Алгебра и анализ
Том 24 (2012), № 6

St. Petersburg Math. J.
Vol. 24 (2013), No. 6, Pages 863–876
S 1061-0022(2013)01270-2
Article electronically published on September 23, 2013

ON INDEPENDENCE OF SOME PSEUDOCHARACTERS ON BRAID GROUPS

I. A. DYNNIKOV AND V. A. SHASTIN

Abstract. It is proved that the pseudocharacter defined on the braid group by the signature of braid closures is linearly independent of all pseudocharacters obtained from the twist number via the Malyutin operators, provided that the number of strands is greater than 4. This pseudocharacter is shown to have a nontrivial kernel part. It is observed that the operators I and R defined by Malyutin on the space of pseudocharacters satisfy the Heisenberg relation, and that some of Malyutin's results are standard consequences of this fact.

§1. Pseudocharacters

Definition 1. A *quasicharacter* on a group G is a map $\chi : G \to \mathbb{R}$ such that the quantity

$$C_\chi = \sup_{g,h\in G} |\chi(gh) - \chi(g) - \chi(h)|,$$

called the *defect* of χ, is finite. A quasicharacter χ is called a *pseudocharacter* if

$$\chi(g^k) = k \cdot \chi(g)$$

for all $g \in G$, $k \in \mathbb{Z}$.

Trivially, each bounded function on a group is a quasicharacter. Since adding a bounded function to a quasicharacter yields a quasicharacter, it is natural to consider quasicharacters up to addition of bounded functions. By a result of [1], this is equivalent to the consideration of pseudocharacters only. Namely, the following is true.

Theorem 1 (see [1])**.** *Every quasicharacter on an arbitrary group G admits a unique presentation as the sum of a pseudocharacter and a function bounded in absolute value by the defect C_χ.*

Thus, for any quasicharacter χ there is a unique pseudocharacter, which we denote by $\bar{\chi}$, that differs from χ by a bounded function, and $C_{\bar{\chi}} \le 4C_\chi$. Its uniqueness follows from the obvious relation

$$\bar{\chi}(g) = \lim_{k\to\infty} \frac{\chi(g^k)}{k}. \tag{1}$$

Obviously, each pseudocharacter is constant on every conjugacy class. Also, we mention the following fact.

Theorem 2 (see [2])**.** *Let H be a normal subgroup of finite index in a group G, and let χ be a pseudocharacter on H such that*

$$\chi(ghg^{-1}) = \chi(h) \quad \textit{for all} \quad g \in G, h \in H. \tag{2}$$

2010 *Mathematics Subject Classification.* Primary 20F36.
Key words and phrases. Braid group, pseudocharacter, twist numbers, signature.
Supported by RFBR (grant no. 10-01-91056-NCNI) and by the Government of RF (grant no. 2010-220-01-077).

Then χ extends uniquely to a pseudocharacter on the entire group G.

The uniqueness of extension is a consequence of the fact that the kth power of any element $g \in G$, where $k = |G/H|$, lies in H, and upon extension we should have

$$\chi(g) = \frac{\chi(g^k)}{k}.$$

Note that if on a normal subgroup $H \subset G$ of finite index we have a pseudocharacter χ for which condition (2) fails, then we can act as follows to get a pseudocharacter χ satisfying (2): take $g_1, \dots, g_k \in G$, where $k = |G/H|$ so that the cosets g_iH are pairwise distinct and average χ by conjugating by the elements g_i:

$$\chi'(g) = \sum_{i=1}^{k} \chi(g_i g g_i^{-1}). \tag{3}$$

In the present paper we shall consider the Artin braid group B_n on n strands; the above theorem will be applied to the case where $H = P_n$, the pure braid subgroup. Thus, if some pseudocharacter is already defined for pure braids on n strands and condition (2) is satisfied, then we tacitly assume it to be extended to the entire braid group.

All pseudocharacters on a fixed group form a vector space over $\mathbb{R}$; in the case of the braid group B_n this vector space will be denoted by X_n. The results of [3, 4] show that for $n \geq 3$ this space X_n is infinite-dimensional.

It is clear that, for two groups G_1 and G_2, the composition of any pseudocharacter χ on G_2 with an arbitrary group homomorphism $\varphi : G_1 \to G_2$ is a pseudocharacter on G_1.

In [5], Malyutin proposed to construct pseudocharacters on B_n, starting with those on B_{n-1}, with the help of the strand removal procedure. This construction yields a linear homomorphism R from X_{n-1} to X_n and can be described as follows.

Let χ be a pseudocharacter on the group B_{n-1}. For an arbitrary pure braid $\beta \in P_n$, let $\operatorname{del}_i \beta$ denote the braid obtained from β by removing the ith strand. Put

$$(R\chi)(\beta) = \sum_{i=1}^{n} \chi(\operatorname{del}_i(\beta)).$$

Since del_i is a homomorphism of P_n to P_{n-1}, the operator R takes pseudocharacters to pseudocharacters. Summation over all strands ensures condition (2). Then the pseudocharacter $R\chi$ extends to the entire group B_n in the standard way.

We denote by ι_{n-1} the standard embedding $B_{n-1} \to B_n$ that takes the Artin generators $\sigma_1, \dots, \sigma_{n-2}$ of B_{n-1} to the similar generators of B_n. The embedding ι_{n-1} gives rise to an operator $I : X_n \to X_{n-1}$. The value taken by the image $I(\chi)$ of a pseudocharacter χ at an arbitrary braid $\beta \in B_{n-1}$ is calculated by the formula

$$I(\chi)(\beta) = \chi(\iota_{n-1}(\beta)).$$

Definition 2. A pseudocharacter χ on B_n is said to be *kernel* if $\chi(\beta) = 0$ for every $\beta \in \iota_{n-1}(B_{n-1})$.

In other words, a pseudocharacter χ on B_n is kernel if $I(\chi) = 0$. The subspace of kernel pseudocharacters in X_n will be denoted by Y_n.

In [2], Malyutin showed how kernel pseudocharacters can be used for proving that some Markov type operations cannot be applied to a braid.

In [5], Malyutin noticed a number of facts about the spaces X_n and the operators I and R that are standard consequences of the following observation, missed in [5].

Theorem 3. *The Malyutin operators I and R on the space*

$$X = \bigoplus_{n=2}^{\infty} X_n$$

of all pseudocharacters on the braid groups satisfy the Heisenberg relation

$$IR - RI = 1.$$

Proof. Let $\chi \in X_n$ be an arbitrary pseudocharacter on B_n and $\beta \in P_n$ an arbitrary pure braid. Then

$$\begin{aligned}((IR)\chi)(\beta) = (R\chi)(\iota_n(\beta)) &= \sum_{i=1}^{n+1} \chi(\mathrm{del}_i(\iota_n(\beta))) \\ &= \sum_{i=1}^{n} \chi(\iota_{n-1}(\mathrm{del}_i(\beta))) + \chi(\beta) = ((RI)\chi)(\beta) + \chi(\beta). \qquad \square\end{aligned}$$

Since, obviously, $I^{n-1}\chi = 0$ for all $\chi \in X_n$, we immediately obtain the following statement.

Corollary 1. *The space X of all pseudocharacters on braid groups carries the structure of a module over the Weyl algebra; this module is isomorphic to*

$$\mathbb{R}[x] \otimes \left(\bigoplus_{n=2}^{\infty} Y_n \right),$$

and on $\mathbb{R} \otimes \left(\bigoplus_{n=2}^{\infty} Y_n\right) = \bigoplus_{n=2}^{\infty} Y_n \subset X$ the isomorphism can be taken identical; if this is done, the operator R converts to multiplication by x and the operator I to the differentiation d/dx. In particular, we have

$$X_n = Y_n \oplus R(Y_{n-1}) \oplus R^2(Y_{n-2}) \oplus \cdots \oplus R^{n-2}(Y_2),$$

the operator R is injective, and I is surjective.

Thus, the space X_n splits into a direct sum of the subspace Y_n of kernel pseudocharacters and the subspace $R(X_{n-1})$, and for each pseudocharacter χ on B_n we can define its *kernel part* χ_{ker}, namely, the projection of χ to Y_n along $R(X_{n-1})$.

Corollary 2. *The kernel part of an arbitrary pseudocharacter χ on B_n is equal to*

$$\chi_{\mathrm{ker}} = \sum_{k=0}^{n-2} \frac{(-1)^k}{k!} R^k I^k \chi. \tag{4}$$

Proof. Since

$$X \cong \mathbb{R}[x] \otimes \left(\bigoplus_{n=2}^{\infty} Y_n \right)$$

(isomorphism of modules over Weyl′s algebra), we can check formula (4) replacing χ with a polynomial $f \in \mathbb{R}[x]$ of degree $n-2$, the operator R with x, and I with d/dx:

$$\sum_{k=0}^{n-2} \frac{(-1)^k}{k!}\, x^k \,\frac{d^k f(x)}{dx^k} = f(x-x) = f(0).$$

The constant term $f(0)$ of the polynomial f is precisely its “kernel part”. $\square$

Note that the operator I^k takes X_n to X_{n-k} for each n, and coincides with the operator of restriction to a subgroup under the standard embedding $B_{n-k} \hookrightarrow B_n$. It is easily seen that the operator R^k acts on $\chi \in X_n$ as follows:

$$(R^k\chi)(\beta) = k! \sum_{\substack{\{i_1,\dots,i_k\}\subset\{1,2,\dots,n+k\},\\ i_1<\dots<i_k}} \chi(\mathrm{del}_{i_1,\dots,i_k}(\beta)), \tag{5}$$

where $\beta \in P_{n+k}$ is a pure braid, and $\mathrm{del}_{i_1,\dots,i_k}$ means that the strands with numbers $i_1, \dots, i_k$ are removed from β. Thus, Corollary 2 is a reformulation of Theorem 2 in [2].

We proceed to the definition of the pseudocharacters to be studied.

1.1. Exponent sum.

Definition 3. The *exponent sum* $\exp(\beta)$ of a braid $\beta \in B_n$ is the image of β under the homomorphism $B_n \to \mathbb{Z}$ that takes each Artin generator σ_i to 1.

The exponent sum is not only a pseudocharacter, but in fact a group homomorphism from B_n to $\mathbb{Z}$. It is easily seen that, for $n > 2$,

$$R(\exp_{n-1}) = (n-2)\exp_n, \quad I(\exp_n) = \exp_{n-1}, \tag{6}$$

where $\exp_n$ denotes the exponent sum on B_n, and $\exp_{n-1}$ – on B_{n-1}. Thus, we cannot obtain new pseudocharacters from exp by applying the operators I and R.

1.2. Twist number. The braid group B_n admits left-invariant orderings, the first of which was invented by Dehornoy [6] and is defined as follows. A braid $\beta \in B_n$ is positive in the Dehornoy order if, for some $i = 1, \dots, n-1$, it can be represented by a word in the Artin generators such that σ_i occurs in this word at least once, but neither σ_i^{-1}, nor σ_j with $j < i$ occur in it. We write $\beta_1 < \beta_2$ if the braid $\beta_1^{-1}\beta_2$ is positive in the Dehornoy order. For our purposes, any Thurston type ordering will fit.

Definition 4. The *Dehornoy floor* $[\beta]_{\mathrm{D}}$ of a braid $\beta \in B_n$ is the largest integer k such that $\beta \geq \Delta_n^{2k}$, where Δ_n is the Garside element, the square of which equals

$$\Delta_n^2 = (\sigma_1\sigma_2\dots\sigma_{n-1})^n.$$

The following statement can be proved by using the fact that Δ_n^2 is a central element of the group B_n.

Proposition 1 (see [8])**.** *The Dehornoy floor is a quasicharacter with defect* 1.

Definition 5. For a braid $\beta \in B_n$, the quantity

$$\omega(\beta) = \lim_{k\to\infty} \frac{[\beta^k]_{\mathrm{D}}}{k} \tag{7}$$

is called the *twist number* of β.

In accordance with Theorem 1 and formula (7), the function ω is well defined on B_n and is a pseudocharacter. The properties of the Dehornoy ordering imply the following statement (see [8]).

Proposition 2. *The pseudocharacter ω is kernel.*

In the sequel, the twist number viewed as a function on B_n will sometimes be denoted by ω_n. The next statement is a consequence of Proposition 2 and the already established structure of X as a module over the Weyl algebra.

Corollary 3. *The pseudocharacters ω_n, $R\omega_{n-1}$, $\dots$, $R^{n-2}\omega_2 \in X_n$ are linearly independent.*

Observe that the pseudocharacter ω_2 coincides with $\frac{1}{2}\exp_2$; together with (6) this yields the formula

$$R^{n-2}\omega_2 = \frac{(n-2)!}{2}\exp_n .$$

1.3. Pseudocharacters of twist type. Yet another way to obtain new pseudocharacters was communicated to us by A. Malyutin. This construction is similar to applying the operator R, but employs another homomorphism from P_n to P_{n-1}, which arose in [9] and is defined as follows. Let $\beta \in P_n$ be a pure braid. It can be represented in the form

$$\beta = \beta_1\sigma_{n-1}^{\pm 2}\beta_2\sigma_{n-1}^{\pm 2}\beta_3 \dots ,$$

where $\beta_1, \beta_2, \dots \in \iota_{n-1}(B_{n-1})$. In this representation, we replace each occurrence of $\sigma_{n-1}^{\pm 2}$ with $(\sigma_{n-2}\sigma_{n-3}\dots\sigma_1\sigma_1\sigma_2\dots\sigma_{n-2})^{\mp 1}$ and then remove the last strand. The resulting braid is denoted by $J(\beta)$ (see Figure 1).

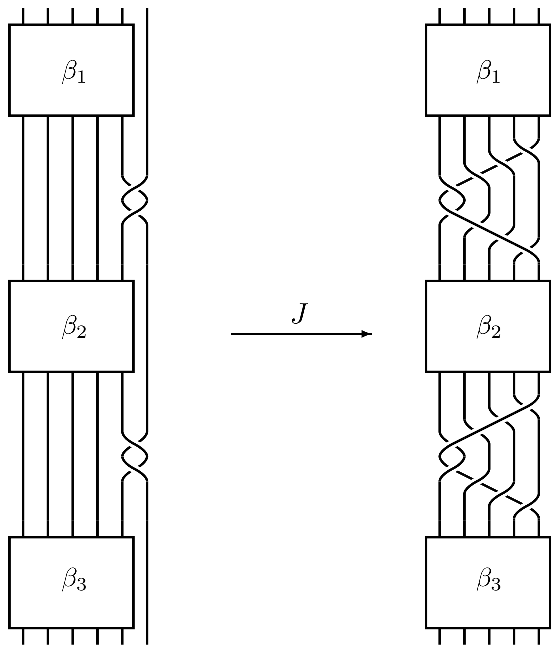

FIGURE 1. The homomorphism J.

Proposition 3 (see [9]). *The map $J : P_n \to P_{n-1}$ defined above is well defined and is a group homomorphism.*

Let $\beta_1, \beta_2, \dots, \beta_{n!}$ be arbitrary representatives of the cosets of the group P_n in B_n. We define an operator $S : X_{n-1} \to X_n$ by the formula

$$(S\chi)(\beta) = \frac{1}{(n-1)!}\sum_{i=1}^{n!}(\chi \circ J)(\beta_i\beta\beta_i^{-1}), \quad \chi \in X_{n-1}.$$

This formula is a particular case of (3) (up to a factor). The resulting pseudocharacter extends to B_n in the standard way. It is not hard to show that the pseudocharacter $\chi \circ J$ on P_n is invariant under conjugation by all braids from $\iota_{n-1}(B_{n-1})$, which implies that $S\chi$ can be calculated by a simple formula:

$$(S\chi)(\beta) = \sum_{i=0}^{n-1}(\chi \circ J)(\gamma_n^i\beta\gamma_n^{-i}), \quad \text{£} \quad \gamma_n = \sigma_1\sigma_2\dots\sigma_{n-1}. \tag{8}$$

Here, the braids γ^i may be replaced with arbitrary representatives of the left cosets of the subgroup generated by $\iota_{n-1}(B_{n-1}) \cup P_n$ in B_n.

Definition 6. By the *pseudocharacters of twist type* we mean the twist numbers ω_n, $n = 2, 3, \dots$ and all pseudocharacters obtained from them by application of the operators S, R, and I, all their products, and all linear combinations of such products.

It has been mentioned above that the pseudocharacter ω is kernel. However, applying S to ω yields a nonkernel pseudocharacter. For example, $(S\omega_2)(\sigma_1^2) = -1$ and $(S\omega_3)(\sigma_1^2) = -2$. In particular, this example shows that $IS\omega_n \neq S\omega_{n-1}$. Thus, from the pseudocharacter $S\omega_n$ we can obtain new pseudocharacters not only via the operator R, but also with the help of I. However, shortly we shall see that the repeated application of I or R cannot yield new pseudocharacters.

Proposition 4. *For any $k, n \geq 2$ we have*

$$I^k S\omega_{n+k-1} - IS\omega_n = -\omega_n.$$

Proof. We employ (8). Let $\beta \in P_n$ be an arbitrary pure braid. We denote by β' and β'' the braids $\iota_n(\beta) \in B_{n+1}$ and $(\iota_{n+k-1} \circ \dots \circ \iota_n)(\beta) \in B_{n+k}$, respectively. For $i = 0, \dots, k-1$, the braid $\gamma^i_{n+k}\beta''\gamma^{-i}_{n+k}$ coincides with the braid β with i strands added on the left and $k-i$ strands added on the right. Therefore, the braid $J(\gamma^i_{n+k}\beta''\gamma^{-i}_{n+k})$, obtained from $\gamma^i_{n+k}\beta''\gamma^{-i}_{n+k}$ by removing the last strand, unlinked with the others, will be split, and its twist number will be zero.

For $i = 1, 2, \dots, n$, the braid $\gamma^{k-1+i}_{n+k}\beta''\gamma^{-(k-1+i)}_{n+k}$ is obtained from $\gamma^i_{n+1}\beta'\gamma^{-i}_{n+1}$ by replacing the ith strand with k parallel strands. The same is true for the braids $J(\gamma^{k-1+i}_{n+k}\beta''\gamma^{-(k-1+i)}_{n+k})$ and $J(\gamma^i_{n+1}\beta'\gamma^{-i}_{n+1})$. Observe that the twist number does not change if in a braid on $n \geq 2$ strands we replace one strand with several parallel strands (or even with any braid). Therefore,

$$(\omega \circ J)\big(\gamma^{k-1+i}_{n+k}\beta''\gamma^{-(k-1+i)}_{n+k}\big) = (\omega \circ J)(\gamma^i_{n+1}\beta'\gamma^{-i}_{n+1}).$$

Finally, the braid $J(\beta')$ coincides with β, whence $(\omega \circ J)(\beta') = \omega(\beta)$. Thus, we have

$$(S\omega_n)(\beta') = (S\omega_{n+k-1})(\beta'') + \omega(\beta),$$

as required. $\square$

We see that no pseudocharacters of the form $= I^k S\omega_n$ are of interest for $k \geq 2$. The next statement shows that iterations of the operator (8) are also of no interest.

Proposition 5. *For any $\chi \in X_{n-2}$, we have*

$$S^2\chi = RS\chi + 4\chi(\Delta_{n-2}) \exp_n.$$

Proof. This follows from the fact that a two-fold application of J to a braid $\beta \in P_n$ yields the same result as the following chain of actions: removal of the last strand; application of J; multiplication by Δ^{2k}_{n-2}, where k is the algebraic number of intersections of the last two strands in β. In its turn, this fact is established directly from the definition of the operation J for β represented as a product of braids in P_{n-1} and braids of the form σ^{2j}_{n-1}, $j \in \mathbb{Z}$. $\square$

Also, nothing new results from applying the operator (8) to the pseudocharacter $IS\omega$; this is seen from the next claim.

Proposition 6. *For any $\chi \in X_{n-1}$, we have*

$$SIS\chi = n\, S\chi + 4\chi(\Delta_{n-1}) \exp_n.$$

Proof. This follows from the fact that for any $j = 0, 1, 2, \dots, n-1$ and any pure braid $\beta \in P_n$ we have

$$J\big(\gamma^j \iota_{n-1}(J(\beta))\gamma^{-j}\big) = J(\gamma^j \beta \gamma^{-j})\Delta_{n-1}^{2k},$$

where k is the algebraic number of intersections of the $(n-j)$th and jth strands if $j > 0$, and $k = 0$ if $j = 0$.

For the generators of the group P_n the above identity can be checked directly, and for the other elements it is valid because both its sides determine homomorphisms from P_n to P_{n-1}. $\square$

Finally, it is easily seen that J commutes with the operation of removing the first strand. This implies the following statement.

Proposition 7. *The operators S and R commute.*

Summarizing Propositions 4–7, we see that the following is true.

Theorem 4. *The space of pseudocharacters of the twist type on B_n is generated by ω_n, $S\omega_{n-1}$, $IS\omega_n$ and pseudocharacters of the form $R\chi$, where χ is a twist type pseudocharacter on B_{n-1}.*

Proof. Let T_n be the subspace in X_n defined recursively as follows:

$$T_2 = X_2 = \langle \exp_2 \rangle, \quad T_n = R(T_{n-1}) + \langle \omega_n, S\omega_{n-1}, IS\omega_n \rangle, \quad n = 3, 4, \dots.$$

By construction, T_n only contains pseudocharacters of twist type. To prove the theorem, it suffices to show that the direct sum of all T_n, $n = 2, 3, \dots$, is invariant under R, I and S.

For R it is true by construction. For S and I, we use induction on n to check that $S(T_n) \subset T_{n+1}$, $I(T_n) \subset T_{n-1}$ (this is obvious for $n = 2$):

$$\begin{aligned} S(T_n) &= SR(T_{n-1}) + \langle S\omega_n, S^2\omega_{n-1}, SIS\omega_n \rangle \\ &= RS(T_{n-1}) + \langle S\omega_n, S^2\omega_{n-1}, SIS\omega_n \rangle \subset RT_n + \langle S\omega_n, RS\omega_{n-1}, \exp_{n+1} \rangle \subset T_{n+1}, \\ I(T_n) &= IR(T_{n-1}) + \langle I\omega_n, IS\omega_{n-1}, I^2S\omega_n \rangle \\ &\subset RI(T_{n-1}) + T_{n-1} + \langle IS\omega_{n-1}, \omega_{n-1} \rangle \subset R(T_{n-2}) + T_{n-1} \subset T_{n-1}. \end{aligned}$$ $\square$

1.4. Signature. We define the *signature* $\operatorname{sign}(\beta)$ of a braid β as the signature of the oriented link represented by β. The definition of the signature of a link can be found, e.g., in [10]. The following result was obtained by Gambaudo and Ghis in [11].

Theorem 5. *The function* sign *on the group B_n is a quasicharacter with defect not exceeding $2n$.*

The pseudocharacter on B_n that corresponds to the signature will be denoted by $\overline{\operatorname{sign}}$. The results of [11, 12] imply the following statement.

Theorem 6. *On the group B_3, the pseudocharacter $\overline{\operatorname{sign}}$ is a linear combination of the twist number and the exponent sum*:

$$\overline{\operatorname{sign}}(\beta) = 2 \cdot \omega(\beta) - \exp_3(\beta).$$

In [5], the question was posed about the relationship between this pseudocharacter and the twist numbers for arbitrary n. Below, we prove the following two facts.

Theorem 7. *The pseudocharacter $\overline{\operatorname{sign}}$ on B_n has a nontrivial kernel part.*

Theorem 8. *For $n \geq 5$, the pseudocharacter $\overline{\operatorname{sign}}_n$ is linearly independent of all pseudocharacters of twist type. Moreover, for $n = 5$ this pseudocharacter is independent of all pseudocharacters of twist type and all other ones obtained by application of the Malyutin operator R to the pseudocharacters on B_4.*

§2. Geometric meaning and calculation of twist numbers

Here we give a geometric interpretation of twist numbers and present a way to calculate them. This interpretation is based on a relationship between the Dehornoy ordering and topology, observed for the first time in [13]. A survey of results on this and similar orderings can be found in [14].

Let β be a braid. It is well known that braids can be regarded as homeomorphisms of n-punctured two-dimensional disk, fixed at the boundary and viewed up to homotopy in the class of such homeomorphisms. As the disk we take a disk D in $\mathbb{R}^2$ centered at the origin, the punctures being n distinct points $P_1, \dots, P_n$ lying on the horizontal diameter and enumerated from left to right; we denote $D \setminus \{P_1, \dots, P_n\}$ by D_n. Let φ_β be a homeomorphism of D_n corresponding to the braid β. Let δ denote the part of the horizontal diameter of D from the leftmost point to P_1; consider the image of δ under φ_β. The freedom in the choice of φ_β allows us to assume that the arc $\varphi_\beta(\delta)$ has the smallest possible number of intersections with the horizontal axis among all paths homotopic to this arc and avoiding the punctures. The Dehornoy floor is responsible for the number of full turns made by the curve $\varphi_\beta(\delta)$ around the segment $[P_1, P_n]$ before crossing it. More precisely, let δ_1 be the part of the curve $\varphi_\beta(\delta)$ from the leftmost point to the first intersection with $[P_1, P_n]$, and let δ_2 be the curve obtained from δ by a small deformation such that the left end is shifted down along the boundary of the disk D_n. Suppose that P_1 is not an end of the curve δ_1. Then the intersection index $\operatorname{ind}(\delta_2, \delta_1)$ is precisely the Dehornoy floor $[\beta]_{\mathrm{D}}$.

Otherwise, if δ_1 ends at P_1 (this is possible only if $\varphi_\beta(\delta)$ does not intersect the interior of $[P_1, P_n]$) then, deforming δ_1 or δ_2 slightly, we get two values of $\operatorname{ind}(\delta_2, \delta_1)$ that differ by 1. One of them is always equal to $[\beta]_{\mathrm{D}}$.

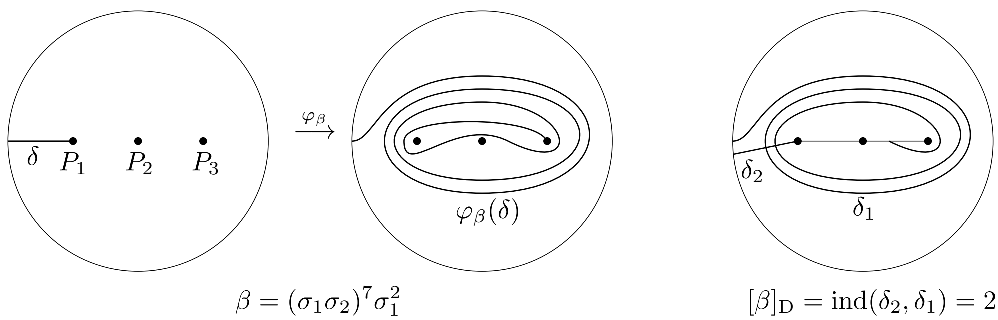

Figure 2. The Dehornoy floor.

The following statement was proved in [8].

Theorem 9. *The twist number of a braid on n strands is a rational number whose denominator does not exceed n.*

Proof. Not going into details, we describe the idea of the proof; this idea is fairly visual. By the result of Thurston [16], for each nontrivial braid β on n strands, one of the following statements is true.

(1) Raised to some power q, the braid β coincides with Δ^{2p}. Such braids are said to be *periodic*.

(2) The braid β is *pseudo-Anosov*. This means that, in the corresponding mapping class of the $n+1$ times punctured sphere (obtained by contracting the boundary of D_n to a point), there is a homeomorphism for which there exists a pair of invariant, mutually transversal, and measurable foliations with singularities (called *stable* and *unstable ones*).

(3) The braid β is *reducible*, i.e., in the punctured disk D_n there is a nonempty and nontrivial family of circles the isotopy class of which is invariant under the homeomorphism φ_β. Nontriviality means that the circles are not homotopic to the boundary of D_n, and inside the disk bounded by each circle there are at least two punctures.

It is well known (see [17] and the references therein) that every periodic braid in B_n is conjugate to some power of one of the braids

$$\sigma_1\sigma_2\dots\sigma_{n-1} \quad \text{or} \quad \sigma_1^2\sigma_2\sigma_3\dots\sigma_{n-1}.$$

The first of these is the nth root of Δ^2, and the second is $(n-1)$st root of Δ^2. It follows that q divides n or $n-1$. For a periodic braid β with $\beta^q = \Delta^{2p}$, obviously, the twist number is equal to p/q.

If β is a pseudo-Anosov braid, then the structure of its stable foliation can be given with the help of an *invariant train track* [18], which, by definition, is a graph T embedded in D_n and satisfying the following conditions. The edges of T are smooth arcs, all edges that emanate from one vertex have a common tangent line, and for each vertex there are two edges going from it in opposite directions. Each component of $D_n \setminus (T \cup \partial D)$ should have one or several "cusps", i.e., subdomains lying in small neighborhoods of the vertices of T and bounded in these neighborhoods by two tangent segments of edges of T. Under an appropriate choice of the homeomorphism φ_β, the graph T should stay in its small neighborhood under the action of φ_β, and the φ_β-images of C^1-smooth paths in T should go "parallel" to smooth paths in T, i.e., should have no local "foldings". The idea is illustrated in Figure 3.

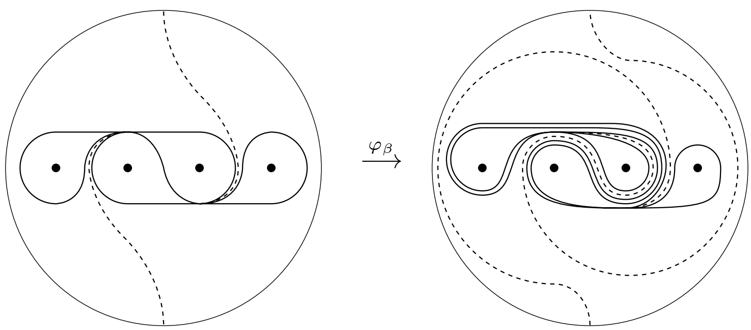

FIGURE 3. The twist number of the braid $\beta = \sigma_3^2\sigma_2\sigma_3\sigma_2\sigma_1$ is equal to p/q with $p = 1$, $q = 2$.

Now, in the component Ω of the set $D_n \setminus T$ that contains ∂D, we draw arcs connecting ∂D with each "cusp" (one arc for each "cusp"; marked by dashed lines in Figure 3). Then the twist number of β will be equal to p/q, where q is the number of "cusps", and p is the difference between the total numbers of full turns about the segment $[P_1P_2]$ made by the chosen arcs before and after the action of φ_β. Thus, the denominator of the twist number is dominated by the number of "cusps" in the domain Ω, which is not greater than $n-2$, for reasons related to calculation of the Euler characteristic of the disk D_n.

If the braid β in question is reducible, then, contracting to a point each of the disks bounded by the circles of the invariant family, we get a homeomorphism of the disk with a smaller number of punctures. The twist number of the braid that corresponds to the new homeomorphism will be the same as for the initial braid. Thus, the twist number of a reducible braid is equal to that of some other braid with a smaller number of strands. $\square$

To calculate the twist number, we can use the algorithm of determining the geometric type of a braid and finding, in the pseudo-Anosov case, an invariant train-track, as described in [18]. Also, we can apply Theorem 11 and Proposition 1, which imply the following statement.

Proposition 8. *The twist number of a braid $\beta \in B_n$ is equal to the rational number p/q with $|q| \le n$ closest to $[\beta^{2n^2}]_{\mathrm{D}}/(2n^2)$.*

A visual interpretation can also be given for the pseudocharacter $\eta_n = IS\omega_n - \omega_n$, which can naturally be called the *internal twist number*. Let β be a pure braid on n strands. Replace its jth strand with two parallel strands that do not intersect each other in projection. Let β_j be the resulting braid, let φ_j be the corresponding homeomorphism of the punctured disk $D\backslash\{P_1, \dots, P_{n+1}\}$, and let δ_j be the part of the horizontal diameter of D from its leftmost point to the jth puncture P_j. We denote by $d_j(\beta)$ the algebraic number of full turns made by $\varphi_j(\delta_j)$ about the pair of punctures P_j, P_{j+1} after going from P_j and up to intersecting some segment $[P_k, P_{k+1}]$ with $|k-j| > 1$ (the number of intersections of $\varphi_j(\delta_j)$ with the horizontal diameter is assumed to be minimized).

It is easy to show that, after the due refinement of the definition, the number $-d_j(\beta)$ coincides with the Dehornoy floor of the braid $J(\gamma_{n+1}^{n-j}\iota_n(\beta)\gamma_{n+1}^{j-n})$, so that it is a quasicharacter on P_n. The corresponding pseudocharacter $\overline{d_j}$ measures the "twist of the braid β around its jth strand", and the pseudocharacter η_n measures the total twist of the braid around all its strands. The subtraction of ω_n in the definition of η_n is due to the fact that $J(\iota_n(\beta)) = \beta$.

§3. Calculating $\overline{\mathrm{sign}}$

We present a handy way to calculate $\overline{\mathrm{sign}}(\beta)$. Let β be an arbitrary braid, and let w be a word representing β in the Artin generators σ_i: $w = \sigma_{i_1}^{\epsilon_1}\sigma_{i_2}^{\epsilon_2}\dots\sigma_{i_m}^{\epsilon_m}$, $\epsilon_1, \dots, \epsilon_m \in \{-1, 1\}$.

Put $(i_{k+2m}, \epsilon_{k+2m}) = (i_{k+m}, \epsilon_{k+m}) = (i_k, \epsilon_k)$ for all $k = 1, \dots, m$ For each $k = 1, \dots, 2m$, denote by k' the smallest integer such that $i_{k'} = i_k$, $k' > k$. We compose a matrix $A_w = (a_{kl})_{k,l=1,\dots,2m}$ by the rule

$$a_{kl} = \begin{cases} 1 & \text{if } \ l \le m, \ k = l', \ \epsilon_k = 1, \\ 1 & \text{if } \ k = l \le m, \ \epsilon_k = \epsilon_{k'} = -1, \\ 1 & \text{if } \ k \le m, \ k < l < k' < l', \ i_k = i_l + 1, \\ -1 & \text{if } \ k \le m, \ l = k', \ \epsilon_l = -1, \\ -1 & \text{if } \ k = l \le m, \ \epsilon_k = \epsilon_{k'} = 1, \\ -1 & \text{if } \ l \le m, \ l < k < l' < k', \ i_k = i_l + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Next, we put

$$B_w(t) = \begin{pmatrix} 1_m & \dfrac{1}{t} \cdot 1_m \end{pmatrix} A_w \begin{pmatrix} 1_m \\ t \cdot 1_m \end{pmatrix},$$

where 1_m stands for the identity matrix of size $m \times m$ and

$$S_w(t) = B_w(t) + B_w(1/t)^{\mathrm{T}}.$$

Obviously, the matrix $S_w(t)$ is Hermitian for $t \in \mathbb{C}$, $|t| = 1$. Let $\mathrm{sign}(S_w(t))$ denote its signature, i.e., the difference between the positive and negative inertia indices.

Proposition 9. *With the above notation, we have*

$$\operatorname{sign}(\beta^s) = \sum_{t^s=1} \operatorname{sign}(S_w(t)), \quad \overline{\operatorname{sign}}(\beta) = \frac{1}{2\pi}\int_0^{2\pi} \operatorname{sign}(S_w(e^{i\phi}))\, d\phi.$$

Proof. The second identity follows from the first, because on the circle $|t| = 1$ the function $\operatorname{sign}(S_w(t))$ is piecewise constant and has finitely many discontinuity points. We prove the first identity. Let $S_w(t) = tS_w^+ + \frac{1}{t}S_w^- + S_w^0$, where S_w^+, S_w^-, and S_w^0 are numerical matrices. Then the matrix

$$T \otimes S_w^+ + T^{-1} \otimes S_w^- + 1_s \otimes S_w^0, \tag{9}$$

where

$$T = \begin{pmatrix} 0 & 1 & 0 & \dots & \dots & 0 \\ \vdots & \ddots & 1 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & \dots & 0 & 1 \\ 1 & 0 & \dots & \dots & \dots & 0 \end{pmatrix}$$

is the matrix of cyclic permutation of length s, and $\otimes$ means the Kronecker product, is the Zeifert matrix for the link represented by the braid β^s, under an appropriate choice of the Zeifert surface and a generating system of cycles. Namely, the Zeifert surface is built via a standard procedure (see, e.g., [10]) from disks whose number equals the number of strands (precisely this will be the number of the Zeifert circles), and from strips connecting these disks, the number of strips being equal to the number of intersections on the braid's diagram. The role of generating cycles is played by curves going around the domains into which the braid closure diagram splits the plane (except for the central and outer domain, see the right drawing in Figure 4). The cycles are enumerated in accordance with the order of intersections at their "summit" (see the left drawing in Figure 4).

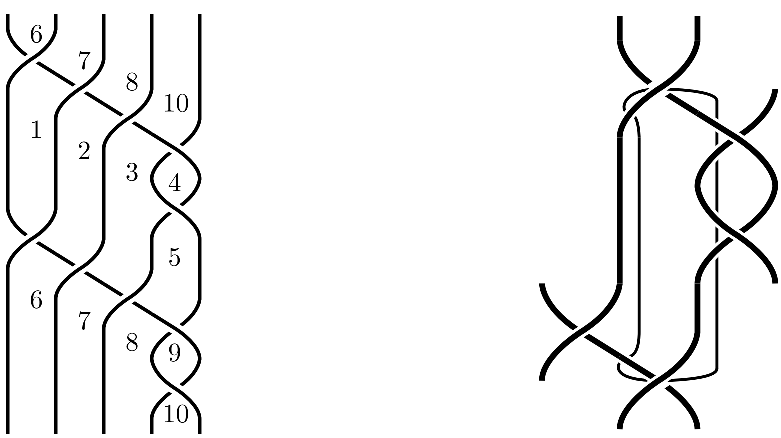

FIGURE 4. Cycles on the standard Zeifert surface.

These cycles do not form a basis in the first homologies of the Zeifert surface, but they generate them. Linear dependence of the cycles does not affect the signature of the intersection matrix.

Replacing the matrix T by the diagonal matrix

$$\begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ 0 & e^{2\pi i/s} & \ddots & & \vdots \\ \vdots & \ddots & e^{4\pi i/s} & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & e^{2(s-1)\pi i/s} \end{pmatrix}$$

similar to T does not change the signature of the matrix (9), reducing it to the form of the direct sum of the matrices $S_w(1)$, $S_w(e^{2\pi i/s})$, …, $S_w(e^{2(s-1)\pi i/s})$, and the required claim follows. □

We present an example of calculating the signature of a braid on five strands. Let a braid β be represented by the word $w = \sigma_1\sigma_2\sigma_3\sigma_4^{-2}$. Then

$$A = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} -1+\frac{1}{t} & 0 & 0 & 0 & 0 \\ -1+t & -1+\frac{1}{t} & 0 & 0 & 0 \\ 0 & -1+t & -1+\frac{1}{t} & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1+t & -t & 1 \end{pmatrix},$$

$$S = \begin{pmatrix} -2+\frac{1}{t}+t & -1+\frac{1}{t} & 0 & 0 & 0 \\ -1+t & -2+\frac{1}{t}+t & -1+\frac{1}{t} & 0 & 0 \\ 0 & -1+t & -2+\frac{1}{t}+t & 0 & -1+\frac{1}{t} \\ 0 & 0 & 0 & 2 & -1-\frac{1}{t} \\ 0 & 0 & -1+t & -1-t & 2 \end{pmatrix}.$$

The signature of the matrix $S_w(e^{i\phi})$ is equal to

$$\operatorname{sign} S_w(e^{i\phi}) = \begin{cases} 1 & \text{if } \phi \in \Big[0, \dfrac{\pi - \arccos(2(\sqrt{2}-1))}{2}\Big) \cup \Big(\dfrac{3\pi + \arccos(2(\sqrt{2}-1))}{2}, 2\pi\Big], \\ -1 & \text{if } \phi \in \Big(\dfrac{\pi - \arccos(2(\sqrt{2}-1))}{2}, \dfrac{3\pi + \arccos(2(\sqrt{2}-1))}{2}\Big). \end{cases}$$

Therefore,

$$\overline{\operatorname{sign}}(\beta) = -\frac{\arccos(2(\sqrt{2}-1))}{\pi}.$$

The above example shows that, unlike twist numbers, the pseudocharacter $\overline{\operatorname{sign}}$ fails to take only rational values, which suggests that $\overline{\operatorname{sign}}$ should be independent of them.

§4. Proofs of theorems on $\overline{\mathrm{sign}}$

Proof of Theorem 7. It suffices to show that the kernel part of the pseudocharacter $\overline{\mathrm{sign}}$ is nonzero at some braid. We shall show that $\Delta_n^2 \in B_n$ fits for that. For $n = 2$, the pseudocharacter $\overline{\mathrm{sign}}$ coincides up to a sign with the exponent sum and is kernel, $\overline{\mathrm{sign}}(\Delta_2^2) = -2$. In the sequel we assume that $n > 2$.

The link represented by the braid Δ_n^{2k} is the (n, kn)-torus link. The signatures of all torus links were calculated in [19], where the following answer was given for the case of our interest:

$$\mathrm{sign}(\Delta_n^{2k}) = \begin{cases} 1 - n^2 & \text{if } k = 2, \\ \mathrm{sign}(\Delta_n^{2k-4}) + 1 - n^2 & \text{if } k > 2,\ n \text{ is odd}, \\ \mathrm{sign}(\Delta_n^{2k-4}) - n^2 & \text{if } k > 2,\ n \text{ is even}. \end{cases}$$

It follows immediately that

$$\overline{\mathrm{sign}}(\Delta_n^2) = \begin{cases} \frac{1-n^2}{2} & \text{if } n \text{ is odd}, \\ \frac{-n^2}{2} & \text{if } n \text{ is even}. \end{cases}$$

Removing any k strands from Δ_n^2 yields the braid Δ_{n-k}^2. On the other hand, given a braid, we shall not change the signature of its closure by adding strands not intersecting the others. Therefore, using formulas (4) and (5), we obtain

$$\begin{aligned} \overline{\mathrm{sign}}_{\mathrm{ker}}(\Delta_n^2) &= \sum_{k=0}^{n-2} (-1)^k \binom{n}{k} \overline{\mathrm{sign}}(\Delta_{n-k}^2) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \overline{\mathrm{sign}}(\Delta_{n-k}^2) \\ &= \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \overline{\mathrm{sign}}(\Delta_k^2) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \frac{-k^2}{2} - \frac{(-1)^n}{2} \sum_{j=0}^{[\frac{n-1}{2}]} \binom{n}{2j+1} \\ &= 0 - \frac{(-1)^n}{2} 2^{n-1} = -(-2)^{n-2} \neq 0. \end{aligned}$$ □

Proof of Theorem 8. Consider the following braids $\beta_1, \beta_2, \beta_3 \in B_5$:

$$\begin{aligned} \beta_1 = {} & \sigma_3^2(\sigma_2\sigma_1^{-1})^2\sigma_2\sigma_3^{-1}\sigma_2\sigma_3\sigma_2^{-1}(\sigma_4\sigma_3^{-1})^3\sigma_2\sigma_3^{-1}\sigma_2^{-1}\sigma_3\sigma_2^{-1}(\sigma_1\sigma_2^{-1})^2\sigma_1\sigma_3^{-2} \\ & \times \sigma_2\sigma_3(\sigma_3\sigma_4^{-1})^3\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}, \\ \beta_2 = {} & \sigma_2\sigma_3^{-1}\sigma_4^{-1}\sigma_2\sigma_1^{-1}\sigma_3^{-1}\sigma_4^{-1}\sigma_2\sigma_3\sigma_4^2\sigma_3^{-1}\sigma_2^{-1}\sigma_4\sigma_3\sigma_1\sigma_2^{-1}\sigma_3\sigma_2^{-1}\sigma_3\sigma_1\sigma_2^{-1}\sigma_4^{-1}\sigma_3\sigma_4^{-2}\sigma_3^{-1}\sigma_4 \\ & \times \sigma_4\sigma_2\sigma_3^{-1}\sigma_3^{-1}\sigma_2\sigma_4\sigma_3^{-1}\sigma_4^2\sigma_3\sigma_1\sigma_2^{-1}\sigma_4^{-1}\sigma_3\sigma_4^{-1}\sigma_3\sigma_2^{-1}\sigma_4^{-1}\sigma_3\sigma_4^{-2}\sigma_3^{-1}\sigma_4\sigma_2\sigma_3^{-1}\sigma_1^{-1}, \\ \beta_3 = {} & \sigma_4^{-1}\sigma_3^{-1}\sigma_1^2\sigma_2\sigma_3^{-2}\sigma_2^{-1}\sigma_3\sigma_4^{-2}\sigma_3^{-1}\sigma_1\sigma_2^2\sigma_1^{-1}\sigma_3\sigma_4^2\sigma_3^{-1}\sigma_2\sigma_3^2\sigma_2^{-1}\sigma_3\sigma_4^{-2}\sigma_3^{-1}\sigma_1\sigma_2^{-2} \\ & \times \sigma_3\sigma_4^4\sigma_3^{-1}\sigma_2^2\sigma_1^{-1}\sigma_3^{-1}\sigma_2^{-2}\sigma_3\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_3^2\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_3\sigma_4^{-1}. \end{aligned}$$

It is not difficult to check that these braids are Brunnian, i.e., each of them becomes a trivial braid after removal of any strand. Therefore, all pseudocharacters belonging to $R(X_4)$ vanish at these braids. The values of the remaining pseudocharacters of interest, calculated with the help of the procedures described above, are presented in the table

	ω_5	$S\omega_4$	$IS\omega_5$	$\overline{\mathrm{sign}}$
β_1	0	3	2	0
β_2	0	-1	-1	-2
β_3	0	-4	-4	-2

,

which shows that $\overline{\mathrm{sign}}$ is linearly independent of these remaining pseudocharacters. This proves the claim for $n = 5$.

To prove it for $n > 5$, observe that the operator I takes $\overline{\mathrm{sign}}_n$ to $\overline{\mathrm{sign}}_{n-1}$, and the pseudocharacters of twist type to those of twist type. Therefore, a linear expression of

$\overline{\mathrm{sign}}_n$ with $n > 5$ in terms of twist type pseudocharacters would imply such an expression for $n = 5$, which is impossible, see above. □

The authors are deeply grateful to Andreĭ Malyutin for reviewing this paper and pointing several possible improvements of the initial text.

References

[1] V. A. Faĭziev, *The stability of the equation $f(xy) - f(x) - f(y) = 0$ on groups*, Acta Math. Univ. Comenian. (N.S.) **69** (2000), no. 1, 127–135. MR1796793 (2003a:39026)
[2] A. V. Malyutin, *Pseudocharacters of braid groups and prime links*, Algebra i Analiz **21** (2009), no. 2, 113–135; English transl., St. Petersburg Math. J. **21** (2010), no. 2, 245–259. MR2549454 (2011b:20105)
[3] M. Bestvina and K. Fujiwara, *Bounded cohomology of subgroups of mapping class groups*, Geom. Topol. **6** (2002), 69–89. MR1914565 (2003f:57003)
[4] D. B. A. Epstein and K. Fujiwara, *The second bounded cohomology of word-hyperbolic groups*, Topology **36** (1997), 1275–1289. MR1452851 (98k:20088)
[5] A. V. Malyutin, *Operators in the spaces of pseudocharacters of braid groups*, Algebra i Analiz **21** (2009), no. 2, 136–165; English transl., St. Petersburg Math. J. **21** (2010), no. 2, 261–280. MR2553045 (2011b:20106)
[6] P. Dehornoy, *Braid groups and left distributive operations*, Trans. Amer. Math. Soc. **345** (1994), no. 1, 115–150. MR1214782 (95a:08003)
[7] H. Short and B. Wiest, *Orderings of mapping class groups after Thurston*, Enseign. Math. (2) **46** (2000), no. 3-4, 279–312. MR1805402 (2003b:57003)
[8] A. V. Malyutin, *Twist number of* (*closed*) *braids*, Algebra i Analiz **16** (2004), no. 5, 59–91; English transl., St. Petersburg Math. J. **16** (2005), no. 5, 791–813. MR2106667 (2005j:57011)
[9] A. J. Berrick, F. R. Cohen, Y. L. Wong, and J. Wu, *Configurations, braids, and homotopy groups*, J. Amer. Math. Soc. **19** (2006), no. 2, 265–326. MR2188127 (2007e:20073)
[10] K. Murasugi, *Knot theory and its applications*, Transl. from the 1993 Japanese original by Bohdan Kurpita, Birkhäuser Boston, Inc., Boston, MA, 1996. MR1391727 (97g:57011)
[11] J.-M. Gambaudo and É. Ghys, *Braids and signatures*, Bull. Soc. Math. France **133** (2005), no. 4, 541–579. MR2233695 (2007d:57014)
[12] K. Honda, W. Kazez, and G. Matić, *Right-veering diffeomorphisms of compact surfaces with boundary*. II, Geom. Topol. **12** (2008), no. 4, 2057–2094. MR2431016 (2009i:57057)
[13] R. Fenn, M. T. Greene, D. Rolfsen, C. Rourke, and B. Wiest, *Ordering the braid groups*, Pacific J. Math. **191** (1999), no. 1, 49–74. MR1725462 (2000j:20064)
[14] P. Dehornoy, I. Dynnikov, D. Rolfsen, and B. Wiest, *Ordering braids*, Math. Surveys Monogr., vol. 148, Amer. Math. Soc., Providence, RI, 2008. MR2463428 (2009j:20046)
[15] J. S. Birman, *Braids, links, and mapping class groups*, Ann. of Math. Stud., No. 82, Princeton Univ. Press, Princeton, NJ; Univ. Tokyo Press, Tokyo, 1974. MR0375281 (51:11477)
[16] W. P. Thurston, *On the geometry and dynamics of diffeomorphisms of surfaces*, Bull. Amer. Math. Soc. (N.S.) **19** (1988), 417–431. MR956596 (89k:57023)
[17] J. González-Meneses, *The nth root of a braid is unique up to conjugacy*, Algebr. Geom. Topol. **3** (2003), 1103–1118. MR2012967 (2004i:20069)
[18] M. Bestvina and M. Handel, *Train-tracks for surface homeomorphisms*, Topology **34** (1995), 109–140. MR1308491 (96d:57014)
[19] C. McA. Gordon, R. A. Litherland, and K. Murasugi, *Signatures of covering links*, Canad. J. Math. **33** (1981), 381–394. MR617628 (83a:57006)

Steklov Mathematical Institute, Russian Academy of Sciences, Gubkin str. 8, Moscow 119991, Russia
E-mail address: dynnikov@mech.math.msu.su

Department of Mechanics and Mathematics, Lomonosov Moscow State University, Moscow 119991, Russia
E-mail address: shast.fds@mail.ru

Received 24/DEC/2011

Translated by A. PLOTKIN