# ON $\delta$-SUPERDERIVATIONS OF SIMPLE SUPERALGEBRAS OF JORDAN BRACKETS 

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#### Abstract

A complete description of $\delta$-derivations and $\delta$-superderivations is given for simple unital superalgebras of Jordan brackets over a field of characteristic different from 2 and for simple unital finite-dimensional Jordan superalgebras over an algebraically closed field of characteristic $p \neq 2$. As a consequence, a criterion for simple unital superalgebras of Jordan brackets to be special is obtained.


## Introduction

The notion of a derivation of an algebra has been generalized by many mathematicians in various directions. In particular, in [1] one can find the definition of a $\delta$-derivation of an algebra. We recall that for fixed $\delta \in F$, a $\delta$-derivation of an $F$-algebra $A$ is a linear map $\phi$ satisfying the condition

$$
\phi(x y)=\delta(\phi(x) y+x \phi(y))
$$

for any $x, y \in A$. In 1 , the $\frac{1}{2}$-derivations were described for an arbitrary prime Lie $F$-algebra $A\left(\frac{1}{6} \in F\right)$ with a nondegenerate symmetric invariant bilinear form. Namely, it was proved that a linear map $\phi: A \rightarrow A$ is a $\frac{1}{2}$-derivation if and only if $\phi \in \Gamma(A)$, where $\Gamma(A)$ is the centroid of $A$. This implies that if $A$ is a central simple Lie algebra over a field of characteristic $p \neq 2,3$ with a nondegenerate invariant bilinear form, then any $\frac{1}{2}$-derivation $\phi$ has the form $\phi(x)=\alpha x$ for some $\alpha \in F$.

In [2] Filippov proved that any prime Lie $\Phi$-algebra admits no nonzero $\delta$-variations if $\delta \neq-1,0, \frac{1}{2}, 1$. Also in [2], it was shown that any prime Lie $\Phi$-algebra $A\left(\frac{1}{6} \in \Phi\right)$ with a nonzero antiderivation is a three-dimensional central simple algebra over the field of fractions of the center $Z_{R}(A)$ for its algebra of right multiplications $R(A)$. In the same paper, a nontrivial $\frac{1}{2}$-derivation was constructed for the Witt algebra $W_{1}$, i.e., a $\frac{1}{2}$-derivation that is not an element of the centroid of $W_{1}$.

The paper 3] contains a description for the $\delta$-derivations of the prime alternative and non-Lie Maltsev $\Phi$-algebras with some restrictions on the operator ring $\Phi$. It was shown that the algebras in these classes have no nonzero $\delta$-derivations if $\delta \neq 0, \frac{1}{2}, 1$. The results of Filippov were partially generalized by Luks and Leger in 9. Those authors considered quasiderivations of Lie algebras, i.e., linear maps $f$ for which there exists a linear map $f^{\prime}$ related to $f$ by the formula $f^{\prime}(x y)=f(x) y+x f(y)$. They proved that

[^0]the space of quasiderivations of a simple finite-dimensional Lie algebra $A$ of rank greater than 1 coincides with the direct sum of the space of derivations and the centroid of $A$.

In 4], a description of $\delta$-derivations was given for the simple finite-dimensional Jordan superalgebras over an algebraically closed field of characteristic zero and for the semisimple finite-dimensional Jordan algebras over an algebraically closed field of characteristic different from 2. Later, in [5], the $\delta$-derivations were described for the classical Lie superalgebras. The paper [6] was devoted to a description of the $\delta$-derivations of Cartan Lie superalgebras. In that paper, the $\delta$-superderivations were described for the simple finite-dimensional Lie superalgebras. Also in [6], the $\delta$-derivations of semisimple finite-dimensional Jordan algebras and $\delta$-superderivations of Jordan superalgebras over an algebraically closed field of characteristic zero were described. For algebras and superalgebras from the papers [4]-[6], the absence of nontrivial $\delta$-derivations and $\delta$-superderivations was proved. Later, the results of [5] were generalized by Zusmanovich in [8]. He described the $\delta$-derivations and $\delta$-superderivations of prime Lie superalgebras. He showed that a prime Lie superalgebra has no nontrivial $\delta$-derivations and $\delta$-superderivations if $\delta \neq-1,0, \frac{1}{2}, 1$. He proved that for a Lie superalgebra $A$ with zero center and a nondegenerate supersymmetric invariant bilinear form satisfying the condition $A=[A, A]$, the space of $\frac{1}{2}$-derivations ( $\frac{1}{2}$-superderivations) coincides with the centroid (supercentroid) of the superalgebra $A$. Also, Zusmanovich gave an affirmative answer to the question of Filippov in [2] concerning the existence of zero divisors in the ring of $\frac{1}{2}$-derivations of a prime Lie algebra. Subsequently, the $\delta$-superderivations of the generalized Kantor double constructed on a prime nonunital associative algebra were considered in [7.

In the present paper, we consider the $\delta$-derivations and $\delta$-superderivations in the case of simple superalgebras of Jordan brackets. We prove the absence of nontrivial $\delta$-derivations and $\delta$-superderivations of simple superalgebras of Jordan brackets, which are not superalgebras of vector type. A description is given for the $\delta$-derivations and $\delta$-superderivations of simple Jordan superalgebras of vector type. As a consequence, with the help of the classification of all simple finite-dimensional Jordan superalgebras, as given in [13], we obtain a description of the $\delta$-derivations and $\delta$-superderivations of the simple unital finite-dimensional Jordan superalgebras over an algebraically closed field of characteristic $p \neq 2$.

## §1. The main facts and definitions

Let $F$ be a field of characteristic $p \neq 2$. An algebra $A$ over the field $F$ is said to be Jordan if it satisfies the identities

$$
x y=y x, \quad\left(x^{2} y\right) x=x^{2}(y x)
$$

Let $G$ be a Grassmann algebra over $F$, given by generators $1, \xi_{1}, \ldots, \xi_{n}, \ldots$ and the defining relations $\xi_{i}^{2}=0, \xi_{i} \xi_{j}=-\xi_{j} \xi_{i}$. The unity 1 and the products $\xi_{i_{1}} \xi_{i_{2}} \cdots \xi_{i_{k}}$, $i_{1}<i_{2}<\cdots<i_{k}$, form a basis of the algebra $G$ over $F$. Denote by $G_{0}$ and $G_{1}$ the subspaces generated by products of even and odd length, respectively; then $G$ is a direct sum of these subspaces: $G=G_{0} \oplus G_{1}$; moreover, we have the inclusions $G_{i} G_{j} \subseteq G_{i+j}$ $(\bmod 2), i, j=0,1$. In other words, $G$ is a $\mathbb{Z}_{2}$-graded algebra (or a superalgebra) over $F$.

Now, let $A=A_{0} \oplus A_{1}$ be an arbitrary superalgebra over $F$. Consider the tensor product $G \otimes A$ of $F$-algebras. Its subalgebra

$$
G(A)=G_{0} \otimes A_{0}+G_{1} \otimes A_{1}
$$

is called the Grassmann envelope of the superalgebra $A$.
Let $\Omega$ be a variety of algebras over $F$. A superalgebra $A=A_{0} \oplus A_{1}$ is called an $\Omega$-superalgebra if its Grassmann envelope $G(A)$ is an algebra in $\Omega$.

In particular, $J=J_{0} \oplus J_{1}$ is a Jordan superalgebra if its Grassmann envelope $G(J)$ is a Jordan algebra. In what follows, for a homogeneous element $x$ of a superalgebra $J=J_{0} \oplus J_{1}$ we put $p(x)=i$ if $x \in J_{i}$. We denote the even part $J_{0}$ of a Jordan superalgebra by $A$ and the odd part $J_{1}$ by $M$.

A classification of the simple finite-dimensional Jordan superalgebras over an algebraically closed field of characteristic zero was given in [10, 11. In [12, 13, all simple finite-dimensional Jordan superalgebras over an algebraically closed field of arbitrary characteristic different from 2 were described.

We give several examples of Jordan superalgebras.
1.1. The Kantor double [10. Let $\Gamma=\Gamma_{0} \oplus \Gamma_{1}$ be an associative supercommutative superalgebra with unity 1 , and let $\{\}:, \Gamma \times \Gamma \rightarrow \Gamma$ be a superskewsymmetric bilinear map, which we shall call a bracket. Starting with the superalgebra $\Gamma$ and the bracket $\{$,$\} , one can construct a superalgebra J(\Gamma,\{\}$,$) . Consider the following direct sum of$ spaces: $J(\Gamma,\{\})=,\Gamma \oplus \Gamma x$, where $\Gamma x$ is an isomorphic copy of the space $\Gamma$. Let $a$ and $b$ be homogeneous elements in $\Gamma$. Then multiplication - on $J(\Gamma,\{\}$,$) is defined by the$ relations

$$
a \cdot b=a b, \quad a \cdot b x=(a b) x, \quad a x \cdot b=(-1)^{p(b)}(a b) x, \quad a x \cdot b x=(-1)^{p(b)}\{a, b\} .
$$

We set $A=\Gamma_{0} \oplus \Gamma_{1} x$ and $M=\Gamma_{1} \oplus \Gamma_{0} x$. Then $J(\Gamma,\{\})=,A \oplus M$ is a $\mathbb{Z}_{2}$-graded algebra.

A bracket $\{$,$\} is said to be Jordan if the superalgebra J(\Gamma,\{\}$,$) is a Jordan superal-$ gebra. It is well known [14] that, for homogeneous elements, the Jordan bracket satisfies the relations

$$
\begin{align*}
\{a, b c\}= & \{a, b\} c+(-1)^{p(a) p(b)} b\{a, c\}-\{a, 1\} b c,  \tag{1}\\
\{a,\{b, c\}\}= & \{\{a, b\}, c\}+(-1)^{p(a) p(b)}\{b,\{a, c\}\}+\{a, 1\}\{b, c\}  \tag{2}\\
& +(-1)^{p(a)(p(b)+p(c))}\{b, 1\}\{c, a\}+(-1)^{p(c)(p(a)+p(b))}\{c, 1\}\{a, b\} .
\end{align*}
$$

Since the superalgebra $J(\Gamma,\{\}$,$) is Jordan, we see that D: a \rightarrow\{a, 1\}$ is a derivation of the superalgebra $\Gamma$.

If $D$ is the zero derivation, then $\{$,$\} is a Poisson bracket, i.e.,$

$$
\{a, b c\}=\{a, b\} c+(-1)^{p(a) p(b)} b\{a, c\},
$$

and $\Gamma$ is a Lie superalgebra with respect to the operation $\{$,$\} . An arbitrary Poisson$ bracket is a Jordan bracket (see [15]).

It is well known [14 that the Jordan superalgebra $J=\Gamma \oplus \Gamma x$ obtained with the help of the Kantor doubling process is simple if and only if $\Gamma$ has no nonzero ideals $B$ satisfying $\{\Gamma, B\} \subseteq B$.
1.2. A superalgebra of vector type $J(\Gamma, D)$. Let $\Gamma=\Gamma_{0} \oplus \Gamma_{1}$ be an associative supercommutative superalgebra with a nonzero even derivation $D$. We define a bracket $\{$,$\} on \Gamma$ by setting

$$
\{a, b\}=D(a) b-a D(b) .
$$

Then the bracket $\{$,$\} is Jordan. We denote by J(\Gamma,\{\}$,$) the resulting superalgebra$ $J(\Gamma, D)$. Multiplication "." in $J(\Gamma, D)$ is defined by

$$
\begin{aligned}
a \cdot b=a b, \quad a \cdot b x & =(a b) x, \quad a x \cdot b=(-1)^{p(b)}(a b) x, \\
a x \cdot b x & =(-1)^{p(b)}(D(a) b-a D(b)),
\end{aligned}
$$

where $a$ and $b$ are homogeneous elements in $\Gamma$ and $a b$ is the product in $\Gamma$. The superalgebra $J(\Gamma, D)$ is called a superalgebra of vector type. If the superalgebra $J(\Gamma, D)$ is simple, then $\Gamma_{1}=0$ (see [14]).
1.3. The Cheng-Kac superalgebra $C K(Z, d)$ [16]. Let $Z$ be an arbitrary unital associative commutative algebra with a nonzero derivation $d: Z \rightarrow Z$. Consider two free $Z$-modules of rank 4:

$$
A=Z+\sum_{i=1}^{3} w_{i} Z, \quad M=x Z+\sum_{i=1}^{3} x_{i} Z
$$

Multiplication on $A$ will be $Z$-linear, $w_{i} w_{j}=0, i \neq j, w_{i}^{2}=-1$. We define

$$
x_{i \times i}=0, \quad x_{1 \times 2}=-x_{2 \times 1}=x_{3}, \quad x_{1 \times 3}=-x_{3 \times 1}=x_{2}, \quad x_{2 \times 3}=-x_{3 \times 2}=x_{1} .
$$

Multiplication $A \times M \rightarrow M$ is defined by the relations

$$
\begin{aligned}
(x f) g & =x(f g), & & \left(x_{i} f\right) g=x_{i}(f g), \\
(x f)\left(w_{j} g\right) & =x_{j}(f d(g)), & & \left(x_{i} f\right)\left(w_{j} g\right)=x_{i \times j}(f g) .
\end{aligned}
$$

Multiplication $M \times M \rightarrow A$ is given in accordance with the rules

$$
\begin{aligned}
(x f)(x g) & =d(f) g-f d(g), & & (x f)\left(x_{j} g\right)=-w_{j}(f g), \\
\left(x_{i} f\right)(x g) & =w_{i}(f g), & & \left(x_{i} f\right)\left(x_{j} g\right)=0 .
\end{aligned}
$$

We also need the definition of a certain superalgebra $B(n, m)$. Let $F$ be an algebraically closed field of characteristic $p>2$. Let $B(m)=F\left[a_{1}, \ldots, a_{m} \mid a_{i}^{p}=0\right]$ be the algebra of truncated polynomials in $m$ even variables. Let $G(n)$ be the Grassmann superalgebra with generators $1, \xi_{1}, \ldots, \xi_{n}$. Then $B(m, n)=B(m) \otimes G(n)$ is an associative supercommutative superalgebra.

The main result concerning classification of simple finite-dimensional unital Jordan superalgebras over algebraically closed fields of characteristic $p>2$ was obtained in the paper [13] by Martinez and Zelmanov.

Theorem 1. Let $J=J_{0}+J_{1}$ be a finite-dimensional simple unital Jordan superalgebra over an algebraically closed field of characteristic $p>2$, where $J_{0}$ is not a semisimple algebra. Then:

1) either there exist integers $m$ and $n$ and a Jordan bracket $\{$,$\} on B(m, n)$ such that $J=J(B(m, n),\{\}$,$) ,$
2) or $J$ is isomorphic to the Cheng-Kac Jordan superalgebra $C K(B(m), d)$, which is defined by the derivation $d: B(m) \rightarrow B(m)$.

As has been mentioned above, for a fixed element $\delta$ of the ground field, by a $\delta$-derivation of a superalgebra $A$ we mean a linear map $\phi: A \rightarrow A$ such that

$$
\phi(x y)=\delta(\phi(x) y+x \phi(y))
$$

for all $x, y \in A$.
The centroid $\Gamma(A)$ of a superalgebra $A$ is the set of all linear maps $\chi: A \rightarrow A$ such that

$$
\chi(x y)=\chi(x) y=x \chi(y)
$$

for all $x, y \in A$.
Note that the 1-derivation is the usual derivation and the 0-derivation is an arbitrary endomorphism $\phi$ of the algebra $A$ such that $\phi\left(A^{2}\right)=0$. It is clear that any element of the centroid of an algebra is a $\frac{1}{2}$-derivation.

A nonzero $\delta$-derivation $\phi$ is said to be nontrivial if $\delta \neq 0,1$ and $\phi \notin \Gamma(A)$.
By a superspace we mean a $\mathbb{Z}_{2}$-graded space. A homogeneous element $\psi$ of the superspace of endomorphisms $A \rightarrow A$ is called a superderivation if

$$
\psi(x y)=\psi(x) y+(-1)^{p(x) p(\psi)} x \psi(y) .
$$

For a fixed element $\delta \in F$, we define the notion of a $\delta$-superderivation of the superalgebra $A=A_{0}+A_{1}$. A homogeneous linear map $\phi: A \rightarrow A$ is called a $\delta$-superderivation if for homogeneous $x, y \in A$ we have

$$
\phi(x y)=\delta\left(\phi(x) y+(-1)^{p(x) p(\phi)} x \phi(y)\right) .
$$

Consider a Lie superalgebra $A=A_{0}+A_{1}$ and fix an element $x \in A_{i}$. Then $R_{x}: y \rightarrow$ $x y$ is an odd superderivation of the superalgebra $A$ and its parity $p\left(R_{x}\right)$ is equal to $i$.

The supercentroid $\Gamma_{s}(A)$ of a superalgebra $A$ is the set of all homogeneous linear maps $\chi: A \rightarrow A$ such that for arbitrary homogeneous elements $a$ and $b$, we have

$$
\chi(a b)=\chi(a) b=(-1)^{p(a) p(\chi)} a \chi(b)
$$

Note that a 1 -superderivation is a usual superderivation; a 0 -superderivation is an arbitrary endomorphism $\phi$ of the superalgebra $A$ such that $\phi\left(A^{2}\right)=0$.

A nonzero $\delta$-superderivation $\phi$ is said to be nontrivial if $\delta \neq 0,1$ and $\phi \notin \Gamma_{s}(A)$.
In accordance with [4, Theorem 2.1] (which is easily generalized to the case of $\delta$-superderivations), for a unital superalgebra $A$, a map $\phi$ can be a nontrivial $\delta$-derivation or $\delta$-superderivation only for $\delta=\frac{1}{2}$. It is easily seen that, in this case, $\phi(x)=\phi(1) x$ for arbitrary $x \in A$.

## §2. $\delta$-DERIVATIONS AND $\delta$-SUPERDERIVATIONS of simple superalgebras of Jordan brackets

In the present section, we consider $\delta$-derivations and $\delta$-superderivations of a simple unital Jordan superalgebra $J=J(\Gamma,\{\}$,$) . We assume that the characteristic of the field$ $F$ is different from 2.

Lemma 2. Let $J=J(\Gamma,\{\}$,$) be a simple unital Jordan superalgebra. Then \Gamma=\Gamma\{\Gamma, \Gamma\}$. In particular, if $z \in \Gamma_{0} \cup \Gamma_{1} \backslash\{0\}$, then $z\{\Gamma, \Gamma\} \neq 0$.

Proof. Consider $I=\Gamma\{\Gamma, \Gamma\}$. It is clear that $I$ is an ideal in $\Gamma(I \triangleleft \Gamma)$. By (11),

$$
\{\Gamma, I\}=\{\Gamma, \Gamma\{\Gamma, \Gamma\}\} \subseteq\{\{\Gamma, \Gamma\},\{\Gamma, \Gamma\}\}+\Gamma\{\Gamma,\{\Gamma, \Gamma\}\}+\{\Gamma, 1\} \Gamma\{\Gamma, \Gamma\} \subseteq \Gamma\{\Gamma, \Gamma\}=I .
$$

By [14, the Jordan superalgebra $J(\Gamma,\{\}$,$) is simple if \Gamma$ contains no nonzero ideals $I$ satisfying the condition $\{\Gamma, I\} \subseteq I$. If $\{\Gamma, \Gamma\}=0$, then $\Gamma x \triangleleft J$. Consequently, $\Gamma x=J$ and $\Gamma_{0}=0$. Hence, $\{\Gamma, \Gamma\} \neq 0$, and since $\Gamma$ is unital, we have $\Gamma=\Gamma\{\Gamma, \Gamma\}$. Now if $z\{\Gamma, \Gamma\}=0$, then $z \Gamma=z \Gamma\{\Gamma, \Gamma\}=0$. Since $\Gamma$ is unital, we obtain $z=0$. The lemma is proved.

Lemma 3. Let $J=J(\Gamma,\{\}$,$) be a simple unital Jordan superalgebra, and let \alpha \in J$. The map $\phi(z)=\alpha z$ is a $\frac{1}{2}$-derivation if and only if $\alpha \in \Gamma_{0}$ and $\{\alpha, b\}=D(\alpha) b-\alpha D(b)$ for any $b \in \Gamma$.

Proof. Let $\alpha=\alpha_{0}+\beta x+\gamma+\mu x$, where $\alpha_{0}, \mu \in \Gamma_{0}$ and $\beta, \gamma \in \Gamma_{1}$. Clearly, the maps $\phi_{1}(z)=(\gamma+\mu x) z$ and $\phi_{2}(z)=\left(\alpha_{0}+\beta x\right) z$ are also $\frac{1}{2}$-derivations of the superalgebra $J$. Moreover, $\phi_{1}(1)=(\gamma+\mu x)$ and $\phi_{2}(1)=\left(\alpha_{0}+\beta x\right)$. For this reason, for arbitrary $z$, $w \in J$ we have

$$
\begin{equation*}
2 \phi_{i}(1)(z w)=\left(\phi_{i}(1) z\right) w+z\left(\phi_{i}(1) w\right) . \tag{3}
\end{equation*}
$$

Setting $i=1, z=x$, and $w=1$ in (3), we get

$$
2 \gamma x=\gamma x+x \gamma=0
$$

i.e., $\gamma=0$.

We prove that $\beta=\mu=0$. For this, we show that $\beta\{\Gamma, \Gamma\}=0$ and $\mu\{\Gamma, \Gamma\}=0$.

Now in (3) we set $i=2, z=a$, and $w=b x$, obtaining

$$
2\{\beta, a b\}=\{\beta a, b\}+(-1)^{p(a)} a\{\beta, b\} .
$$

By (1), we have

$$
\begin{align*}
2\{\beta, a b\}=-(-1)^{p(b)+p(a) p(b)} 2\{b, \beta\} a & -(-1)^{p(a) p(b)} \beta\{b, a\} \\
& +(-1)^{p(b)+p(b) p(a)} D(b) \beta a . \tag{4}
\end{align*}
$$

Substituting $i=2, z=a x$, and $w=b$ in (3), we get

$$
2\{\beta, a b\}=\{\beta, a\} b-(-1)^{p(a)}\{a, \beta b\} .
$$

By (1), we have

$$
\begin{equation*}
2\{\beta, a b\}=2\{\beta, a\} b-\beta\{a, b\}-\beta D(a) b . \tag{5}
\end{equation*}
$$

Substituting $i=2, z=a x$, and $w=b x$ in (3), we get

$$
\begin{equation*}
2 \beta\{a, b\}=\{\beta, a\} b-(-1)^{p(a)} a\{\beta, b\} . \tag{6}
\end{equation*}
$$

Comparison of (4) and (5) yields

$$
\begin{align*}
2 \beta\{a, b\} & =2\{\beta, a\} b-\beta D(a) b-(-1)^{p(b)+p(b) p(a)} D(b) \beta a+(-1)^{p(b)+p(b) p(a)} 2\{b, \beta\} a \\
& =2\{\beta, a\} b-\beta D(a) b-\beta a D(b)-(-1)^{p(a)} 2 a\{\beta, b\}  \tag{7}\\
& =2\{\beta, a\} b-\beta D(a b)-(-1)^{p(a)} 2 a\{\beta, b\} .
\end{align*}
$$

This relation and (6) imply that

$$
2 \beta\{a, b\}=\beta D(a b)
$$

Putting $b=1$, we get $\beta D(a)=2 \beta D(a)$. Hence, $\beta D(a)=0$ and $\beta\{\Gamma, \Gamma\}=0$. Thus, $\beta=0$ by Lemma 2 .

Substituting $i=1, z=a, w=b x$ in (3), we obtain

$$
\begin{equation*}
2\{\mu, a b\}=-\{\mu a, b\}+(-1)^{p(a)} a\{\mu, b\} . \tag{8}
\end{equation*}
$$

Choosing $i=1, z=a x, w=b$ in (3), we get

$$
\begin{equation*}
2\{\mu, a b\}=\{\mu, a\} b+(-1)^{p(a)}\{a, \mu b\} . \tag{9}
\end{equation*}
$$

For $i=1, z=a x, w=b x$ in (3), we have

$$
\begin{equation*}
2 \mu\{a, b\}=\{\mu, a\} b+(-1)^{p(a)} a\{\mu, b\} . \tag{10}
\end{equation*}
$$

For $a=b=1$, relation (10) implies that $D(\mu)=0$. Substituting $b=1$ in (8), we obtain

$$
2\{\mu, a\}=\{\mu a, 1\}=D(\mu a)=\mu D(a) .
$$

Taking $a=1$ in (9), we have

$$
2\{\mu, b\}=\{1, \mu b\}=-D(\mu b)=-\mu D(b)
$$

Comparing the expressions obtained, we get $\{\mu, \Gamma\}=0$. Therefore, $\mu\{a, b\}=0$ by (10).
Using Lemma 2, we see that $\mu=0$.
Thus, we have proved that $\phi(z)=\alpha z$, where $\alpha \in \Gamma_{0}$.
Putting $i=2, z=a x$, and $w=b x$ in (3), we obtain

$$
\begin{equation*}
2 \alpha\{a, b\}=\{\alpha a, b\}+\{a, \alpha b\} . \tag{11}
\end{equation*}
$$

Note that identity (1) yields the relation $\{\alpha a, b\}+\{a, \alpha b\}=-(-1)^{p(b) p(a)}(\{b, \alpha\} a+\alpha\{b, a\}-D(b) \alpha a)+\{a, \alpha\} b+\alpha\{a, b\}-D(a) \alpha b$.
Hence, (11) shows that

$$
a\{b, \alpha\}-\{a, \alpha\} b=(a D(b)-D(a) b) \alpha .
$$

Consequently, for $b=1$,

$$
\{\alpha, a\}=D(\alpha) a-\alpha D(a) .
$$

It is easy to verify that, for any $\alpha \in \Gamma_{0}$ such that $\{\alpha, a\}=D(\alpha) a-\alpha D(a), \phi(z)=\alpha z$ is a $\frac{1}{2}$-derivation of the superalgebra $J$. The lemma is proved.

Thus, a $\delta$-derivation of a simple unital superalgebra $J=J(\Gamma,\{\}$,$) is an even \delta$-superderivation of the superalgebra $J=J(\Gamma,\{\}$,$) . For this reason, in the sequel we deal only$ with $\delta$-superderivations.

Remark 4. Let $J=J(\Gamma,\{\}$,$) be a simple unital Jordan superalgebra. The map \phi(z)=$ $\alpha z$ is an odd $\frac{1}{2}$-superderivation if and only if $\alpha \in \Gamma_{1}$ and

$$
\{\alpha, a\}=D(\alpha) a-\alpha D(a)
$$

for any $a \in \Gamma$.
Proof. This is proved by straightforward calculations similar to those in the proof of Lemma 3.

Corollary 5. If $J$ is a simple unital superalgebra of vector type, then the map $\phi(z)=\alpha z$ is a $\frac{1}{2}$-superderivation if and only if $\alpha \in \Gamma_{0}$. If $J$ is a superalgebra of the Poisson bracket, then the map $\phi(z)=\alpha z$ is a $\frac{1}{2}$-superderivation if and only if $\alpha \in \Gamma_{0} \cup \Gamma_{1}$ and $\{\alpha, \Gamma\}=0$.

Let $J=J(\Gamma, D)$ be a superalgebra of vector type. The map $\phi(z)=\alpha z$ with $\alpha \in \Gamma$ is the trivial $\frac{1}{2}$-superderivation if $\phi \in \Gamma_{s}(J)$, i.e., if

$$
\alpha((b x)(c x))=(-1)^{p(\alpha) p(b x)}(b x)(\alpha(c x))
$$

which is equivalent to $D(\alpha) b c=0$. Consequently, $\phi$ is the trivial $\frac{1}{2}$-superderivation if $D(\alpha) \neq 0$.

Let $J=J(\Gamma,\{\}$,$) be a superalgebra of the Poisson bracket, and let \phi(z)=\alpha z$ be a $\frac{1}{2}$-superderivation of $J$. By Remark 4, we have

$$
\begin{aligned}
(a x)(\alpha(b x)) & =(-1)^{p(b)+p(\alpha)}\{a, \alpha b\}=(-1)^{p(b)+p(\alpha)}\left(\{a, \alpha\} b+(-1)^{p(\alpha) p(a)} \alpha\{a, b\}\right) \\
& =(-1)^{p(b)+p(\alpha)+p(a) p(\alpha)} \alpha\{a, b\}=(-1)^{p(\alpha)(p(a)+1)} \alpha((a x)(b x)) .
\end{aligned}
$$

This easily implies that $\phi$ is the trivial $\frac{1}{2}$-superderivation.
We set

$$
\Phi=\left\{\alpha \in \Gamma_{0} \cup \Gamma_{1} \mid\{\alpha, a\}=D(\alpha) a-\alpha D(a), a \in \Gamma\right\}
$$

Lemma 6. Suppose $J=J(\Gamma,\{\}$,$) is a simple Jordan superalgebra, \alpha \in \Phi \backslash\{0\}$, and $D(\alpha)=0$. Then $\alpha$ is invertible in $\Gamma$ and $\alpha \in \Gamma_{0}$. In particular, if $J(\Gamma,\{\}$,$) is a$ superalgebra of the Poisson bracket, then $\alpha$ is invertible in $\Gamma$ and $\alpha \in \Gamma_{0}$.

Proof. Consider $I=\alpha \Gamma$. It is clear that $I \triangleleft \Gamma$. By the definition of $\Phi$, we have

$$
\{I, \Gamma\}=\{\alpha \Gamma, \Gamma\} \subseteq\{\Gamma, \alpha\} \Gamma+\alpha\{\Gamma, \Gamma\}+D(\Gamma) \alpha \Gamma \subseteq \alpha \Gamma=I
$$

Using [14, we conclude that $I=\Gamma$. Since $\Gamma$ is unital, it follows that $\alpha$ is invertible. The lemma is proved.

Lemma 7. Let $J=J(\Gamma,\{\}$,$) be a Jordan superalgebra. Then \Phi$ is closed with respect to the derivation $D$, i.e., $D(\Phi) \subseteq \Phi$. In particular, $D^{k}(\Phi) \subseteq \Phi, k>0$.

## Proof. Using (2), we obtain

$$
\begin{aligned}
&\{D(b), c\}+\{b, D(c)\}=-(-1)^{p(b) p(c)}\{c,\{b, 1\}\}+\{b,\{c, 1\}\} \\
&=-(-1)^{p(b) p(c)}\{\{c, b\}, 1\}-(-1)^{p(b) p(c)}(-1)^{p(c) p(b)}\{b,\{c, 1\}\} \\
&-(-1)^{p(b) p(c)}\{c, 1\}\{b, 1\}-(-1)^{p(b) p(c)}(-1)^{p(c) p(b)}\{b, 1\}\{1, c\}+\{\{b, c\}, 1\} \\
&+(-1)^{p(b) p(c)}\{c,\{b, 1\}\}+\{b, 1\}\{c, 1\}+(-1)^{p(b) p(c)}\{c, 1\}\{1, b\} \\
&= 2 D(\{b, c\})-\{D(b), c\}-\{b, D(c)\} .
\end{aligned}
$$

Consequently,

$$
\begin{equation*}
D(\{b, c\})=\{D(b), c\}+\{b, D(c)\} . \tag{12}
\end{equation*}
$$

Therefore,

$$
\begin{aligned}
\{D(\alpha), a\} & =D(\{\alpha, a\})-\{\alpha, D(a)\} \\
& =D(D(\alpha)) a+D(\alpha) D(a)-D(\alpha) D(a)-\alpha D(D(a))-D(\alpha) D(a)+\alpha D(D(a)) \\
& =D(D(\alpha)) a-D(\alpha) D(a)
\end{aligned}
$$

This completes the proof of the lemma.
Lemma 8. Let $J=J(\Gamma,\{\}$,$) be a Jordan superalgebra. Then for arbitrary b, c \in J$ and $\alpha \in \Phi$ we have

$$
D^{k}(\alpha)\{b, c\}=D^{k}(\alpha)(D(b) c-b D(c))
$$

Proof. From the definition of $\Phi$ and (12), it follows that

$$
\begin{equation*}
\{\alpha,\{b, c\}\}=D(\alpha)\{b, c\}-\alpha D(\{b, c\})=D(\alpha)\{b, c\}-\alpha\{D(b), c\}-\alpha\{b, D(c)\} \tag{13}
\end{equation*}
$$

Now we use relations (11), (2) and Lemmas 3 and 7 to write

$$
\begin{aligned}
&\{\alpha,\{b, c\}\}=\{\{\alpha, b\}, c\}+(-1)^{p(\alpha) p(b)}\{b,\{\alpha, c\}\}+D(\alpha)\{b, c\} \\
&+(-1)^{p(\alpha)(p(b)+p(c))} D(b)\{c, \alpha\}+(-1)^{p(c)(p(\alpha)+p(b))} D(c)\{\alpha, b\} \\
&=\{ D(\alpha) b, c\}-\{\alpha D(b), c\}+(-1)^{p(\alpha) p(b)}\{b, D(\alpha) c\}-(-1)^{p(\alpha) p(b)}\{b, \alpha D(c)\} \\
&+D(\alpha)^{\{b, c\}-(-1)^{p(\alpha) p(b)} D(b)(D(\alpha) c-\alpha D(c))} \\
&+(-1)^{p(c)(p(\alpha)+p(b))} D(c)(D(\alpha) b-\alpha D(b)) \\
&=-(-1)^{p(c)(p(\alpha)+p(b))}\{c, D(\alpha) b\}+(-1)^{p(c)(p(b)+p(\alpha))}\{c, \alpha D(b)\} \\
&+(-1)^{p(\alpha) p(b)}\{b, D(\alpha) c\}-(-1)^{p(\alpha) p(b)}\{b, \alpha D(c)\}+D(\alpha)\{b, c\} \\
&-(-1)^{p(\alpha) p(b)} D(b) D(\alpha) c+(-1)^{p(\alpha) p(b)} D(b) \alpha D(c) \\
&+(-1)^{p(c)(p(\alpha)+p(b))} D(c) D(\alpha) b-(-1)^{p(c)(p(\alpha)+p(b))} D(c) \alpha D(b) \\
&=-(-1)^{p(c)(p(b)+p(\alpha))}\{c, D(\alpha)\} b-(-1)^{p(b) p(c)} D(\alpha)\{c, b\} \\
&+(-1)^{p(c)(p(\alpha)+p(b))} D(c) D(\alpha) b \\
&+(-1)^{p(c)(p(b)+p(\alpha))}\{c, \alpha\} D(b)+(-1)^{p(b) p(c)} \alpha\{c, D(b)\} \\
&+(-1)^{p(c)(p(\alpha)+p(b))} D(c) \alpha D(b)+(-1)^{p(\alpha) p(b)}\{b, D(\alpha)\} c+D(\alpha)\{b, c\} \\
&-(-1)^{p(\alpha) p(b)} D(b) D(\alpha) c-(-1)^{p(\alpha) p(b)}\{b, \alpha\} D(c)-\alpha\{b, D(c)\} \\
&+(-1)^{p(\alpha) p(b)} D(b) \alpha D(c)+D(\alpha)\{b, c\}-(-1)^{p(\alpha) p(b)} D(b) D(\alpha) c \\
&+(-1)^{p(\alpha) p(b)} D(b) \alpha D(c)+(-1)^{p(c)(p(\alpha)+p(b))} D(c) D(\alpha) b \\
&-(-1)^{p(c)(p(\alpha)+p(b))} D(c) \alpha D(b)
\end{aligned}
$$

$$
\begin{aligned}
=( & -1)^{p(b) p(c)} D(D(\alpha)) c b-(-1)^{p(c) p(b)} D(\alpha) D(c) b \\
& +D(\alpha)\{b, c\}+(-1)^{p(c)(p(\alpha)+p(b))} D(c) D(\alpha) b \\
& -(-1)^{p(b) p(c)} D(\alpha) c D(b)+(-1)^{p(b) p(c)} \alpha D(c) D(b)-\alpha\{D(b), c\} \\
& +(-1)^{p(c)(p(\alpha)+p(b))} D(c) \alpha D(b)-D(D(\alpha)) b c+D(\alpha) D(b) c+D(\alpha)\{b, c\} \\
& -(-1)^{p(\alpha) p(b)} D(b) D(\alpha) c+D(\alpha) b D(c)-\alpha D(b) D(c)-\alpha\{b, D(c)\} \\
& +(-1)^{p(\alpha) p(b)} D(b) \alpha D(c)+D(\alpha)\{b, c\}-(-1)^{p(\alpha) p(b)} D(b) D(\alpha) c \\
& +(-1)^{p(\alpha) p(b)} D(b) \alpha D(c)+(-1)^{p(c)(p(\alpha)+p(b))} D(c) D(\alpha) b \\
& -(-1)^{p(c)(p(\alpha)+p(c))} D(c) \alpha D(b) \\
= & 3 D(\alpha)\{b, c\}-\alpha\{D(b), c\}-\alpha\{b, D(c)\}-2 D(\alpha)(D(b) c-b D(c)) .
\end{aligned}
$$

Therefore,

$$
\begin{equation*}
3 D(\alpha)\{b, c\}-\alpha\{D(b), c\}-\alpha\{b, D(c)\}-2 D(\alpha)(D(b) c-b D(c))=\{\alpha,\{b, c\}\} . \tag{14}
\end{equation*}
$$

Comparing (13) and (14), we obtain

$$
D(\alpha)\{b, c\}=D(\alpha)(D(b) c-b D(c))
$$

Lemma 7 allows us to generalize this relation as follows:

$$
D^{k}(\alpha)\{b, c\}=D^{k}(\alpha)(D(b) c-b D(c))
$$

The lemma is proved.
Lemma 9. Let $J=J(\Gamma,\{\}$,$) be a simple unital Jordan superalgebra, and let \alpha \in \Phi$. If $D(\alpha) \neq 0$, then $J$ is a superalgebra of vector type. In particular, if $J$ is not a superalgebra of vector type, then $D(\alpha)=0$ and $\alpha$ is invertible in $\Gamma$.
Proof. We set $I=\Gamma D(\alpha)+\Gamma D^{2}(\alpha)+\cdots$. Note that $I \triangleleft \Gamma$. Using (1), the definition of $\Phi$, and Lemma 7, it is easy to deduce that

$$
\begin{aligned}
\left\{\Gamma, \Gamma D^{k}(\alpha)\right\} & \subseteq\left\{\Gamma, D^{k}(\alpha)\right\} \Gamma+D^{k}(\alpha)\{\Gamma, \Gamma\}+D(\Gamma) D^{k}(\alpha) \Gamma \\
& \subseteq D(\Gamma) D^{k}(\alpha) \Gamma+\Gamma D^{k+1}(\alpha) \Gamma+D^{k}(\alpha)\{\Gamma, \Gamma\}+D(\Gamma) D^{k}(\alpha) \Gamma \\
& \subseteq \Gamma D^{k}(\alpha)+\Gamma D^{k+1}(\alpha),
\end{aligned}
$$

whence $\{\Gamma, I\} \subseteq I$. By [14], $\Gamma$ contains no nonzero ideals $I$ such that $\{\Gamma, I\} \subseteq I$. If $D(\alpha) \neq 0$, then $I=\Gamma$. Therefore, $1=\gamma_{1} D(\alpha)+\cdots+\gamma_{l} D^{l}(\alpha)$. Consequently, by Lemma 8, for arbitrary $b, c \in \Gamma$ we get

$$
\begin{aligned}
\{b, c\} & =\left(\gamma_{1} D(\alpha)+\cdots+\gamma_{l} D^{l}(\alpha)\right)\{b, c\} \\
& =\gamma_{1} D(\alpha)(D(b) c-b D(c))+\cdots+\gamma_{l} D^{l}(\alpha)(D(b) c-b D(c))=D(b) c-b D(c)
\end{aligned}
$$

Thus, $J$ is a superalgebra of vector type.
If $J$ is not a superalgebra of vector type, then the said above implies $D(\alpha)=0$. By Lemma 6, $\alpha$ is an invertible element in $\Gamma$. The lemma is proved.

The above results are generalized in the following theorem.
Theorem 10. Let $J=J(\Gamma,\{\}$,$) be a simple unital superalgebra of a Jordan bracket over$ a field of characteristic different from 2. Then either $J$ has no nontrivial $\delta$-derivations and $\delta$-superderivations, or $J$ is a superalgebra of vector type. If $J$ is a superalgebra of vector type, then $\Gamma_{1}=0$ and the superalgebra $J$ has no nontrivial odd $\delta$-superderivations. For $\delta \neq \frac{1}{2}$, the superalgebra J has no nontrivial $\delta$-derivations. The space of $\frac{1}{2}$-derivations coincides with $R^{*}(J)=\left\{R_{z} \mid z \in \Gamma_{0}\right\}$, and if $D(z) \neq 0$, then the map $R_{z}$ is a nontrivial $\frac{1}{2}$-derivation.

Let $A=A_{0} \oplus A_{1}$ be an associative superalgebra. On the vector space $A$, we define a supersymmetric product $\circ$ by the rule

$$
a \circ b=\frac{1}{2}\left(a b+(-1)^{p(a) p(b)} b a\right) .
$$

We denote by $A^{(+)}$the algebra obtained. A Jordan superalgebra $B$ is said to be special if it is isomorphically embedded in the superalgebra $A^{(+)}$for a suitable associative superalgebra $A$.

Using Theorem 10 and the well-known fact that the unital Jordan superalgebras of vector type are special (see [14), we obtain the following statement.

Corollary 11. If a simple unital superalgebra of a Jordan bracket $J=J(\Gamma,\{\}$,$) has a$ nontrivial $\delta$-derivation, then $J$ is special.

## §3. The $\delta$-derivations and $\delta$-Superderivations

 of simple unital finite-dimensional Jordan superalgebrasNow we proceed to a description of the $\delta$-derivations and $\delta$-superderivations of simple unital finite-dimensional Jordan superalgebras over an algebraically closed field of characteristic $p \neq 2$. We recall that the $\delta$-derivations and $\delta$-superderivations of simple unital finite-dimensional Jordan superalgebras over an algebraically closed field of characteristic zero were described in [4, 6,

Recall that an algebra $A$ is said to be alternative if the identities

$$
(x, x, y)=0, \quad(x, y, y)=0
$$

are valid, where $(x, y, z)=(x y) z-x(y z)$ is the associator of elements $x, y, z \in A$. The algebra $O$ of octonions or Cayley numbers (see [17]) is a classical example of an alternative nonassociative algebra.

Now we give examples of simple nontrivial nonassociative alternative superalgebras of characteristic 3. Below $B$ denotes an alternative superalgebra over a field $F$, and $C$ and $M$ are the even and the odd part of $B$, respectively.
3.1. The superalgebra $B(1,2)$. Let $F$ be a field of characteristic 3 , and let $B(1,2)=$ $C+M$ be a supercommutative superalgebra over $F$ in which $C=F \cdot 1$ and $M=F \cdot x+F \cdot y$, where 1 is the unity of $B$ and $x y=-y x=1$. Note that the superalgebra $B(1,2)$ is precisely the simple Jordan superalgebra of the supersymmetric bilinear form $f(s, r)=s r$ on the odd vector space $M$.
3.2. The superalgebra $B(4,2)$. Let $F$ be a field of characteristic 3 , let $C=M_{2}(F)$ be the algebra of $(2 \times 2)$-matrices over $F$, and let $M=F \cdot m_{1}+F \cdot m_{2}$ be a two-dimensional irreducible Cayley bimodule over $C$, i.e., $C$ acts on $M$ in the following way:

$$
\begin{aligned}
e_{i j} \cdot m_{k} & =\delta_{i k} m_{j}, \quad i, j, k \in\{1,2\}, \\
m \cdot a & =\bar{a} \cdot m
\end{aligned}
$$

where $a \in C, m \in M$, and $a \mapsto \bar{a}$ is the symplectic involution in $C=M_{2}(F)$. Odd multiplication on $M$ is defined by the relations

$$
m_{1}^{2}=-e_{21}, \quad m_{2}^{2}=e_{12}, \quad m_{1} m_{2}=e_{11}, \quad m_{2} m_{1}=-e_{22}
$$

It is known (see [18) that $B(1,2)$ and $B(4,2)$ are simple alternative superalgebras with the superinvolutions

$$
(a+m)^{*}=a-m \text { for } B(1,2), \quad(a+m)^{*}=\bar{a}-m \text { for } B(2,4) .
$$

An even linear transformation $*$ of a superalgebra $A$ is called a superinvolution if

$$
\left(a^{*}\right)^{*}=a,(a b)^{*}=(-1)^{p(a) p(b)} b^{*} a^{*}, \quad a, b \in A_{0} \cup A_{1} .
$$

In [18] it was shown that the superalgebras $B(1,2)$ and $B(4,2)$ give rise to the simple Jordan superalgebras $H_{3}(B(1,2))$ and $H_{3}(B(4,2))$.
Lemma 12. The superalgebras $H_{3}(B(1,2))$ and $H_{3}(B(2,4))$ have no nontrivial $\delta$-derivations and $\delta$-superderivations.

Proof. We denote by $e_{i j}$ the matrix units of the algebras $B(1,2)_{3}$ and $B(2,4)_{3}$. Let $\phi$ be a nontrivial $\delta$-derivation or $\delta$-superderivation. Clearly, $\delta=\frac{1}{2}$.

Let

$$
\phi\left(e_{i i}\right)=\sum_{j=1}^{3} \alpha_{j}^{i} e_{j j}+\sum_{k, l, k \neq l} x_{k l}^{i} e_{k l},
$$

where $x_{k l}^{i}=\overline{x_{l k}^{i}}$; then

$$
2 \phi\left(e_{i i}\right)=2 e_{i i} \circ \phi\left(e_{i i}\right)=2 \alpha_{i}^{i} e_{i i}+\sum_{k \neq i}\left(x_{i k}^{i} e_{i k}+x_{k i}^{i} e_{k i}\right) .
$$

Thus, $\phi\left(e_{i i}\right)=\alpha_{i}^{i} e_{i i}$. If $\beta \in F$ and

$$
\phi\left(\beta\left(e_{21}+e_{12}\right)\right)=\left(\begin{array}{ccc}
\frac{\gamma_{1}}{\overline{x_{12}}} & x_{12} & x_{13} \\
\overline{x_{13}} & \overline{x_{23}} & x_{23} \\
\gamma_{3}
\end{array}\right)
$$

then

$$
\begin{gathered}
\left(\begin{array}{ccc}
0 & x_{12}+\frac{1}{2} \beta \alpha_{2}^{2} & 0 \\
\overline{x_{12}}+\frac{1}{2} \beta \alpha_{2}^{2} & 2 \gamma_{2} & x_{23} \\
0 & \overline{x_{23}} & 0
\end{array}\right)=\phi\left(\beta\left(e_{12}+e_{21}\right)\right) \circ e_{22}+\beta\left(e_{12}+e_{21}\right) \circ \phi\left(e_{22}\right) \\
\\
=2 \phi\left(\left(\begin{array}{ccc}
0 & \beta & 0 \\
\beta & 0 & 0 \\
0 & 0 & 0
\end{array}\right) \circ e_{22}\right)=2 \phi\left(\left(\begin{array}{ccc}
0 & \beta & 0 \\
\beta & 0 & 0 \\
0 & 0 & 0
\end{array}\right) \circ e_{11}\right) \\
\\
=\left(\begin{array}{ccc}
2 \gamma_{1} & x_{12}+\frac{1}{2} \beta \alpha_{1}^{1} & x_{13} \\
\overline{x_{12}}+\frac{1}{2} \beta \alpha_{1}^{1} & 0 & 0 \\
\overline{x_{13}} & 0 & 0
\end{array}\right) .
\end{gathered}
$$

This implies that $\alpha_{1}^{1}=\alpha_{2}^{2}=\alpha$. Similarly, we can show that $\alpha_{3}^{3}=\alpha$. Let $e=e_{11}+e_{22}+e_{33}$ be the unity of the superalgebra $H_{3}(B(1,2))$ (or $H_{3}(B(2,4))$ ). Thus, in the case of a $\frac{1}{2}$-derivation and an even $\frac{1}{2}$-superderivation, we have $\phi(e)=\alpha e$, and in the case of an odd $\frac{1}{2}$-superderivation we have $\phi(e)=0$. This implies the triviality of $\phi$. The lemma is proved.

Lemma 13. Let $F$ be a field of characteristic $p>2$, and let $J=J(B(m, n),\{\}$,$) be a$ Jordan superalgebra that is not a superalgebra of vector type. Then $J$ has no nontrivial $\delta$-derivations and $\delta$-superderivations.

Proof. Lemma 3 shows that every $\delta$-derivation is an even $\delta$-superderivation. Let $\phi$ be a nontrivial $\delta$-superderivation. Clearly, $\delta=\frac{1}{2}$ and, by Lemma 3, $\phi(x)=\alpha x$, where $\alpha \in \Phi$. We see that $\alpha=\beta \cdot 1+r$, where $r$ is nilpotent and $\beta \in F$. Assume that $r \neq 0$. We may assume that $\alpha=r$, so that $\alpha$ is not invertible. By Lemma $6, D(\alpha) \neq 0$. Consequently, by Lemma $9, J$ is a superalgebra of vector type. This contradiction implies that $r=0$ and $\alpha \in F$. The lemma is proved.

Lemma 14. The superalgebra $C K(Z, d)$ has no nontrivial $\delta$-derivations and $\delta$-superderivations.

Proof. Obviously, the even $\delta$-superderivations are $\delta$-derivations. Let $\phi_{0}$ be a nontrivial $\delta$-derivation and $\phi_{1}$ a nontrivial odd $\delta$-superderivation of the superalgebra $C K(Z, d)$. It is clear that $\delta=\frac{1}{2}$ and $\phi_{i}(x)=\phi_{i}(1) x$ for any $x \in C K(Z, d)$. We set

$$
\phi_{j}(1)=\alpha^{j}+\sum_{i=1}^{3} w_{i} \alpha_{i}^{j}+x \beta^{j}+\sum_{i=1}^{3} x_{i} \beta_{i}^{j}
$$

Note that, since $\phi_{1}$ is homogeneous, we have $\alpha^{1}=\alpha_{i}^{1}=0$. We show that $\beta^{j}=\beta_{i}^{j}=$ $\alpha_{i}^{j}=0$.

It is easily seen that

$$
\begin{aligned}
0=\phi_{j}\left(x w_{k}\right)= & \frac{1}{2}\left(\left(\left(\alpha^{j}+\sum_{i=1}^{3} w_{i} \alpha_{i}^{j}+\beta^{j} x+\sum_{i=1}^{3} x_{i} \beta_{i}^{j}\right) x\right) w_{k}\right. \\
& \left.+(-1)^{j} x\left(\left(\alpha^{j}+\sum_{i=1}^{3} w_{i} \alpha_{i}^{j}+\beta^{j} x+\sum_{i=1}^{3} x_{i} \beta_{i}^{j}\right) w_{k}\right)\right) \\
= & \frac{1}{2}\left(-\beta_{k}^{j}-(-1)^{j}\left(x \alpha_{k}^{j}-\sum_{i=1}^{3} w_{i \times k} \beta_{i}\right)\right)
\end{aligned}
$$

This implies $\alpha_{i}^{j}=\beta_{i}^{j}=0$ and $\phi(1)=\alpha^{j}+\beta^{j} x$.
Now we have

$$
x_{i \times k} \alpha^{j}-w_{i \times k} \beta^{j}=\phi_{j}\left(x_{i} w_{k}\right)=\frac{1}{2}\left(\left(\phi(1) x_{i}\right) w_{k}+(-1)^{j} x_{i}\left(\phi(1) w_{k}\right)\right)=w_{i \times k} \alpha^{j},
$$

whence the required statement follows: $\phi_{0}(x)=\alpha^{0} x, \alpha^{0} \in Z$, and $\phi_{1}=0$, i.e., $\phi_{j}$ is trivial. The lemma is proved.

By the results of 12, 18, the simple unital finite-dimensional Jordan superalgebras with a semisimple even part over an algebraically closed field of characteristic $p>2$ are exhausted by the superalgebras $H_{3}(B(1,2))$ and $H_{3}(B(2,4))$, which are considered over fields of characteristic 3. By [13], the simple unital finite-dimensional Jordan algebras with a nonsemisimple even part over an algebraically closed field of characteristic $p>2$ are exhausted by the superalgebras $J=J(B(m, n),\{\}$,$) and C K(B(m), d)$. Thus, combining the above classification of simple unital Jordan superalgebras over an algebraically closed field of characteristic $p>2$, the results of [4, 6, Corollary 5, and Lemmas 12-14, we see that the following is true.

Theorem 15. Let $J$ be a simple unital finite-dimensional Jordan superalgebra over an algebraically closed field of characteristic $p \neq 2$. Then either $J$ has no nontrivial $\delta$ derivations and $\delta$-superderivations, or $J$ is a superalgebra of vector type over a field of characteristic $p>2$. If $J=J(B(m, n),\{\}$,$) is a superalgebra of vector type, then$ $n=0$ and the superalgebra $J$ has no nontrivial odd $\delta$-superderivations. For $\delta \neq \frac{1}{2}$, the superalgebra $J$ has no nontrivial $\delta$-derivations. The space of $\frac{1}{2}$-derivations coincides with $R^{*}(J)=\left\{R_{z} \mid z \in B(m)\right\}$, and for $D(z) \neq 0$ the map $R_{z}$ is a nontrivial $\frac{1}{2}$-derivation.

In conclusion, it should be noted that, while this paper was being published, the derivations of the Cheng-Kac superalgebras [19] and of the Kantor double of a simple unital Poisson superalgebra [20] were described. Also, a complete description was obtained for the $\delta$-(super)derivations of simple nonunital Jordan superalgebras over an algebraically closed field and, as a consequence, for the $\delta$-(super)derivations of semisimple finite-dimensional Jordan superalgebras over an algebraically closed field of characteristic different from 2 (see [21).

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