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ON A METHOD OF APPROXIMATION BY GRADIENTS

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Abstract. The subject of this study is the possibility of approximation of a continuous vector field on a compact set $K \subset \mathbb{R}^n$ by gradients of smooth functions defined on the entire $\mathbb{R}^n$. A method is obtained that yields either approximation or an obstruction for it. This method does not involve the Hahn–Banach theorem and is based on solving a quasilinear elliptic equation in partial derivatives. A discrete analog of the above problem is studied, namely, the problem of approximation by gradients on a finite oriented graph. A stepwise algorithm is suggested in this case.

§1. Introduction

1.1. Statements of problems of uniform approximation by gradients. In what follows, K is a compact subset of $\mathbb{R}^d$, $d \geq 2$, and $C(K)$, $\vec{C}(K)$ denote the spaces of all real-valued functions and of all vector fields (respectively) that are continuous on K. These spaces are endowed with the norms $\|\cdot\|_{C(K)}$, $\|\cdot\|_{\vec{C}(K)}$: $\|f\|_{C(K)} = \max_K |f|$, $\|\vec{f}\|_{\vec{C}(K)} = \max_K |\vec{f}|$ for $f \in C(K)$, $\vec{f} \in \vec{C}(K)$ ($|v|$ is the Euclidean norm of a vector $v \in \mathbb{R}^d$ and $\langle v_1, v_2\rangle$ is the scalar product of vectors $v_1, v_2 \in \mathbb{R}^d$).

The following definition plays a crucial role in this paper.

Definition 1.1. Let $\vec{f}$ be a vector field in $\vec{C}(K)$ and ε a positive number. A function u defined and continuously differentiable near K is called an *ε-primitive* of the vector field $\vec{f}$ if $\|\nabla u - \vec{f}\|_{\vec{C}(K)} < \varepsilon$.

We are interested in conditions that provide either the existence of an ε-primitive of a given field $\vec{f}$, or an obstruction to its existence. We note immediately that the existence of a closed rectifiable Lipschitz curve γ of length $l(\gamma)$ supported on K and such that the circulation of $\vec{f}$ along γ is at least $\varepsilon\, l(\gamma)$ provides an obstruction to the existence of an ε-primitive of $\vec{f}$ on K; it turns out that for $d = 2$ this is the only kind of obstructions, but if $d \geq 3$, then the situation is more complicated (see Subsection 1.3). In §3 we shall describe an approach to construction of an ε-primitive when it exists.

The notion of an ε-primitive of a vector field is related to the following definition.

Definition 1.2. Suppose that a field $\vec{f} \in \vec{C}(K)$ has an ε-primitive for any $\varepsilon > 0$. Then we say that $\vec{f}$ is a *quasigradient* (on the set K).

Note that, in general, a quasigradient $\vec{f}$ is not a gradient: it may happen that there is no function u that is smooth near K and satisfies $\nabla u|_K = \vec{f}$ (see [6]).

Definition 1.3. A compact set $K \subset \mathbb{R}^d$ is called a *gradient set* if every field $\vec{f} \in \vec{C}(K)$ is a quasigradient.

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We are interested in the following questions. First: *what are conditions on a compact set $K \subset \mathbb{R}^d$, a vector field $\vec{f} \in \vec{C}(K)$, and a number $\varepsilon > 0$ that ensure the existence of an ε-primitive of $\vec{f}$ on the set K*? Second: *under what conditions a given vector field $\vec{f} \in \vec{C}(K)$ is a quasigradient*? Third: *which compact sets $K \subset \mathbb{R}^d$ are gradient sets*?

The notions introduced above arise naturally in the study of approximation properties of harmonic vector fields. The concept of a *harmonic vector field* generalizes the notion of an analytic function of a complex variable. Recall that a vector field $\vec{v}\colon O \to \mathbb{R}^d$ continuously differentiable on an open set $O \subset \mathbb{R}^d$ is said to be *harmonic* if its Jacobi matrix is symmetric and has zero trace. This means that $\vec{v}$ locally coincides with the gradient of a harmonic function. In the case where $d = 2$, a vector field $\vec{v} = (v_1, v_2)$ is harmonic if and only if the function $v_1 - iv_2$ is an analytic function of the complex variable. If $d = 3$, then harmonicity for a vector field $\vec{v}$ means that $\operatorname{curl}\vec{v} = 0$, $\operatorname{div}\vec{v} = 0$ in O.

Approximation properties of analytic functions of the complex variable (or, which is the same, of harmonic vector fields in $\mathbb{R}^2$) are described rather completely (see [3, 4]). For $d \geq 3$, there is only a little progress in the study of this problem (see [5, 6, 7]), and a principal distinction from the case of $d = 2$ arose.

We mention the following fact, which illustrates the role of quasigradients in the above-listed problems on harmonic approximation. Suppose that $\lambda_d(K) = 0$ ($\lambda_d(E)$ is the Lebesgue measure of a set $E \subset \mathbb{R}^d$) and $\vec{f}$ is a quasigradient on K. Then $\vec{f}$ is a limit in $\vec{C}(K)$ of fields of the form $\nabla u|_K$, where u is a function *harmonic* near K (see [8, 6]). Thus, if K is a gradient set of zero Lebesgue measure, then *any* field $\vec{f} \in \vec{C}(K)$ is a uniform limit of gradients of *harmonic* functions.

In order to answer the stated questions on uniform approximation by gradients, we introduce the notion of a *solenoidal vector charge*.

1.2. Solenoidal vector charges. Denote by $\vec{M}(K)$ the space of all Borel vector-valued charges supported on K, i.e., the $\mathbb{R}^d$-valued countably-additive set functions defined on the σ-algebra of all Borel subsets of K; next, $M(K)$ is the space of all real-valued Borel charges supported on K. We identify a charge $\vec{\mu} \in \vec{M}(K)$ with the charge in $\mathbb{R}^d$ that coincides with $\vec{\mu}$ on K and vanishes outside of K; the same convention concerns charges of class $M(K)$. The spaces $\vec{M}(K)$ and $M(K)$ are normed by the total variations of charges.

The space $M(K)$ is isometrically isomorphic to the space dual to $C(K)$, and the space $\vec{M}(K)$ is isometrically isomorphic to the space dual to $\vec{C}(K)$. Therefore, we introduce the following notation: for $\mu \in M(K)$ and $f \in C(K)$ we set $\mu[f] = \int_K f\,d\mu$; for $\vec{f} \in \vec{C}(K)$ and $\vec{\mu} \in \vec{M}(K)$ we set $\vec{\mu}[\vec{f}] = \int_K \langle \vec{f}, d\vec{\mu}\rangle = \sum_{i=1}^d \int_K f_i\,d\mu_i$, where the f_i and μ_i ($i = 1, \ldots, d$) are the components of the field $\vec{f}$ and the charge $\vec{\mu}$ in some fixed orthogonal coordinate system (the value of $\sum_{i=1}^d \int_K f_i\,d\mu_i$ does not depend on the choice of a coordinate system). The numbers $\mu[f]$ and $\vec{\mu}[\vec{f}]$ can be understood as results of applying the functionals $\mu \in M(K) = (C(K))^*$ and $\vec{\mu} \in \vec{M}(K) = (\vec{C}(K))^*$ to the elements $f \in C(K)$ and $\vec{f} \in \vec{C}(K)$, respectively.

Definition 1.4. Let $\vec{\mu} \in \vec{M}(K)$ be a charge supported on K. We call $\vec{\mu}$ a *solenoidal vector charge* if for any (scalar) function u defined and continuously differentiable near K we have

$$\vec{\mu}[\nabla u] = 0.$$

It is easily seen that a vector charge $\vec{\mu}$ is solenoidal if and only if $\operatorname{div}\vec{\mu} = 0$, where div is the divergence of a charge (in the sense of distributions). The space $\vec{M}(K)$ can be treated as a subspace of the space $\vec{\mathcal{D}}'$ of all vector distributions, i.e., roughly speaking, of all vector fields whose components are scalar distributions of class $\mathcal{D}'$; also, the space $\vec{\mathcal{D}}'$ can be defined as the space of all continuous functionals on the space $\vec{\mathcal{D}}$ of all test vector fields (i.e., the C^∞-smooth vector fields with compact support) endowed with the standard topology. The operator div maps $\vec{\mathcal{D}}'$ to $\mathcal{D}'$ by the following rule: $\operatorname{div}\vec{T}[u] = -\vec{T}[\nabla u]$ for $\vec{T} \in \vec{\mathcal{D}}'$ and $u \in \mathcal{D}$ (where $\cdot[\cdot]$ stands for the result of applying a functional to a test function or to a test vector field, respectively).

The set of all solenoidal vector charges supported on a compact set $K \subset \mathbb{R}^d$ is denoted by $\operatorname{sol}(K)$.

The solenoidal charges in $\vec{M}(K)$ are the charges orthogonal to the subspace of $\vec{C}(K)$ that consists of the restrictions to K of all gradients of C^1-smooth scalar functions (i.e., $\vec{\mu}[\nabla u] = 0$ for $\vec{\mu} \in \operatorname{sol}(K)$, $u \in C^1(\mathbb{R}^d)$). A standard application of the Hahn–Banach theorem leads us to the following conclusion:

$$\inf\{\|\vec{f} - \nabla u\|_{\vec{C}(K)} :\ u \in C^1(\mathbb{R}^d)\} = \sup\{\vec{\mu}[\vec{f}\,] :\ \vec{\mu} \in \operatorname{sol}(K), \|\vec{\mu}\|_{\vec{M}(K)} \le 1\} \tag{1}$$

(the supremum on the right-hand side is attained by a weak compactness argument). Thus, we can give necessary and sufficient solvability conditions for the problems stated in Subsection 1.1 in terms of solenoidal charges.

Theorem 1.5. (1) *A vector field $\vec{f} \in \vec{C}(K)$ has an ε-primitive on the set K if and only if*

$$\vec{\mu}[\vec{f}\,] < \varepsilon\|\vec{\mu}\|_{\vec{M}(K)} \quad \textit{for any} \quad \vec{\mu} \in \operatorname{sol}(K). \tag{2}$$

(2) *A field $\vec{f} \in \vec{C}(K)$ is a quasigradient (on the set K) if and only if*

$$\vec{\mu}[\vec{f}\,] = 0 \quad \textit{for any} \quad \vec{\mu} \in \operatorname{sol}(K).$$

(3) *A compact set K is a gradient set if and only if* $\operatorname{sol}(K) = \{0\}$.

These conclusions characterize the properties of approximation by gradients by duality. It turns out that there are conditions characterizing this approximation in terms of geometric measure theory. These conditions are possible due to theorem on decomposition of solenoidal vector charges into what is called *elementary solenoids*, proved by S. K. Smirnov.

1.3. S. K. Smirnov theorem. The theorem of S. K. Smirnov (see [1]) states the possibility of decomposition of solenoidal vector charges with, in general, noncompact supports into the so-called elementary solenoids. S. K. Smirnov also obtained a result on decomposition of *normal* charges (i.e., those whose divergence understood in the sense of distributions is also a charge). We restrict ourselves to giving the statement for solenoidal charges with compact supports.

Definition 1.6 (Elementary solenoid)**.** Suppose that $K \subset \mathbb{R}^d$ is a compact set. A charge $\vec{T} \in \vec{M}(K)$ is called an *elementary solenoid* if there exists a vector-valued function $\vec{\gamma} : \mathbb{R} \to \mathbb{R}^d$ such that

1. $\vec{\gamma}$ is a Lipschitz function with $\|\vec{\gamma}'\|_{L^\infty(\mathbb{R})} \le 1$;
2. for any C^∞-smooth vector field $\vec{\varphi}$ with a compact support (a test field), we have
$$\vec{T}[\vec{\varphi}] = \lim_{k\to\infty} \frac{1}{2k} \int_{-k}^{k} \langle \vec{\gamma}'(s), \vec{\varphi}\,(\vec{\gamma}(s))\rangle\, ds$$
(in particular, the limit on the right-hand side exists for any test field $\vec{\varphi}$);
3. $\|\vec{T}\|_{\vec{M}(K)} = 1$;
4. $\vec{\gamma}(\mathbb{R}) \subset \operatorname{supp}\vec{T}$.

Thus, an elementary solenoid is the Cesaro mean of the integrals of the tangent component for a test field along arcs of some Lipschitz curve. The circulation along a cycle is a particular case of an elementary solenoid.

Denote by $\mathfrak{c}(K)$ the set of all elementary solenoids supported on a compact set $K \subset \mathbb{R}^d$; $\mathfrak{c}(K)$ is endowed with the $\vec{\mathcal{D}}$-weak topology (determined by the fields from $\vec{\mathcal{D}}$). The theorem by S. K. Smirnov for solenoidal charges with compact supports can be formulated as follows.

Theorem 1.7 (S. K. Smirnov)**.** *Suppose that $K \subset \mathbb{R}^d$ is a compact set. Every charge $\vec{\mu} \in \mathrm{sol}(K)$ admits a convex decomposition into elementary solenoids, namely, there exists a Borel* (*with respect to the above-mentioned topology*) *measure α on $\mathfrak{c}(K)$ such that*

$$\vec{\mu} = \int_{\mathfrak{c}(K)} \vec{T}\, d\alpha(\vec{T}),$$
$$\|\vec{\mu}\|_{\vec{M}(K)} = \alpha(\mathfrak{c}(K)).$$

The integral in the first formula is understood in the weak sense:

$$\vec{\mu}[\vec{f}\,] = \int_{\mathfrak{c}(K)} \vec{T}[\vec{f}\,]\, d\alpha(\vec{T})$$

for any field $\vec{f} \in \vec{C}(K)$; in particular, the mapping $\vec{T} \mapsto \vec{T}[\vec{f}\,]$, $\vec{T} \in \mathfrak{c}(K)$, is measurable with respect to the Borel σ-algebra generated by the above-mentioned $\vec{\mathcal{D}}$-weak topology.

The convex decomposition of solenoidal vector charges allows us to check the conditions of Theorem 1.5 for elementary solenoids supported on K only.

Theorem 1.8. (1) *A field $\vec{f} \in \vec{C}(K)$ has an ε-primitive on the set K if and only if*

$$\vec{T}[\vec{f}\,] < \varepsilon \quad \textit{for any} \quad \vec{T} \in \mathfrak{c}(K). \tag{3}$$

(2) *A field $\vec{f} \in \vec{C}(K)$ is a quasigradient on the set K if and only if*

$$\vec{T}[\vec{f}\,] = 0 \quad \textit{for any} \quad \vec{T} \in \mathfrak{c}(K).$$

(3) *A compact set $K \subset \mathbb{R}^d$ is a gradient set if and only if $\mathfrak{c}(K) = \varnothing$.*

This theorem is a result that characterizes the uniform approximation by gradients in terms of geometric measure theory. For example, the last assertion of the theorem implies that if there is no rectifiable curve of positive length supported on K, then K is a gradient set (see [6]; note that this property was established in [8] for nonrectifiable Jordan curves). If, moreover, $\lambda_d(K) = 0$, then any field $\vec{f} \in \vec{C}(K)$ can be approximated by gradients of functions harmonic near K with arbitrary precision.

In the case where $d = 2$, any elementary solenoid can be decomposed into simple closed Lipschitz curves (see [6]). A charge $\vec{T} \in \mathrm{sol}(K)$ is called a *simple closed Lipschitz curve* if there exists a number $L > 0$ and an injective Lipschitz function $\vec{\gamma} : [0, L] \to \mathbb{R}^2$, $\vec{\gamma}(0) = \vec{\gamma}(L)$, such that the charge $\vec{T}$ understood as a functional on $\vec{C}(K)$ is the circulation of a test field around the curve $\vec{\gamma}$, that is,

$$\vec{T}[\vec{f}\,] = \int_0^L \langle \vec{\gamma}\,'(s), \vec{f}(\vec{\gamma}(s)) \rangle\, ds$$

for any infinitely smooth vector field $\vec{f}$ with compact support in $\mathbb{R}^2$. Thus, for $d = 2$, it suffices to check the conditions of Theorem 1.8 only for closed Lipschitz curves. Note that for $d \geq 3$ the situation is, in general, more complicated: for example, the support of an elementary solenoid in $\mathbb{R}^3$ may have positive two-dimensional Hausdorf measure (see [1]).

1.4. A direct approach to the problem of uniform approximation by gradients. Theorem 1.5 delivers dual solvability conditions for problems of uniform approximation by gradients, while Theorem 1.8 gives such conditions in terms of geometric measure theory. The conclusions of Theorem 1.5 were obtained by using the Hahn–Banach theorem. In other words, equation (1) was derived *via duality*.

Our main goal in the present paper is *to obtain a conclusion like* (1) *without invocation of the Hahn–Banach theorem*. We say some word why we want to do this.

Duality is a nice technique that allows us to obtain approximation results by exploration of sets polar to subspaces consisting of the objects by which we want to approximate arbitrary functions or vector fields. In our situation these approximating objects are fields of the form $\nabla u|_K \in \vec{C}(K)$, while the charges $\vec{\mu}$ arising on the right-hand side of (1) can be treated as *obstructions to approximation*.

However, application of duality gives no sensible answer to the question as to *how to construct an approximating field of the form ∇u, or a charge $\vec{\mu} \in \mathrm{sol}(K)$ that hampers approximation.* Our study is intended to search for a method to construct one of these two objects as explicitly as it is possible. Our approach is not entirely constructive, as this term is understood in constructive mathematics. Indeed, we shall need a solution of a certain partial derivative equation; the existence theorem will be established by a "nonconstructive" variational argument. Moreover, a charge that cancels our approximation (if it exists) will be obtained by using the "nonconstructive" Banach–Alaoglu theorem. Thus, *our approach is intended to discovering a method that avoids application of the Hahn–Banach theorem.* We call such an approach *direct*.

Note that the desire to find a direct approach to constructing ε-primitives or charges that deny their existence is also provoked by the following circumstance: the theorems by Mergelyan and Vitushkin that characterize the possibility of analytic approximation on the complex plane can be proved by duality (see [4]) as well as without application of the Hahn–Banach theorem (see [3]). The same observation is true for the analog of the Runge theorem for harmonic differential forms (a proof by duality can be found in [5], while a direct proof was given in [7]). So, since we have a dual method of exploring problems of uniform approximation by gradients (that is, the method based on the Hahn–Banach theorem, the theorem of S. K. Smirnov, and the Riesz representation theorem), it is natural to ask whether it is possible to find a direct approach to approximation by gradients.

Relation (1) obtained by the use of the Hahn–Banach theorem is a precise identity. Having an intention to find a direct method of approximation by gradients, we allow ourselves a gap between the left-hand side and the right-hand side in (1) (like in constructive mathematics and computer sciences). Now we state the problem of uniform approximation by gradients in the form to be dealt with below.

So, suppose that $d \geq 2$ and that $K \subset \mathbb{R}^d$ is a compact set. Let $\vec{f}$ be a vector field of class $\vec{C}(K)$. Suppose also that two numbers $\varepsilon_*, \varepsilon^* (0 < \varepsilon_* < \varepsilon^*)$ are given. Our goal is *to construct explicitly either a function $u \in C^1(\mathbb{R}^d)$ such that $|\nabla u - \vec{f}| < \varepsilon^*$ on K, or a vector charge $\vec{\mu} \in \mathrm{sol}(K)$ for which*

$$\vec{\mu}[\vec{f}] \geq \varepsilon_* \, \|\vec{\mu}\|_{\vec{M}(K)}. \tag{4}$$

Since a discrete analog of this problem will also be treated, in order to distinguish these two problems we shall call the above-stated problem the *continuous problem* of uniform approximation by gradients.

Note that we want to find explicitly merely a *solenoidal* charge $\vec{\mu}$ satisfying (4), while we could ask a question on explicit construction of an *elementary solenoid* for which (4) is true. In other words, we are going to avoid application of the Hahn–Banach theorem, but not the combination "Hahn–Banach theorem + theorem by S. K. Smirnov": unfortunately, we have not found an appropriate method for this. Note that in the discrete analog of the problem of uniform approximation by gradients, which will be treated below, we shall find a discrete analog of the elementary solenoid in question in an explicit way.

1.5. Organization of the paper. We start with the study of a discrete analog of the problem of uniform approximation by gradients, namely, the problem of approximation by gradients on a finite graph (§2). This problem arose from attempts to find an approach to the continuous problem, but the problem of approximation by gradients on a graph is also of independent interest. The method obtained for solution of this problem is a step-by-step process that leads either to a "discrete ε-primitive" or to a "discrete charge" that cancels approximation. We call this method the *algorithm of rebuildings* — in the spirit of numerical analysis and computer sciences. So, in §2 we state the problem of approximation by gradients on a graph (before that we demonstrate its analogy with the above-stated continuous problem), and then we describe the method of solution of the graph problem — the algorithm of rebuildings. After that we investigate this method by duality and, finally, show without duality how this algorithm leads either to a discrete ε-primitive or to an obstruction to its existence. Note that in the discrete problem we shall only study the case where $\varepsilon^* = 3\varepsilon_*$.

§3 is devoted to the continuous problem of approximation by gradients. Taking the method of solution of the discrete problem into account, we suggest a method of solution of the continuous approximation problem. This method will require a solution of a quasilinear partial derivative equation, thus first we prove the existence theorem for this equation and then show how to use its solution for our problem.

Now we pass to the graph problem.

§2. Discrete analog: approximation by gradients on graphs

2.1. Heuristic considerations: statement of the graph problem. Let K be a compact subset of $\mathbb{R}^d$. Suppose that we know that some field $\vec{f} \in \vec{C}(K)$ has an ε-primitive on K. Extend our field $\vec{f}$ from K to $\mathbb{R}^d$ by continuity. The resulting field (still denoted by $\vec{f}$) has an ε-primitive on the set of the form $\operatorname{clos}\Omega$, where Ω is some neighborhood of K (in what follows clos denotes closure of a set in a topological space); assume that the set Ω is given. Suppose that we need to find an ε-primitive of the field $\vec{f}$ on the set $\operatorname{clos}\Omega$ (denote it by u) in an explicit way. How to do this? If we are going to do this by using numerical methods it is natural to start with a cubic lattice with a sufficiently small step in Ω and take the values of u at the nodes of the lattice as the unknown quantities. Let us write out the system of inequalities that arises under such discretization. Suppose that our lattice is oriented by an orthogonal coordinate system whose basis vectors are $(e_1, \dots, e_d)$, and let δ denote the step of our lattice. Denote by $\{x_l\}_{l=1}^{L}$ the set of nodes of the lattice. The condition $|\nabla u(x) - \vec{f}(x)| < \varepsilon$, $x \in \operatorname{clos}\Omega$, can be replaced, for example, by the following system of inequalities:

$$\left| \frac{u(x_l + \delta e_j) - u(x_l)}{\delta} - \left\langle \vec{f}(x_l + \frac{1}{2}\delta e_j), e_j \right\rangle \right| \le \varepsilon_1. \tag{5}$$

We write such inequalities for all pairs (x_l, j), where $x_l \in \operatorname{clos}\Omega$ is a node of our lattice and j is an index of a basis vector such that $x_l + \delta e_j$ is also a node of the lattice; ε_1 is any real number greater than ε. It can be shown that if $\vec{f}$ has an ε-primitive on $\operatorname{clos}\Omega$, then for every $\varepsilon_1 > \varepsilon$, system (5) will have a solution if the step of the lattice is sufficiently small (depending on the modulus of continuity of $\vec{f}$). After we find $u(x_l)$

for all nodes x_l, the function u can be obtained by interpolation by these nodes to some neighborhood of K, e.g., to the union of all cubes of the lattice that lie in Ω: the values of u in the interior of some cube can be constructed by linear interpolation by vertices of this cube; after that we should regularize the resulting function by convolution with an approximate identity. It can be shown that for any $\varepsilon_2 > \varepsilon_1$, the function obtained in such a way will satisfy the condition $|\nabla u - \vec{f}\,| < \varepsilon_2$ near K. We do not give rigorous proofs of these technical propositions: for us it is important that we arrive at the statement of a problem on the lattice. We present these arguments to demonstrate the analogy of the continuous problem of approximation by gradients and its discrete version.

Clearly, we may assume that $\delta = 1$, by scaling. Next, for final statement of discrete problem, we forget the rectangular structure of our lattice and work with a system on an arbitrary graph, writing inequalities of the form (5) for all its edges.

Before we pass to formalization of the graph problem, we say about our intentions. Our purpose is not to produce a good computative method to find a solution of the discrete system in order to obtain the solution of the continuous problem by interpolation by nodes of the lattice: *we state the discrete problem of approximation by gradients in order to find a method of its solution that can be transferred to continuous problem.* We can characterize problem (5) immediately: this is a finite system of linear inequalities or, in other words, a problem of linear programming. The well-known simplex algorithm used to solve such problems seems to be nontransferable to our continuous problem. Thus, we need to search for another approach.

Now we pass to formalization of the problem of approximation by gradients on an arbitrary graph.

2.2. Formalization of the graph problem: discrete analogs of some notions of vector analysis. Let G be a finite oriented graph, V the set of its vertices and E the set of its edges (loops and multiple edges are allowed). Denote by $\operatorname{begin} e$ the head of an edge $e \in E$ and denote by $\operatorname{end} e$ its tail; in this situation we write $e = (\operatorname{begin} e, \operatorname{end} e)$ (in general, in the case of multiple edges this representation does not determine the edge uniquely, but this will not confuse us).

We introduce two spaces:

$$U = U(G) = \{u : V \to \mathbb{R}\}$$

is the set of real-valued functions of vertices, and

$$F = F(G) = \{f : E \to \mathbb{R}\}$$

is the set of all functions of edges. These two spaces can be endowed with the following norms: put $\|f\|_\infty = \max_{e\in E} |f(e)|$, $\|f\|_1 = \sum_{e\in E} |f(e)|$ for $f \in F$; and U is normed by

$$\|u\|_\infty = \max_{v\in V} |u(v)|, \quad u \in U. \tag{6}$$

The elements of U are analogs of scalar functions of d variables, the elements of the space F endowed with $\|\cdot\|_\infty$-norm are analogs of vector fields (and we call the elements of F endowed with this norm *discrete fields*), finally, the elements of the space F endowed with $\|\cdot\|_1$-norm are analogs of vector charges (and we call them *discrete charges*). Note that the range of discrete fields and discrete charges is one-dimensional. This is quite natural: a continuous vector field $\vec{f}$ delivers a scalar value to inequalities (5).

For $u_1, u_2 \in U$ put $u_1[u_2] = \sum_{v\in V} u_1(v)\, u_2(v)$, and for $f_1, f_2 \in F$ put $f_1[f_2] = \sum_{e\in E} f_1(e)\, f_2(e)$.

Now we define operators $\nabla : U \to F$ and $\operatorname{div} : F \to U$ in the following way: for $u \in U$ put

$$(\nabla u)\,(e) = u(\operatorname{end} e) - u(\operatorname{begin} e), \quad e \in E, \tag{7}$$

and for $f \in F$ put

$$(\operatorname{div} f)(v) = \sum_{\text{begin}\, e = v} f(e) - \sum_{\text{end}\, e = v} f(e), \quad v \in V. \tag{8}$$

The operators ∇ and div are linear, and a *discrete analog of the divergence theorem is true*: $f[\nabla u] = -\operatorname{div} f[u]$ for $f \in F$, $u \in U$ (the proof is obtained by immediate computation). An element $f \in F$ is said to be *solenoidal* if $\operatorname{div} f = 0$.

The problem of the search of an ε-primitive of a discrete field f is stated as follows. Suppose that we are given a discrete field $f : E \to \mathbb{R}$ and a number $\varepsilon > 0$. It is required to find a vertex function $u : V \to \mathbb{R}$ such that the following relation is true for any $e \in E$:

$$|u(\operatorname{end} e) - u(\operatorname{begin} e) - f(e)| \le \varepsilon,$$

or, in a compact form,

$$\|\nabla u - f\|_\infty \le \varepsilon.$$

Such a function u is called a (*discrete*) *ε-primitive of the discrete field f*.

It is clear that our problem will not change if to the graph G we add an edge of the form $e' = (\operatorname{end} e, \operatorname{begin} e)$ for some $e \in E$ and put $f(e') = -f(e)$ (it may happen that this new edge e' will be a multiple edge if there have already been an edge from $\operatorname{end} e$ to $\operatorname{begin} e$ in the graph G; this will not lead to confusion). Thus, we can suppose that the following condition is satisfied.

Assumption 2.1. *If $e = (v_1, v_2) \in E$, then the graph G has the edge $e' = (v_2, v_1)$, and $f(e') = -f(e)$. This edge e' is said to be dual to the edge e.*

Under this assumption, a function $u \in U$ is an ε-primitive of a field f if and only if

$$u(\operatorname{end} e) - u(\operatorname{begin} e) - f(e) \ge -\varepsilon \quad \text{for all} \quad e \in E$$

(for us it is convenient to formulate the condition on an ε-primitive precisely in this way).

The standard argument based on the Hahn–Banach theorem immediately leads us to the following conclusion: a discrete field f admits an ε-primitive if and only if

$$|\mu[f]| \le \varepsilon \, \|\mu\|_1 \tag{9}$$

for any discrete charge $\mu \in F$ such that $\operatorname{div} \mu = 0$. We say that a discrete charge $\mu \in F$ is *positive* if $\mu(e) \ge 0$ for any edge $e \in E$. Suppose that inequality (9) fails for some solenoidal charge μ. *Then, under Assumption* 2.1, *there is a positive solenoidal charge μ for which* (9) *is not true.* Indeed, suppose that $\mu(e) < 0$ for some $e \in E$. Then replace the charge μ by the charge $\widetilde{\mu}$ defined by $\widetilde{\mu}(e) = 0, \widetilde{\mu}(e') = \mu(e') - \mu(e)$, where e' is the edge in the graph G dual to e (this edge exists by Assumption 2.1) and $\widetilde{\mu}(\widetilde{e}) = \mu(\widetilde{e})$, $\widetilde{e} \in E$, $\widetilde{e} \ne e, e'$. It is obvious that if (9) fails for μ, then it fails for the charge $\widetilde{\mu}$ obtained by such a replacement; moreover, this charge $\widetilde{\mu}$ stays solenoidal. Now, if $\widetilde{\mu}(e') < 0$, then we make a similar replacement for the edge e'. If we perform such replacements for all edges $e \in E$ with $\mu(e) < 0$, we obtain a positive solenoidal charge $\widetilde{\mu} \in F$ for which (9) is not true.

We single out an elementary example of a positive solenoidal discrete charge. A cycle Γ in G that goes along directions of its edges is said to be *simple* if it does not pass any vertex twice. Let Γ be a nonempty simple cycle in G *that goes along the directions of its edges*. We associate a discrete charge $\widetilde{\Gamma} \in F$ with this cycle: for an edge $e \in E$, we put $\widetilde{\Gamma}(e) = 1$ if Γ passes this edge and $\widetilde{\Gamma}(e) = 0$ otherwise. In what follows we identify Γ with the charge $\widetilde{\Gamma}$ generated by Γ and write simply Γ instead of $\widetilde{\Gamma}$. In our notation, $\|\Gamma\|_1$ is the number of edges (or of vertices) in Γ, and $\Gamma[f] = \sum_{e \in \Gamma} f(e)$. Denote by $\mathfrak{c} = \mathfrak{c}(G)$ the set of all nonempty simple cycles in G. If Assumption 2.1 is fulfilled, and furthermore,

$E \neq \varnothing$, then $\mathfrak{c} \neq \varnothing$: in this case, the graph has at least a cycle consisting of some edge $e \in E$ and its dual edge e'.

Now we formulate a discrete analog of the S. K. Smirnov decomposition theorem (see Theorem 1.7) for positive solenoidal discrete charges.

Claim 2.2. Let $\mu \in F$ be a positive solenoidal charge. Then μ admits a decomposition into cycles belonging to $\mathfrak{c}$, i.e., there exist cycles $\Gamma_1, \dots, \Gamma_l \in \mathfrak{c}$ and numbers $a_1, \dots, a_l \geq 0$ for which

$$\mu = a_1\Gamma_1 + \dots + a_l\Gamma_l.$$

Proof. The required decomposition of μ will be obtained by the following step-by-step process. If $\mu = 0$, then the desired decomposition is null. Otherwise, we take any edge $e_1 \in E$ for which $\mu(e_1) > 0$. Since $\operatorname{div}\mu(\operatorname{end} e_1) = 0$, there exists an edge $e_2 \in E$ such that $\operatorname{begin} e_2 = \operatorname{end} e_1$ and $\mu(e_2) > 0$ (see (8)). Next, we proceed by taking edges e_k in such a way that $\operatorname{begin} e_{k+1} = \operatorname{end} e_k$ and $\mu(e_k) > 0$ for all k. Since the number of edges in G is finite, we have $e_k = e_{k'}$ for some $k \neq k'$. Passing to new indices, we obtain a set of edges $e_1, e_2, \dots, e_K \in E$ for which $\mu(e_k) > 0$ $(k = 1, 2, \dots, K)$, $\operatorname{end} e_k = \operatorname{begin} e_{k+1}$ $(k = 1, 2, \dots, K-1)$, and $\operatorname{end} e_K = \operatorname{begin} e_1$. Let Γ be the cycle consisting of all these edges. Put $a = \min_{k=1,\dots,K} \mu(e_k)$; we may assume that Γ is a simple cycle. Replace the charge μ by $\widetilde{\mu} = \mu - a\Gamma$ and carry out the same procedure for the resulting charge (which is, of course, solenoidal together with μ). After several steps we should obtain the null charge: indeed, after the described replacement of μ by $\widetilde{\mu}$ the number of edges e with $\mu(e) \neq 0$ decreases. Summation of the charges of the form $a\Gamma$ obtained at all the steps of our process gives us the required decomposition. $\square$

Claim 2.2 together with the Hahn–Banach theorem yields the following relation:

$$\min\{\|\nabla u - f\|_\infty \,:\, u \in U\} = \max\left\{\frac{\Gamma[f]}{\|\Gamma\|_1} \,:\, \Gamma \in \mathfrak{c}\right\}. \tag{10}$$

Note, by the way, that a primitive of a field f (i.e., a function $u \in U$ for which $\nabla u = f$) exists if and only if $\Gamma[f] = 0$ for every $\Gamma \in \mathfrak{c}$.

Now we want to prove an identity like (10) without using duality. This will be done with the help of the step-by-step *rebuilding algorithm*, which we describe below. As in the continuous problem, we allow a gap between the left-hand and right-hand sides of (10): given two numbers ε_*, ε^* $(\varepsilon_* < \varepsilon^*)$, we are going to construct either an ε^*-primitive of a given discrete field f or a cycle $\Gamma \in \mathfrak{c}$ for which $\Gamma[f] \geq \varepsilon_*\|\Gamma\|_1$. We shall consider the case where $\varepsilon^* = 3\varepsilon_*$, the general case is treated in the same way. So, finally we state the discrete problem of approximation by gradients in the following way:

given a finite graph G, a discrete field $f \in F = F(G)$ and a number $\varepsilon > 0$,
we need to find *in an explicit way* either a function $u \in U$
such that $\|\nabla u - f\|_\infty \leq 3\varepsilon$, or a cycle $\Gamma \in \mathfrak{c}$ satisfying $\Gamma[f] \geq \varepsilon\|\Gamma\|_1$.

2.3. Rebuilding algorithm. We define a sequence of functions $u_n : V \to \mathbb{R}$, $n = 0, 1, 2 \dots$. The function u_0 is defined arbitrarily. Suppose that u_n is defined for some n. If $\nabla u_n(e) - f(e) = u_n(\operatorname{end} e) - u_n(\operatorname{begin} e) - f(e) \geq -3\varepsilon$ for all $e \in E$, then the function u_n is a 3ε-primitive of the field f by Assumption 2.1, and in this case the process of constructing the functions u_n is stopped. Otherwise, we take an edge e for which $\nabla u_n(e) - f(e) = u_n(\operatorname{end} e) - u_n(\operatorname{begin} e) - f(e) < -3\varepsilon$. Define the function $u_{n+1} \in U$ as follows:

$$u_{n+1}(v) = \begin{cases} u_n(\operatorname{end} e) + \varepsilon, & v = \operatorname{end} e; \\ u_n(\operatorname{begin} e) - \varepsilon, & v = \operatorname{begin} e; \\ u_n(v), & v \neq \operatorname{begin} e, \operatorname{end} e. \end{cases}$$

(if e is a loop, then $u_{n+1} = u_n$). We call this operation a *rebuilding* of a discrete function u_n (more precisely, we call this operation a *rebuilding of* u_n *on the edge* e). Then we apply the same operation to the resulting function u_{n+1} and so on, either infinitely many times or until our process stops. Note immediately that the value $\sum_{v\in V} u(v)$ does not change under the operation described.

2.4. Semiinvariant of the rebuilding algorithm. First, we prove that if the discrete field f has an ε-primitive, then our process must stop after several steps. The argument essentially uses the existence of an ε-primitive. We call such an argument *implicit.*

Theorem 2.3. *Suppose that there exists a function $u^* \in U$ for which $\|\nabla u^* - f\|_\infty \le \varepsilon$. In this case the process described above should stop necessarily.*

Proof. Consider the following quantity

$$w(u) = \sum_{v\in V} \left(u(v) - u^*(v)\right)^2.$$

It turns out that $w(u)$ is a semiinvariant of our process. Indeed, we shall show that $w(u_{n+1}) < w(u_n)$. Suppose that u_{n+1} is obtained from u_n by rebuilding on the edge $e = (v_1, v_2)$. Then $\nabla u_{n+1}(e) = \nabla u_n(e) + 2\varepsilon$. We evaluate $w(u_n) - w(u_{n+1})$ by using the identities $a^2 + b^2 = \frac{(a+b)^2}{2} + \frac{(a-b)^2}{2}$ and $u_n(v_1) + u_n(v_2) = u_{n+1}(v_1) + u_{n+1}(v_2)$:

$$\begin{aligned} w(u_n) - w(u_{n+1}) &= (u_n(v_1) - u^*(v_1))^2 + (u_n(v_2) - u^*(v_2))^2 \\ &\quad - (u_{n+1}(v_1) - u^*(v_1))^2 (u_{n+1}(v_2) - u^*(v_2))^2 \\ &= \frac{1}{2}((u_n(v_1) + u_n(v_2) - u^*(v_1) - u^*(v_2))^2 + (u_n(v_1) \\ &\quad - u_n(v_2) - u^*(v_1) + u^*(v_2))^2) - \frac{1}{2}((u_{n+1}(v_1) + u_{n+1}(v_2) - u^*(v_1) \\ &\quad - u^*(v_2))^2 + (u_{n+1}(v_1) - u_{n+1}(v_2) - u^*(v_1) + u^*(v_2))^2) \\ &= \frac{1}{2}(\nabla u^*(e) - \nabla u_n(e))^2 - \frac{1}{2}(\nabla u^*(e) - \nabla u_{n+1}(e))^2 \\ &= \frac{1}{2}(\nabla u_{n+1}(e) - \nabla u_n(e))(2\nabla u^*(e) - \nabla u_n(e) - \nabla u_{n+1}(e)) \\ &= \varepsilon(2\nabla u^*(e) - 2\nabla u_n(e) - 2\varepsilon) \\ &= 2\varepsilon(\nabla u^*(e) - f(e) - (\nabla u_n(e) - f(e)) - \varepsilon). \end{aligned}$$

The last expression is greater than $2\varepsilon^2$: indeed, we have $\nabla u_n(e) - f(e) < -3\varepsilon$ by the choice of e, and since the function u^* is an ε-primitive of the field f, we have $\nabla u^*(e) - f(e) \ge -\varepsilon$. Thus, $w(u_n) - w(u_{n+1}) > 2\varepsilon^2$ for all n, whence $w(u_n) < w(u_0) - 2n\varepsilon^2$, $n = 1, 2, \ldots$. Since $w(u_n) \ge 0$, we conclude that if there exists an ε-primitive of the field f, then our process must stop. □

2.5. A direct examination of the algorithm. The theorem proved above does not reach our goal: namely, it does not avoid application of the Hahn–Banach theorem. Now we prove that if

$$\Gamma[f] \le \varepsilon \,\|\Gamma\|_1 \tag{11}$$

for all cycles $\Gamma \in \mathfrak{c}$, then our process necessarily stops, and the proof avoids the Hahn–Banach theorem. In the case where condition (11) is fulfilled not for all cycles $\Gamma \in \mathfrak{c}$, our argument allows us to find in an explicit way a cycle Γ for which this condition fails.

Now we are going to prove that the functions u_n defined in our algorithm are uniformly bounded. This statement is true independently of condition (11).

Lemma 2.4. *There exists a constant $C > 0$ depending on u_0, f, and ε such that $\|u_n\|_\infty \le C$ for all $n = 0, 1, 2 \dots$.*

This lemma will be deduced from an assertion concerning a process of a more general form.

Let V be some finite set and, as above, let $U = U(V) = \{u : V \to \mathbb{R}\}$ be the set of all functions defined on V. We endow the space U with the norm $\|\cdot\|_\infty$ given by (6). For $u \in U$ and $I \subset V$, put $S_I(u) = \sum_{v\in I} u(v)$. Fix a number $M > 0$. Consider a sequence $u_0, u_1, u_2, \dots$ of functions in U. Suppose that for any $n = 0, 1, 2 \dots$, the function u_{n+1} can be obtained from u_n by changing its values at two elements of the set V, with some additional restrictions. Namely, there exists a two-element set $(v_1, v_2) = (v_1^{(n)}, v_2^{(n)})$ such that the following two conditions are fulfilled. First, $u_{n+1}(v) = u_n(v)$ for $v \in V$, $v \ne v_1, v_2$, and $u_{n+1}(v_1)+u_{n+1}(v_2) = u_n(v_1)+u_n(v_2)$. Second, if $|u_n(v_1)-u_n(v_2)| > M$, then $|u_{n+1}(v_1) - u_{n+1}(v_2)| \le |u_n(v_1) - u_n(v_2)|$ (in this case we say that *u_{n+1} is obtained from u_n by a transformation of the first type*); if, otherwise, $|u_n(v_1) - u_n(v_2)| \le M$, then $|u_{n+1}(v_1) - u_{n+1}(v_2)| \le M$ (in this case we say that *u_{n+1} is obtained from u_n by a transformation of the second type*).

Lemma 2.5. *There exists a constant $C > 0$ depending on u_0 and M such that $\|u_n\|_\infty \le C$ for all $n = 0, 1, 2, \dots$.*

Proof. Let a be a real number to be specified later. Consider the following quantity:

$$\nu(I) = \nu_a(I) = a + (a-1) + (a-2) + \dots + (a - |I| + 1), \quad I \subset V \tag{12}$$

($|A|$ denotes the number of elements in some finite set A). We say that a function $u \in U$ is *subordinate* to the number a if

$$S_I(u) \le M\nu(I) \tag{13}$$

for any $I \subset V$. Now take the number a so large that u_0 is subordinate to a. We are going to show that then all the functions u_n are subordinate to a. It suffices to prove that if inequalities (13) are true for u_n (and for any set $I \subset V$), then these inequalities are true for u_{n+1}.

Suppose that u_{n+1} is obtained from the function u_n by changing its values at the elements $v_1, v_2 \in V$. Let I be some subset of V; we must prove that $S_I(u_{n+1}) \le M\nu(I)$. If $v_1, v_2 \in I$ or $v_1, v_2 \notin I$, then $S_I(u_{n+1}) = S_I(u_n) \le M\nu(I)$. Therefore, we may assume that $|\{v_1, v_2\} \cap I| = 1$, say, $v_1 \in I$, $v_2 \in V \setminus I$.

Suppose that u_{n+1} is obtained from u_n by a transformation of the first type. Then the numbers $u_{n+1}(v_1)$ and $u_{n+1}(v_2)$ lie between the numbers $u_n(v_1)$ and $u_n(v_2)$, whence $u_{n+1}(v_1), u_{n+1}(v_2) \le \max\{u_n(v_1), u_n(v_2)\}$; if the right-hand side of the last inequality is $u_n(v_1)$, then $S_I(u_{n+1}) \le S_I(u_n) \le M\nu(I)$. But if $\max\{u_n(v_1), u_n(v_2)\} = u_n(v_2)$, then

$$\begin{aligned} S_I(u_{n+1}) &= S_{I\setminus\{v_1\}}(u_{n+1}) + u_{n+1}(v_1) \le S_{I\setminus\{v_1\}}(u_n) + u_n(v_2) \\ &= S_{(I\setminus\{v_1\})\cup\{v_2\}}(u_n) \le M\nu((I \setminus \{v_1\}) \cup \{v_2\}) = M\nu(I). \end{aligned}$$

Thus, in both cases the required inequality is true.

Now suppose that u_{n+1} is obtained from u_n by a transformation of the second type. Suppose that

$$S_I(u_{n+1}) > M\nu(I). \tag{14}$$

Put $I_+ = I \cup \{v_2\}$, $I_- = I \setminus \{v_1\}$. Since the function u_{n+1} is obtained from u_n by a transformation of the second type, we have $u_{n+1}(v_2) \geq u_{n+1}(v_1) - M$, and (14) implies that $S_{I_- \cup \{v_2\}}(u_{n+1}) > M(\nu(I) - 1)$. Adding this inequality and (14), we get

$$2S_{I_-}(u_{n+1}) + u_{n+1}(v_1) + u_{n+1}(v_2) = S_{I_+}(u_n) + S_{I_-}(u_n) > M(2\nu(I) - 1). \tag{15}$$

On the other hand, since u_n is subordinate to a, we have

$$S_{I_+}(u_n) \leq M\nu(I_+), \tag{16}$$

$$S_{I_-}(u_n) \leq M\nu(I_-). \tag{17}$$

But $\nu(I_+) = \nu(I) + (a - |I|)$, $\nu(I_-) = \nu(I) - (a - |I| + 1)$. Adding these identities and also (16) and (17), we obtain

$$S_{I_-}(u_n) + S_{I_+}(u_n) \leq M(2\nu(I) - 1),$$

which contradicts (15). Thus, $S_I(u_{n+1}) \leq M\nu(I)$.

By the choice of a, the function u_0 is subordinate to a, so that the functions u_n are subordinate to a for all $n = 0, 1, 2\dots$; from this we inter that $u_n(v) \leq aM$ for all $n = 0, 1, 2, \dots$ and $v \in V$, it suffices to apply (13) with $u = u_n$ and $I = \{v\}$. On the other hand, $u_n(v) = S_V(u_n) - S_{V\setminus\{v\}}(u_n) \geq S_V(u_n) - M\nu(V \setminus \{v\})$, and the last quantity does not depend on n because $S_V(u_n)$ does not depend on n. The boundedness of the functions u_n is proved. □

Proof of Lemma 2.4. Put $M = \max\{\|f\|_\infty, 2\varepsilon\}$ (recall that $\|f\|_\infty = \max_{e\in E} |f(e)|$). Let $u_0, u_1, u_2, \dots$ be the functions constructed in the rebuilding algorithm. We check that for any $n = 0, 1, 2\dots$ the function u_{n+1} is obtained from u_n by a transformation of the first or of the second type described before Lemma 2.5 (with respect to our M). Suppose that u_{n+1} is obtained from u_n by rebuilding on the edge $e \in E$. Taking $v_1 = \operatorname{begin} e$, $v_2 = \operatorname{end} e$, we conclude that $u_{n+1}(v) = u_n(v)$ for $v \in V$, $v \neq v_1, v_2$, and $u_{n+1}(v_1) + u_{n+1}(v_2) = u_n(v_1) + u_n(v_2)$.

Now we check the second condition in the definition of transformations of the first and second type. In accordance with the rebuilding algorithm, $\nabla u_n(e) = u_n(\operatorname{end}(e)) - u_n(\operatorname{begin}(e)) - f(e) < -3\varepsilon$, and we perform the following transformation: $u_{n+1}(\operatorname{end} e) = u_n(\operatorname{end} e) + \varepsilon$, $u_{n+1}(\operatorname{begin} e) = u_n(\operatorname{begin} e) - \varepsilon$, whence

$$\nabla u_{n+1}(e) = \nabla u_n(e) + 2\varepsilon. \tag{18}$$

We have

$$\nabla u_n(e) < f(e) - 3\varepsilon \leq \|f\|_\infty - 3\varepsilon \leq M - 3\varepsilon. \tag{19}$$

Suppose that $|\nabla u_n(e)| = |u_n(\operatorname{end} e) - u_n(\operatorname{begin} e)| > M$. Then, by (19), we obtain $\nabla u_n(e) < -M$. Application of (18) gives

$$\nabla u_n(e) < \nabla u_{n+1}(e) = \nabla u_n(e) + 2\varepsilon \leq -M + 2\varepsilon \leq 0,$$

whence $|u_{n+1}(\operatorname{end} e) - u_{n+1}(\operatorname{begin} e)| < |u_n(\operatorname{end} e) - u_n(\operatorname{begin} e)|$, and the function u_{n+1} is obtained from u_n by a transformation of the first type.

Now suppose that $|u_n(\operatorname{end} e) - u_n(\operatorname{begin} e)| \leq M$. In this case (18) implies

$$\nabla u_{n+1}(e) = \nabla u_n(e) + 2\varepsilon > -M,$$

and, by (19), we have

$$\nabla u_{n+1}(e) = \nabla u_n(e) + 2\varepsilon < M - \varepsilon < M,$$

so that $|u_{n+1}(\operatorname{end} e) - u_{n+1}(\operatorname{begin} e)| < M$. In this case the function u_{n+1} is obtained from u_n by a transformation of the second type.

Thus, we are in the setting of Lemma 2.5, and the functions u_n are bounded uniformly in $n = 0, 1, 2, \dots$. □

Theorem 2.6. *Suppose that the process of rebuildings of functions of vertices described above does not stop. Then there is a cycle* $\Gamma \in \mathfrak{c}$ *for which* $\Gamma[f] > \varepsilon \|\Gamma\|_1$. *Such a cycle* Γ *can be constructed explicitly in terms of the sequence of edges on which the rebuildings were done and the functions* u_n *obtained in the algorithm.*

Proof. Observe that if the functions u_n are bounded and the values of $u_n(v)$ for fixed $v \in V$ and varying n differ from each other by an integral multiple of ε, then the vector $(u_n(v_1), \dots, u_n(v_{|V|}))$ (where $v_1, \dots, v_{|V|}$ are all the vertices of G) can take only finitely many values. Thus, if our process does not stop, then there are two numbers n, n', $n \neq n'$, for which $u_n = u_{n'}$. We may assume that $n' = 0$ and $u_0 = u_n$. Suppose that the function u_n is obtained from u_0 by rebuildings on the edges e_1, e_2, …, e_n, namely, suppose that u_k is obtained from u_{k-1} by its rebuilding on the edge e_k, $k = 1, \dots, n$.

We say that an edge e_l is *appropriate* for an edge e_k whenever $\operatorname{begin} e_l = \operatorname{end} e_k$ and $u_k(\operatorname{end} e_k) \geq u_l(\operatorname{begin} e_l)$.

For any edge e_k*, there exists at least one edge that is appropriate for* e_k. Indeed, put $v = \operatorname{end} e_k$. Let l be an index such that $u_l(v) = \min_{l'} u_{l'}(v)$. We may assume that $u_l(v) = u_{l-1}(v) - \varepsilon$, otherwise we shall pass to $\widetilde{l} = l - 1$ (or to $\widetilde{l} = n$ if $l = 0$); since the value of $u_m(v)$ was changed at least once, this procedure will give us an index l for which $u_l(v) = u_{l-1}(v) - \varepsilon$. It is obvious that in this case the edge e_l will be appropriate for e_k.

Choose an edge e_{i_1} arbitrarily; then take an edge e_{i_2} appropriate for e_{i_1}, and, next, let $e_{i_{k+1}}$ be an edge appropriate for e_{i_k}. Since the number of edges in G is finite, we have $e_l = e_{l'}$ for some l, l' $(l < l')$. Passing to the indices l, $l+1$, …, l', we obtain a set of edges e_{i_1}, e_{i_2}, …, e_{i_N} for which $e_{i_{k+1}}$ is appropriate for e_{i_k}, $k = 1, \dots, N$ (we put $i_{N+1} = i_1$ to simplify the notation). Let Γ be a cycle consisting of these edges; obviously, we may assume that the cycle Γ is simple. We prove that condition (11) fails for this cycle.

By the definition of our algorithm, the following inequalities are true:

$$u_{i_k-1}(\operatorname{end} e_{i_k}) - u_{i_k-1}(\operatorname{begin} e_{i_k}) - f(e_{i_k}) < -3\varepsilon, \quad k = 1, \dots, N.$$

Summation over all $k = 1, \dots, N$ yields

$$\begin{aligned} -3N\varepsilon &> -\Gamma[f] + \sum_{k=1}^{N} (u_{i_k-1}(\operatorname{end} e_{i_k}) - u_{i_k-1}(\operatorname{begin} e_{i_k})) \\ &= -\Gamma[f] + \sum_{k=1}^{N} \left(u_{i_k-1}(\operatorname{end} e_{i_k}) - u_{i_{k+1}-1}(\operatorname{begin} e_{i_{k+1}})\right) \\ &= -\Gamma[f] - 2N\varepsilon + \sum_{k=1}^{N} \left(u_{i_k}(\operatorname{end} e_{i_k}) - u_{i_{k+1}}(\operatorname{begin} e_{i_{k+1}})\right). \end{aligned}$$

By the choice of the edges e_{i_k}, all terms in the last sum are nonnegative, and this implies that $\Gamma[f] > N\varepsilon = \varepsilon \|\Gamma\|_1$. □

Remark 2.7. Our construction can be interpreted as follows. Suppose that $\varepsilon = 1$ and that $u_0(v) \in \mathbb{N}$ for all $v \in V$. Then $u_n(v) \in \mathbb{Z}$ for all $n = 0, 1, 2, \dots$ and $v \in V$. By Lemma 2.4, the functions u_n are uniformly bounded; therefore, replacing the function u_0 by $\widetilde{u}_0(v) = u_0(v) + N$ with $N \in \mathbb{N}$ sufficiently large, we may assume that $u_n(v) \in \mathbb{N}$, $n = 0, 1, 2, \dots$, $v \in V$. Suppose that after the nth step of our process there is a pile on $u_n(v)$ coins lying at every vertex $v \in V$. We interpret the rebuilding on the edge e in the following way: we take the coin from the top of the pile in $\operatorname{begin} e$ and put this coin to the top of the pile in $\operatorname{end} e$. Consider the path of some (individual) coin. Suppose that it was moved in rebuildings on the edges e_1, e_2, …. It can easily be seen that then the edge e_{k+1} is appropriate for the edge e_k. Suppose that some coin returned to the place where it had been before in the course of our process, i.e., to the same vertex and to the same altitude where it had been before — if the rebuildings do not stop, then such a situation will necessarily occur for at least one coin. The argument in the proof of Theorem 2.6 shows that, in this case, if Γ is the cyclic path of this coin, then $\Gamma[f] > \varepsilon \|\Gamma\|_1$. In other words, the closed paths of individual coins give us cycles that are obstructions to approximation.

We summarize our study of the discrete problem of approximation by gradients. We have obtained a method that not only avoids application of the Hahn–Banach theorem, but also, in the case where a function u does not admit the required approximation, allows us to construct a cycle Γ for which (11) fails, that is, in fact, to avoid the theorem on decomposition of discrete solenoidal charges. Now we are going to use this approach as heuristic arguments on the study of the continuous problem of approximation by gradients.

§3. Continuous problem of uniform approximation by gradients

The following subsection is purely heuristic. The formal arguments concerning the continuous problem of approximation by gradients start from Subsection 3.2 and can be understood independently of Subsection 3.1. Subsection 3.2 is devoted to the solvability of a certain quasilinear partial derivative equation, to which we reduce the continuous problem of approximation by gradients in Subsection 3.3. The main result of the §3 is Theorem 3.4 (see Subsection 3.3).

3.1. Heuristic considerations: approach to the continuous problem. To start with, assume that K is a closure of an open set, $K = \operatorname{clos}\Omega$, where the boundary of Ω is sufficiently smooth. Given a vector field $\vec{f} \in \vec{C}(K)$, we want to find a function $u \in C^1(\mathbb{R}^d)$ such that $\|\nabla u - \vec{f}\|_{\vec{C}(\operatorname{clos}\Omega)} < \varepsilon$.

The algorithm of rebuildings has two features which make it impossible to generalize its outline to the continuous case immediately. First, our algorithm is a step-by-step process, i.e., its time is discrete. Second, *rebuildings are local transformations* involving only two vertices of the graph. This means that if we try to apply the same pattern in the continuous problem, then we shall need to make "continuous rebuildings" in some infinitely small sets.

In order to outline the general principle of a "continuous modification" of the rebuilding algorithm, we write the rebuilding in terms of the formalism of differential operators on a graph that we introduced in Subsection 2.2. Suppose that u_{n+1} is obtained from u_n by rebuilding on an edge e. We define a discrete field $\Phi = \Phi_n$ as follows: put $\Phi(e) = \varepsilon$, $\Phi(\widetilde{e}) = 0$ for $\widetilde{e} \neq e$, $\widetilde{e} \in E$. Now we can write this rebuilding in the form

$$u_{n+1} = u_n - \operatorname{div}\Phi$$

(see (8)). We can interpret this representation as follows: suppose that $u_n(v)$ grams of some substance is placed at every vertex $v \in V$; then the rebuilding moves ε grams of this substance from $\operatorname{begin} e$ to $\operatorname{end} e$; so Φ is a flow of this substance. Thus, we arrive at an idea to find an ε-primitive u in the continuous problem as a result of a process that satisfies an equation of the form

$$\frac{\partial u}{\partial t} = -\operatorname{div}\vec{\Phi}, \tag{20}$$

where $u = u(x,t)$ depends on a coordinate $x \in \Omega$ and on time t, while a field $\vec{\Phi}$ depends on u and on the field $\vec{f}$ to be approximated. The field $\vec{\Phi}$ should be chosen in such a way that the argument similar to that presented in Subsection 2.4 be possible for the solution of equation (20). Namely, we put $W(t) = \int_\Omega (u(x,t) - u^*(x))^2\, d\lambda_d(x)$, where u

is the solution of (20) and u^* is an ε-primitive of $\vec{f}$ on K (if it exists), and try to define $\vec{\Phi}$ in such a way that $W(t)$ decreases as t increases (cf. the argument in Subsection 2.4, where a similar quantity $w(u_n)$ was introduced). A formal application of the divergence theorem gives

$$\frac{dW}{dt} = 2\int_\Omega \big(\langle\vec{\Phi}, \nabla u - \vec{f}\rangle - \langle\vec{\Phi}, \nabla u^* - \vec{f}\rangle\big)\, d\lambda_d - 2\int_{\partial\Omega} \langle\vec{\Phi}, \vec{n}\rangle (u - u^*)\, dS, \tag{21}$$

where $\vec{n}$ is the outward unit normal field of $\partial\Omega$ and dS denotes the surface measure on $\partial\Omega$. We take into consideration the *residual vector* $\vec{\xi} = \vec{\xi}(x,t) = \vec{f}(x) - \nabla u(x,t)$. We know that $|\nabla u^* - \vec{f}\,| \le \varepsilon$ on Ω, so that W will be a monotone decreasing function of t if $\langle\vec{\Phi}, \vec{n}\rangle = 0$ on $\partial\Omega$ and

$$\langle\vec{\Phi}, \vec{\xi}\,\rangle \ge \varepsilon|\vec{\Phi}| \tag{22}$$

in Ω. At the points where $|\vec{\xi}\,| < \varepsilon$, the last inequality can be fulfilled only if $\vec{\Phi} = 0$, while at the points where $|\vec{\xi}\,| \ge \varepsilon$ the angle between $\vec{\Phi}$ and the residual field $\vec{\xi}$ should be acute. It is seen that inequality (22) will be fulfilled if we take $\vec{\Phi} = 0$ at the points where $|\vec{\xi}\,| < \varepsilon$ and $\vec{\Phi} = \vec{\xi}$ at the points where $|\vec{\xi}\,| \ge \varepsilon$. But such a field $\vec{\Phi}$, nevertheless, depends on u and $\vec{f}$ in a nonsmooth way. It is easy to correct this by a *slight* modification of the dependence of $\vec{\Phi}$ on u and $\vec{f}$. Taking a smooth function $\zeta : [0,+\infty) \to [0,+\infty)$ which is in some sense close to $\mathbb{1}_{[\varepsilon,+\infty)}$, consider the following equation:

$$\begin{cases} u = u(x,t) \, : \, \Omega \times \mathbb{R} \to \mathbb{R} \\ \dfrac{\partial u}{\partial t}(x,t) = -\operatorname{div} \vec{\Phi}(x,t) \\ \vec{\Phi}(x,t) = \zeta(|\vec{f}(x) - \nabla u(x,t)|)\big(\vec{f}(x) - \nabla u(x,t)\big) \\ \langle\vec{\Phi}(x,t), \vec{n}(x)\rangle = 0 \quad \text{on} \quad \partial\Omega \quad \text{(Neumann condition)}; \end{cases} \tag{23}$$

the function $u(\cdot,0)$ can be defined arbitrarily with only one restriction: if should be compatible with the Neumann boundary condition on $\partial\Omega$; the function ζ should satisfy the following restrictions: it should be a monotone nondecreasing bounded function with $\zeta(s) = 0$ for $s \in [0,\varepsilon]$, and $\zeta(s) > 0$ for $s > \varepsilon$. (Note that should we take $\zeta \equiv 1$ on $[0,+\infty)$, we would obtain the usual Neumann problem for the heat equation.)

Suppose that equation (23) has a sufficiently smooth solution u. The field $\vec{\Phi} = \zeta(|\vec{\xi}\,|)\vec{\xi}$ satisfies (22), so the quantity $W(t)$ does not increase as t increases. Moreover, suppose that ε^* is some number greater than ε; then, by (21),

$$\left|\frac{dW}{dt}\right| \ge \int_{|\vec{\xi}\,|>\varepsilon^*} \zeta(\varepsilon^*)|\vec{\xi}\,|(|\vec{\xi}\,| - \varepsilon)\, d\lambda_d.$$

The left-hand side of this inequality takes arbitrarily small values when $t \in [0,+\infty)$. Thus, in the case where $\vec{f}$ has an ε-primitive on K, the quantity $\int_{|\vec{\xi}\,|>\varepsilon^*} |\vec{\xi}\,|\, d\lambda_d$ must reach a sufficiently small value for some $t \in [0,+\infty)$. In this case we can approximate $\vec{f}$ by a gradient via regularization of the function $u(\cdot,t)$ with this t. (More precisely, this will give us an ε^{**}-primitive of $\vec{f}$ on an arbitrary compact subset of Ω with any $\varepsilon^{**} > \varepsilon^*$.)

But if the quantities $\int_{|\vec{\xi}\,|>\varepsilon^*} |\vec{\xi}\,|\, d\lambda_d$ are separated away from zero for all t, our equation allows us to outline an obstruction to approximation. For this, consider the fields

$$\vec{\Phi}^T = \vec{\Phi}^T(x) = \frac{\int_0^T \vec{\Phi}(x,t)\, dt}{\int_\Omega \int_0^T |\vec{\Phi}(x,t)|\, dt\, d\lambda_d(x)} \tag{24}$$

and, further, a sequence $\{T_n\}_{n=1}^{\infty}$ such that $T_n \to \infty$ and the charges $\vec{\Phi}^{T_n}\lambda_d \mathbb{1}_\Omega$ converge weakly to some charge $\vec{\mu} \in \vec{M}(K)$. Obviously, $\|\vec{\mu}\|_{\vec{M}(\operatorname{clos}\Omega)} \le 1$. Since our equation is degenerate parabolic, it is natural to expect the functions $u(\cdot,t)$ to be bounded in $L^2(\Omega)$ uniformly in t. To prove this, consider the function $B(t) = \int_\Omega u^2(x,t)\,d\lambda_d(x)$; evaluation of its derivative gives

$$B'(t) = 2\int_\Omega \zeta(|\vec{f} - \nabla u|)\langle \vec{f} - \nabla u, \nabla u\rangle\,d\lambda_d \le -\int_\Omega \zeta(|\vec{\xi}\,|)|\nabla u|^2\,d\lambda_d + \int_\Omega \zeta(|\vec{\xi}\,|)|\vec{f}\,|^2\,d\lambda_d.$$

So, there exist two constants c_1, $c_2 > 0$ depending on the field $\vec{f}$, the function ζ, and the set Ω, for which $B'(t) \le -c_1\int_\Omega |\nabla u|^2\,d\lambda_d + c_2$ (because the function $\zeta(|\vec{\xi}\,|)$ is upper bounded and at the same time is small only at the points where $|\nabla u|$ is bounded). Next, note that $\int_\Omega u\,d\lambda_d$ does not depend on t, so that there exist two constants c_3, $c_4 > 0$ (depending only on Ω and the value of $\int_\Omega u(x,0)\,d\lambda_d(x)$) for which $B(t) \le c_3\int_\Omega |\nabla u|^2\,d\lambda_d + c_4$. Thus, the function $B(t)$ admits an estimate of the form $B'(t) \le -c_5 B(t) + c_6$, $c_5 > 0$. The boundedness of $B(t)$ is derived from this estimate by the standard technique of integration of linear differential inequalities.

Now we can show that $\vec{\mu} \ne 0$, $\vec{\mu}[\vec{f}\,] \ge \varepsilon$, and $\operatorname{div}\vec{\mu} = 0$. Indeed, denote by A_T the denominator in (24). If the values of $\int_{|\vec{\xi}(x,T_n)|>\varepsilon^*} |\vec{\xi}(x,T_n)|\,d\lambda_d(x)$ are lower bounded for $n = 1, 2\dots$, then

$$\inf_{n\in\mathbb{N}} \frac{A_{T_n}}{T_n} > 0. \tag{25}$$

A formal evaluation gives

$$\int_\Omega \langle \vec{\Phi}^{T_n}(x), \vec{f}(x)\rangle\,d\lambda_d(x) = \frac{1}{A_{T_n}}\int_\Omega\int_0^{T_n} \zeta(|\vec{\xi}(x,t)|)|\vec{\xi}(x,t)|^2\,dt\,d\lambda_d(x) + \frac{1}{2A_{T_n}}\int_\Omega \bigl(u^2(x,T_n) - u^2(x,0)\bigr)\,d\lambda_d(x).$$

By (25) and the boundedness of the norms $\|u(\cdot,T_n)\|_{L^2(\Omega)}$, the second term tends to zero as $n \to \infty$. The first term is less than ε for any n, because $\zeta(|\vec{\xi}(x,t)|)|\vec{\xi}(x,t)|^2 \ge \varepsilon\zeta(|\vec{\xi}(x,t)|)|\vec{\xi}(x,t)| = \varepsilon|\vec{\Phi}(x,t)|$ for all $x \in \Omega$ and $t \ge 0$. Thus, $\vec{\mu}[\vec{f}\,] \ge \varepsilon$. In order to prove that $\operatorname{div}\vec{\mu} = 0$, we take a test function $\eta \in C^1(\mathbb{R}^d)$. It is easily seen that

$$\int_\Omega \eta(x)\operatorname{div}\vec{\Phi}^{T_n}(x)\,d\lambda_d(x) = -\frac{1}{A_{T_n}}\int_\Omega (u(x,T_n) - u(x,0))\,\eta(x)\,d\lambda_d(x).$$

The last expression tends to zero as $n \to \infty$ (again by (25) and the boundedness of the norms $\|u(\cdot,T_n)\|_{L^2(\Omega)}$), whence the limit charge $\vec{\mu}$ is solenoidal. Thus, in the case where the values $\int_{|\vec{\xi}\,|>\varepsilon^*} |\vec{\xi}\,|\,d\lambda_d$ are lower bounded for all t, the solution of equation (23) allows us to construct a solenoidal vector charge that cancels the required approximation.

It could be natural to try to transfer the arguments of Theorem 2.6 to our continuous problem. Nevertheless, this way leads us to the conclusion that the consideration of equation (23) is needless. Namely, in the case where the process (23) does not converge (in some sense), an attempt to construct an elementary solenoid $\vec{T}$ for which $\vec{T}[\vec{f}\,] > \varepsilon$ finishes with the following intuitive conclusion: such a solenoid will be an integral curve of the field $\vec{\Phi}(x,\infty)$ that corresponds to a "limit" function $u(x,\infty)$. It is natural to expect that the limit function $u(x,\infty)$ satisfies the equation $\operatorname{div}\bigl(\zeta(|\vec{f} - \nabla u|)(\vec{f} - \nabla u)\bigr) = 0$ in Ω, with the Neumann boundary condition $\langle \zeta(|\vec{f} - \nabla u|)(\vec{f} - \nabla u), \vec{n}\rangle = 0$ on $\partial\Omega$. Thus, finally we shall use an equation of this form to solve the problem of uniform approximation by gradients. If this approximation is possible, then an ε-primitive will be obtained by regularization of the solution u of our equation. And if the field $\vec{f}$ cannot be approximated by gradients on $\operatorname{clos}\Omega$, then an obstruction to approximation will be the charge $\vec{\mu} = \vec{\Phi}\, \mathbb{1}_\Omega\, \lambda_d$, where $\vec{\Phi} = \zeta(|\vec{f} - \nabla u|)(\vec{f} - \nabla u)$. (Observe that our equation ensures that the charge $\vec{\Phi}\, \mathbb{1}_\Omega\, \lambda_d$ be solenoidal if $\vec{\Phi}$ is specified as above: indeed, any charge of the form $\vec{F}\, \mathbb{1}_\Omega\, \lambda_d$, where $\vec{F}$ is a smooth field, is solenoidal if and only if $\operatorname{div}\vec{F} = 0$ in Ω and $\langle \vec{F}, \vec{n}\rangle = 0$ on $\partial\Omega$.) If $\vec{\mu} \neq 0$, then an elementary solenoid $\vec{T}$ for which $\vec{T}[\vec{f}\,] > \varepsilon$ can be obtained by application of the S. K. Smirnov theorem to this charge $\vec{\mu}$. Thus, *the argument that will be given below does not avoid the S. K. Smirnov decomposition theorem, but only allows us to avoid application of the Hahn–Banach theorem.*

The equation that we shall deal with is an elliptic quasilinear partial derivative equation. Under our choice of the function ζ, this equation is degenerate, because $\zeta(s) = 0$ when s is small. But this will not cause any problems for our proof of the existence theorem (see below).

Now we pass to the subsection where we shall finally state our equation and prove the theorem on the existence of its (weak) solution. The existence theorem will be obtained by a standard variational argument.

3.2. The existence theorem for the quasilinear equation. Let $\Omega \subset \mathbb{R}^d$ be some open set. Suppose for simplicity that the number of connected components of Ω is finite. We assume that Ω has a sufficiently smooth boundary $\partial\Omega$, and denote by $\vec{n}$ the outward unit normal field of $\partial\Omega$. Let $\vec{f}$ be a vector field of class $\vec{C}(\operatorname{clos}\Omega)$. Fix a number $\varepsilon > 0$. Let $\zeta : [0, +\infty) \to [0, +\infty)$ be a bounded, infinitely smooth, and monotone nondecreasing function such that $\zeta(t) = 0$ for $t \in [0, \varepsilon]$. We shall assume only that ζ is not equal to zero identically on $[0, +\infty)$, say, there is some $t_0 > \varepsilon$ for which $\zeta(t_0) = a > 0$.

Consider the equation

$$\begin{cases} \operatorname{div}\bigl(\zeta(|\nabla u - \vec{f}\,|)(\nabla u - \vec{f}\,)\bigr) = 0 & \text{in } \Omega, \\ \bigl\langle \zeta(|\nabla u - \vec{f}\,|)(\nabla u - \vec{f}\,), \vec{n}\bigr\rangle = 0 & \text{on } \partial\Omega. \end{cases} \tag{26}$$

For definiteness, to this equation we add additional conditions on the function u : $\int_{\Omega_k} u\, d\lambda_d = 0$, where the Ω_k ($k = 1, 2, \ldots, K$) are the connected components of Ω. We shall search a solution of (26) in the Sobolev space $W^{1,2}(\Omega)$.

Since, in general, gradients of functions in the Sobolev space $W^{1,2}(\Omega)$ have no boundary values, the boundary condition in (26) makes no sense. Therefore, we shall say that a function $u \in W^{1,2}(\Omega)$ is a *weak solution* of (26) if for any (test) function $\eta \in W^{1,2}(\Omega)$ the following relation is true:

$$\int_\Omega \bigl\langle \zeta(|\nabla u - \vec{f}\,|)(\nabla u - \vec{f}\,), \nabla\eta\bigr\rangle\, d\lambda_d = 0. \tag{27}$$

The standard argument shows that a C^2-smooth function u is a (classical) solution of equation (26) if and only if u is its weak solution.

Theorem 3.1. *Suppose that a bounded open set $\Omega \subset \mathbb{R}^d$ has finitely many connected components and that its boundary $\partial\Omega$ is sufficiently smooth. Let ζ be the function defined as above. Then for any vector field $\vec{f} \in \vec{C}(\operatorname{clos}\Omega)$ there exists a weak solution of equation* (26)*, i.e., a function $u \in W^{1,2}(\Omega)$ such that* (27) *is true for any $\eta \in W^{1,2}(\Omega)$.*

Proof. Obviously, we may assume that Ω is connected (i.e., that Ω is a domain). Our goal is to choose a functional on the space $W^{1,2}(\Omega)$ whose Euler equation coincides with our equation. For convenience of computations, we define a function ζ_1 by the relation $\zeta_1(t^2) = \zeta(t)$, $t \geq 0$. Now (27) takes the form

$$\int_\Omega \bigl\langle \zeta_1(|\nabla u - \vec{f}\,|^2)(\nabla u - \vec{f}\,), \nabla\eta\bigr\rangle\, d\lambda_d = 0, \quad \eta \in W^{1,2}(\Omega).$$

We put $\mathcal{X} = \left\{u \in W^{1,2}(\Omega) : \int_\Omega u\, d\lambda_d = 0\right\}$ and endow $\mathcal{X}$ with the norm

$$\|u\|_{\mathcal{X}} = \left(\int_\Omega |\nabla u|^2\, d\lambda_d\right)^{1/2}.$$

Then $\mathcal{X}$ is a separable Hilbert space. (The space $\mathcal{X}$ is complete: the smoothness of $\partial\Omega$ implies that $\left\{u \in W^{1,2}_{\mathrm{loc}}(\Omega) : \int_\Omega |\nabla u|^2\, d\lambda_d < \infty\right\} \subset W^{1,2}(\Omega)$).

Putting $Z(t) = \int_0^t \zeta_1(s)\, ds$, $t \geq 0$, we define a (nonlinear) functional I on $\mathcal{X}$:

$$I(u) = \int_\Omega Z(|\nabla u - \vec{f}\,|^2)\, d\lambda_d$$

and pose the problem of minimization of $I(u)$ over all $u \in \mathcal{X}$.

The functional I is well defined and finite for any function $u \in \mathcal{X}$. Indeed, $Z(t) \leq t \sup_{s\geq 0} \zeta(s)$ for any $t \geq 0$, whence $I(u) \leq \sup_{s\geq 0} \zeta(s) \cdot \int_\Omega |\nabla u - \vec{f}\,|^2\, d\lambda_d < +\infty$ for all $u \in \mathcal{X}$. On the other hand, $Z(t^2) \geq a(t^2 - t_0^2)$ for any $t \geq 0$, which implies

$$I(u) \geq a\|\nabla u - \vec{f}\,\|^2_{L^2(\Omega)} - a t_0^2 \lambda_d(\Omega);$$

thus, *the functional I is coercive on the space $\mathcal{X}$.*

An argument from [2, V §2] leads to the following conclusion.

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set, and let $F = F(x, p)$ be a function in $2d$ variables ($x = (x_1, \dots, x_d) \in \operatorname{clos}\Omega$, $p = (p_1, \dots, p_d) \in \mathbb{R}^d$). Suppose that F is continuous on $\operatorname{clos}\Omega \times \mathbb{R}^d$ and has continuous derivatives $\frac{\partial F}{\partial p_i}$ and $\frac{\partial^2 F}{\partial p_i \partial p_j}$, $i, j = 1, \dots, d$, on this set. Suppose that $F(x, p) \geq 0$ for all $x \in \operatorname{clos}\Omega$, $p \in \mathbb{R}^d$. Finally, suppose that at any point $(x, p) \in \operatorname{clos}\Omega \times \mathbb{R}^d$ the following convexity condition is satisfied:*

$$\sum_{i,j=1}^d \frac{\partial^2 F}{\partial p_i \partial p_j} \xi_i \xi_j \geq 0 \tag{28}$$

for any $\xi_1, \dots, \xi_d \in \mathbb{R}$.

Under these conditions, the functional $I(u) = \int_\Omega F(x, \nabla u(x))\, d\lambda_d(x)$ is weakly lower semicontinuous on $W^{1,2}(\Omega)$.

Thus, in order to check the weak lower semicontinuity of our functional I, it suffices to check the above convexity condition for the function $F(x, p) = Z(|p - \vec{f}(x)|^2)$. We evaluate the second order derivatives of this function with respect to the coordinates of the vector $p = (p_1, \dots, p_d)$:

$$\frac{\partial}{\partial p_i}\Big(Z(|p - \vec{f}\,|^2)\Big) = \frac{\partial}{\partial p_i} \int_0^{|p - \vec{f}\,|^2} \zeta_1(s)\, ds = 2(p_i - f_i)\zeta_1(|p - \vec{f}\,|^2),$$

$$\frac{\partial^2}{\partial p_i \partial p_j}\Big(Z(|p - \vec{f}\,|^2)\Big) = 2\delta_{ij}\zeta_1(|p - \vec{f}\,|^2) + 4(p_i - f_i)(p_j - f_j)\zeta_1'(|p - \vec{f}\,|^2),$$

where δ_{ij} is equal to one when $i = j$ and to zero otherwise ($i, j = 1, \dots, d$), and f_1, …, f_d are the components of $\vec{f}$. Next,

$$\sum_{i,j=1}^d \frac{\partial^2}{\partial p_i \partial p_j}\big(Z(|p - \vec{f}\,|^2)\big)\xi_i\xi_j = 2\zeta_1(|p - \vec{f}\,|^2)\sum_{i=1}^d \xi_i^2 + 4\zeta_1'(|p - \vec{f}\,|^2)\Big(\sum_{i=1}^d \xi_i(p_i - f_i)\Big)^2 \geq 0$$

for any real $\xi_1, \dots, \xi_d$, because the function ζ_1 is monotone nondecreasing. Thus, our functional I is weakly lower semicontinuous on $W^{1,2}(\Omega)$ (and also on $\mathcal{X}$).

A standard argument shows that the weak lower semicontinuity of I together with coercivity gives the existence of a function $u_0 \in \mathcal{X}$ such that $I(u_0) = \inf_{u\in\mathcal{X}} I(u)$.

Suppose that $\eta \in \mathcal{X}$. We show that the derivative $\frac{d}{dt}I(u_0+t\eta)$ exists for $t=0$ and evaluate this quantity. For this, consider the difference ratio

$$\frac{I(u_0+t\eta)-I(u_0)}{t} = \int_\Omega \frac{Z(|\nabla u_0(x)+t\nabla\eta(x)-\vec{f}(x)|^2)-Z(|\nabla u_0(x)-\vec{f}(x)|^2)}{t}\,d\lambda_d(x), \tag{29}$$

assuming that $t\in[-1,1]$. Observe that for any $t\in[-1,1]$ and any point $x\in\Omega$ where $\nabla u_0(x)$ and $\nabla\eta(x)$ are finite, we have

$$\frac{Z(|\nabla u_0(x)+t\nabla\eta(x)-\vec{f}(x)|^2)-Z(|\nabla u_0(x)-\vec{f}(x)|^2)}{t}$$
$$= 2(\langle\nabla u_0(x)-\vec{f}(x),\nabla\eta(x)\rangle+\theta|\nabla\eta(x)|^2)\cdot\zeta_1(|\nabla u_0(x)+\theta\nabla\eta(x)-\vec{f}(x)|^2)$$

with $\theta\in[-1,1]$ depending on x and t ($|\theta|\le|t|$). Therefore, the integrand on the right-hand side of (29) is bounded above by $2|\nabla u_0-\vec{f}\,|^2+4|\nabla\eta|^2$ and tends to $2(\langle\nabla u_0(x)-\vec{f}(x),$ $\nabla\eta(x)\rangle\zeta_1(|\nabla u_0(x)-\vec{f}(x)|^2)$ as $t\to0$ at any point where ∇u_0 and $\nabla\eta$ are finite. By the dominated convergence theorem, the derivative $\frac{d}{dt}I(u_0+t\eta)$ at $t=0$ exists and is equal to

$$\int_\Omega\big\langle\zeta_1(|\nabla u_0-\vec{f}\,|^2)(\nabla u_0-\vec{f}\,),\nabla\eta\big\rangle\,d\lambda_d.$$

By the choice of u_0, the last expression is zero for any $\eta\in\mathcal{X}$; obviously, then it is equal to zero also for any $\eta\in W^{1,2}(\Omega)$. Thus, the function u_0 that gives the minimum for the functional I is a weak solution of (26). The theorem is proved. □

Remark 3.3. We shall apply this theorem to "good" sets Ω. Notice, nevertheless, that our argument is valid for arbitrary bounded open sets Ω. Indeed, the distributional setting (27) makes sense for such Ω. In general, for arbitrary open sets it is not true that $\nabla u\in L^2(\Omega)$ implies $u\in L^1(\Omega)$. So the space $\mathcal{X}$ should be defined as the space $\{u\in W^{1,2}_{\mathrm{loc}}(\Omega) : \nabla u\in L^2(\Omega)\}$ factorized by its subspace consisting of all locally constant functions; as above, we put $\|u\|_{\mathcal{X}}=\big(\int_\Omega|\nabla u|^2\,d\lambda_d\big)^{1/2}$. Theorem 3.1 is now valid is the following form: there exists a function $u\in\mathcal{X}$ for which identity (27) is true for any $\eta\in\mathcal{X}$. The proof is carried out without any essential changes: the weak lower semi-continuity of the functional I on the space $\mathcal{X}$ (with respect to the weak convergence in $\mathcal{X}$) can be obtained by application of Theorem 3.2 to open sets with smooth boundaries that approximate Ω from below.

3.3. A direct construction of either an approximating gradient or an obstruction to approximation. Now we can finally describe our main construction. Our goal is to approximate a field $\vec{f}\in\vec{C}(K)$ by a gradient (or to find an obstruction to such approximation). We may assume that $\vec{f}$ is extended to the entire space $\mathbb{R}^d$ saving its continuity. Moreover, we may assume that the extended field $\vec{f}$ is compactly supported and that its modulus of continuity $\omega_{\vec{f}}(\rho)=\sup_{|x-y|\le\rho}|\vec{f}(x)-\vec{f}(y)|$ is finite for any ρ and tends to zero as $\rho\to0$.

Theorem 3.4. *Let K be a compact subset of $\mathbb{R}^d$, and let $\vec{f}$ be a vector field of class $\vec{C}(\mathbb{R}^d)$. Let ε be some positive number.*

One of the following assertions is true:

(1) *There exists an open neighborhood Ω of the set K and a function $u\in W^{1,\infty}(\Omega)$ such that $|\nabla u-\vec{f}\,|\le\varepsilon$ almost everywhere in Ω.*
(2) *There exists a nonzero vector charge $\vec{\mu}\in\mathrm{sol}(K)$ for which $\vec{\mu}[\vec{f}\,]=\int_K\langle\vec{f},d\vec{\mu}\rangle\ge\varepsilon\|\vec{\mu}\|_{\vec{M}(K)}$.*

In the first case we can take a weak solution of equation (26) *for the set* Ω *for the role of* u. *In the second case, the charge* $\vec{\mu}$ *can be constructed as the weak limit of the sequence of certain charges that can be obtained with the help of solutions of this equation for a decreasing sequence of open sets* Ω_m *whose intersection is* K.

Proof. Take bounded open sets Ω_m, $m = 1, 2, \dots$, with sufficiently smooth boundaries in such a way that $\bigcap_{m=1}^{\infty} \Omega_m = K$, $\operatorname{clos} \Omega_{m+1} \subset \Omega_m$, $m = 1, 2, \dots$.

Take a C^∞-smooth bounded monotone nondecreasing function $\zeta : [0, +\infty) \to [0, +\infty)$ such that $\zeta = 0$ on $[0, \varepsilon]$ and $\zeta(t) > 0$ on $(\varepsilon, +\infty)$.

Suppose that a function $u_m \in W^{1,2}(\Omega_m)$ is a weak solution of equation (26) in Ω_m, i.e., that the identity

$$\int_\Omega \big\langle \zeta(|\nabla u_m - \vec{f}\,|)(\nabla u_m - \vec{f}\,), \nabla \eta \big\rangle \, d\lambda_d = 0 \tag{30}$$

is true for any $\eta \in W^{1,2}(\Omega_m)$ (the existence of such a function is guaranteed by Theorem 3.1).

Denote by $\vec{\xi}_m$ the residual vector field, $\vec{\xi}_m = \vec{f} - \nabla u_m$, and put $\vec{\Phi}_m = \zeta(|\vec{\xi}_m|)\vec{\xi}_m$. In accordance with (30) the charge $\vec{\Phi}_m \mathbb{1}_{\Omega_m} \lambda_d$ is a solenoid (as a charge in $\mathbb{R}^d$). In other words, $\int_{\Omega_m} \langle \vec{\Phi}_m, \nabla \eta \rangle \, d\lambda_d = 0$ for any $\eta \in C^1(\mathbb{R}^d)$ (because (30) is true even for any $\eta \in W^{1,2}(\Omega_m)$).

We show that

$$\int_{\Omega_m} \big\langle \vec{\Phi}_m, \vec{f}\, \big\rangle \, d\lambda_d \geq \varepsilon \int_{\Omega_m} |\vec{\Phi}_m| \, d\lambda_d. \tag{31}$$

Put $A_1 = \{x \in \Omega_m : |\vec{\xi}_m(x)| \geq \varepsilon\}$, $A_2 = \{x \in \Omega_m : |\vec{\xi}_m(x)| < \varepsilon\}$. By (30), the left-hand side of (31) is equal to

$$\int_{\Omega_m} \big\langle \vec{\Phi}_m, \vec{f} - \nabla u_m \big\rangle \, d\lambda_d = \int_{\Omega_m} \big\langle \zeta(|\vec{\xi}_m|)\vec{\xi}_m, \vec{\xi}_m \big\rangle \, d\lambda_d, \tag{32}$$

so that the difference of the two sides in (31) is

$$\int_{\Omega_m} \big(\zeta(|\vec{\xi}_m|)|\vec{\xi}_m|^2 - \varepsilon |\vec{\Phi}_m|\big) \, d\lambda_d = \left(\int_{A_1} + \int_{A_2} \right) \big(\zeta(|\vec{\xi}_m|)|\vec{\xi}_m|\big(|\vec{\xi}_m| - \varepsilon\big) \, d\lambda_d\big) = I_1 + I_2.$$

We have $|\vec{\xi}_m| \geq \varepsilon$ in A_1, whence $I_1 \geq 0$; at the same time, $\zeta(|\vec{\xi}_m|) = 0$ in A_2, whence $I_2 = 0$. Thus, $I_1 + I_2 \geq 0$, and (31) is proved.

Suppose that the field $\vec{\Phi}_m$ vanishes almost everywhere in Ω_m for some m. Then $\zeta(|\vec{\xi}_m|) = 0$ almost everywhere in Ω_m, i.e., $|\nabla u_m - \vec{f}\,| \leq \varepsilon$ almost everywhere in Ω_m, and in this case the first assertion of the theorem is true. Clearly, then we have $u_m \in W^{1,\infty}(\Omega_m)$.

If, on the contrary, the fields $\vec{\Phi}$ do not vanish almost everywhere in Ω_m for all $m = 1, 2, \dots$, then we consider the vector charges

$$\vec{\mu}_m = \frac{\mathbb{1}_{\Omega_m} \vec{\Phi}_m \lambda_d}{\big\| \mathbb{1}_{\Omega_m} \vec{\Phi}_m \big\|_{L^1(\mathbb{R}^d)}} \in \vec{M}(\Omega_1).$$

These charges are supported on $\operatorname{clos} \Omega_1$, and $\|\vec{\mu}_m\|_{\vec{M}(\Omega_1)} = 1$. Relation (30) implies that all the charges $\vec{\mu}_m$ are solenoids; moreover, (31) implies that

$$\vec{\mu}_m[\vec{f}\,] \geq \varepsilon, \quad m = 1, 2, \dots. \tag{33}$$

Now we take a subsequence $\vec{\mu}_{m_k}$ of the sequence $\vec{\mu}_m$ that is weak* convergent to some vector charge $\vec{\mu} \in \vec{M}(\operatorname{clos} \Omega_1)$ (i.e., $\vec{\mu}_{m_k}[\vec{\eta}\,] \to \vec{\mu}[\vec{\eta}\,]$ as $k \to \infty$ for any vector field $\vec{\eta} \in \vec{C}(\operatorname{clos} \Omega_1)$). Since $\bigcap_{m=1}^{\infty} \Omega_m = K$ and $\operatorname{supp} \vec{\mu}_m \subset \Omega_m$, we have $\operatorname{supp} \vec{\mu} \subset K$. Next, $\|\vec{\mu}_m\|_{\operatorname{clos} \Omega_1} = 1$ implies $\|\vec{\mu}\|_{\vec{M}(K)} \leq 1$, and the solenoidality of the charges $\vec{\mu}_m$ implies that of $\vec{\mu}$ (see Definition 1.4: the solenoidality condition is preserved under taking a weak limit). Finally, (33) yields

$$\vec{\mu}[\vec{f}\,] \geq \varepsilon \geq \varepsilon \|\vec{\mu}\|_{\vec{M}(K)}.$$

Thus, in the case where $\|\vec{\Phi}_m\|_{L^1(\Omega_m)} \neq 0$ for all $m = 1, 2, \ldots$, the resulting charge $\vec{\mu}$ satisfies the conditions of the second assertion of the theorem. □

Finally, we prove a result on approximation by smooth gradients.

Theorem 3.5. *Let K be a compact subset of $\mathbb{R}^d$ and $\vec{f}$ a vector field of class $\vec{C}(K)$. Suppose that two numbers ε_* and ε^* are given, $0 < \varepsilon_* < \varepsilon^*$.*

One of the following assertions is true:

(1) *There exists a function $u \in C^\infty(\mathbb{R}^d)$ such that $\|\nabla u - \vec{f}\,\|_{\vec{C}(K)} \leq \varepsilon^*$.*

(2) *There exists a nonzero vector charge $\vec{\mu} \in \mathrm{sol}(K)$ such that $\vec{\mu}[\vec{f}\,] = \int_K \langle \vec{f}, d\vec{\mu}\rangle \geq \varepsilon_* \|\vec{\mu}\|_{\vec{M}(K)}$.*

In the first case the function u can be obtained by regularization of the weak solution of equation (26) *for some domain Ω that includes K. In the second case, the charge $\vec{\mu}$ can be constructed as the weak limit of the sequence of certain charges that can be obtained with the help of solutions of this equation for a decreasing sequence of open sets Ω_m whose intersection is K.*

Proof. We extend our field $\vec{f}$ to the entire $\mathbb{R}^d$ by continuity (we assume that the resulting field is compactly supported) and apply Theorem 3.4 to the field $\vec{f}$ and the set K with $\varepsilon = \varepsilon_*$. If the second assertion of Theorem 3.4 is true, then the charge $\vec{\mu}$ obtained there satisfies the conditions of the second assertion of our theorem. Suppose that the first assertion of Theorem 3.4 is true and there exists a neighborhood Ω of K and a function $u \in W^{1,\infty}(\Omega)$ for which $|\nabla u - \vec{f}\,| \leq \varepsilon_*$ almost everywhere in Ω. In this case, we take a number $\rho > 0$ such that $\rho < \mathrm{dist}(K, \partial\Omega)$ and $\omega_{\vec{f}}(\rho) \leq \varepsilon^* - \varepsilon_*$ (recall that $\omega_{\vec{f}}$ is the modulus of continuity of $\vec{f}$). Take a C^∞-smooth nonnegative function $\psi : \mathbb{R}^d \to \mathbb{R}$ supported in the ball in $\mathbb{R}^d$ with center at zero and having radius ρ such that

$$\int_{\mathbb{R}^d} \psi \, d\lambda_d = 1.$$

Then $|\vec{f} - \vec{f} * \psi| \leq \varepsilon^* - \varepsilon_*$ in K, and

$$|\nabla(u * \psi) - \vec{f}\,| = |\nabla u * \psi - \vec{f}\,| \leq |(\nabla u - \vec{f}\,) * \psi| + |\vec{f} * \psi - \vec{f}\,| \leq \varepsilon_* + (\varepsilon^* - \varepsilon_*) = \varepsilon^*,$$

by the choice of u and ρ (all the convolutions in the above chain of inequalities are well defined on K because $\rho < \mathrm{dist}(K, \partial\Omega)$). Thus, in this case the function $u * \psi$ is an ε^*-primitive of $\vec{f}$ on K, and the first assertion of the theorem is true. □

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