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FORMATIONS OF FINITE C_π-GROUPS

E. P. VDOVIN, D. O. REVIN, AND L. A. SHEMETKOV

Abstract. It is proved that the class of all finite C_π-groups is closed under subdirect products. Conditions for a given formation of C_π-groups to be saturated or partially saturated are found.

§1. Introduction

By a group we always mean a finite group. Throughout, a (fixed) set of primes is denoted by π and p is a prime. The set of primes not belonging to π is denoted by π'. Given a natural number n, we denote by $\pi(n)$ the set of all prime divisors of n, and $\pi(G)$ equals $\pi(|G|)$ for a group G.

Recall that a group G with $\pi(G) \subseteq \pi$ is called a π-group. A subgroup H of G is called a *π-Hall subgroup* if $\pi(H) \subseteq \pi$ and $\pi(|G : H|) \subseteq \pi'$.

In accordance with [1], we say that *G satisfies E_π* if G possesses a π-Hall subgroup. If G satisfies E_π and every two π-Hall subgroups are conjugate, then we say that *G satisfies C_π*. If G satisfies C_π and each π-subgroup of G is included in a π-Hall subgroup, then we say that *G satisfies D_π*. A group satisfying E_π (C_π, D_π) is also called an E_π-group (respectively, C_π-group, D_π-group). The symbols E_π, C_π, and D_π will be used also for the classes of all E_π-, C_π-, and D_π-groups, respectively. If $\mathfrak{X}$ is a class of groups, then by $E_\pi\mathfrak{X}$ we denote the class of all groups possessing a π-Hall subgroup from $\mathfrak{X}$, while $C_\pi\mathfrak{X}$ and $D_\pi\mathfrak{X}$ are $C_\pi \cap (E_\pi\mathfrak{X})$ and $D_\pi \cap (E_\pi\mathfrak{X})$, respectively.

Recall that a class of groups $\mathfrak{F}$ is called a *formation* if it is closed under homomorphic images and finite subdirect products (in other words, for each G and its normal subgroups M and N, $G \in \mathfrak{F}$ imply $G/M \in \mathfrak{F}$, while $G/M \in \mathfrak{F}$ and $G/N \in \mathfrak{F}$ imply $G/(M\cap N) \in \mathfrak{F}$).

A formation $\mathfrak{F}$ is said to be:

1) *p-saturated* if $G/N \in \mathfrak{F}$ and $N \le O_p(\Phi(G))$ imply $G \in \mathfrak{F}$ (here and below $O_\pi(G)$ is the largest π-subgroup of G for a fixed set of primes π, while $\Phi(G)$ is the Frattini subgroup of G, i.e., the intersection of all maximal subgroups);

2) *saturated* if $\mathfrak{F}$ is p-saturated for each p;

3) *p-solubly saturated* if the conditions $G/N \in \mathfrak{F}$ and $N \le O_p(\Phi(G_{p-\mathfrak{S}}))$, where $G_{p-\mathfrak{S}}$ is a p-solvable radical of G, always imply $G \in \mathfrak{F}$;

4) *solubly saturated* if the conditions $G/N \in \mathfrak{F}$ and $N \le \Phi(G_\mathfrak{S})$, where $G_\mathfrak{S}$ is a solvable radical of G, always imply $G \in \mathfrak{F}$.

In [2, Corollary 8], the classification of finite simple group was used to show that the class E_π is a formation (notice that the proof of this statement in [19], which appeared

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earlier than [2], had a gap). In the series of papers [3, 15–18], by using the classification of finite simple groups, it was proved that a group satisfies D_π if and only if each its composition factor satisfies D_π, see [3, Theorem 7.7]. It follows that D_π is a formation. The two formations E_π and D_π are saturated because the Frattini subgroup is nilpotent.

In [4] it was proved that if $\mathfrak{F}$ is a saturated formation, then the class of all solvable $C_\pi\mathfrak{F}$-groups is also a formation. In 1978, L. A. Shemetkov proposed the following conjecture.

Conjecture [5, Problem 19]**.** *Let π be a set of primes and $\mathfrak{F}$ a saturated formation. Then $C_\pi\mathfrak{F}$ is a saturated formation.*

Later in [6, Theorem 2], a criterion for $C_\pi\mathfrak{F}$ to be saturated was obtained, where $\mathfrak{F}$ is a formation such that $C_\pi\mathfrak{F}$ is a formation. As a corollary, it was proved that $C_\pi\mathfrak{N}$ may fail to be saturated, $\mathfrak{N}$ being the formation of all nilpotent groups, so that the original Shemetkovs' conjecture was disproved. However, the cases where this conjecture remains valid have not been investigated. By [6, Theorem 2], this question reduces to the following: given a formation $\mathfrak{F}$, is $C_\pi\mathfrak{F}$ a formation, in particular, is C_π a formation? Moreover, conditions ensuring that $C_\pi\mathfrak{F}$ is a solubly saturated, p-saturated, or p-solubly saturated formation are of interest.

In this paper we prove that $C_\pi\mathfrak{F}$ is a formation for every formation $\mathfrak{F}$ (see Theorem 1 below). We use the results of [7], where a criterion for a group to satisfy C_π was obtained. This criterion was based on the classification of finite simple groups, and it was formulated in terms of the normal structure of the group. Moreover, it was proved that the C_π property is inherited by some subgroups of index not divisible by primes from π, and is inherited by the factor groups. We complete the analysis started in [6] by finding the cases where $C_\pi\mathfrak{F}$ turns out to be p-saturated or p-solubly saturated.

We formulate the main results of the paper.

Theorem 1. 1) *C_π is a formation*;

2) *$C_\pi\mathfrak{F}$ is a formation for each formation $\mathfrak{F}$.*

Corollary 1.1. *Each finite group admits a C_π-coradical, i.e., the smallest normal subgroup such that the factor group by this subgroup is a C_π-group.*

Theorem 2. *If $\mathfrak{F}$ is a p-solubly saturated formation, then $C_\pi\mathfrak{F}$ is also a p-solubly saturated formation.*

Corollary 2.1. *Let $\mathfrak{F}$ be a p-solubly saturated formation including all p-groups. Assume that G possesses a normal subgroup A and a normal p-subgroup B such that $B \leq A$ and $A/B \in C_\pi\mathfrak{F}$. If B is included in the Frattini subgroup of the p-solvable radical of G, then $A \in C_\pi\mathfrak{F}$.*

Corollary 2.2. *If $\mathfrak{F}$ is a solubly saturated formation, then $C_\pi\mathfrak{F}$ is also a solubly saturated formation.*

Corollary 2.3. *Let $\mathfrak{F}$ be a solubly saturated formation. Assume that G possesses normal subgroups A and B such that $B \leq A$ and $A/B \in C_\pi\mathfrak{F}$. If B is included in the Frattini subgroup of the solvable radical of G, then $A \in C_\pi\mathfrak{F}$.*

In two last claims given below we use the following notation and definitions. By $O_{p',p}(G)$ we denote the p-nilpotent radical of G (i.e., the largest normal subgroup possessing a normal p'-Hall subgroup). If $\mathfrak{X}$ is a class of groups, then the smallest formation containing $\mathfrak{X}$ (the intersection of all formations containing $\mathfrak{X}$) is denoted by *form* $(\mathfrak{X})$. A group is said to be *monolitic* if it has a unique minimal normal subgroup. The class of all p-groups is denoted by $\mathfrak{N}_p$. If $\mathfrak{X}$ is a formation, then $\mathfrak{N}_p\mathfrak{X}$ is the class of all extensions of p-groups by $\mathfrak{X}$-groups.

Theorem 3. *Let $\mathfrak{F}$ be a p-saturated formation, and let $p \in \pi$. Then $C_\pi\mathfrak{F}$ is p-saturated if and only if any π-Hall subgroup of each monolitic $C_\pi\mathfrak{F}$-group with nonabelian socle of order divisible by p belongs to $\mathfrak{N}_p\,form(G/O_{p',p}(G) \mid G \in \mathfrak{F})$.*

Corollary 3.1. *Let $\mathfrak{F}$ be a saturated formation. Then $C_\pi\mathfrak{F}$ is saturated if and only if any π-Hall subgroup of each monolitic $C_\pi\mathfrak{F}$-group H with nonabelian socle N belongs to $\mathfrak{N}_p\,form(G/O_{p',p}(G) \mid G \in \mathfrak{F})$ for every $p \in \pi(N) \cap \pi$.*

§2. Preliminary results

A natural number divisible by p (not divisible by p) is called a *pd*-number (p'-number, respectively). A group is called a *pd-group* if its order is a *pd*-number. We denote by $\mathcal{K}(X)$ the class of all simple groups isomorphic to composition factors of X.

The following statement is well known (see, e.g., [1, Lemma 1] or [8, Chapter IV, (5.11)]).

Lemma 1. *Let G be a group and A its normal subgroup. If H is a π-Hall subgroup of G, then $H \cap A$ is a π-Hall subgroup of A, while the factor group HA/A is a π-Hall subgroup of G/A.*

Lemma 2 (S. A. Chunikhin, see [9])**.**

(1) *Let G be a group and A its normal subgroup. If G/A and A satisfy C_π, then G satisfies C_π.*[1]

(2) *If all composition factors of a subnormal series of G are either π-groups or π'-groups, then G satisfies D_π (see also* [8, Chapter V, Theorem 3.7]).

Lemma 3 [2, Corollary 7]**.** *Let G be a group, and let M and N be its normal subgroups. If G/M and G/N are E_π-groups, then $G/(M \cap N) \in E_\pi$.*

Lemma 4 [7, Lemma 9]**.** *Let G be a C_π-group and A its normal subgroup. Then G/A satisfies C_π.*

Lemma 5 [7, Lemma 7]**.** *Let G be a C_π-group, A its normal subgroup, and H a π-Hall subgroup of G. Then $N_G(HA)$ and $N_G(H \cap A)$ satisfy C_π.*

Lemma 6 [7, Theorem 1]**.** *Let G be a C_π-group, A its normal subgroup, and H a π-Hall subgroup of G. Then HA satisfies C_π.*

Following [10]–[12], we recall the terminology and some results on partially saturated formations.

Definition 1. Let H/K be a normal section of G. *The small centralizer $c_G(H/K)$* of H/K is the subgroup generated by all normal subgroups N of G such that $\mathcal{K}(NK/K) \cap \mathcal{K}(H/K) = \varnothing$.

Definition 2. Let f be a function that maps each *pd*-number to one and the same formation $f(p)$ and each p'-number to one and the same formation $f(p')$. Such f is called a *p-local satellite*.

Definition 3. Let $\mathfrak{F}$ be a formation and f a p-local satellite. We say that f is a *p-local satellite of* $\mathfrak{F}$ if $\mathfrak{F}$ coincides with the class of all groups G such that each composition factor H/K of G has the following properties:

1) if H/K is a *pd*-group, then $G/C_G(H/K) \in f(p)$;

2) if H/K is a p'-group, then $G/c_G(H/K) \in f(p')$.

Definition 4. Let f be a p-local satellite of $\mathfrak{F}$. We say that f is a *canonical p-local satellite of* $\mathfrak{F}$ if $f(p) = \mathfrak{N}_p f(p) \subseteq \mathfrak{F}$ and $f(p') = \mathfrak{F}$.

[1] See also [1, Theorems C1 and C1] or [8, Chapter V, (3.12)].

Lemma 7 [11, Theorem 1]**.** *A nonempty formation is p-saturated if and only if it possesses a p-local satellite. Each p-saturated formation $\mathfrak{F}$ containing $\mathfrak{N}_p$ possesses a canonical p-local satellite f, and $f(p) = \mathfrak{N}_p\, form(G/O_{p',p}(G) \mid G \in \mathfrak{F})$.*

Definition 5. Let f be a function that maps each nonzero power of p to one and the same formation $f(p)$, while each number that is not equal to a nonzero power of p is mapped to one and same formation $f(p')$. Such f is called a *p-composite satellite*.

Definition 6. Let $\mathfrak{F}$ be a formation and f a p-composite satellite. We say that f is a *p-composite satellite of* $\mathfrak{F}$ if $\mathfrak{F}$ coincides with the class of all groups G such that each chief composition factor H/K of G has the following properties:

1) if H/K is a p-group, then $G/C_G(H/K) \in f(p)$;

2) if H/K is not a p-group, then $G/c_G(H/K) \in f(p')$.

We note that the class of all groups G satisfying conditions 1) and 2) of Definition 6 is a formation (see [10, Example 6]).

Definition 7. Let f be a p-composite satellite of a formation $\mathfrak{F}$. We say that f is a *canonical p-composite satellite of* $\mathfrak{F}$ if $f(p) = \mathfrak{N}_p f(p) \subseteq \mathfrak{F}$ and $f(p') = \mathfrak{F}$.

Lemma 8 (see [10, Lemma 7; 12, Theorem 1])**.** *A nonempty formation is p-solubly saturated if and only if it possesses a p-composite satellite. Each p-solubly saturated formation possesses a canonical p-composite satellite.*

Lemma 9 (see [10, Theorem 1])**.** *Let $\mathfrak{F}$ be a formation containing all p-groups. Assume that G possesses a normal subgroup A and a normal p-subgroup B such that $B \le A$ and $A/B \in \mathfrak{F}$. Then $A \in \mathfrak{F}$ whenever one of the following condition is fulfilled:*

1) $\mathfrak{F}$ is p-saturated and $B \le \Phi(G)$;

2) $\mathfrak{F}$ is p-solubly saturated and B is included in the Frattini subgroup of the p-solvable radical of G.

The next lemma follows from the properties of the Frattini p-elementary extension of a pd-group (see [14, p. 853]).

Lemma 10. *Let G be a monolitic group with a nonabelian socle whose order is divisible by p. Then there exists a group E and an elementary Abelian normal p-subgroup $A \neq 1$ of E such that $E/A \simeq G$, $A \le \Phi(E)$, and $O_{p'}(H) = 1$ for each subgroup H with $A \le H \le E$.*

Lemma 11. *Let f be a canonical p-local satellite of a formation $\mathfrak{F}$. If a pd-group G lies in $\mathfrak{F}$ and $O_{p'}(G) = 1$, then $G \in f(p)$.*

Proof. If H/K is a chief pd-factor of G, then $G/C_G(H/K) \in f(p)$ by Definition 3. Since $f(p)$ is a formation, it follows that $G/R \in f(p)$, where R is the intersection of all centralizers of all chief pd-factors. It is known that $R = O_{p',p}(G)$. Since $O_{p'}(G) = 1$, we obtain $G \in \mathfrak{N}_p f(p)$. Consequently, $G \in f(p)$, because for a canonical p-local satellite f we have $\mathfrak{N}_p f(p) = f(p)$. □

Lemma 12 (P. Shmid, see. [5, Theorem 7.11] and [20, Lemma 4])**.** *If $\Phi(G) = 1$ and S is the socle of G, then $C_G(S) \le S$.*

Lemma 13. *Let f be a canonical p-local satellite of a formation $\mathfrak{F}$. Let K be a normal Abelian p-subgroup of G such that $G/K \in \mathfrak{F}$ and $G/C_G(K) \in f(p)$. Then $G \in \mathfrak{F}$.*

Proof. If $O_{p'}(G) \neq 1$, then $G/O_{p'}(G) \in \mathfrak{F}$ by induction. Since $G/K \in \mathfrak{F}$, we obtain $G \in \mathfrak{F}$. If $O_{p'}(G) = 1$, then $G/O_p(G) \in f(p)$, and we use the condition $\mathfrak{N}_p f(p) = f(p)$. □

Lemma 14. *Let M and N be normal subgroups of G and let $M \cap N = 1$. Then for any two subgroups A and B of M the following is true*:

(1) $N \leq C_G(A)$;

(2) *if* $AN = BN$, *then* $A = B$;

(3) $N_G(A)/N = N_{G/N}(AN/N)$.

Proof. Since the product of M and N is direct, we have (1). Applying the coordinate projection from $M \times N$ onto M to the subgroup $AN = BN$, we obtain $A = B$, and (2) follows. Now we prove (3). The inclusion $N_G(A)/N \leq N_{G/N}(AN/N)$ is obvious. If $Ng \in N_{G/N}(AN/N)$ for some $g \in G$, then

$$A^g N = (AN)^g = AN$$

and $A^g \leq M$ because M is normal. Applying (2), we obtain $A^g = A$ and $g \in N_G(A)$. Statement (3) is proved. □

§3. Proof of Theorem 1 and its corollary

First, we prove statement 1) of Theorem 1. By Lemma 4, C_π is closed under factor groups. Assume that it is not closed under subdirect products. Then there exists a group G and its subgroups M and N such that G/M and G/N satisfy C_π, while $G/(M \cap N) \notin C_\pi$. Assume that G has a minimal possible order. We show that the conjecture about the existence of such a group leads to a contradiction. We split the proof into a series of steps.

By Lemma 3, we have

(1) $G/(M \cap N) \in E_\pi$.

The minimality of the order of G implies that

(2) $M \cap N = 1$ *and* $G \simeq G/(M \cap N)$. *In particular,* $[M, N] = 1$.

Therefore,

(3) $G \in E_\pi$ *and* $G \notin C_\pi$.

It follows that

(4) G *possesses two nonconjugate* π*-Hall subgroups* H *and* K.

Since $G/N \in C_\pi$, and NH/N and NK/N are π-Hall subgroups of G/N, there is no loss of generality in assuming that

(5) $NH = NK$;

(6) $G = MNH = MNK$.

Indeed, suppose $MNH < G$. We have

$$MNH/M = (MN/M)(MH/M),$$

while $MN/M \trianglelefteq G/M$ and HM/M is a π-Hall subgroup of $G/M \in C_\pi$. Applying Lemmas 1, 2, and 6, we see that $MNH/M \in C_\pi$. The same arguments imply $MNH/N \in C_\pi$. By the minimality of the order of G, we have

$$MNH \simeq MNH/(M \cap N) \in C_\pi.$$

But H and K are π-Hall subgroups of MNH by (5), so that they are conjugate in contrast to (4), a contradiction. The fact that $MNH = MNK$ follows from (5).

(7) $(M \cap H)N/N = (HN/N) \cap (MN/N)$,
$(M \cap K)N/N = (KN/N) \cap (MN/N)$.

Indeed, $M \cap H$ is a π-Hall subgroup of M by Lemma 1, whence $(M \cap H)N/N$ is a π-Hall subgroup of MN/N. Since HN/N is a π-Hall subgroup of G/N and $MN/N \trianglelefteq G/N$, Lemma 1 implies that $(HN/N) \cap (MN/N)$ is a π-Hall subgroup of MN/N. Since, clearly,

$$(M \cap H)N/N \leq (HN/N) \cap (MN/N),$$

the coincidence of orders on the right and on the left shows that

$$(M \cap H)N/N = (HN/N) \cap (MN/N).$$

The relation

$$(M \cap K)N/N = (KN/N) \cap (MN/N)$$

is checked in the same way. Statement (7) is proved.

From (5) and (7) it follows that

$$(M \cap H)N/N = (HN/N) \cap (MN/N) = (KN/N) \cap (MN/N) = (M \cap K)N/N.$$

Hence, $(M \cap H)N = (M \cap K)N$, and by Lemma 14(2) we obtain

(8) $M \cap H = M \cap K$;

(9) $M \cap H = M \cap K \trianglelefteq G$.

Suppose that (9) fails. Then $N_G(M \cap H) < G$. By (2) and Lemma 14(1) it follows that $N \leq N_G(M \cap H)$. By (7) and Lemma 14(3),

$$N_G(M \cap H)/N = N_{G/N}((HN/N) \cap (MN/N)),$$

and this group satisfies C_π by Lemma 5. Since $H, N \leq N_G(M \cap H)$, (6) implies $G = MN_G(M \cap H)$. Now $N_M(M \cap H) \trianglelefteq N_G(M \cap H)$ and

$$N_G(M \cap H)/N_M(M \cap H) \simeq MN_G(M \cap H)/M = G/M \in C_\pi.$$

Thus, $N_G(M \cap H)$ possesses two normal subgroups $N_M(M \cap H)$ and N with trivial intersection, and the factors by these subgroups satisfy C_π. The minimality of G implies that $N_G(M \cap H) \in C_\pi$, and since $H, K \leq N_G(M \cap H)$ by (8), the subgroups H and K are conjugate in contrast to (4), a contradiction.

Now $M \cap H \trianglelefteq H$ follows from (9), so that $M \cap H$ is a unique π-Hall subgroup of M. In particular,

(10) $M \in C_\pi$.

We conclude that G is an extension of the C_π-group M by the C_π-group G/M; then $G \in C_\pi$ by Lemma 2, which contradicts (3).

Statement 1) of Theorem 1 is proved.

We prove statement 2) of Theorem 1. By Lemmas 1–4, $C_\pi\mathfrak{F}$ is closed under homomorphic images. We show that it is closed under subdirect products. Let G be a group such that $M, N \trianglelefteq G$ and $G/M, G/N \in C_\pi\mathfrak{F}$. We may assume that $M \cap N = 1$. Since statement 1) of Theorem 1 shows that $G \in C_\pi$, it remains to show that a π-Hall subgroup H of G is an $\mathfrak{F}$-group. Since $H/(H \cap M) \simeq MH/M$, $H/(H \cap N) \simeq NH/N$, and MH/M and NH/N are π-Hall subgroups of G/M and G/N, it follows that $H/(H \cap M)$ and $H/(H \cap N)$ are $\mathfrak{F}$-groups. Since $\mathfrak{F}$ is a formation, $H \simeq H/(H \cap M \cap N) \in \mathfrak{F}$.

Theorem 1 is proved.

Corollary 1.1 follows from item 1) of Theorem 1 and [14, Chapter II, Lemma (2.4)].

§4. Proof of Theorem 2 and its corollaries

For applications, it is not sufficient to know that a formation is p-soluble saturated; the form of its p-composite satellite is also important. For this reason, we prove a refined version of Theorem 2.

Theorem 2*. *Let $p \in \pi$ and let f be a canonical p-composite satellite of $\mathfrak{F}$. Then $C_\pi\mathfrak{F}$ is a p-solubly saturated formation with a p-composite satellite h such that*

$$h(x) = \begin{cases} C_\pi f(p) & \text{if } x = p, \\ C_\pi\mathfrak{F} & \text{if } x = p'. \end{cases}$$

Proof. This is obvious for $f(p) = \varnothing$. Assume that $f(p) \neq \varnothing$, i.e., $\mathfrak{F}$ contains all p-groups. Using Definition 6, we construct a formation $\mathfrak{H}$ with a p-composite satellite h as in the theorem. We prove that $C_\pi\mathfrak{F} = \mathfrak{H}$.

Suppose $C_\pi\mathfrak{F} \not\subseteq \mathfrak{H}$. Let G be a group of minimal order in $(C_\pi\mathfrak{F}) \setminus \mathfrak{H}$. Clearly, G possesses a unique minimal normal subgroup K, and moreover, $G/K \in \mathfrak{H}$. If K is not a p-group, then $c_G(K) = 1$, $h(p') = C_\pi\mathfrak{F}$, and, by the definition of the satellite of a formation, we have $G/c_G(K) \simeq G \in C_\pi\mathfrak{F}$. Let K be a p-group. Clearly, K is included in a π-Hall subgroup R of G. Since $K \leq R \in \mathfrak{F}$ and f is a canonical p-composite satellite of $\mathfrak{F}$, the group R induces a group of automorphisms on each R-chief factor of K, and this group of automorphisms is included in $f(p)$. Then $R/C_R(K) \in \mathfrak{N}_p f(p) = f(p)$. Consequently, $G/C_G(K) \in C_\pi f(p)$. Thus, we have $G \in \mathfrak{H}$ by Definition 6. The inclusion $C_\pi\mathfrak{F} \subseteq \mathfrak{H}$ follows.

Now we assume that G is a group of minimal order in $\mathfrak{H} \setminus C_\pi\mathfrak{F}$. G possesses a unique minimal normal subgroup N, and moreover, $G/N \in C_\pi\mathfrak{F}$. If N is not a p-group, then $c_G(N) = 1$, $h(p') = C_\pi\mathfrak{F}$, and $G/c_G(N) \simeq G \in h(p') = C_\pi\mathfrak{F}$. Let N be a p-group. Clearly, $G \in C_\pi$ and N is included in a π-Hall subgroup T of G. Since $G/C_G(N) \in C_\pi f(p)$, we have $TC_G(N)/C_G(N) \simeq T/C_T(N) \in f(p)$. By Lemma 13, it follows that $T \in \mathfrak{F}$. Thus, G is a C_π-group with a π-Hall subgroup $R \in \mathfrak{F}$. The inclusion $\mathfrak{H} \subseteq C_\pi\mathfrak{F}$ follows. Theorem 2* and, by Lemma 8, also Theorem 2 are proved. □

Corollary 2.1 follows from Theorem 2 and Lemma 9.

Proof of Corollary 2.2. A formation is said to be $\mathfrak{N}_p$-*saturated* if $G \in \mathfrak{F}$ whenever $G/N \in \mathfrak{F}$, $N \leq \Phi(O_p(G))$. A formation is known to be a p-solubly saturated if and only if it is $\mathfrak{N}_p$-saturated (see [12, Theorem 1; 13, Theorem 3.2]). It follows that a formation is solubly saturated if and only if it is p-solubly saturated for every prime p. It remains to apply Theorem 2. □

Corollary 2.3 follows from Corollary 2.2 and Lemma 9.

§5. Proof of Theorem 3 and its corollary

The conclusion of the theorem is obvious if $\mathfrak{F}$ consists of p'-groups. So, we assume that $\mathfrak{F}$ contains all p-groups. Let f denote a canonical p-local satellite of $\mathfrak{F}$ (its existence follows from Lemma 7). If the order of a socle is divisible by p, we call this socle a *pd-socle* for brevity.

The "only if" part. Assume that $C_\pi\mathfrak{F}$ is p-saturated and G is a monolitic $C_\pi\mathfrak{F}$-group with nonabelian *pd*-socle. Assume that E and A are chosen as in Lemma 10. Since $A \leq \Phi(E)$ and $C_\pi\mathfrak{F}$ is a p-saturated formation, we have $E \in C_\pi\mathfrak{F}$. Let H be a π-Hall subgroup of E. By Lemma 10, $O_{p'}(H) = 1$. Thus, $H \in f(p)$ by Lemma 11. It remains to recall that $f(p) = \mathfrak{N}_p\, form(G/O_{p',p}(G) \mid G \in \mathfrak{F})$ by Lemma 7.

The "if" part. Assume that the following condition (*) is fulfilled:
any π-Hall subgroup of each monolitic $C_\pi\mathfrak{F}$-group with a nonabelian pd-socle belongs to $\mathfrak{N}_p\,\mathit{form}(G/O_{p',p}(G) \mid G \in \mathfrak{F})$.

We prove that $C_\pi\mathfrak{F}$ is p-saturated. Using Definition 6, we construct a formation $\mathfrak{G}$ with p-local satellite g such that

$$g(x) = \begin{cases} C_\pi f(p) & \text{if } x = p, \\ C_\pi\mathfrak{F} & \text{if } x = p'. \end{cases}$$

Then $\mathfrak{G}$ is p-saturated by Lemma 7. We prove that $C_\pi\mathfrak{F} = \mathfrak{G}$.

Assume that $C_\pi\mathfrak{F} \not\subseteq \mathfrak{G}$. Let G be a group of minimal order in $C_\pi\mathfrak{F} \setminus \mathfrak{G}$. Clearly, G possesses a unique minimal normal subgroup K; moreover, $G/K \in \mathfrak{G}$ and $c_G(K) = 1$. If K is a p'-group, then $g(p') = C_\pi\mathfrak{F}$ and $G/c_G(K) \simeq G \in C_\pi\mathfrak{F}$. Since $G/K \in \mathfrak{G}$, we obtain $G \in \mathfrak{G}$ by Definition 3.

Now, assume that K is a nonabelian pd-group. It is clear that $C_G(K) = 1$. By (*), any π-Hall subgroup of G belongs to $f(p)$. Hence, $G/C_G(K) \simeq G \in C_\pi f(p)$. Therefore, $G \in g(p)$, and $G \in \mathfrak{G}$ by Definition 3.

Finally, assume that K is an Abelian p-group. Since $\mathfrak{G}$ is p-saturated (see Lemma 7), we have $\Phi(G) = 1$, whence $K = C_G(K)$ by Lemma 12. Hence, if H is a π-Hall subgroup of G, then $O_{p'}(H) = 1$, and, by Lemma 11, we have $H \in f(p)$. Thus, $G/C_G(K) = G/K \in g(p)$. Since $G/K \in \mathfrak{G}$, it follows that $G \in \mathfrak{G}$. The inclusion $C_\pi\mathfrak{F} \subseteq \mathfrak{G}$ is proved.

Now, let G be a group of minimal order in $\mathfrak{G} \setminus C_\pi\mathfrak{F}$. G possesses a unique minimal normal subgroup N, and $G/N \in C_\pi\mathfrak{F}$. If N is a p'-group, then $c_G(N) = 1$ and $G/c_G(N) \simeq G \in g(p') = C_\pi\mathfrak{F}$. If N is a nonabelian pd-group, then $C_G(N) = 1$ and $G/C_G(N) \simeq G \in g(p) \subseteq C_\pi\mathfrak{F}$. Assume that N is a p-group. Then $G/C_G(N) \in C_\pi f(p)$ because $G \in \mathfrak{G}$. Obviously, $G \in C_\pi$. It is clear that N is included in a π-Hall subgroup R of G. Then $G/C_G(N) \in C_\pi f(p)$, $G/N \in C_\pi\mathfrak{F}$, whence $R/N \in \mathfrak{F}$ and $R/C_R(N) \in f(p)$. Lemma 13 shows that $R \in \mathfrak{F}$. Thus, G is a C_π-group with a π-Hall subgroup $R \in \mathfrak{F}$. The inclusion $\mathfrak{G} \subseteq C_\pi\mathfrak{F}$ follows.

So, $C_\pi\mathfrak{F}$ coincides with a p-saturated formation $\mathfrak{G}$. Theorem 3 is proved.

Corollary 3.1 follows from the definition of a saturated formation and Theorem 3.

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Sobolev Institute of mathematics, pr. Academica Koptyuga 4, Novosibirsk 630090, Russia
E-mail address: `vdovin@math.nsc.ru`

Sobolev Institute of mathematics, pr. Academica Koptyuga 4, Novosibirsk 630090, Russia
E-mail address: `revin@math.nsc.ru`

Francisk Skorina Gomel State University, ul. Sovetskaya 104, Gomel 246019, Belarus
E-mail address: `shemetkov@gsu.by`

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