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ASYMPTOTICS OF EIGENVALUES IN A PROBLEM OF HIGH EVEN ORDER WITH DISCRETE SELF-SIMILAR WEIGHT

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Abstract. Spectral asymptotics for the boundary problem $(-1)^n y^{(2n)} - \lambda\rho y = 0$, $y^{(k)}(0) = y^{(k)}(1) = 0$, $0 \le k < n$, is studied in the case where the order $2n$ of the equation satisfies the inequality $n > 1$, and the weight $\rho \in W_2^{-1}[0,1]$ is the generalized derivative of a self-similar function $P \in L_2[0,1]$ of zero spectral order.

§1. Introduction

1.1. Our aim in this paper is to extend the results of [1] about the spectral asymptotics of the boundary-value problem

$$-y'' - \lambda\rho y = 0,$$
$$y(0) = y(1) = 0,$$

where the weight $\rho \in W_2^{-1}[0,1]$ is the generalized derivative of a self-similar function $P \in L_2[0,1]$ of zero spectral order, to the case of the boundary-value problem

$$(-1)^n y^{(2n)} - \lambda\rho y = 0, \tag{1.1}$$

$$y^{(k)}(0) = y^{(k)}(1) = 0, \quad 0 \le k < n, \tag{1.2}$$

which corresponds to the same class of weight functions for $n > 1$. In essence, the techniques of the present paper coincide with those of [1]. However, this does not mean that the arguments in the general case are mechanical repetition of those in the partial case of $n = 1$. Important additional difficulties that emerge in higher orders (compared to the case of the Sturm–Liouville problem) will be indicated below.

It should be noted that the problem in question differs radically from the classical problem [2, §4.11], which corresponds to a smooth weight function ρ. In particular, the eigenvalue asymptotics obtained in §4 are of exponential rather than power order.

Throughout the paper, we reserve the symbol n to denote half the order of equation (1.1).

1.2. Finding eigenvalue asymptotics for problems like (1.1), (1.2) is interesting not only for its own sake. For example, in [3] similar spectral estimates[1] were used for obtaining logarithmic asymptotics of small deviations for some random processes with respect to certain self-similar measures. As to the context of the problem studied here, some spectral

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[1]Related, however, to the case of a weight of positive spectral order, which is treated by quite different techniques; see [4, 5].

asymptotics[2] (cruder than ours) were used also in the recent paper [6] for the study of small deviations.

1.3. The paper is organized as follows. In §2, we present the information needed below about self-similar functions of zero spectral order. In §3, an operator viewpoint to problem (1.1), (1.2) is presented and some auxiliary statements are proved. In §4, the main results about spectral asymptotics for problem (1.1), (1.2) are discussed. Finally, in §5 the constructive nature of the results is under study and some outcomes of numerical experiments that illustrate the results are exposed.

§2. Square integrable self-similar functions of zero spectral order

2.1. We fix a natural number $N > 1$. Suppose also that a collection of $3N$ real numbers $a_k > 0$, β_k, and d_k, where $k = 1, \dots, N$, satisfies the inequality

$$\sum_{k=1}^{N} a_k = 1.$$

To this collection of numbers, we relate a continuous mapping $G : L_2[0,1] \to L_2[0,1]$ of the form

$$G(f) \rightleftharpoons \sum_{k=1}^{N} \left\{ \beta_k \cdot \chi_{(\alpha_{k-1},\alpha_k)} + d_k \cdot G_k(f) \right\}, \tag{2.1}$$

where the following notation was used:

(1) by α_k, we denote the numbers $\alpha_0 \rightleftharpoons 0$ and $\alpha_k \rightleftharpoons \alpha_{k-1} + a_k$ for $k = 1, \dots, N$;
(2) by χ_Δ we denote the characteristic function of the interval Δ, viewed as an element of $L_2[0,1]$;
(3) By G_k, $k = 1, \dots, N$, we denote continuous linear mappings in the space $L_2[0,1]$ that operate by the following rule: given $f \in L_2[0,1]$, we put

$$[G_k(f)](x) \rightleftharpoons \begin{cases} f((x - \alpha_{k-1})/a_k) & \text{if } x \in (\alpha_{k-1}, \alpha_k), \\ 0 & \text{otherwise.} \end{cases}$$

In what follows, mappings of the form (2.1) will be called *similarity operators*. We mention two simple facts.

Proposition 1. *A similarity operator G in $L_2[0,1]$ is a contraction if and only if*

$$\sum_{k=1}^{N} a_k \, |d_k|^2 < 1. \tag{2.2}$$

Proposition 2. *If* (2.2) *is true, then the equation $G(f) = f$ possesses a unique solution $f \in L_2[0,1]$.*

The functions satisfying the equation $G(f) = f$ with a certain contractive similarity operator G are said to be *affine self-similar* or simply self-similar. The collections of numbers

$$\left\{N, \, \{a_k\}_{k=1}^N, \, \{\beta_k\}_{k=1}^N, \, \{d_k\}_{k=1}^N\right\}$$

that determine G will be called the collections of *self-similarity parameters* for f.

[2]Obtained by direct application of the techniques developed in [1].

2.2. A nontrivial self-similar function[3] is called a function *of zero spectral order* if its self-similarity parameters satisfy the following conditions:

(1) at least one among the numbers β_k, $k = 1, \dots, N$, is nonzero;
(2) precisely one among the numbers d_k, $k = 1, \dots, N$, is nonzero.

Below, when talking of self-similar functions of zero spectral order, we shall denote by m the natural number in $[1, N]$ for which $d_m \neq 0$. Then inequality (2.2) turns into the inequality $a_m |d_m|^2 < 1$, which, in its turn, clearly implies the inequality $a_m^{2n-1} |d_m| < 1$.

2.3. More information on square-integrable self-similar functions can be found in [7, 8].

§3. Operator treatment of the problem and some auxiliary statements

3.1. In what follows, we denote by $\mathfrak{H}$ the Sobolev space $W_2^n[0,1]$ endowed with the scalar product

$$\langle y, z \rangle \rightleftharpoons \int_0^1 y^{(n)} \overline{z^{(n)}}\, dx.$$

We denote by $\mathfrak{H}'$ the dual of $\mathfrak{H}$ relative to $L_2[0,1]$, i.e., the completion of $L_2[0,1]$ with respect to the norm

$$\|y\|_{\mathfrak{H}'} \rightleftharpoons \sup_{\|z\|_{\mathfrak{H}}=1} \left| \int_0^1 y \bar{z}\, dx \right|.$$

The definition of $\mathfrak{H}'$ shows immediately that the adjoint $J^* : L_2[0,1] \to \mathfrak{H}$ to the embedding operator $J : \mathfrak{H} \to L_2[0,1]$ can be extended by continuity up to an isometry $J^+ : \mathfrak{H}' \to \mathfrak{H}$.

As was done in [1] in the case of the Sturm–Liouville problem, for the role of a spectral model for the problem (1.1), (1.2) we take a linear pencil $T_\rho : \mathbb{C} \to \mathcal{B}(\mathfrak{H}, \mathfrak{H}')$ of bounded operators satisfying

$$(\forall \lambda \in \mathbb{C})\, (\forall y \in \mathfrak{H}) \quad \langle J^+ T_\rho(\lambda) y, y \rangle = \int_0^1 \left(|y^{(n)}|^2 + \lambda P \cdot (|y|^2)' \right) dx. \tag{3.1}$$

Here, as before, we denote by P the square integrable generalized primitive of the weight function $\rho \in W_2^{-1}[0,1]$.

3.2. In the sequel, we denote by $\operatorname{ind} D$ the negative inertia index of a bounded Hermitian operator D acting in a Hilbert space $\mathfrak{E}$, i.e., the lowest upper bound of the dimensions of subspaces $\mathfrak{M} \subseteq \mathfrak{E}$ satisfying

$$(\exists\, \varepsilon > 0)\, (\forall y \in \mathfrak{M}) \quad \langle Dy, y \rangle_{\mathfrak{E}} \leq -\varepsilon \, \|y\|_{\mathfrak{E}}^2.$$

3.3. Next, by $\mathfrak{H}_0$ we shall denote the closure of the linear span of the system of eigenvalues of the pencil T_ρ. We mention three facts.

Proposition 3. *The orthogonal complement $\mathfrak{H} \ominus \mathfrak{H}_0$ of the subspace $\mathfrak{H}_0$ is the set of functions $y \in \mathfrak{H}$ vanishing on the support $\operatorname{supp} \rho \subseteq [0,1]$ of the weight function ρ.*

Proof. The definition (3.1) and the general theory of selfadjoint operators in a Hilbert space show that the orthogonal complement in question is the set of functions $y \in \mathfrak{H}$ satisfying

$$(\forall z \in \mathfrak{H}) \qquad \int_0^1 P \cdot (y\bar{z})'\, dx = 0.$$

This is equivalent to the condition claimed. □

[3] That is, a self-similar function that is not piecewise constant with finitely many jumps.

Proposition 4. *Let λ be a real number. Then there exists a subspace $\mathfrak{M} \subseteq \mathfrak{H}_0$ of dimension* ind $J^+T_\rho(\lambda)$ *and satisfying the condition*

$$(\exists\, \varepsilon > 0)\, (\forall y \in \mathfrak{M}) \quad \langle J^+T_\rho(\lambda)y, y\rangle \le -\varepsilon\, \|y\|^2_{\mathfrak{H}}.$$

For the proof, it suffices to observe that any function $y \in \mathfrak{H}$ and its orthogonal projection $z \in \mathfrak{H}_0$ obey the inequality

$$\langle J^+T_\rho(\lambda)z, z\rangle \le \langle J^+T_\rho(\lambda)y, y\rangle.$$

Proposition 5. *For every isolated point $\xi \in \operatorname{supp}\rho$ there exists a unique function $\varphi_\xi \in \mathfrak{H}_0$ that vanishes on $\operatorname{supp}\rho \setminus \{\xi\}$ and satisfies $\varphi_\xi(\xi) = 1$.*

Proof. Take an arbitrary function $f \in \mathfrak{H}$ with $f(\xi) = 1$ that vanishes on $\operatorname{supp}\rho \setminus \{\xi\}$. By statement 3, the orthogonal projection of f to $\mathfrak{H}_0$ can be chosen for the role of φ_ξ. To complete the proof, it suffices to observe that an arbitrary function $y \in \mathfrak{H}_0$ is uniquely determined by its restriction to $\operatorname{supp}\rho$. □

3.4. We introduce two subspaces $\mathfrak{H}_1 \subseteq \mathfrak{H}_0$ and $\mathfrak{H}_2 \subseteq \mathfrak{H}_0$ as follows:

(1) $\mathfrak{H}_1$ is formed by all functions $y \in \mathfrak{H}_0$ that vanish on $\operatorname{supp}\rho \setminus (\alpha_{m-1}, \alpha_m)$;
(2) $\mathfrak{H}_2$ is the linear hull of the functions φ_ξ corresponding to arbitrary points $\xi \in \operatorname{supp}\rho \setminus (\alpha_{m-1}, \alpha_m)$.

Consider two linear pencils $A : \mathbb{C} \to \mathcal{B}(\mathfrak{H}_1, \mathfrak{H}_1)$ and $C : \mathbb{C} \to \mathcal{B}(\mathfrak{H}_2, \mathfrak{H}_2)$ of bounded operators such that

$$(3.2)\qquad \begin{aligned} &(\forall \lambda \in \mathbb{C})\, (\forall y \in \mathfrak{H}_1) \quad \langle A(\lambda)y, y\rangle = \int_0^1 \big(|y^{(n)}|^2 + \lambda P \cdot (|y|^2)'\big)\, dx,\\ &(\forall \lambda \in \mathbb{C})\, (\forall y \in \mathfrak{H}_2) \quad \langle C(\lambda)y, y\rangle = \int_0^1 \big(|y^{(n)}|^2 + \lambda P \cdot (|y|^2)'\big)\, dx, \end{aligned}$$

and also an operator $B : \mathfrak{H}_1 \to \mathfrak{H}_2$ satisfying

$$(3.3)\qquad (\forall y \in \mathfrak{H}_1)\, (\forall z \in \mathfrak{H}_2) \quad \langle By, z\rangle = \int_0^1 y^{(n)}\overline{z^{(n)}}\, dx.$$

Proposition 6. *There exists a nonnegative operator $E : \mathfrak{H}_1 \to \mathfrak{H}_1$ of finite rank such that, independently of $\lambda > 0$, we have*

$$\operatorname{ind}[A(\lambda) + E] = \operatorname{ind} J^+T_\rho(a_m^{2n-1} d_m\, \lambda).$$

Proof. Consider an operator $S : \mathfrak{H} \to \mathfrak{H}$ of the form

$$[Sy](x) \rightleftharpoons \begin{cases} y((x - \alpha_{m-1})/a_m) & \text{for } x \in (\alpha_{m-1}, \alpha_m),\\ 0 & \text{otherwise,} \end{cases}$$

and also the operator $R : \mathfrak{H}_1 \to \mathfrak{H}$ that takes any function $y \in \mathfrak{H}_1$ to the orthogonal projection to $\mathfrak{H}_0$ of a function of the form

$$z(x) \rightleftharpoons \psi(x) y(\alpha_{m-1} + a_m x),$$

where $\psi \in W_2^n[0, 1]$ is an arbitrary fixed function identically equal to 1 on some neighborhood of the set $\operatorname{supp}\rho$ and vanishing on some neighborhood of $\{0, 1\} \setminus \operatorname{supp}\rho$. Proposition 3 and the fact that P is self-similar show that, for every $y \in \mathfrak{H}_1$, the following two statements hold true.

(1) The orthogonal projection of SRy to the subspace $\mathfrak{H}_0$ coincides with y.
(2) The function Ry is a unique element of $\mathfrak{H}_0$ whose S-image has orthogonal projection to this subspace equal to y.

Accordingly, the operator $E \rightleftharpoons R^*S^*SR - 1$ is nonnegative. Proposition 3 and the fact that P is self-similar also imply that $SRy = y$ on the finite-codimensional subspace of functions $y \in \mathfrak{H}_1$ that vanish outside the interval (α_{m-1}, α_m). Thus, the operator E is of finite rank.

Finally, the definitions (3.1) and (3.2), Proposition 3, and the fact that P is self-similar easily imply the identity

$$(\forall y \in \mathfrak{H}_1) \quad \langle [A(\lambda) + E]y, y\rangle = a_m^{1-2n} \cdot \big\langle J^+T_\rho(a_m^{2n-1} d_m\, \lambda) Ry, Ry\big\rangle.$$

Combining this identity with Proposition 4 and the relation $\operatorname{im} R = \mathfrak{H}_0$ (which follows from the above discussion), we prove the claim. □

Proposition 7. *Let λ be a real number not belonging to the spectrum of the pencil C. Then*

$$\operatorname{ind} J^+T_\rho(\lambda) = \operatorname{ind}[A(\lambda) - B^*C^{-1}(\lambda)B] + \operatorname{ind} C(\lambda).$$

Proof. A direct calculation easily shows that for every $y \in \mathfrak{H}_1$ and $z \in \mathfrak{H}_2$ we have

$$\big\langle J^+T_\rho(\lambda)\,(y+z), (y+z)\big\rangle = \big\langle [A(\lambda) - B^*C^{-1}(\lambda)B]y, y\big\rangle + \langle C(\lambda)u, u\rangle,$$

where we have put $u \rightleftharpoons z + C^{-1}(\lambda)By$. Combining this with Proposition 4 and the relation $\mathfrak{H}_0 = \mathfrak{H}_1 \dotplus \mathfrak{H}_2$, we prove the claim. □

3.5. Consider some quantities ζ_k, where $k = 1, \dots, N-1$, of the form

$$\zeta_k \rightleftharpoons \begin{cases} \beta_m - \beta_{m-1} + d_m\beta_1 & \text{for} \quad k = m-1, \\ \beta_{m+1} - \beta_m - d_m\beta_N & \text{for} \quad k = m, \\ \beta_{k+1} - \beta_k & \text{otherwise.} \end{cases}$$

Also, denote by $\mathrm{Z}_\pm$ two quantities

$$\mathrm{Z}_\pm \rightleftharpoons \#\{k \in [1, N-1] \,:\, \pm\zeta_k > 0\}.$$

Proposition 8. *For every sufficiently large real number $\lambda > 0$, we have*

$$\operatorname{ind} C(\lambda) = \mathrm{Z}_+.$$

This is an immediate consequence of the following identity, the proof of which is easy

$$(\forall\lambda \in \mathbb{R})\,(\forall y \in \mathfrak{H}_2) \quad \langle C(\lambda)y, y\rangle = \|y\|^2_{\mathfrak{H}} - \lambda \sum_{k=1}^{N-1} \zeta_k \cdot |y(\alpha_k)|^2. \tag{3.4}$$

Proposition 9. *Suppose that $\mathrm{Z}_+ + \mathrm{Z}_- = N - 1$. Then for every sufficiently large real number $\lambda > 0$, the operator $C(\lambda)$ is boundedly invertible*; *moreover, for $\lambda \to +\infty$ we have the following asymptotics*:

$$\|C^{-1}(\lambda)\| = O(\lambda^{-1}).$$

This is also an easy consequence of (3.4).

3.6.

Proposition 10. *Let $\mathfrak{E}$ be a finite-dimensional Hilbert space, $D : \mathfrak{E} \to \mathfrak{E}$ a nonnegative operator, and $F : \mathfrak{E} \to \mathfrak{E}$ a Hermitian operator. Denote by $\{\mu_k\}_{k=1}^r$ and $\{\lambda_k\}_{k=1}^r$ (where $r \rightleftharpoons \operatorname{rank} F$) the lists of eigenvalues (counted with multiplicities) of the pencils $1 - \lambda F$ and $1 + D - \lambda F$, respectively. Then*

$$\prod_{k=1}^r \frac{\lambda_k}{\mu_k} \le \det(1 + D).$$

Proof. We decompose $\mathfrak{E}$ in the direct sum $\ker F \oplus \operatorname{im} F$ and consider block-matrix representations corresponding to these decompositions:

$$D = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & 0 \\ 0 & F_{22} \end{pmatrix}.$$

Observe that the spectrum of the pencil $1 - \lambda F$ coincides with the spectrum of the pencil $1 - \lambda F_{22}$, whereas the spectrum of the pencil $1 + D - \lambda F$ coincides with that of $1 + D_{22} - D_{21}(1 + D_{11})^{-1} D_{12} - \lambda F_{22}$. Accordingly, we have the identities

$$\prod_{k=1}^{r} \mu_k = \frac{1}{\det F_{22}}, \qquad \prod_{k=1}^{r} \lambda_k = \frac{\det[1 + D_{22} - D_{21}(1 + D_{11})^{-1} D_{12}]}{\det F_{22}},$$

whence it follows that

$$\prod_{k=1}^{r} \frac{\lambda_k}{\mu_k} = \det[1 + D_{22} - D_{21}(1 + D_{11})^{-1} D_{12}] = \frac{\det(1 + D)}{\det(1 + D_{11})}.$$

The claim readily follows from this and the fact that the operators D and D_{11} are nonnegative. □

Proposition 11. *Let $\mathfrak{E}$ be a separable Hilbert space, $D : \mathfrak{E} \to \mathfrak{E}$ a nonnegative operator of finite rank, and $F : \mathfrak{E} \to \mathfrak{E}$ a compact Hermitian operator. Let also $\{\mu_k\}_{k=1}^{\infty}$ and $\{\lambda_k\}_{k=1}^{\infty}$ be the sequences of positive eigenvalues (counted with multiplicities and enumerated in the increasing order) of the pencils $1 - \lambda F$ and $1 + D - \lambda F$, respectively. Then the sequence of partial products of the infinite product*

$$\prod_{k=1}^{\infty} \frac{\lambda_k}{\mu_k} \tag{3.5}$$

is monotone nondecreasing and bounded.

Proof. Note that $\operatorname{ind}[1 + D - \lambda F] \le \operatorname{ind}[1 - \lambda F]$ for every $\lambda \in \mathbb{R}$. Note also that all positive eigenvalues of the pencils $1 - \lambda F$ and $1 + D - \lambda F$ are of negative type (see, e.g., [9, 10]). General variation principles for selfadjoint operator-valued functions (see the above reference) show that, therefore, for every $k \ge 1$ we have $\lambda_k \ge \mu_k$. Thus, the sequence of partial products of the infinite product (3.5) is monotone nondecreasing.

Next, we fix a sequence $\{Q_l\}_{l=1}^{\infty}$ of finite rank orthogonal projections convergent pointwise to the identity and satisfying $Q_l D Q_l \equiv D$. Denote by $\{\mu_{k,l}\}_{k=1}^{\infty}$ and $\{\lambda_{k,l}\}_{k=1}^{\infty}$ sequences[4] of positive eigenvalues (enumerated in the increasing order and counted with multiplicities) of the pencils $1 - \lambda Q_l F Q_l$ and $1 + D - \lambda Q_l F Q_l$, respectively. Similarly, we denote by $\{\mu_{-k,l}\}_{k=1}^{\infty}$ and $\{\lambda_{-k,l}\}_{k=1}^{\infty}$ sequences of negative eigenvalues (enumerated in the decreasing order and counted with multiplicities) of the same pencils. The variation principles mentioned above show that the number $r_{+,l}$ of positive eigenvalues and the number $r_{-,l}$ of negative eigenvalues of the pencil $1 - \lambda Q_l F Q_l$ coincide with the similar quantities for $1 + D - \lambda Q_l F Q_l$ and, moreover, we have the inequalities

$$\begin{aligned} (\forall k \in [1, r_{+,l}]) &\quad \lambda_{k,l} \ge \mu_{k,l}, \\ (\forall k \in [1, r_{-,l}]) &\quad \lambda_{-k,l} \le \mu_{-k,l}. \end{aligned}$$

Taking this into account, we see that the relation

$$\prod_{k=1}^{r_{+,l}} \frac{\lambda_{k,l}}{\mu_{k,l}} \times \prod_{k=1}^{r_{-,l}} \frac{\lambda_{-k,l}}{\mu_{-k,l}} \le \det(1 + D)$$

[4]They are partial, i.e., we do not suppose that their terms are determined for every index $k \ge 1$.

(which is a consequence of Proposition 10) means that

$$\prod_{k=1}^{r} \frac{\lambda_{k,l}}{\mu_{k,l}} \le \det(1+D)$$

for every choice of $r \le r_{+,l}$. A limit passage readily implies that, independently of the choice of $r \ge 1$, we have

$$\prod_{k=1}^{r} \frac{\lambda_k}{\mu_k} \le \det(1+D).$$

Thus, the partial products of (3.5) are bounded. □

§4. Principal results

4.1. The following three statements hold.

Theorem 1. *Suppose that* $d_m > 0$, $\mathrm{Z}_+ > 0$, *and* $\mathrm{Z}_+ + \mathrm{Z}_- = N-1$. *Then there exist real numbers* $\tau_l > 0$, $l = 1, \dots, \mathrm{Z}_+$, *such that the sequence* $\{\lambda_k\}_{k=1}^{\infty}$ *of positive eigenvalues* (*enumerated in the increasing order and counted with multiplicities*) *for problem* (1.1), (1.2) *satisfies the following asymptotic relation as* $k \to \infty$:

$$\lambda_{l+k\mathrm{Z}_+} = \tau_l \cdot (a_m^{2n-1} d_m)^{-k} \cdot (1+o(1)).$$

Proof. By Proposition 9, there exists a real number $\lambda_0 > 0$ such that for any $\lambda > \lambda_0$ we have $\|C^{-1}(\lambda)\| < \lambda_0/(3\lambda)$. Combined with the estimate $\|B\| \le 1$ (which is clear from the definition (3.3)), this shows that for every $\lambda > \lambda_0$ the following inequalities are true:

$$\text{(4.1)} \qquad \operatorname{ind}[A(\lambda) + \lambda_0/(3\lambda)] \le \operatorname{ind}[A(\lambda) - B^* C^{-1}(\lambda) B] \le \operatorname{ind}[A(\lambda) - \lambda_0/(3\lambda)].$$

Now, the definition (3.2) readily implies the identities

$$A(\lambda \pm \lambda_0/2) = (1 \pm \lambda_0/(2\lambda)) \cdot \left[A(\lambda) \mp \frac{\lambda_0}{2\lambda \pm \lambda_0} \right];$$

using them and (4.1), we easily deduce from the above that

$$\text{(4.2)} \qquad \operatorname{ind} A(\lambda - \lambda_0/2) \le \operatorname{ind}[A(\lambda) - B^* C^{-1}(\lambda) B] \le \operatorname{ind} A(\lambda + \lambda_0/2).$$

We denote by $\{\mu_k\}_{k=1}^{\infty}$ the sequence of positive eigenvalues of the pencil A enumerated in the increasing order and counted with multiplicities. From (4.2), Propositions 7, 8, and variational principles for selfadjoint operator-valued functions (see, e.g., [9, 10]), it follows that for all $k \gg 1$ we have

$$\mu_k - \lambda_0/2 \le \lambda_{k+\mathrm{Z}_+} \le \mu_k + \lambda_0/2;$$

therefore, the following asymptotic relation holds true:

$$\text{(4.3)} \qquad \frac{\lambda_{k+\mathrm{Z}_+}}{\mu_k} = 1 + O(\lambda_{k+\mathrm{Z}_+}^{-1}).$$

On the other hand, by Propositions 6 and 11, the sequence of partial products of the infinite product

$$\text{(4.4)} \qquad \prod_{k=1}^{\infty} \frac{(a_m^{2n-1} d_m) \cdot \mu_k}{\lambda_k}$$

is monotone and bounded. Combining this fact with the asymptotics (4.3), we prove that, independently of the choice of the parameters $l \ge 1$ and $\varepsilon > 0$, we have the asymptotics

$$\lambda_{l+k\mathrm{Z}_+}^{-1} = O\big([(1+\varepsilon) \cdot a_m^{2n-1} d_m]^k\big),$$

which means that the infinite product

$$\prod_{k=1}^{\infty} \frac{\lambda_{l+k\mathrm{Z}_+}}{\mu_{l+(k-1)\mathrm{Z}_+}}.$$

converges. To finish the proof, it suffices to combine this with the fact that the sequence of partial products of the infinite product (4.4) is monotone and bounded (which has already been mentioned). □

Theorem 2. *Suppose that $d_m > 0$, $\mathrm{Z}_- > 0$, and $\mathrm{Z}_+ + \mathrm{Z}_- = N - 1$. Then there exist real numbers $\tau_l > 0$, $l = 1, \dots, \mathrm{Z}_-$, such that the sequence $\{\lambda_{-k}\}_{k=1}^{\infty}$ of negative eigenvalues (enumerated in the decreasing order and counted with multiplicities) for problem* (1.1), (1.2) *satisfies the following asymptotic relation as $k \to \infty$:*

$$\lambda_{-(l+k\mathrm{Z}_-)} = -\tau_l \cdot (a_m^{2n-1} d_m)^{-k} \cdot (1 + o(1)).$$

Theorem 2 is proved much as the preceding Theorem 1.

Theorem 3. *Suppose that the relations $d_m < 0$ and $\mathrm{Z}_+ + \mathrm{Z}_- = N - 1$ are fulfilled. Then there exist real numbers $\tau_l > 0$, $l = 1, \dots, N-1$, such that the sequence $\{\lambda_k\}_{k=1}^{\infty}$ of positive eigenvalues (enumerated in the increasing order and counted with multiplicities) of problem* (1.1), (1.2) *satisfies the following asymptotic relation as $k \to \infty$:*

$$\lambda_{l+k(N-1)} = \tau_l \cdot (a_m^{2n-1} |d_m|)^{-2k} \cdot (1 + o(1)),$$

and the sequence $\{\lambda_{-k}\}_{k=1}^{\infty}$ of negative eigenvalues (enumerated in the decreasing order and counted with multiplicities) of problem (1.1), (1.2) *satisfies the following asymptotic relation as $k \to \infty$:*

$$\lambda_{-(l+\mathrm{Z}_-+k(N-1))} = -\tau_l \cdot (a_m^{2n-1} |d_m|)^{-2k-1} \cdot (1 + o(1)).$$

The proof of Theorem 3 is also similar to that of Theorem 1.

4.2. The theorems stated above do not cover the case where

$$\mathrm{Z}_+ + \mathrm{Z}_- < N - 1, \tag{4.5}$$

which means that some of the quantities ζ_k vanish. However, it is easily seen that if all indices k with the property $\zeta_k = 0$ satisfy $k \notin \{m-1, m\}$, then the identity $\mathrm{Z}_+ + \mathrm{Z}_- = N - 1$ can be ensured by redefining the collection of self-similarity parameters. Thus, substantially, the case (4.5) corresponds to the situation when at least one point of the set $\{\alpha_{m-1}, \alpha_m\} \cap (0, 1)$ is a continuity point for P. This case will be left beyond the scope of the present paper.

§5. Examples and discussion

5.1. In Table 1, we present the results of numerical evaluation of the first eight positive eigenvalues of the fourth order problem in which the weight function is the generalized derivative of the square integrable function with self-similarity parameters

$$N = 3, \quad a_1 = a_2 = a_3 = 1/3, \quad m = 3, \quad d_3 = 1/2, \quad \beta_1 = 0, \quad \beta_2 = 2/3, \quad \beta_3 = 1.$$

In this case we have $\zeta_1 = 2/3$, $\zeta_2 = 1/3$, $\mathrm{Z}_+ = 2$, $\mathrm{Z}_- = 0$. Table 1 illustrates Theorem 1.

In Table 2, we present the results of numerical evaluation of the first four negative eigenvalues of the fourth order problem in which the weight function is the generalized derivative of the square-integrable function with self-similarity parameters

$$N = 3, \quad a_1 = a_2 = a_3 = 1/3, \quad m = 3, d_3 = 1/2, \quad \beta_1 = \beta_3 = 0, \quad \beta_2 = -1.$$

In this case we have $\zeta_1 = -1$, $\zeta_2 = 1$, $\mathrm{Z}_+ = \mathrm{Z}_- = 1$. Table 2 illustrates Theorem 2.

TABLE 1. Estimates of first eigenvalues in the case where $n = 2$, $N = 3$, $a_1 = a_2 = a_3 = 1/3$, $m = 3$, $d_3 = 1/2$, $\beta_1 = 0$, $\beta_2 = 2/3$, $\beta_3 = 1$.

l	k	λ_{l+2k}	$\lambda_{l+2k}/54^k$
1	0	$2,86 \cdot 10^2 \pm 1\%$	$286{,}10 \pm 10^{-2}$
2	0	$1,38 \cdot 10^3 \pm 1\%$	$1377{,}99 \pm 10^{-2}$
1	1	$1,48 \cdot 10^4 \pm 1\%$	$273{,}71 \pm 10^{-2}$
2	1	$6,83 \cdot 10^4 \pm 1\%$	$1265{,}31 \pm 10^{-2}$
1	2	$7,91 \cdot 10^5 \pm 1\%$	$271{,}33 \pm 10^{-2}$
2	2	$3,69 \cdot 10^6 \pm 1\%$	$1264{,}04 \pm 10^{-2}$
1	3	$4,27 \cdot 10^7 \pm 1\%$	$271{,}32 \pm 10^{-2}$
2	3	$1,99 \cdot 10^8 \pm 1\%$	$1264{,}04 \pm 10^{-2}$

TABLE 2. Estimates of first eigenvalues in the case of $n = 2$, $N = 3$, $a_1 = a_2 = a_3 = 1/3$, $m = 3$, $d_3 = 1/2$, $\beta_1 = \beta_3 = 0$, $\beta_2 = -1$.

l	k	$-\lambda_{-(l+k)}$	$-\lambda_{-(l+k)}/54^k$
1	0	$3,70 \cdot 10^2 \pm 1\%$	$369{,}75 \pm 10^{-2}$
1	1	$8,51 \cdot 10^3 \pm 1\%$	$157{,}53 \pm 10^{-2}$
1	2	$4,58 \cdot 10^5 \pm 1\%$	$157{,}20 \pm 10^{-2}$
1	3	$2,48 \cdot 10^7 \pm 1\%$	$157{,}20 \pm 10^{-2}$

TABLE 3. Estimates of first eigenvalues in the case of $n = 2$, $N = 3$, $a_1 = a_2 = a_3 = 1/3$, $m = 3$, $d_3 = -1/2$, $\beta_1 = \beta_3 = 0$, $\beta_2 = -1$.

l	k	λ_{l+2k}	$\lambda_{l+2k}/54^{2k}$
1	0	$3,04 \cdot 10^2 \pm 1\%$	$304{,}08 \pm 10^{-2}$
2	0	$1,38 \cdot 10^4 \pm 1\%$	$13820{,}11 \pm 10^{-2}$
1	1	$8,72 \cdot 10^5 \pm 1\%$	$299{,}00 \pm 10^{-2}$
2	1	$4,01 \cdot 10^7 \pm 1\%$	$13764{,}02 \pm 10^{-2}$
1	2	$2,54 \cdot 10^9 \pm 1\%$	$299{,}00 \pm 10^{-2}$
2	2	$1,17 \cdot 10^{11} \pm 1\%$	$13764{,}02 \pm 10^{-2}$

l	k	$-\lambda_{-(l+1+2k)}$	$-\lambda_{-(l+1+2k)}/54^{2k+1}$
1	0	$1,61 \cdot 10^4 \pm 1\%$	$299{,}04 \pm 10^{-2}$
2	0	$7,43 \cdot 10^5 \pm 1\%$	$13764{,}22 \pm 10^{-2}$
1	1	$4,71 \cdot 10^7 \pm 1\%$	$299{,}00 \pm 10^{-2}$
2	1	$2,17 \cdot 10^9 \pm 1\%$	$13764{,}02 \pm 10^{-2}$
1	2	$1,37 \cdot 10^{11} \pm 1\%$	$299{,}00 \pm 10^{-2}$
2	2	$6,32 \cdot 10^{12} \pm 1\%$	$13764{,}02 \pm 10^{-2}$

In Table 3, we present the results of numerical evaluation of the first six positive and seven negative (except that with index 1) eigenvalues for the fourth order problem in which the weight function is the generalized derivative of the square integrable function with self-similarity parameters

$$N = 3, \quad a_1 = a_2 = a_3 = 1/3, \quad m = 3, \quad d_3 = -1/2, \quad \beta_1 = \beta_3 = 0, \quad \beta_2 = -1.$$

In this case we have $\zeta_1 = -1$, $\zeta_2 = 1$, $\mathrm{Z}_+ = \mathrm{Z}_- = 1$. Table 3 illustrates Theorem 3.

To obtain the above illustrative material, we used the calculation method described in [11], with slight modifications.

5.2. Certain arguments in the preceding sections are not appropriate from a constructive mathematics viewpoint, see [12, 13]. In particular, this applies to the proofs in Subsection 3.3, involving orthogonal projections in Hilbert space. A slightly bulkier argument (not presented here) could be used to avoid the consequences of the orthogonal projection theorem without influencing the essence. However, the use of the Bolzano–Weierstrass theorem on the convergence of a bounded monotone sequence in §4 seems to be indispensable in the present framework. Accordingly, the coefficients τ_l of the asymptotic formulas obtained here turn out to be so-called *pseudonumbers* from a constructive viewpoint[5].

In the case of the Sturm–Liouville problem, the operator E in Proposition 6 is zero, which allows one to completely renounce the use of Proposition 11. The question of further detail about the constructive nature of the asymptotic formulas obtained here remains open.

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[5]This means that an algorithm of construction of arbitrarily close rational approximations for these numbers is either unknown or impossible.

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