alpha,\beta) \le \pi/3$. Formula (3.2) shows that all coefficients of the matrix g can be expressed polynomially for a fixed $(\alpha)_g$ via those variables.

3. It remains to show that those variables can take arbitrary values. We need to construct a root element for a given set of values of those variables. Let f be a root element and β any root such that $\angle(\alpha,\beta) = \pi/3$. Consider the conjugation of f with a root element $h = x_{\beta-\alpha}\big(N_{\beta-\alpha,\alpha}\frac{(\beta)_g}{(\alpha)_g}\big)$. By Lemma 3.2, the coefficient $(\gamma)_f$ can be changed only if $\angle(\gamma,\beta-\alpha) \le \pi/3$. Since $\angle(\alpha,\beta-\alpha) = 2\pi/3$, the coefficient of the root α does not change, whereas the only changed coefficient for roots at an angle of $\pi/3$ with α is the coefficient of the root β. Namely, $(\beta)_{hfh^{-1}} = (\beta)_f + \frac{(\beta)_g(\alpha)_f}{(\alpha)_g}$. Consider the element $f = x_\alpha((\alpha)_g)$ and conjugate it with $x_{\beta-\alpha}\big(N_{\beta-\alpha,\alpha}\frac{(\beta)_g}{(\alpha)_g}\big)$ for all roots β forming angle $\pi/3$ with α. It is easy to check that after these conjugations we have $(\gamma)_f = (\gamma)_g$ for any root γ such that $\angle(\gamma,\alpha) \le \pi/3$. Hence, we have obtained the first 21 variables.

By Lemma 3.2 the coordinates at the root α and the roots β forming angle $\pi/3$ with α are invariant under the conjugation of f with $x_{-\alpha}(a)$ (i.e., $(\alpha)_f = (\alpha)_{x_{-\alpha}(a)fx_{-\alpha}(-a)}$, $(\beta)_f = (\beta)_{x_{-\alpha}(a)fx_{-\alpha}(-a)}$). At the same time, this conjugation adds $h_\alpha(ab)$ to the sum $\sum a_i h_i$, where b does not depend on a. Conjugating the result obtained in the previous paragraph with $x_{-\alpha}$ and some coefficient, we get the required value of all 22 variables. $\square$

Remark. It is easily seen that for any nonidentity root element g there is a root α such that $(\alpha)_g \neq 0$. Indeed, since g is conjugate to an elementary root element, all the eigenvalues of g are equal to 1. Therefore, if g is diagonal, it is identity.

We shall use of the following fact. Strictly speaking, it is a consequence of the proof of Theorem 1, but we prefer to give a short and nice proof rather than formally deduce it from Theorem 1.

Lemma 3.4. *Let g be a root element, and let α be a root. Assume that $(\beta)_g = 0$ for all β such that $\angle(\alpha, \beta) > \pi/3$. Then $g_{\rho\rho} = 1$ for all $\rho \in \Lambda$.*

Proof. Suppose that this statement fails. Note that $g_{\phi\psi}$ can be nonzero only if one of the following is true: 1) $\phi \in I_1^\alpha$ while $\psi \in I_2^\alpha \cup I_3^\alpha$; 2) $\phi \in I_2^\alpha$ while $\psi \in I_3^\alpha$; 3) $\phi = \psi$. Consider a permutation matrix $f \in \mathrm{GL}(27, K)$ that maps I_1^α to I_1, I_2^α to I_2, and I_3^α to I_3. It is clear that the matrix fgf^{-1} is upper-triangular. Since g is a root element, it is conjugate to an elementary root element. Hence, all the eigenvalues of the matrix g (and fgf^{-1}) are equal to 1. We have a contradiction. □

Denote $\mathrm{Im}(g-E)$ by V^g. In other words, this is the six-dimensional subspace generated by all columns of the matrix $g - E$. In what follows we shall use the six-dimensional subspace $V_g < V^*$ generated by all rows of the matrix $g - E$. It is well known (see, for example, [98]) that $c_{\phi\psi} = c_{\psi\phi}$ for all weights ϕ and ψ. Hence, $x_\alpha(a)^T = x_{-\alpha}(a)$. Therefore, if g is a root element, then g^T is also a root element; if $g \in G_{\mathrm{sc}}(\mathrm{E}_6, K)$, then $g^T \in G_{\mathrm{sc}}(\mathrm{E}_6, K)$. This means that we can 'transpose' all the subsequent propositions (and their proofs) by transposing all notation, equalities, etc, and substituting rows for columns (and *vice versa*). The roots are taken to their opposites, while the weights are taken to themselves (we identify weights with coweights). In particular, the sets $I_1^\alpha, I_2^\alpha, I_3^\alpha$ are invariant under this operation. In what follows we shall freely use the 'transposed' versions of the propositions concerning V^g without any explicit mention. In other cases we shall state the transposed version explicitly.

Lemma 3.5. *Let g be a root element, and let α be any root. The following statements are equivalent*:

(1) $(\alpha)_g \neq 0$;
(2) *the six columns $(g-E)_{*,\mu_i}$ for $\mu_i \in I_3^\alpha$ and $1 \leq i \leq 6$ generate a six-dimensional subspace V^g*;
(3) *there exists a base $\{v^i\}_{i=1}^6$ of V^g such that $v^i_{\lambda_j} = \delta_{i,j}$, where $1 \leq i \leq 6$, $\lambda_j \in I_1^\alpha$, and $1 \leq j \leq 6$.*

Proof. Suppose $\lambda_i \in I_1^\alpha$, $\nu_k \in I_2^\alpha$, $\mu_i \in I_3^\alpha$ for $1 \leq i \leq 6$, $1 \leq k \leq 15$, and $\lambda_i = \mu_i + \alpha$. Suppose (1) is true. As was noted in Theorem 1, if $(\alpha)_g \neq 0$, then the submatrix $\{g_{\lambda_i,\mu_j}\}_{i,j=1}^6$ is diagonal and invertible. This proves (2) and (3). We prove that (2) implies (1). Suppose (1) fails, i.e., $(\alpha)_g = 0$. Then $g_{\lambda_i,\mu_j} = 0$ for all $1 \leq i, j \leq 6$. Moreover, suppose (2) is true, i.e., the columns $(g-E)_{*,\mu_j}$, $1 \leq j \leq 6$, generate the subspace V^g that contains all the columns of the matrix $g - E$. Therefore, all the rows of $g - E$ with labels λ_i, $1 \leq i \leq 6$, are zero. In particular, $g_{\lambda_j,\nu_k} = 0$ for all $1 \leq j \leq 6, 1 \leq k \leq 15$. Hence, $(\beta)_g = 0$ for all β such that $\angle(\alpha, \beta) = \pi/3$ and $g_{\nu_k,\mu_i} = 0$ for all $1 \leq i \leq 6, 1 \leq k \leq 15$. Since (2) is valid, we have $g_{\nu_k,\rho} = \delta_{\nu_k,\rho}$ for any $1 \leq k \leq 15$ and $\rho \in \Lambda$. In particular, $g_{\nu_k,\nu_l} = \delta_{\nu_k,\nu_l}$ for all $1 \leq k, l \leq 15$. Furthermore, as was noted before, $g_{\lambda_i,\rho} = \delta_{\lambda_i,\rho}$ for all $1 \leq i \leq 6$, $\rho \in \Lambda$; hence, by Lemma 3.3, we have $g_{\rho\rho} = 1$ for any $\rho \in \Lambda$. Note that for any $1 \leq i \neq j \leq 6$ there exist $\nu_k \neq \nu_l \in I_2^\alpha$ such that $g_{\mu_i,\mu_j} = \pm g_{\nu_k,\nu_l}$, which equals 0. Therefore, $g_{\rho,\mu_i} = \delta_{\rho,\mu_i}$, which contradicts (2).

It remains to prove that (3) implies (1). Suppose that (3) holds true while (1) does not. Then $g_{\lambda_i,\mu_j} = 0$ for $1 \leq i, j \leq 6$. From (3) it follows that if $u \in V^g$ and $u_{\lambda_j} = 0$ for

all $1 \le j \le 6$, then the vector u can be expressed as a linear combination of v^i with zero coefficients, i.e., $u = 0$. Hence, $g_{\rho,\mu_j} = \delta_{\rho,\mu_j}$. It follows that $(\gamma)_g = 0$ for all γ such that $\angle(\gamma, -\alpha) > \pi/3$, so we can apply the previous lemma. This means that $g_{\rho\rho} = 1$ for all $\rho \in \Lambda$. Therefore, $g_{\lambda_j,\rho} = \delta_{\lambda_i,\rho}$, which contradicts (3). □

In Theorem 1 and Lemma 3.5 we constructed a map from the set of root subgroups to the set of six-dimensional subspaces, taking an element g to the subspace V^g. Now we prove that this map is injective. In §4 we shall prove that the restriction of this map to the set of singular six-dimensional subspaces is a bijection.

Corollary. *Each root subgroup is mapped to a six-dimensional subspace*; *root elements from distinct root subgroups are mapped to distinct subspaces.*

Proof. We must show that if $V^g = V^h$ for root elements g and h, then g and h lie in a common root subgroup; in other words, that $g' = g - E$ is proportional to $h' = h - E$. Let α be a root such that $(\alpha)_g \neq 0$. Now, by part (3) of the previous lemma, $(\alpha)_h \neq 0$. It is clear that the basis $\{v^i\}_{i=1}^6$ mentioned there is unique. Notice that $g'_{*,\mu_i} = (\alpha)_g c_{\lambda_i,\mu_i} v^i$ for all $1 \le i \le 6$ and in the same way $h'_{*,\mu_i} = (\alpha)_h c_{\lambda_i,\mu_i} v^i$. Using Theorem 1, we get $g' = \frac{(\alpha)_g}{(\alpha)_h} h'$. This completes the proof of the corollary. □

We conclude this section with two easy but interesting and useful propositions.

Proposition 9. *Let $g \in G_{\mathrm{sc}}(\mathrm{E}_6, K)$, and let α be a root. Then there exists a root β such that the matrix $\{g_{\lambda^\beta,\mu^\alpha}\}_{\lambda^\beta \in I_1^\beta, \mu^\alpha \in I_3^\alpha}$ is invertible.*

Proof. Consider a root element $f = gx_{-\alpha}(a)g^{-1}$. Note that $V^{x_{-\alpha}(a)}$ is generated by the columns $(x_{-\alpha}(a) - E)_{*,\lambda^\alpha} = \pm a e^{\lambda^\alpha - \delta}$, where λ^α runs over I_1^α. Therefore, V^f is generated by the six columns g_{*,μ^α} for $\mu^\alpha \in I_3^\alpha$. Since the root element f is not identity, by the remark after Theorem 1 there exists a root β such that $(\beta)_f \neq 0$. Now we can apply part (3) of Lemma 3.5 and finish the proof. □

Proposition 10. *Let g be a root element. If $(\alpha)_g = 0$, then condition* (3.2) *is fulfilled.*

Proof. By the remark after Theorem 1, it can be assumed that there exists a root β such that $(\beta)_g \neq 0$. Hence, by Theorem 1, all the entries of the matrix g can be expressed in terms of $(\beta)_g, (\gamma)_g$ for $\angle(\beta, \gamma) = \pi/3$, and $g_{\rho\rho}$ for $\rho, \rho - \beta \in \Lambda$. Therefore, (3.2) can be viewed as equality of polynomials in 22 variables. Moreover, the coefficients of these polynomials are integers and do not depend of the ground field. This means that in order to show equality of the polynomials in (3.2) we may assume that we work over the field of complex numbers. The condition $(\alpha)_g = 0$ determines a subvariety of codimension 1. Equality of polynomials everywhere except a subvariety of codimension 1 implies equality of their coefficients. Thus, our polynomials are equal over any field. □

§4. Root elements and singular subspaces

Our aim in this section is to study singular subspaces and their relationship with root elements. In particular, we prove that a root subgroup can be put into correspondence to any six-dimensional singular subspace (we constructed the inverse mapping in §3). We also explain how to determine the angle between two root subgroups by looking at the corresponding six-dimensional singular subspaces.

1. Singular subspaces. We recall the definition from §2: a vector v is *singular* if for any vector x we have $Q(x, v) = 0$. A subspace is said to be singular if every one of its vectors is singular.

Proposition 11.

(1) *Let g be any element of $G_{\mathrm{sc}}(\mathrm{E}_6, K)$, and let α be a root. Then the six-dimensional subspace $\langle g_{*\phi} \mid \phi \in I_3^\alpha \rangle$ is singular.*

(2) *If g is a root element, then the six-dimensional subspace $V^g < V$ is singular.*

Proof. As was noted in §2, g maps singular vectors to singular vectors. Note that every e^ρ is singular; hence, all columns of the matrix g are singular. The expression for F shows that for any two distinct weights ρ and σ we have $F(e^\rho, e^\sigma, x) = 0$ for all x if and only if $\rho - \sigma$ is a root. Applying Proposition 1, we get part (1).

In order to prove (2), we need to find an element $g' \in G_{\mathrm{sc}}(\mathrm{E}_6, K)$ and a root α such that $\langle g'_{*\phi} \mid \phi \in I_3^\alpha \rangle = V^g$. Let α be a root such that $(\alpha)_g \neq 0$. Next, put $f = x_{-\alpha}\big(-\frac{1}{(\alpha)_g}\big)$ and consider $g' = fgf^{-1}$. It is easily seen that these elements are as we need. □

In this subsection, we prove that every six-dimensional subspace corresponds to some root subgroup. We start with the following lemma.

Lemma 4.1.

(1) *Suppose $1 \leq n \leq 6$. Then the set of n-tuples of pairwise adjacent weights under the action of the Weyl group forms one orbit for $n \neq 5$, and two orbits for $n = 5$.*

(2) *Let $n = 1, 2, 3, 4,$ or 6. Then the n-tuple of basis vectors generating an n-dimensional singular subspace can be mapped into the first n basis vectors by the action of the group $\widetilde{W} < G_{\mathrm{sc}}(\mathrm{E}_6, K)$.*

Remark. Suppose $n = 5$. One orbit in part (1) of Lemma 4.1 consists of all 5-tuples of pairwise adjacent weights that are contained in some 6-tuple. For example, the first five weights have this property. The other orbit consists of exceptional 5-tuples of pairwise distinct weights. These 5-tuples are not contained in any 6-tuple of pairwise adjacent weights. For example, the first four weights and the seventh weight form an exceptional 5-tuple.

Proof. To prove part (1), suppose $\rho_1, \rho_2, \ldots, \rho_n$ are pairwise adjacent weights. Consider the action of the Weyl group W; for clarity, we may look only at $w_i = w_{\alpha_i}$ and use the same notation for the images of the weights ρ_j. Since the weight diagram is connected, we can map ρ_1 into the first weight. It is readily seen that the weights at a distance of 1 from the first weight (there are exactly 16 of them) form a connected subgraph after we remove the edges marked α_1 from the weight diagram. This means that we can map ρ_2 to the second weight (naturally, if $n > 2$), while the first weight stays untouched (because, among all the elements w_i, only w_1 moves the first weight). Likewise, the weights at a distance of 1 from the first two weights (there are exactly 10 of them) form a connected subgraph after we remove the edges marked α_1 and the edges marked α_3 from the weight diagram. This allows us to map ρ_3 to the third weight (if $n > 2$) while leaving the first two weights untouched. Next, the weights at a distance of 1 from the first three weights (there are 6 of them) form a connected subgraph after we remove the edges marked α_1, α_3, and α_4 from the weight diagram. Hence, we can map ρ_4 to the fourth weight (if $n > 3$), while the first three stay untouched. Finally, there are exactly three weights (5th, 6th, and 7th) adjacent to the first four weights. The 5th and 6th weights are adjacent, while the 7th is adjacent to none of them. If $n = 5$, we can map ρ_5 either to the 5th weight or to the 7th weight. On the other hand, if $n = 6$, we can map ρ_5 to the 5th weight and ρ_6 to the 6th weight. The first four weights stay untouched.

We prove the second part. As was noted in §2, the action of the Weyl group W on the set of weights Λ corresponds to the action of the extended Weyl group $\widetilde{W}$ on the set of plus or minus weights $\Lambda^\pm = \{\pm e^\rho; \rho \in \Lambda\}$. We also noticed that the distance

between basis vectors is equal to the distance between the corresponding weights. Thus, n pairwise adjacent basis vectors correspond to n pairwise adjacent weights. In particular, we have proved that if $n = 1, 2, 3, 4$, or 6, then an n-tuple $(a^1, a^2, \dots, a^n)$, where the a^i are pairwise adjacent basis weights and $\dim\langle a^1, a^2, \dots, a^n\rangle = n$, can be mapped to $(\pm e^1, \pm e^2, \dots, \pm e^n)$ by the action of the group $\widetilde{W}$. As was noted in §2, if $\lambda, \lambda+\alpha \in \Lambda$, then $(w_\alpha(1))^2 e^\lambda = -e^\lambda$ and $(w_\alpha(1))^2 e^{\lambda+\alpha} = -e^{\lambda+\alpha}$; on the other hand, if $\lambda \in \Lambda$ and $\lambda+\alpha, \lambda-\alpha \notin \Lambda$, then $(w_\alpha(1))^2 e^\lambda = (w_\alpha(1)) e^\lambda = e^\lambda$. In other words, the matrix $g = (w_\alpha(1))^2 \in G_{\mathrm{sc}}(\mathrm{E}_6, K)$ is diagonal; $g_{\lambda\lambda} = -1$ if $\lambda+\alpha$ or $\lambda-\alpha$ is a weight, and $g_{\lambda\lambda} = 1$ otherwise. By Proposition 4, if $\angle(\alpha, \delta) = \pi/2$, then $(w_\alpha(1))^2$ changes the sign of two of the first six basis vectors (and these can be two arbitrary vectors). Likewise, if $\angle(\alpha, \delta) = \pi/3$, then $(w_\alpha(1))^2$ changes the sign of three of the first six basis vectors (and these can be three arbitrary vectors). This implies that any collection of basis vectors can be mapped to a 'positive' one by matrices of the form $(w_\alpha(1))^2$. □

Before we proceed to the main proposition of this section, we need to prove two more lemmas.

Lemma 4.2. *Suppose $n = 1, 2, 3, 4$, or 6. Let $\{u^i\}_{i=1}^n$ be singular vectors that form an n-dimensional singular subspace. Suppose there exists a root α such that $u^i_{\lambda_j} = c_{\lambda_j, \lambda_j-\alpha}\delta_{i,j}$, where $\lambda_j \in I_1^\alpha$, $1 \le i \le n, 1 \le j \le 6$. Then there exists a root element h such that $h_{*,\lambda_i-\alpha} = u^i$.*

Proof. Suppose $\lambda_i \in I_1^\alpha$, $\nu_k \in I_2^\alpha$, and $\mu_i \in I_3^\alpha$ for $1 \le i \le 6$ and $1 \le k \le 15$, while $\lambda_i = \mu_i + \alpha$. We need to construct a root element h. By Theorem 1, such an h exists and is unique if $(\alpha)_h \ne 0$, $(\beta)_h$ for $\angle(\beta, \alpha) = \pi/3$, and h_{λ_1,λ_1} are defined. The proof of Theorem 1 shows that we can define h_{μ_1,μ_1} instead of h_{λ_1,λ_1}. Put $(\alpha)_h = 1$, $(\beta)_h = u^i_{\nu_k} c_{\nu_k,\mu_i}$ for $\angle(\beta, \alpha) = \pi/3$ if $\nu_k - \mu_i = \beta$. Finally, put $h_{\mu_1,\mu_1} = u^1_{\mu_1}$. If some $(\beta)_h$ is not fixed yet (this can happen if $n < 4$), we can choose them arbitrarily. We need to show that the result is well defined (note that $(\beta)_h$ may fail to be unique) and that $u^i = h_{*,\mu_i}$.

Let g be a root element. As was noted in §3, the columns with labels μ_i of the matrix of g can be described as follows:

(1) $g_{\lambda_i,\mu_i} = c_{\lambda_i,\mu_i}(\alpha)_g$ for all $1 \le i \le 6$;
(2) $g_{\lambda_j,\mu_i} = 0$ for all $1 \le j \ne i \le 6$;
(3) $g_{\nu_k,\mu_i} = 0$ for all $1 \le i \le 6$ and $1 \le k \le 15$ if $d(\nu_k, \mu_i) = 2$;
(4) for all $1 \le i \ne j \le 6$, the element g_{μ_j,μ_i} can be expressed uniquely in terms of g_{ν_k,μ_i} and g_{λ_i,μ_i} for $1 \le k \le 15$;
(5) $g_{\nu_k,\mu_i} = c_{\nu_k,\mu_i}(\nu_k - \mu_i)_g$ for all $1 \le i \le 6$ and $1 \le k \le 15$ such that $\nu_k - \mu_i \in \Phi$;
(6) for all $1 \le i, j \le 6$ the element g_{μ_i,μ_i} can be expressed uniquely in terms of g_{ν_k,μ_i}, g_{λ_i,μ_i}, g_{μ_j,μ_j}, and g_{ν_k,μ_j} for $1 \le k \le 15$.

We shall prove that the columns u^i satisfy these conditions. It will then follow immediately that h is well defined and $u^i = h_{*,\mu_i}$ (because, by Proposition 11, the columns h_{*,μ_i} with $1 \le i \le n$ also span an n-dimensional singular subspace)

Note that the u^i satisfy the first two conditions because $u^i_{\lambda_j} = c_{\lambda_j,\lambda_j-\alpha}\delta_{i,j}$ by assumption. We prove that $u^i_{\nu_k} = 0$ if $d(\nu_k, \mu_i) = 2$ (this is condition (3) for the columns u^i). Consider $Q(e_{\mu_j}, u^i)$ for $i \ne j$ and $d(\mu_j, \nu_k) = 2$ (the weight μ_j with this property exists by part (4) of Proposition 6). On the one hand, it can be expressed as a sum of five terms, each of them having some λ_l, $l \ne j$, as a multiplier. Therefore, all the summands are 0, except for $\pm u^i_{\nu_k} u^i_{\lambda_i}$. Since u^i is a singular vector, the whole sum equals zero. Thus, $u^i_{\nu_k} = 0$. In the same way, consider $Q(e_{\nu_m}, u^i)$ for $d(\nu_m, \mu_i) = d(\nu_m, \mu_j) = 2$. By the same argument, $u^i_{\mu_j}$ for $i \ne j$ can be expressed uniquely in terms of $u^i_{\nu_k}$, $1 \le k \le 15$ (this is condition (4)).

Now suppose $n > 1$. We need to show that $u^j_{\mu_j}$ can be expressed uniquely in terms of $u^i_{\mu_i}$, $u^i_{\nu_k}$, and $u^j_{\nu_k}$, $1 \le k \le 15$ (condition (6)). Also we need to show that $u^i_{\nu_k} c_{\nu_k,\mu_i} = u^j_{\nu_l} c_{\nu_l,\mu_j}$ for $\nu_k - \mu_i = \nu_l - \mu_j \in \Phi$ (this is condition (5) together with the fact that h is well defined). Suppose $u = u^i + u^j$. Since the vector u is singular, $Q(x,u) = 0$ for all $x \in V$. In order to prove the first statement, consider $x = e_{\nu_k}$, where $d(\nu_k, \mu_i) = d(\lambda_j, \nu_k) = 2$. Then $Q(x,u)$ is a sum of five terms: $\pm u_{\mu_j} u_{\lambda_i}$, $\pm u_{\mu_i} u_{\lambda_j}$, and three terms of the form $\pm u_{\nu_l} u_{\nu_m}$. Since the the whole sum equals 0, the first statement follows.

Suppose $\nu_k - \mu_i = \nu_l - \mu_j$. Then $\nu_k - \nu_l = \mu_i - \mu_j \in \Phi$, i.e., $d(\nu_k, \nu_l) = 1$. By the corollary to Proposition 2, $d(\mu_i, \nu_l) = d(\mu_j, \nu_k) = 2$. Thus, we need to show that $u_{\nu_k} c_{\nu_k,\mu_i} = u_{\nu_l} c_{\nu_l,\mu_j}$. By part (5) of Proposition 6, if $d(\nu_k, \nu_l) = 1$, then there exists a weight μ_m such that $d(\nu_k, \mu_m) = d(\nu_l, \mu_m) = 2$. Hence, $m \ne i, j$. By the corollary to Proposition 2, $d(\lambda_i, \mu_m) = d(\lambda_j, \mu_m) = 2$. Since $u_{\lambda_s} = 0$ for $s \ne i, j$, the sum $Q(e_{\mu_m}, u)$ consists of two terms: $\pm u_{\nu_k} u_{\lambda_j}$ and $\pm u_{\nu_l} u_{\lambda_i}$. Therefore, $u_{\nu_k} = \pm u_{\nu_l}$. We do not need to know the exact sign in this expression; it suffices to say that it is uniquely determined and does not depend on u. Indeed, the columns labeled by μ_i and μ_j of the root element h span a two-dimensional singular subspace, so that we have $u_{\nu_k} = \pm u_{\nu_l}$ for $u = h_{*,\mu_i} + h_{*,\mu_j}$. Since $u_{\nu_k} c_{\nu_k,\mu_i} = u_{\nu_l} c_{\nu_l,\mu_j}$ for $u = h_{*,\mu_i} + h_{*,\mu_j}$, it follows that $\pm 1 = c_{\nu_k,\mu_i} c_{\nu_l,\mu_j}$. Therefore, the identity we need is true for arbitrary u. □

Lemma 4.3. *Suppose $n = 1, 2, 3, 4$, or 6. Let $\{u^i\}_{i=1}^n$ be singular vectors that span an n-dimensional singular subspace. Moreover, suppose there exists a root α such that the matrix $(u^i_{\lambda_j})$, where $\lambda_j \in I_1^\alpha$, $1 \le i \le n$, $1 \le j \le 6$, has rank n. Then there exists a matrix $g \in G_{\mathrm{sc}}(\mathrm{E}_6, K)$ such that $g_{*i} = u^i$ for $i \le \min(n, 5)$. If $n = 6$, then $g_{*6} = au^6, a \in K^*$.*

Proof. Since the matrix $(u^i_{\lambda_j})$ has rank n for $1 \le i \le n$, $1 \le j \le 6$, it can be completed to an invertible matrix A of size 6×6. Choose a matrix B such that $AB_{ij} = c_{\lambda_j,\mu_j} \delta_{i,j}$, where $\mu_j = \lambda_j - \alpha \in I_3^\alpha$, $1 \le i \le 6, 1 \le j \le 6$. If $n < 6$, we may assume that B lies in $\mathrm{SL}(6, K)$. If $n = 6$, we may assume the same after multiplying u^6 by some scalar. In what follows, we assume that $B \in \mathrm{SL}(6, K)$. Now, consider a matrix $C \in D_\alpha$ such that $C_{\lambda_i,\lambda_j} = B_{ij}$ for $1 \le i, j \le 6$. Note that we can apply the previous lemma to the columns $v^i = Cu^i$; thus, there exists a root element h such that $h_{*,\mu_i} = v^i = Cu^i$ for $\mu_i = \lambda_i - \alpha \in I_3^\alpha$, $1 \le i \le n$. By Lemma 4.1, there exists a matrix X such that $Xe_i = e_{\mu_i}$ for $1 \le i \le 6$. Hence, $u^i = C^{-1} h_{*,\mu_i} = \{C^{-1}h\}_{*,\mu_i} = \{C^{-1}hX\}_{*,i}$. Now we put $g := C^{-1}hX$ and recall that we have divided u^6 by some nonzero determinant; this finishes the proof. □

Theorem 2. *Suppose $n = 1, 2, 3, 4$, or 6. Suppose that $\{u^i\}_{i=1}^n$ and $\{v^i\}_{i=1}^n$ are two n-tuples of singular vectors that span two n-dimensional singular subspaces. Then there exists a matrix $g \in G_{\mathrm{sc}}(\mathrm{E}_6, K)$ such that $u^i = gv^i$ for $i \le \min(n, 5)$. If $n = 6$, then $u^6 = agv^6$ for some $a \in K^*$.*

Proof. Note that it suffices to prove the theorem for $\{v^i\}_{i=1}^n = \{e^i\}_{i=1}^n$. Suppose $n = 1$. There exists a weight ρ such that $u^1_\rho \ne 0$. Now we take a root α such that $\rho - \alpha \in \Lambda$ and apply the previous lemma.

Suppose $1 < n < 5$. As before, we can map u^i to e^i for $1 \le i < n$. Denote by u the image of u^n under this map. We show that $u_\rho = 0$ if $d(i, \rho) = 2$ for some i, $1 \le i < n$. Indeed, let σ be a weight that is adjacent to neither i nor ρ. Consider $F(e^i, u, e^\sigma) = \pm u_\rho$. Since the vectors u and e^i are adjacent, we have $F(e^i, u, x) = 0$ for any vector x, whence $u_\rho = 0$. Therefore, there exists a weight ρ such that $d(i, \rho) = 1, 1 \le i < n$, and $u_\rho \ne 0$. From the first part of Lemma 4.1 it easily follows that if $d(i, \rho) = 1$ for $1 \le i < n$, then for $n < 5$ there exists a root α such that $\rho - \alpha, i - \alpha \in \Lambda, 1 \le i < n$. By the definition of ρ, now we can apply the previous lemma.

Finally, suppose $n = 6$. As before, we can map u^i to e^i for $1 \le i \le 4$; we shall denote the images of the vectors u^5 and u^6 by the same letters. As in the previous case, we have $u^5_\rho = 0$ and $u^6_\rho = 0$ whenever $d(i, \rho) = 2$ for some i, $1 \le i \le 4$. Furthermore, we may assume that $u^5_i, u^6_i = 0$ for $1 \le i \le 4$. Then the vectors u^5 and u^6 have no more than three nonzero entries: they correspond to the 5th, 6th, and 7th weights. Note that if one of these vectors have nonzero 7th entry, then both of them have zeroes in the 5th and 6th entries (because they are singular and their sum is singular), which is impossible. Therefore, all nonzero entries of these vectors are among the first six entries, and now we can apply the previous lemma, putting $\alpha = \delta$. □

Corollary. *Any six-dimensional singular subspace corresponds to some root element. Any four-, three-, two- or one-dimensional singular subspace is contained in some six-dimensional singular subspace; however, there exist exceptional five-dimensional singular subspaces that are not contained in any six-dimensional singular subspace.*

Proof. Let V be a six-dimensional singular subspace. By Theorem 2, there exists a matrix g that maps V to the subspace spanned by the first six basis vectors. Put $h = g^{-1}x_\delta(1)g$. It can easily be checked that $V = V^h$. The second statement follows immediately from Theorem 2. □

2. Angle between root elements. Our aim in this subsection is to prove several well-known facts concerning the geometry of two long root subgroups. The only new (although trivial) statement is the description of angles in terms of six-dimensional singular subspaces (Proposition 12). We tried to preserve the spirit of §3 in the following proofs, operating at the level of individual elements and without appealing to external facts. Namely, here we only use Theorem 1, Lemma 3.4, and (implicitly) the corollary to Theorem 2 (in the proof of the corollary to Proposition 12).

Recall that $X_\alpha = \{x_\alpha(a); a \in K\}$ is called an elementary root subgroup of the group $G_{\mathrm{sc}}(\mathrm{E}_6, K)$. If we do not care about a particular (nonidentity) element of an elementary root subgroup X_α, we denote it by $x_\alpha(\cdot)$. We say that a pair of root elements (g, h) is conjugate to a pair (g_1, h_1) if there exists an element x such that $xgx^{-1} = g_1$ and $xhx^{-1} = h_1$.

Lemma 4.4. *Any pair of root elements is conjugate to some pair of elementary root elements.*

Proof. Consider an arbitrary pair of root elements. By definition, we may assume that one of them equals $x_\alpha(\cdot)$. Denote the other by g. The third part of the proof of Theorem 1 shows that if g is any root element with $(\beta)_g \ne 0$, then $g = fx_\beta((\beta)_g)f^{-1}$, where f is a product of the elementary root elements $x_\gamma\big(N_{\gamma\beta}\frac{(\beta+\gamma)_g}{(\beta)_g}\big)$ over all roots γ such that $\angle(\gamma, \beta) = 2\pi/3$ and $x_{-\beta}(\cdot)$. The coefficient in the root element $x_{-\beta}(\cdot)$ depends on the order of the $x_\gamma(\cdot)$ in this product.

Suppose $(-\alpha)_g$ is nonzero. Substituting $-\alpha$ for β in the previous paragraph, we obtain $g = fx_{-\alpha}(\cdot)f^{-1}$, where f is a product of the elementary root elements $x_\alpha(\cdot)$ and $x_\gamma(\cdot)$, $\angle(\alpha, \gamma) = \pi/3$. Therefore, $x_{-\alpha}(\cdot) = f^{-1}gf$. Finally, note that $x_\alpha(\cdot) = f^{-1}x_\alpha(\cdot)f$, because $x_\alpha(\cdot)$ commutes with each factor of f.

Next, suppose $(-\alpha)_g = 0$, and suppose there exists a root β with $\angle(\alpha, \beta) = 2\pi/3$ such that $(\beta)_g \ne 0$. Then $g = fx_\beta(\cdot)f^{-1}$, where f is a product of the elementary root elements $x_\gamma\big(N_{\gamma\beta}\frac{(\beta+\gamma)_g}{(\beta)_g}\big)$ over all roots γ such that $\angle(\gamma, \beta) = 2\pi/3$, and $x_{-\beta}(\cdot)$. Note that if $\angle(\alpha, \gamma) \le \pi/2$, then $x_\gamma(\cdot)$ commutes with $x_\alpha(\cdot)$. It is clear that in this case there is exactly one factor of f that does not commute with $x_\alpha(\cdot)$, namely, $x_{-\alpha-\beta}\big(N_{-\alpha-\beta,\beta}\frac{(-\alpha)_g}{(\beta)_g}\big)$. At the

same time, $(-\alpha)_g = 0$; thus, $x_{-\alpha-\beta}(\cdot)$ appears with zero coefficient in f. Therefore, $x_\alpha(\cdot)$ again commutes with each factor of f, and we can argue as in the previous paragraph.

If all the coefficients of the roots that form the angles π or $2\pi/3$ with α are zero, while some root with a nonzero coefficient is orthogonal to α, the proof is similar.

Now suppose that we have $(\beta)_g = 0$ for any β with $\angle(\alpha, \beta) > \pi/3$, but there exists a root $\gamma \neq \alpha$ such that $(\gamma)_g \neq 0$. Arguing as above, we can show easily that f has nonzero coordinates only at the roots β such that $\angle(\alpha, \beta) \leq \pi/2$, and at the root $-\gamma$. By Lemma 3.4, the diagonal entries of the matrix g are equal to 1. By Theorem 1, the coordinate of f at $-\gamma$ is also equal to 0, so that $x_\alpha(\cdot)$ again commutes with each factor in f. □

Lemma 4.5. *Suppose $\angle(\beta_1, \gamma_1) = \angle(\beta_2, \gamma_2)$. Then there exists a matrix $g \in G_{\mathrm{sc}}(\mathrm{E}_6, K)$ such that $gX_{\beta_1}g^{-1} = X_{\beta_2}$ and $gX_{\gamma_1}g^{-1} = X_{\gamma_2}$.*

Proof. As was noted in §2, conjugation with $w_\alpha(1)$ maps X_β to $X_{w_\alpha\beta}$. Now the statement follows from the fact that we can map any pair of roots to any other pair with the same angle between them by the action of the Weyl group. □

Definition. Suppose a pair of root elements (g, h) is conjugate to a pair of elementary root elements $(x_\alpha(\cdot), x_\beta(\cdot))$. The angle between the roots α and β is called the angle between the root elements g and h.

We show that the angle between root elements is well defined together with the proof of the following easy but useful statement: the angle between root elements is uniquely determined by the relative position of the corresponding six-dimensional singular subspaces.

Proposition 12. *Let g, h be two root elements. The angle between g and h is well defined, and exactly one of the following conditions is fulfilled*:

(1) *if $V^g = V^h$, then $\angle(g, h) = 0$;*
(2) *if $\dim(V^g \cap V^h) = 3$, then $\angle(g, h) = \pi/3$;*
(3) *if $\dim(V^g \cap V^h) = 1$, then $\angle(g, h) = \pi/2$;*
(4) *if $V^g \cap V^h = 0$ and there exists a six-dimensional singular subspace W such that $\dim(V^g \cap W) = \dim(V^h \cap W) = 3$, then $\angle(g, h) = 2\pi/3$;*
(5) *if $V^g \cap V^h = 0$ and for any vector $v \in V^g$ there is a unique (up to scalar multiplication) vector $u \in V^h$ such that $v + u$ is singular, then $\angle(g, h) = \pi$.*

Proof. Note that the conditions in all five cases are invariant under conjugation. By Lemma 4.4, we may assume that g and h are elementary root elements. By Lemma 4.5, we can map g to $x_\delta(\cdot)$ and h to $x_\delta(\cdot), x_{\alpha_2}(\cdot), x_{\alpha_1}(\cdot), x_{-\alpha_2}(\cdot)$, or $x_{-\delta}(\cdot)$, where α_1 and α_2 are the first two simple roots. It is easily seen that each of these cases corresponds to exactly one case of the statement. Hence, the angle is well defined and the proof is finished. □

Corollary 1. *All possibilities for a relative position of two six-dimensional singular subspaces are described by Proposition* 12.

Proof. Trivial. □

The description of the orbits of pairs of root subgroups in all algebraic groups is well known. We state a particular case of this description, which follows from the discussion above.

Proposition 1. *A pair of root subgroups is conjugated to another pair by an element of $G_{\mathrm{sc}}(\mathrm{E}_6, K)$ if and only if the angles between the elements of each pair are the same.*

Proof. The condition on angles is necessary by Proposition 12, and sufficient by Lemmas 4.4 and 4.5. □

§5. Triples of root subgroups

In the paper [21], N. Vavilov and the author considered the problem of generation for three root subgroups two of which are opposite, in an arbitrary simply connected group $G(\Phi, K)$ with $\operatorname{char} K \neq 2$. Our aim in the present section is to give a geometric answer to the following problem: given three root subgroups in $\mathrm{SO}(2n, K)$ (the general case was reduced to this one in [21]) and in $G_{\mathrm{sc}}(\mathrm{E}_6, K)$, determine which subgroup on the list in [21] they generate.

1. Triples of root subgroups in $\mathrm{SO}(2n, K)$**.** We state the main result of [21]. Consider the Heisenberg group

$$H_1 = \left\{ \begin{pmatrix} 1 & a & b & c \\ . & 1 & . & b \\ . & . & 1 & -a \\ . & . & . & 1 \end{pmatrix}, \ a, b, c \in K \right\}.$$

It is the unipotent radical of the standard parabolic subgroup P_1 of the symplectic group $\mathrm{Sp}(4, K)$. The action of the Levi subgroup on the unipotent radical of P_1 induces the action of $\mathrm{SL}(2, K)$ on H_1. This action is used in part (5) of the next theorem. Part (8) is similar, but we need to consider the action of the Levi subgroup on the unipotent radical H_2 of the parabolic subgroup P_2 in the Chevalley group of type G_2. Parts (3) and (4) refer to the natural action of $SL(2, K)$ on 2-dimensional vectors, while part (7) refers to the natural action on pairs of 2-dimensional vectors.

Theorem 3. *Let X, Y, Z be root subgroups in $G(\Phi, K)$. Suppose X and Y are opposite. Put $H = \langle X, Y, Z \rangle$. Suppose* $\operatorname{char} K \neq 2$. *Then one of the following statements holds true*:

(1) $H \cong \mathrm{SL}(2, K)$;
(2) $H \cong \mathrm{SL}(2, K) \times K$;
(3) $H \cong \mathrm{SL}(2, K) \cdot K^2$;
(4) $H \cong (\mathrm{SL}(2, K) \cdot K^2) \times K$;
(5) $H \cong \mathrm{SL}(2, K) \cdot H_1$;
(6) $H \cong \mathrm{SL}(3, K)$;
(7) $H \cong (\mathrm{SL}(2, K) \cdot K^4) \times K$;
(8) $H \cong \mathrm{SL}(2, K) \cdot H_2$;
(9) $H \cong \mathrm{SU}(3, L)$, *where L is a quadratic extension of K.*

We shall use the following result from [21].

Lemma 5.1. *Let $G = G(\Phi, K)$. Suppose X, Y, Z are three long root subgroups. Then there exists a subsystem $\Delta \subset \Phi$, which is a twisted form of a subsystem of δ_4, and an element $u \in G(\Phi, K)$ such that*

$$\langle X, Y, Z \rangle \leq uG(\Delta, K)u^{-1}.$$

All possibilities for Δ are

$$\Delta = \alpha_1, \ \alpha_2, \ \alpha_3, \ \beta_2, \ \beta_3, \ \delta_4, \ \mathrm{G}_2.$$

This reduction was proved in [10, 11] for individual long semisimple root elements, and later appeared in the final form in [23, 24, 39]. From a technical point of view, this corresponds to Borel orbits of the group $B(\Phi \setminus \{\alpha_i\}, K)$ in certain representations, where α_i is a simple root connected to a maximal root in the extended Dynkin diagram. In a

similar situation, this reduction was discovered independently by Röhrle [80]. Groups of type δ_4 also play a key role in the work of Eugeny Bashkirov [3, 4, 5, 51, 52, 53], which can be also explained by the same fundamental reasons.

Lemma 5.1 allows us to restrict our attention to the group $\mathrm{SO}(8, K)$. In this subsection, we put $G = \mathrm{SO}(8, K)$ and $V = K^8$. Fixing a Witt basis $e_1, e_2, e_3, e_4, e_{-4}, e_{-3}, e_{-2}, e_{-1}$, we let X, Y, and Z be three root subgroups such that X is opposite to Y. Recall that V possesses a nonsingular symmetric bilinear form B preserved by G; it can be expressed as $B(u, v) = u_1 v_{-1} + \cdots + u_{-1} v_1$ in our basis. A vector u is said to be *isotropic* if $B(u, u) = 0$; a subspace U is *totally isotropic* if any vector in U is isotropic. Since g acts transitively on pairs of opposite root subgroups, without loss of generality we may assume that $X = X_{1,-2}$ and $Y = X_{-2,1}$. Clearly, the root subgroups in $\mathrm{SO}(2n, K)$ correspond to two-dimensional totally isotropic subspaces, and *vice versa*. For a two-dimensional isotropic subspace W, denote by X_W the corresponding root subgroup. Let $U = \mathrm{Im}(Z - E)$, and let (u, v) be a basis of U; then $Z = X_U$.

Let F be the normalizer of the subgroup $\langle X, Y \rangle$. It is clear that F can be expressed as a direct product of two copies of the group $\mathrm{SO}(4, K)$, the first copy acting on the subspace $V_1 = \langle e_1, e_2, e_{-2}, e_{-1} \rangle$, and the second copy acting on the complement $V_2 = \langle e_3, e_4, e_{-4}, e_{-3} \rangle$. Obviously, V is the *orthogonal* direct sum of V_1 and V_2. In [21] we described the orbits of nonzero isotropic vectors under the action of F.

Lemma 5.2. *Any nonzero isotropic vector $u \in V$ is equivalent under the action of F to one of the following*:

$$e_1, \quad e_3, \quad e_1 + e_3, \quad e_1 + e_3 - a e_{-3} + a e_{-1}$$

for some $a \neq 0$.

The following four lemmas were proved in [21].

Lemma 5.3. *Let $Z = X_U$, where $U \cap V_1 \neq \varnothing$. Then, after conjugation by an element of F and a choice of a basis in $U = \langle u, v \rangle$, we may assume that*

(A) $u = e_1$, $v = e_2$;
(B) $u = e_1$, $v = e_{-2}$;
(C) $u = e_1$, $v = e_3$;
(D) $u = e_1$, $v = e_2 + e_3$;
(E) $u = e_1$, $v = e_{-2} + e_3$;
(F) $u = e_1$, $v = e_2 + e_3 - a e_{-3} + a e_{-2}$ *for some* $a \neq 0$.

Remark. The case (F) in [21] contains a misprint.

Lemma 5.4. *Let $Z = X_U$ for $U \cap V_1 = \varnothing$, $U \cap V_2 \neq \varnothing$. After conjugation with an element of F and a choice of a basis in $U = \langle u, v \rangle$, we may assume that*

(G) $u = e_3$, $v = e_4$;
(H) $u = e_3$, $v = e_{-4}$;
(I) $u = e_3$, $v = e_1 + e_4$;
(J) $u = e_3$, $v = e_1 + e_{-4}$;
(K) $u = e_3$, $v = e_1 + e_4 - a e_{-4} + a e_{-1}$ *for some* $a \neq 0$.

Lemma 5.5. *Let $Z = X_U$ for $U \cap V_1 = U \cap V_2 = \varnothing$, where U contains a vector with an isotropic projection to V_1. After conjugation with an element of F and a choice of a basis in $U = \langle u, v \rangle$, we may assume that*

(L) $u = e_1 + e_3$, $v = e_{-1} - e_{-3}$;
(M) $u = e_1 + e_3$, $v = e_2 + a e_4$ *for some* $a \neq 0$;
(N) $u = e_1 + e_3$, $v = e_2 + a e_{-4}$ *for some* $a \neq 0$;

(O) $u = e_1 + e_3$, $v = e_{-2} + ae_4$ *for some* $a \neq 0$;
(P) $u = e_1 + e_3$, $v = e_{-2} + ae_{-4}$ *for some* $a \neq 0$;
(Q) $u = e_1 + e_3$, $v = e_2 + be_4 + ce_{-4} - bce_{-2}$ *for some* $b, c \neq 0$.

Lemma 5.6. *Suppose* $\operatorname{char} K \neq 2$. *Let* $Z = X_U$, *where* U *contains no vector with an isotropic projection to* V_1. *After conjugation with an element of* F *and a choice of a basis in* $U = \langle u, v\rangle$, *we may assume that*

(R) $u = e_1 + e_3 - ae_{-3} + ae_{-1}$, $v = e_2 + be_4 + ce_{-4} - bce_{-2}$ *for some* $a, b, c \neq 0$ *such that* $abc \notin K^2$.

Furthermore, in [21] it was noted that in cases (G) and (H), (I) and (J), (M) and (N), (O) and (P) the groups are not only isomorphic, but also are equivalent under the action of an outer automorphism of $\mathrm{SO}(8, K)$, while failing to be conjugate in the group. The group generated by X, Y, and Z was computed in the proof of Theorem 3. Namely, the following isomorphisms hold.

In case (A), H is isomorphic to $\mathrm{SL}(2, K)$.

In case (B), H is isomorphic to $\mathrm{SL}(2, K) \times K$.

In case (C), H is isomorphic to $\mathrm{SL}(2, K) \cdot K^2$.

In case (D), H is isomorphic to $\mathrm{SL}(2, K) \cdot K^2$.

In case (E), H is isomorphic to $(\mathrm{SL}(2, K) \cdot K^2) \times K$.

In case (F), H is isomorphic to $\mathrm{SL}(2, K) \cdot H_1$.

In case (G) $\sim$ (H), H is isomorphic to $\mathrm{SL}(2, K) \times K$.

In case (I) $\sim$ (J), H is isomorphic to $(\mathrm{SL}(2, K) \cdot K^2) \times K$.

In case (K), H is isomorphic to $(\mathrm{SL}(2, K) \cdot K^4) \times K$.

In case (L), H is isomorphic to $\mathrm{SL}(3, K)$.

In case (M) $\sim$ (N), H is isomorphic to $\mathrm{SL}(2, K) \cdot H_1$.

In case (O) $\sim$ (P), H is isomorphic to $(\mathrm{SL}(2, K) \cdot K^4) \times K$.

In case (Q), H is isomorphic to $\mathrm{SL}(2, K) \cdot H_2$.

In case (R), H is isomorphic to $\mathrm{SU}(3, L)$, where L is a quadratic extension of K.

In order to obtain an invariant geometric description of these cases, we need to find an alternative description of them in Lemmas 5.3–5.6. First, we note that in cases (A) and (B) the subspace U is contained in V_1, in cases (C), (D), (E), and (F) their intersection has dimension 1, while in cases (G) $\sim$ (H), (I) $\sim$ (J), (K), (L), (M) $\sim$ (N), (O) $\sim$ (P), (Q), and (R) their intersection is trivial.

Let U' denote a projection of U onto V_1, and let U'' be the subspace spanned by all isotropic vectors of U'. It follows easily that the subspace U' is trivial in case (G) $\sim$ (H), one-dimensional in cases (C), (I) $\sim$ (J), and (K), and two-dimensional in all the remaining

cases. The subspace U'' is trivial in cases (G) $\sim$ (H), (K), and (R), one-dimensional in cases (C), (F), (I) $\sim$ (J), and (Q), and two-dimensional in all the remaining cases. Hence, we have constructed the following table.

	$\dim(U \cap V_1)$	$\dim U'$	$\dim U''$
(A)	2	2	2
(B)	2	2	2
(C)	1	1	1
(D)	1	2	2
(E)	1	2	2
(F)	1	2	1
(G) $\sim$ (H)	0	0	0
(I) $\sim$ (J)	0	1	1
(K)	0	1	0
(L)	0	2	2
(M) $\sim$ (N)	0	2	2
(O) $\sim$ (P)	0	2	2
(Q)	0	2	1
(R)	0	2	0

Therefore, it remains to distinguish (A) from (B), (D) from (E), and the three cases (L), (M) $\sim$ (N), (O) $\sim$ (P); however, in case (L) the subspace U'' contains exactly two isotropic vectors (up to scalar multiplication), while in all other cases U'' is totally isotropic, so (L) can be distinguished from the other cases. In order to deal with the rest, we must look at the relative position of $U' = U''$ in V_1 (in all these cases U' coincides with U''). It is easily shown that the two-dimensional isotropic subspaces of V_1 form two orbits under the action of F: one consists of the subspaces corresponding to the root elements in $\langle X, Y\rangle$, the other consists of all the remaining subspaces. For example, U' lies in the first orbit in cases (A), (D), and (M) $\sim$ (N), and in the second orbit in cases (B), (E), and (O) $\sim$ (P). In order to determine the orbit containing U', consider the dimension of the intersection of U' with $V_X = \mathrm{Im}(X - E)$. If U' lies in the first orbit, this dimension equals 0 or 2. If U' lies in the second orbit, this dimension equals 1. This allows us to reformulate the classification from the paper [21] in the invariant geometric terms. This description can easily be transferred to the case of $G = \mathrm{SO}(2n, K)$.

Remark. Let us explain why the two-dimensional isotropic subspaces of V_1 form exactly two orbits. Let X, Y be two root elements, and V_X, V_Y the corresponding two-dimensional isotropic subspaces. The map $\langle X, Y\rangle \to \langle V_X, V_Y\rangle$ (used implicitly) is well defined but not injective. Namely, for any group $\langle X, Y\rangle$ there exists exactly one more group $\langle X', Y'\rangle$ such that $\langle V_X, V_Y\rangle = \langle V_{X'}, V_{Y'}\rangle$. In particular, for the case of $\langle X_{1,-2}, X_{-2,1}\rangle$ there exists a group $\langle X_{1,2}, X_{2,1}\rangle$ that corresponds to the same four-dimensional subspace V_1. The root elements of the first group correspond to two-dimensional isotropic subspace of type $\langle \alpha e_1 + \beta e_{-2}, \alpha e_2 - \beta e_{-1}\rangle$, whereas the root elements of the second group correspond to $\langle \alpha e_1 + \beta e_2, \alpha e_{-2} - \beta e_{-1}\rangle$.

2. Triples of root subgroups in $G_{\mathrm{sc}}(E_6, K)$. In this subsection $G = G_{\mathrm{sc}}(\mathrm{E}_6, K)$, and V is a 27-dimensional subspace with a G-action (see §1). Let X, Y, Z be three root

subgroups in G such that X is opposite to Y. By Lemma 5.1, there exists a subsystem $\Delta \subset \mathrm{E}_6$ of type δ_4 and an element $u \in G$ such that $\langle X, Y, Z\rangle \leq uG(\Delta, K)u^{-1}$. It is well known that all subgroups $G(\Delta, K)$ are conjugate to each other in G. Therefore, we may assume that $\langle X, Y, Z\rangle \leq G_1 = \langle X_{\alpha_2}, X_{\alpha_3}, X_{\alpha_4}, X_{\alpha_5}\rangle$. Note that G_1 is a spinore group. Under the action of G_1, the 27-dimensional subspace V splits into six invariant subspaces, three one-dimensional ($\langle e^1\rangle$, $\langle e^{11}\rangle$, and $\langle e^{27}\rangle$), and three eight-dimensional, namely,

$$\begin{aligned} V^1 &= \langle e^2, e^3, e^4, e^5, e^7, e^8, e^9, e^{10}\rangle, \\ V^2 &= \langle e^6, e^{12}, e^{13}, e^{14}, e^{16}, e^{17}, e^{19}, e^{22}\rangle, \\ V^3 &= \langle e^{15}, e^{18}, e^{20}, e^{21}, e^{23}, e^{24}, e^{25}, e^{26}\rangle. \end{aligned}$$

The action of G_1 on the one-dimensional subspaces is trivial, and the action on V^1, V^2, and V^3 leads to three well-known eight-dimensional representations of a spinor group: the natural one and two half-spinor ones, related to each other by an outer automorphism. These representations are not faithful; their kernels are isomorphic to $\mathbb{Z}/2\mathbb{Z}$. The quotient of G_1 by each of the kernels is the group $\mathrm{SO}(8, K)$. In order to distinguish its images acting on V^1, V^2, V^3, we shall denote them by $\mathrm{SO}(V^1), \mathrm{SO}(V^2)$, and $\mathrm{SO}(V^3)$ respectively. More information on these representations and on their *triviality* can be found in [82].

It is readily seen that the trilinear form F on V induces a bilinear form f on V^1: $f(x, y) = F(x, y, e^{27})$ for $x, y \in V^1$. Similarly, Q induces a quadratic form q: $q(x) = Q(e^{27}, x)$ for $x \in V^1$. If we choose an appropriate basis of V^1, these forms coincide with the standard ones. Namely, put $e_1 = e^2$, $e_2 = e^3$, $e_3 = e^4$, $e_4 = e^5$, $e_{-4} = -e^7$, $e_{-3} = e^8$, $e_{-2} = -e^9$, and $e_{-1} = e^{10}$, where the e^i are the basis vectors of V, while the e_i form a new basis of V_1. The bases of V^2 and V^3 can be chosen similarly. Thus, singular vectors in V^i with respect to Q are isotropic vectors with respect to q.

Despite the fact that two elements of G_1 correspond to each element of $\mathrm{SO}(V^1)$, we can reconstruct a root element of G_1 by its projection in $\mathrm{SO}(V^1)$. Let E^i be a trivial action on V^i. Then a root element $g \in G_1$ is determined by a six-dimensional singular subspace $U = \mathrm{Im}(g - E)$. At the same time, its restrictions $g^i = g|_{V^i}$, $1 \leq i \leq 3$, are determined by the two-dimensional isotropic subspaces $U^i = \mathrm{Im}(g^i - E^i)$; moreover, $U = U^1 \oplus U^2 \oplus U^3$. In addition, for a given two-dimensional isotropic subspace $U^1 < V^1$ there exists a unique six-dimensional singular subspace U such that $U^1 < U$ and $\dim U \cap V^1 = \dim U \cap V^2 = \dim U \cap V^3 = 2$

This allows us to use the results of [21] described in the previous subsection. Namely, we can reduce the study of triples of root subgroups two of which are opposite to one of 18 cases. The elements $X_{1,-2}$ and $X_{-2,1}$ described above go to X_α and $X_{-\alpha}$, where $\alpha = \begin{smallmatrix} 01210 \\ 1 \end{smallmatrix}$; the four-dimensional subspace V_1 goes to a twelve-dimensional one, which will be denoted by W: $W = \langle e^2, e^3, e^6, e^9, e^{10}, e^{12}, e^{15}, e^{18}, e^{19}, e^{22}, e^{25}, e^{26}\rangle$. Let us list the six-dimensional singular subspace V^Z for each of the 18 cases. In every case the first two basis elements lie in V^1, the second two lie in V^2, and the last two lie in V^3:

(A) $V^Z = \langle e^2, e^3, e^6, e^{12}, e^{15}, e^{18}\rangle$;

(B) $V^Z = \langle e^2, e^9, e^{13}, e^{16}, e^{20}, e^{23}\rangle$;

(C) $V^Z = \langle e^2, e^4, e^6, e^{13}, e^{15}, e^{20}\rangle$;

(D) $V^Z = \langle e^2, e^3 + e^4, e^6, e^{12} + e^{13}, e^{15}, e^{18} + e^{20}\rangle$;

(E) $V^Z = \langle e^2, e^4 - e^9, e^6 + e^{16}, e^{13}, e^{15} + e^{23}, e^{20}\rangle$;

(F) $V^Z = \langle e^2, e^3 + e^4 + ae^8 + ae^9, e^6 - ae^{16}, e^{12} + e^{13}, e^{15} - ae^{23}, e^{18} + e^{20}\rangle$;

(G) $V^Z = \langle e^4, e^5, e^6, e^{19}, e^{20}, e^{21}\rangle$;

(H) $V^Z = \langle e^4, e^7, e^{13}, e^{14}, e^{15}, e^{25}\rangle$;

(I) $V^Z = \langle e^2 + e^5, e^4, e^6, e^{13} - e^{19}, e^{15} + e^{21}, e^{20}\rangle$;

(J) $V^Z = \langle e^2 - e^7, e^4, e^6 - e^{14}, e^{13}, e^{15}, e^{20} + e^{25}\rangle$;

(K) $V^Z = \langle e^2 + e^5 + ae^7 + ae^{10}, e^4, e^6 + ae^{14}, e^{13} - e^{19}, e^{15} + e^{21}, e^{20} - ae^{25}\rangle$;

(L) $V^Z = \langle e^2 + e^4, -e^8 + e^{10}, e^{12} - e^{14}, e^{16} + e^{19}, e^{18} - e^{21}, e^{23} + e^{25}\rangle$;

(M) $V^Z = \langle e^2 + e^4, e^3 + ae^5, e^6, e^{12} - e^{14} + ae^{16} + ae^{19}, e^{15} - ae^{20}, e^{18} - e^{21}\rangle$;

(N) $V^Z = \langle e^2 + e^4, e^3 - ae^7, e^6 + ae^{13}, e^{12} - e^{14}, e^{15}, e^{18} - e^{21} - ae^{23} - ae^{25}\rangle$;

(O) $V^Z = \langle e^2 + e^4, ae^5 - e^9, ae^6 - e^{13}, e^{16} + e^{19}, ae^{18} - ae^{21} + e^{23} + e^{25}, e^{20}\rangle$;

(P) $V^Z = \langle e^2 + e^4, ae^7 + e^9, ae^{12} - ae^{14} - e^{16} - e^{19}, e^{13}, ae^{15} + e^{20}, e^{23} + e^{25}\rangle$;

(Q) $V^Z = \langle e^2 + e^4, e^3 + be^5 - ce^7 + bce^9, e^6 + ce^{13}, e^{12} - e^{14} + be^{16} + be^{19},$
$e^{15} - be^{20}, e^{18} - e^{21} - ce^{23} - ce^{25}\rangle$;

(R) $V^Z = \langle e^2 + e^4 - ae^8 + ae^{10}, e^3 + be^5 - ce^7 + bce^9, e^6 + ce^{13} - ae^{17} + ace^{22},$
$e^{12} - e^{14} + be^{16} + be^{19}, e^{15} - be^{20} - ae^{24} - abe^{26}, e^{18} - e^{21} - ce^{23} - ce^{25}\rangle$.

Besides these cases, related to a representation of G_1 on V^1, we can consider the cases related to representations on V^2 and V^3. Often we get the same cases, but sometimes they are different. Note that the groups $\langle X, Y, Z\rangle$ stay the same; moreover, these cases are indistinguishable from a geometric point of view. This allows us to merge these triples of cases. It follows that we have 10 cases instead of 18: (A), (B) $\sim$ (G) $\sim$ (H), (C), (D), (E) $\sim$ (I) $\sim$ (J), (F) $\sim$ (M) $\sim$ (N), (K) $\sim$ (O) $\sim$ (P), (L), (Q), and (R).

We proceed to finding an invariant geometric description of these cases. As above, first we consider the dimension of the intersection $W \cap V^Z$. In case (A), V^Z is contained in W, in case (B) $\sim$ (G) $\sim$ (H) the subspace $W \cap V^Z$ is two-dimensional, in cases (C) and (D) it is three-dimensional, in cases (E) $\sim$ (I) $\sim$ (J) and (F) $\sim$ (M) $\sim$ (N) it is one-dimensional, and in cases (K) $\sim$ (O) $\sim$ (P), (L), (Q), (R) the intersection of W with V^Z is trivial.

There is no natural projection of V to an arbitrary subspace. However, for the subspace W one can geometrically define a 'standard' complementary subspace (invariant under the choice of a basis). Hence, one can define a projection of an arbitrary subspace onto W. Before proceeding to a definition, we need to establish some simple facts.

Proposition 13. *Let u be a singular vector in V, and let $\rho \in \Lambda$ be a weight. Suppose $u_\rho \neq 0$ and $u_\sigma = 0$ for all weights σ adjacent to ρ; then $u = u_\rho e^\rho$.*

Proof. Suppose $u_\sigma \neq 0$ for some weight σ nonadjacent to ρ. Let τ be a weight such that τ is adjacent to neither ρ nor σ. Since u is singular, we have $Q(e^\tau, u) = 0$. On the other hand, by the definition of Q, it equals $\sum \pm u_\phi u_\psi$, where the sum is over all pairs $\{\phi, \psi\}$ forming a triad with τ. By the definition of u and parts (1) and (2) of Proposition 5, we see that this sum equals $\pm u_\rho u_\sigma$, whence $u_\sigma = 0$, a contradiction. $\square$

Let W_k denote the subspaces $\langle e^\phi; \phi \in I_k\rangle$ for $1 \leq k \leq 3$ (in §2 these subspaces were denoted by V_k, but here we choose another notation in order to avoid confusion). Recall that by D we denote the group $\langle X_\beta; \beta \perp \delta\rangle$.

Lemma 5.7. *All nonzero singular vectors in W_2 form a single orbit under the action of the subgroup D.*

Proof. First, note that all the basis vectors e^i, $i \in I_2$, belong to one orbit, because the weights of I_2 lie in one connected component after we remove the second root from the Dynkin diagram. Moreover, let $i, j \in I_2$, $i - j \in \pm\Pi$. Multiplying by

$$x_{j-i}\Big(-\frac{1}{a}\Big)x_{i-j}(a),$$

we see that the vectors ae^i belong to the same orbit for all $i \in I_2$ and all nonzero $a \in K$.

Let v be an arbitrary singular vector in W_2, and $i \in I_2$ a weight such that $v_i \neq 0$. It suffices to show that we can choose an element $g \in D$ such that gv has nonzero coefficient at the weight i and has zero coefficients at all the weights adjacent to i; the proof then follows from Proposition 13. Let $j \in I_2$ be a weight adjacent to i and such that $v_j \neq 0$. Multiply v by $x_{j-i}(\frac{-c_{ij}v_j}{v_i})$; the coefficient v_j then goes to 0. By Proposition 2 and its corollary, v_i and all v_k for $k \neq j$, $d(k,i) = 1$, stay the same. Repeating this for all weights $j \in I_2$ adjacent to i, we obtain the result. □

Combining the previous lemma with Lemma 2.2, we get the following corollaries.

Corollary 1. (1) *All vectors in W_1 are singular and adjacent to each other.*
(2) *For any singular vector $v \in W_1 \oplus W_2$ not in W_1, the singular vectors in W_1 adjacent to it form a four-dimensional subspace.*
(3) *For any singular vector $v \notin W_1 \oplus W_2$, the singular vectors in W_1 adjacent to it form a one-dimensional subspace.*

Corollary 2. *Let $W = \langle V^X, V^Y\rangle$, where X and Y are two opposite subgroups. Let Ω denote the set of singular vectors $v \in V$ such that the singular vectors in W adjacent to v span an eight-dimensional subspace. Then the set Ω spans a fifteen-dimensional subspace $\overline{W}$, and any singular vector in $\overline{W}$ belongs to Ω. Moreover, any vector in V can be expressed uniquely as a sum of a vector in $\overline{W}$ and a vector in W.*

Remark 1. We can extend the notion of adjacent vectors: u and v are called adjacent vectors if $F(au, bv, z) = 0$ for all $a, b \in K$, $z \in V$. Then $\overline{W}$ consists of all vectors $v \in V$ such that the vectors in W adjacent to v form an eight-dimensional subspace.

Remark 2. Another interesting question is as follows: what is the dimension of the subspace generated by all the singular vectors in W adjacent to a vector $v \in V$. From Lemma 5.7 it follows that this dimension equals 2, 5, 7, or 8. One can give an invariant description of all possible cases by using this language, because the set of the corresponding dimensions is different in different cases. But the author feels that the description below is somewhat simpler.

Corollary 2 allows one to talk about the projection of arbitrary vectors to W (along $\overline{W}$). Let W' be the projection of the subspace V^Z to W; we compute its dimension in each of the 10 cases. In cases (A), (D), (F) $\sim$ (M) $\sim$ (N), (L), (Q), and (R) it equals 6, in case (B) $\sim$ (G) $\sim$ (H) it equals 2, in case (C) it equals 3, in the case (E) $\sim$ (I) $\sim$ (J) it equals 4, and in case (K) $\sim$ (O) $\sim$ (P) it equals 5. Finally, as in the first subsection, we look at the dimension of the subspace W'' generated by all singular vectors in W'. In cases (A), (D), (L) it equals 6, in case (B) $\sim$ (G) $\sim$ (H) it equals 2, in cases (C) and (Q) it equals 3, in cases (E) $\sim$ (I) $\sim$ (J) and (K) $\sim$ (O) $\sim$ (P) it equals 4, in case (F) $\sim$ (M) $\sim$ (N) it equals 5, and in case (R) it equals 0. All these results are presented in the

following table:

	$\dim(V^Z \cap W)$	$\dim W'$	$\dim W''$	$\langle X, Y, Z\rangle$
(A)	6	6	6	$\mathrm{SL}(2, K)$
(B) $\sim$ (G) $\sim$ (H)	2	2	2	$\mathrm{SL}(2, K) \times K$
(C)	3	3	3	$\mathrm{SL}(2, K) \cdot K^2$
(D)	3	6	6	$\mathrm{SL}(2, K) \cdot K^2$
(E) $\sim$ (I) $\sim$ (J)	1	4	4	$(\mathrm{SL}(2, K) \cdot K^2) \times K$
(F) $\sim$ (M) $\sim$ (N)	1	6	5	$\mathrm{SL}(2, K) \cdot H_1$
(K) $\sim$ (O) $\sim$ (P)	0	5	4	$(\mathrm{SL}(2, K) \cdot K^4) \times K$
(L)	0	6	6	$\mathrm{SL}(3, K)$
(Q)	0	6	3	$\mathrm{SL}(2, K) \cdot H_2$
(R)	0	6	0	$\mathrm{SU}(3, L), [L : K] = 2$

Remark. It is easily seen that (unlike in the first subsection) the subspace $W = \langle V^X, V^Y\rangle$ uniquely determines the subgroup $\langle X, Y\rangle$.

Comparing the results of the first and the second subsections, we see that from a geometric point of view the classification of the subgroups in $G_{\mathrm{sc}}(\mathrm{E}_6, K)$ generated by three root subgroups two of which are opposite is simpler than that of the subgroups in $\mathrm{SO}(8, K)$; it has a lesser number of cases, and the cases can be distinguished easier. On the other hand, explicit calculations are easier for $\mathrm{SO}(8, K)$.

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