Алгебра и анализ
Том 25 (2013), № 2

St. Petersburg Math. J.
Vol. 25 (2014), No. 2, Pages 327–337
S 1061-0022(2014)01293-9
Article electronically published on March 12, 2014

SCATTERING PROBLEM FOR THE ORDINARY DIFFERENTIAL OPERATOR OF ORDER FOUR ON THE HALF-LINE. I. DIRECT PROBLEM

R. SHTERENBERG AND V. SUKHANOV

Dedicated to the memory of V. S. Buslaev

Abstract. Direct and inverse spectral problems are studies for ordinary differential operators of order four on the half-line. This first part of the text is devoted to the study of the direct problem.

§1. Introduction

Our ultimate goal is to study inverse and direct spectral problems for ordinary differential operators of order four on the the half-line. We consider the operator

$$L = \frac{d^4}{dx^4} + \frac{d}{dx}u(x)\frac{d}{dx} + v(x), \quad x \in [0, +\infty), \tag{1.1}$$

with the boundary conditions

$$\Psi(0, k) = 0, \quad \Psi'(0, k) = 0. \tag{1.2}$$

Here $u(x)$ and $v(x)$ are real-valued smooth potentials,

$$u(x), \quad v(x) \in C^\infty[0, \infty),$$

admitting smooth continuation to the interval $[-\varepsilon, \infty)$, $\varepsilon > 0$, and decaying rapidly as $x \to +\infty$:

$$\frac{d^l}{dx^l}u(x) = O(x^{-n}), \quad \frac{d^l}{dx^l}v(x) = O(x^{-n})$$

for any natural l and n.

Remark 1.1. It is possible to consider different boundary conditions for this operator. However, every boundary condition requires special algebraic calculations for the Riemann–Hilbert problem (see §4). We consider a simplest version of these calculations. For these boundary conditions, most of algebraic relations are similar to the corresponding algebraic relations for the inverse problem on the entire line.

On the entire line, direct and inverse problems for differential operators of order $n > 2$ were investigated in the book [1]. We consider similar problems for a differential operator on the half-line. We assume that the potentials are real-valued. In contrast to [1], we use two symmetries for the operator L: selfadjoint and real (on the entire line such an operator was studied in [6]). So, we consider operators with more specific potentials possessing some new algebraic properties, which simplifies the analysis of the problem. We use a Riemann–Hilbert problem as a main tool for investigating the inverse problem.

2010 *Mathematics Subject Classification.* Primary 34L25.

Key words and phrases. Ordinary differential operator, scattering problem on the half-axis, inverse problem, operator of order four.

Supported by RFBR (grant no. 05-01-01076).

Such an approach turned out to be very fruitful in the study of higher order operators [1, 4]. Notice that the Riemann–Hilbert approach was also used in a number of other related papers, see [2, 3, 6].

In the present paper, we consider the direct problem and state the Riemann–Hilbert problem to be used in the study of the inverse problem.

§2. Solutions of the spectral equation

We introduce a basis $\{\Psi_j\}_{j=1}^4$ of solutions of the spectral equation

$$\left(\frac{d^4}{dx^4} + \frac{d}{dx}u(x)\frac{d}{dx} + v(x)\right)\Psi = k^4\Psi.$$

The functions $\{\Psi_j\}_{j=1}^4$ satisfy special boundary conditions. Consider the following set of rays:

$$\{\gamma_m\}_{m=0}^7, \quad \gamma_m = \{k \,:\, \arg k = m\pi/4\}.$$

These rays split the complex plane into eight domains

$$\Omega_m, \quad m = 1, 2, \dots, 8; \quad \Omega_m = \{k \,:\, (m-1)\pi/4 < \arg k < m\pi/4\}.$$

The boundary conditions and the construction of solutions are different for different domains. They depend on the "strength" of the exponentials corresponding to the solutions of the free equation (with zero potentials). We start with the domain Ω_1. If $k \in \Omega_1$, then

$$|e^{-kx}| < |e^{ikx}| < |e^{-ikx}| < |e^{kx}|, \quad x > 0.$$

We fix the functions $\{\Psi_j\}_{j=1}^4$ with the help of the following conditions:

$$\begin{aligned} \Psi_j &= \exp(k_j x)(1 + o(1)), \quad x \to +\infty; \\ k_j &= k\exp(i\pi(j-1)/2), \end{aligned} \tag{2.1}$$

and

$$\Psi_1(0,k) = \Psi_1'(0,k) = \Psi_1''(0,k) = 0, \tag{2.2}$$

$$\Psi_2(0,k) = 0, \tag{2.3}$$

$$\Psi_4(0,k) = \Psi_4'(0,k) = 0. \tag{2.4}$$

A similar procedure can be used for the domain Ω_2. If $k \in \Omega_2$, we have

$$|e^{ikx}| < |e^{-kx}| < |e^{kx}| < |e^{-ikx}|, \quad x > 0.$$

For $k \in \Omega_2$, we define $\{\Psi_j\}_{j=1}^4$ by the following conditions:

$$\Psi_j = \exp(k_l x)(1 + o(1)), \quad x \to +\infty, \tag{2.5}$$

and

$$\begin{aligned} &\Psi_1(0,k) = \Psi_1'(0,k) = 0, \\ &\Psi_3(0,k) = 0, \\ &\Psi_4(0,k) = \Psi_4'(0,k) = \Psi_4''(0,k) = 0. \end{aligned}$$

Consider the vector-valued function

$$\mathbf{\Psi}(x,k) = (\Psi_1, \Psi_2, \Psi_3, \Psi_4)^T.$$

For all other domains Ω_m, the functions Ψ_j, will be fixed by the following relations:

$$\mathbf{\Psi}(x, ik) = \sigma\mathbf{\Psi}(x,k). \tag{2.6}$$

Here

$$\sigma = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

We shall prove that the solutions $\Psi_j(x,\cdot)$ exist and are meromorphic functions in the sectors Ω_m, $m = 1, \dots 8$,

$$\Omega_m = \{k \,:\, \pi(m-1)/4 < \arg k < \pi m/4\},$$

and that they have jumps on the set of rays γ_m.

We consider some algebraic properties of $\Psi_l(x,k)$. The Wronskian of three functions is introduced by the formula

$$\mathcal{W}_3[f,g,h] = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix}.$$

Lemma 2.1. *If $f(x,k)$, $g(x,k)$, $h(x,k)$ are solutions of the spectral equation, then so is the Wronskian*

$$\mathcal{W}_3[f(x,k), g(x,k), h(x,k)].$$

This well-known statement follows from Liouville's formula.

Lemma 2.2. *The solutions $\Psi_l(x,k)$ satisfy the following relations*:

$$\Psi_1(x,k) = \overline{\Psi_1(x,\bar{k})}, \qquad \Psi_3(x,k) = \overline{\Psi_3(x,\bar{k})},$$
$$\Psi_2(x,k) = \overline{\Psi_4(x,\bar{k})}, \qquad \Psi_1(x,k) = \overline{\Psi_4(x,i\bar{k})}, \qquad \Psi_2(x,k) = \overline{\Psi_3(x,i\bar{k})}.$$

Proof. The left-hand sides of all these relations are solutions of the spectral equation and satisfy the same conditions at $+\infty$ and at 0 as the functions on the right-hand side. This implies that they must be equal. □

Now we consider the construction of the functions Ψ_l. We need the following two simple statements.

Lemma 2.3. *There exist solutions $\{h_l(x,k)\}$, $l = 1,2,3,4$, of the spectral equation that are analytic in $k \in \Omega_1$ and satisfy the boundary conditions*

$$h_l(x,k) = e^{k_l x}(1 + o(1)), \quad x \to +\infty, \quad l = 1,2,3,4. \tag{2.7}$$

For a proof of this lemma see, e.g., [3].

The next fact is trivial.

Lemma 2.4. *Let $k \in \Omega_1$. Then for any solution $\Phi(x,k)$ of the spectral equation there exists $l \in \{1,2,3,4\}$ such that $\Phi(x,k)$ satisfies the boundary conditions*

$$\Phi(x,k) = e^{k_l x}(\varrho(k) + o(1)), \quad x \to +\infty,$$

with some $\varrho \neq 0$.

Let $k \in \Omega_1$. Then the solution Ψ_3 satisfies the Volterra integral equation

$$\Psi_3(x,k) = e^{-kx} + \frac{1}{4k^3}\int_x^{+\infty} G(x,y)\Psi_3(y,k)\,dy.$$

Here

$$G(x,y) = G_0(x-y)v(y) - (u'(y)G_0(x-y))'_y + (u(y)G_0(x-y))''_{yy},$$
$$G_0(x) = e^{kx} + ie^{ikx} - e^{-kx} - ie^{-ikx}.$$

Since the above integral equation is of Volterra type, it not difficult to show that the function $\Psi_3(x,\cdot)$ is analytic for $k \in \Omega_1$.

In order to obtain the other functions Ψ_l, we introduce another set of solutions $g_l(x,k)$, $l = 1,2,3,4$, of the spectral equation that satisfy the following Volterra integral equations:

$$g_1(x,k) = e^{kx} + ie^{ikx} - e^{-kx} - ie^{-ikx} - \frac{1}{4k^3}\int_0^x G(x,y)g_1(y,k)\,dy,$$
$$g_2(x,k) = e^{ikx} - e^{-kx} - \frac{1}{4k^3}\int_0^x G(x,y)g_2(y,k)\,dy,$$
$$g_3(x,k) = e^{-kx} - \frac{1}{4k^3}\int_0^x G(x,y)g_3(y,k)\,dy,$$
$$g_4(x,k) = ie^{ikx} - (1+i)e^{-kx} + e^{-ikx} - \frac{1}{4k^3}\int_0^x G(x,y)g_4(y,k)\,dy.$$

The functions $g_l(x,k)$, $l = 1,2,3,4$, are analytic in $k \in \Omega_1$.

In particular, we see that the functions $g_l(x,k)$ satisfy the same conditions (2.2)–(2.4) at $x = 0$ as the functions $\Psi_l(x,k)$. Consider the asymptotics of the functions $g_l(x,k)$ as $x \to +\infty$:

$$g_l(x,k) = e^{kx}(c_l(k) + o(1)).$$

The function $c_l(k)$ has the same analytical properties as the function $g_l(x,k)$. They are analytic in the sector Ω_1.

We have

$$\Psi_1(x,k) = g_1(x,k)/c_1(k). \tag{2.8}$$

Thus, $\Psi_1(x,\cdot)$ is meromorphic in the sector Ω_1. The poles of $\Psi_1(x,\cdot)$ are zeros of $c_1(k)$. If the coefficient $c_1(k)$ equals zero at the point $k_0 \in \Omega_1$, then the asymptotics of the function $g_1(x,k)$ as $x \to +\infty$ has the form

$$g_1(x,k_0) = e^{-ik_0x}(d_0 + o(1)).$$

If $d_0 = 0$, then k_0^4 is an eigenvalue of the initial operator, which is impossible for a formally selfadjoint operator L. Thus,

$$d_0 \neq 0.$$

Let $c_1(k) \neq 0$. Then the function

$$\widetilde{g} = g_4 - \frac{c_4}{c_1}g_1$$

satisfies the boundary conditions (1.2) and has the following asymptotics as $x \to +\infty$:

$$\widetilde{g}(x,k) = e^{-ikx}(\widetilde{c}(k) + o(1)).$$

Therefore,

$$\Psi_4(x,k) = \widetilde{g}(x,k)/\widetilde{c}(k). \tag{2.9}$$

For any $k \in \Omega_1$ we have

$$\widetilde{c}(k) \neq 0,$$

because a zero of the function $\widetilde{c}(k)$ gives us an eigenvalue of the initial operator L. If $c_1(k_0) = 0$, then

$$\Psi_4(x,k_0) = g_1(x,k_0)/d_0. \tag{2.10}$$

So, the function $\Psi_4(x,k)$ is analytic in Ω_1.

Finally, the function Ψ_2 can be found with the help of Lemma 2.3. We have a function $h_2(x,k)$ analytic in Ω_1,

$$h_2(x,k) = e^{ikx}(1 + o(1)), \quad x \to +\infty,$$

which is a solution of the spectral equation.

Now

$$\Psi_2(x,k) = h_2(x,k) - h_2(0,k)\Psi_3(x,k)(\Psi_3(0,k))^{-1}. \tag{2.11}$$

This function is meromorphic in Ω_1. The poles of $\Psi_2(x,\cdot)$ are zeros of $\Psi_3(0,k)$.

§3. Properties of the coefficients $a_j(k)$

We consider the behavior of the functions Ψ_l for $x = 0$. We introduce the following functions $\{a_j(k)\}_{j=1}^4$ $(k \in \Omega_1)$:

$$\Psi_1'''(0,k) = 4k^3 a_1(k), \tag{3.1}$$

$$\Psi_2'(0,k) = (1+i)k a_2(k), \tag{3.2}$$

$$\Psi_3(0,k) = a_3(k), \tag{3.3}$$

$$\Psi_4''(0,k) = -2(1+i)k^2 a_4(k). \tag{3.4}$$

Lemma 3.1. *The functions $\{a_j(k)\}_{j=1}^4$ satisfy the relations*

$$a_1(k) = \overline{a_1(\bar{k})}, \qquad a_3(k) = \overline{a_3(\bar{k})},$$

$$a_2(k) = \overline{a_4(\bar{k})}, \qquad a_1(k) = \overline{a_4(i\bar{k})}, \qquad a_2(k) = \overline{a_3(i\bar{k})}.$$

Lemma 3.2. *The functions $\{a_j(k)\}_{j=1}^4$ satisfy the asymptotic relations*

$$a_l(k) = 1 + o(1), \quad |k| \to \infty.$$

Lemma 3.3. *The functions $a_l(k)$ satisfy the identities*

$$a_1(k)\, a_3(k) = 1, \quad a_2(k)\, a_4(k) = 1. \tag{3.5}$$

Proof. The proofs of the two identities in (3.5) are similar, so we consider only the first. Now we introduce the following Wronskian of two functions:

$$\mathcal{W}_2[f,g] = f'''g - f''g' + f'g'' - fg'''. \tag{3.6}$$

Using the spectral equation, we obtain

$$(\mathcal{W}_2[\Psi_1,\Psi_3])' = \Psi_1''''\Psi_3 - \Psi_1\Psi_3'''' = (u\Psi_1\Psi_3' - u\Psi_1'\Psi_3)',$$

whence

$$\mathcal{W}_2[\Psi_1,\Psi_3](c) - \mathcal{W}_2[\Psi_1,\Psi_3](0) = (u\Psi_1\Psi_3' - u\Psi_1'\Psi_3)|_0^c.$$

The limit of the right-hand side as $c \to +\infty$ equals zero. Thus,

$$\mathcal{W}_2[\Psi_1,\Psi_3](0,k) = \lim_{x\to+\infty} \mathcal{W}_2[\Psi_1,\Psi_3](x,k).$$

Combined with relations (3.1), (3.3), and (2.1) (which are true even together with the derivatives with respect to x), this completes the proof of the lemma. □

Similar relations were found for the first time in [5] for the problem on the entire line.

As a result, from Lemma 3.3 and the analytic properties of the functions Ψ_l we obtain the following statement.

Lemma 3.4. *The functions $a_l(k)$ possess the following properties*:

1) *$a_3(k)$ and $a_4(k)$ are analytic in Ω_1;*

2) *$a_1(k)$ and $a_2(k)$ are meromorphic in Ω_1;*

3) *$a_1(k) \neq 0$ and $a_2(k) \neq 0$, $k \in \Omega_1$;*

4) *if k_0 is a zero of $a_3(k)$, then k_0 is simultaneously a pole of $a_1(k)$, a zero of $a_4(k)$, and a pole of $a_2(k)$.*

Finally, we arrive at the following theorem.

Theorem 3.1. *The solutions $\Psi_l(x,\cdot)$ of the spectral equation possess the following properties*:
1) $\Psi_3(x,\cdot)$ *and* $\Psi_4(x,\cdot)$ *are analytic in* Ω_1;
2) $\Psi_1(x,\cdot)$ *and* $\Psi_2(x,\cdot)$ *are meromorphic in* Ω_1;
3) *the poles* $\Psi_1(x,\cdot)$ *and* $\Psi_2(x,\cdot)$ *in* Ω_1 *are zeros of* $a_3(k)$.

This theorem, Lemma 2.2, and formula (2.6) give us full information about analytic properties of the functions $\Psi_l(x,k)$ in all domains Ω_m.

The poles of $\Psi_l(x,k)$ can accumulate to the boundaries of domains. Unfortunately, we have no efficient procedure to investigate the direct and inverse problems with an infinite number of singularities. We shall consider only finitely many singularities. Moreover, we impose some technical restrictions on the spectral data (in particular, we consider only simple poles for the function a_j). So, in the further constructions we restrict ourselves to the following class of operators.

Definition 3.1. The data of the inverse problem belong to the generic class $\mathcal{R}$ if:
1) the function $a_3(k)$ has finitely many zeros $k_1^*,\dots,k_N^*$ in Ω_1;
2) the zeros of $a_3(k)$ in Ω_1 are simple;
3) the limits of $a_3(k)$ on the boundary of the domain Ω_1 have no zeros for $k\in\gamma_0$ and $k\in\gamma_1$;
4) the functions $a_{1,2,3,4}(k)$ have finite nonzero limits as $k\to 0$;
5) the function $a_2(k)$ is smooth for $k\in\gamma_1$ and has finitely many simple zeros $k_1,\dots k_M$, $k_j\in\gamma_1$ (respectively, the function $a_4(k)$ has singularities at these points)
6) the functions $a_{2,4}(k)$ have nonzero smooth limit for $k\in\gamma_0$

Remark 3.1. It is not hard to see that the zeros k_j of the function $a_2(k)$ correspond to the negative eigenvalues of the operator L.

§4. Riemann–Hilbert problem

Denote by $\mathbf{\Psi}^{+m}$ ($\mathbf{\Psi}^{-m}$) the limits of $\mathbf{\Psi}$ as k approaches the ray γ_m in such a way that $\arg k\to\pi m/4\pm 0$. Let $G_m(k)$ be the matrices that relate the vector-valued functions $\mathbf{\Psi}^{+m}(x,k)$ and $\mathbf{\Psi}^{-m}(x,k)$:

$$\mathbf{\Psi}^{+m}(x,k)=G_m(k)\mathbf{\Psi}^{-m}(x,k).$$

If $m=0$, then

$$\mathbf{\Psi}^{(+0)}(x,k)=\begin{pmatrix} e^{kx}\\ e^{ikx}\\ e^{-kx}\\ e^{-ikx}+r_0(k)e^{ikx}\end{pmatrix}(1+o(1)),\quad x\to+\infty,$$

and respectively,

$$\mathbf{\Psi}^{(-0)}(x,k)=\begin{pmatrix} e^{kx}\\ e^{ikx}+\overline{r_0(k)}e^{-ikx}\\ e^{-kx}\\ e^{-ikx}\end{pmatrix}(1+o(1)),\quad x\to+\infty.$$

Lemma 4.1. *The coefficient r_0 can be found by the formula*

$$r_0(k)=-ia_4/\overline{a}_4.$$

Proof. Consider the function $\widetilde{\Psi}_4(x,k)$ that satisfies the spectral equation and the following boundary conditions:

$$\widetilde{\Psi}_4(0,k) = \widetilde{\Psi}'_4(0,k) = 0, \quad \widetilde{\Psi}''_4(0,k) = 1,$$
$$\widetilde{\Psi}_4(x,k) = O(|\exp(-ikx)|), \quad x \to +\infty, \quad k \in \Omega_1.$$

It is easily seen that $\widetilde{\Psi}_4^{(+0)}(x,k)$ is real for $k \in \mathbb{R}$ ($\overline{\widetilde{\Psi}_4(x,k)}$ is a solution of the spectral equation and satisfies the same conditions as $\widetilde{\Psi}_4(x,k)$). On the other hand,

$$\Psi_4(x,k) = -2(1+i)k^2 a_4(k)\widetilde{\Psi}_4(x,k)$$

and

$$\Psi_4^{(+0)}(x,k) = e^{-ikx}(1+o(1)) + r_0(k)e^{ikx}, \quad x \to +\infty.$$

Consequently,

$$r_0(k) = -ia_4/\bar{a}_4.$$ □

We have

$$G_0(k) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\bar{r}_0 \\ 0 & 0 & 1 & 0 \\ 0 & r_0 & 0 & 1-|r_0|^2 \end{pmatrix}.$$

If $m = 1$, then

$$\mathbf{\Psi}^{(+1)}(x,k) = \begin{pmatrix} e^{kx} \\ e^{ikx} \\ e^{-kx} + r_1(k)e^{ikx} \\ e^{-ikx} + r_2(k)e^{kx} \end{pmatrix} (1+o(1)), \quad x \to +\infty,$$

and

$$\mathbf{\Psi}^{(-1)}(x,k) = \begin{pmatrix} e^{kx} + \overline{r_2(k)}e^{-ikx} \\ e^{ikx} + \overline{r_1(k)}e^{-kx} \\ e^{-kx} \\ e^{-ikx} \end{pmatrix} (1+o(1)), \quad x \to +\infty,$$

Lemma 4.2. *The coefficient r_1 can be found by the formula*

$$r_1(k) = -\frac{\bar{a}_2}{a_2}.$$

Proof. Consider the function $\widetilde{\Psi}_2(x,k)$ that satisfies the following boundary conditions:

$$\widetilde{\Psi}_2(0,k) = 0, \quad \widetilde{\Psi}'_2(0,k) = 1,$$
$$\widetilde{\Psi}_2(x,k) = O(|\exp(ikx)|), \quad x \to +\infty, \quad k \in \Omega_1.$$

Let $k \in \gamma$ be not among the eigenvalues of the operator L. It is easily seen that $\widetilde{\Psi}_2^{(-1)}(x,k)$ is real for $k \in \gamma_1$ ($\overline{\widetilde{\Psi}_2(x,k)}$ is the solution of the spectral equation and satisfies the same conditions as $\widetilde{\Psi}_2(x,k)$). On the other hand,

$$\Psi_2(x,k) = (1+i)ka_2(k)\widetilde{\Psi}_2(x,k)$$

and

$$\Psi_2^{(-1)}(x,k) = e^{ikx}(1+o(1)) + \bar{r}_1(k)e^{-kx}, \quad x \to +\infty.$$

Consequently,

$$r_1(k) = -\frac{\bar{a}_2}{a_2}.$$ □

Lemma 4.3. *The coefficients r_1 and r_2 satisfy*

$$r_2 = -ir_1.$$

Proof. From Lemma 2.1 we know that

$$\mathcal{W}_3[\Psi_1(x,k), \Psi_2(x,k), \Psi_3(x,k)]$$

is a solution of the spectral equation. The boundary conditions for the functions Ψ_l imply that

$$\mathcal{W}_3[\Psi_1(x,k), \Psi_2(x,k), \Psi_3(x,k)] = -4k^3\Psi_2(x,k).$$

We consider this relation for k such that $\arg k = \pi/4 - 0$ in the limit as $x \to +\infty$. On the one hand,

$$\mathcal{W}_3[\Psi_1^{(-1)}(x,k), \Psi_2^{(-1)}(x,k), \Psi_3^{(-1)}(x,k)] = -4k^3e^{ikx}(1+o(1)) + 4ik^3\bar{r}_2e^{-kx}, \quad x \to +\infty.$$

On the other hand,

$$-4k^3\Psi_2^{(-1)}(x,k) = -4k^3(e^{ikx}(1+o(1) + \bar{r}_1e^{-kx}) \quad x \to +\infty.$$

Consequently,

$$r_2 = -ir_1.$$ □

We have

$$G_1(k) = \begin{pmatrix} 1 & 0 & 0 & -\bar{r}_2 \\ 0 & 1 & -\bar{r}_1 & 0 \\ 0 & r_1 & 1-|r_1|^2 & 0 \\ r_2 & 0 & 0 & 1-|r_2|^2 \end{pmatrix}.$$

By using the invariance of the spectral equation under the transformation $k \mapsto ik$, we can find all matrices G_m:

$$G_{m+2}(ik) = \sigma G_m(k)\sigma^{-1}, \quad \sigma = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

We have the following asymptotics as $|k| \to \infty$:

$$\mathbf{\Psi}(x,k) = \exp(Jxk)(\mathbf{i}_0 + \mathbf{o}(1)).$$

Here

$$J = \operatorname{diag}(i, -1, -i, 1), \quad \mathbf{i}_0 = (1,1,1,1)^T.$$

Lemma 4.4. *The singularities of the limit of Ψ_4 on γ_1 satisfy the following relations*:

$$(k-k_j)\Psi_4^{(-1)}(x,k) = D_j\Psi_2^{(-1)}(x,k_j) + o(1), \quad k \to k_j \quad j = 1,2,\dots,M.$$

Here D_j is some constant.

Proof. Let $k = k_j$, $k_j \in \gamma_1$, be a simple zero of a_2. Then k_j is a singularity of the the function $a_4(k) = (a_2(k))^{-1}$. Definition 3.1 and Lemma 3.3 show that $a_1(k_j) \neq 0$, $a_3(k_j) \neq 0$. Consider the function $\widetilde{\Psi}_4(x,k)$ that satisfies the boundary conditions

$$\widetilde{\Psi}_4(0,k) = \widetilde{\Psi}_4'(0,k) = 0, \quad \widetilde{\Psi}_4''(0,k) = 1,$$
$$\widetilde{\Psi}_4(x,k) = O(\exp(-ikx)), \quad x \to +\infty, \quad k \in \Omega_1.$$

It differs from the function $\Psi_4(x,k)$:

$$\Psi_4(x,k) = -2(1+i)k^2a_4(k)\widetilde{\Psi}_4(x,k).$$

Then for $k = k_j$ we have

$$\lim_{k\to k_j}(k-k_j)\Psi_4^{(-1)}(x,k) = -2(1+i)k_j^2\widetilde{\Psi}_4^{(-1)}(x,k_j)\lim_{k\to k_j}(k-k_j)a_4(k)$$
$$= -2(1+i)k_j^2\widetilde{\Psi}_4^{(-1)}(x,k_j)(a_2'(k_j))^{-1}.$$

On the other hand,

$$\widetilde{\Psi}_4^{(-1)}(x,k_j) = \alpha_j\Psi_2^{(-1)}(x,k_j).$$

Here α_j is some constant. Thus,

$$D_j = -2(1+i)k_j^2(a_2'(k_j))^{-1}\alpha_j. \qquad \square$$

Lemma 4.5. *The residues of the functions Ψ_1 and Ψ_2 satisfy*

$$\operatorname{Res}_{k=k_j^*}\Psi_1(x,k) = C_j^*\Psi_4(x,k_j^*), \quad j=1,2,\dots,N,$$
$$\operatorname{Res}_{k=k_j^*}\Psi_2(x,k) = -iC_j^*\Psi_3(x,k_j^*), \quad j=1,2,\dots,N.$$

Proof. Consider an inverse problem belonging to the generic class $\mathcal{R}$. Let $k = k_j^*$, $k_j^* \in \Omega_1$, be a simple zero of a_3. Then k_j^* is a simple pole of the function a_1 and the function $\Psi_1(x,k)$ (see Lemma 3.3), so that it is a simple zero of $c_1(k)$. From (2.8) and (2.10) we immediately see that

$$\operatorname{Res}_{k=k_j^*}\Psi_1(x,k) = C_j^*\Psi_4(x,k_j^*).$$

Here C_j^* is some complex constant.

By (2.11),

$$\operatorname{Res}_{k=k_j^*}\Psi_2(x,k) = \operatorname{Res}_{k=k_j^*}(h_2(x,k)-h_2(0,k)\Psi_3(x,k)(a_3(k))^{-1})\widetilde{C}_j^*\Psi_3(x,k_j^*)$$

with some constant $\widetilde{C}_j^*$. On the other hand, from Lemma 2.1 we know that

$$\mathcal{W}_3[\Psi_1(x,k),\Psi_2(x,k),\Psi_3(x,k)]$$

is a solution of the spectral equation. The boundary conditions for Ψ_l imply that

$$\mathcal{W}_3[\Psi_1(x,k),\Psi_2(x,k),\Psi_3(x,k)] = -4k^3\Psi_2(x,k).$$

Similarly,

$$\mathcal{W}_3[\Psi_4(x,k),\Psi_2(x,k),\Psi_3(x,k)] = 4ik^3\Psi_3(x,k).$$

Thus,

$$\operatorname{Res}_{k=k_j^*}\mathcal{W}_3[\Psi_1(x,k),\Psi_2(x,k),\Psi_3(x,k)] = \mathcal{W}_3[C_j^*\Psi_4(x,k_j^*),\Psi_2(x,k_j^*),\Psi_3(x,k_j^*)]$$
$$= 4i(k_j^*)^3C_j^*\Psi_3(x,k_j^*) = -4(k_j^*)^3\widetilde{C}_j^*\Psi_3(x,k_j^*).$$

Therefore,

$$\widetilde{C}_j^* = -iC_j^*. \qquad \square$$

Lemma 4.6. *The function $r_1(k)$ satisfies the following conditions*:

1) $r_1(k) \in C^\infty(\gamma_1)$;
2) $r_1(k)$ *admits the asymptotic expansion* $r_1(k) \sim -1 + \sum_{l=1}^\infty r_l^1 k^{-l}$, $k \to +\infty$;
3) $|r_1(k)| = 1$;
4) $r_1(0) = -1$;
5) $\Delta\arg_{\gamma_1} r_1(k) = -N$.

Proof. Statements 1)–4) are obvious. For the function r_1 we have

$$r_1(k) = i\bar{r}_2(k) = \frac{a_1(k)}{\bar{a}_1(k)} = \frac{\bar{a}_3(k)}{a_3(k)}.$$

Here the function $a_3(k)$ is analytic for k such that $|\arg k| < \pi/4$. Therefore, the function $a_3(\sqrt{-i\xi})$, $|\arg(\sqrt{-i\xi})| < \pi/4$, is analytic for $\xi \in \mathbb{C}^+$. Then statement 5) is a consequence of the argument principle for this function. □

Lemma 4.7. *The function $r_0(k)$ satisfies the following conditions*:

1) $r_0(k) \in C^\infty[0, \infty)$;
2) $r_0(k)$ *admits the asymptotic expansion*

$$r_0(k) \sim -i + \sum_{l=1}^{\infty} r_l^0 k^{-l}, \quad k \to +\infty;$$

3) $|r_0(k)| = 1$;
4) $r_0(0) = -i$;
5) $\Delta \arg_{\gamma_0} r_0(k) = M$.

Proof. Statements 1)–4) are obvious. For the function r_0 we have

$$r_0(k) = -i\frac{a_4(k)}{\bar{a}_4(k)} = -i\frac{\bar{a}_2(k)}{a_2(k)}.$$

Consider the function $\alpha(k) = a_2(k)a_3(k)$. It is easily seen that this function is analytic for k such that $0 < \arg(k) < \pi/2$. Therefore, the function $\alpha(\sqrt{\xi})$, $0 < \arg(\sqrt{\xi}) < \pi/2$, is analytic for $\xi \in \mathbb{C}^+$. Then statement 5) is a consequence of the argument principle for this function. □

Finally, we arrive at the following problem.

Riemann–Hilbert problem 4.1. The functions Ψ_l possess the following properties.

1) All components of the vector-valued function $\mathbf{\Psi}(x, \cdot)$ are analytic in all domains Ω_m and have continuous limits on the boundaries of these domains except for following points.

For the domain Ω_1, the functions Ψ_1 and Ψ_2 have simple poles with the residues

$$\operatorname{Res}_{k=k_j^*} \Psi_2(x, k) = -iC_j^* \Psi_3(x, k_j^*), \quad j = 1, 2, \ldots, N, \quad k_j^* \in \Omega_1,$$
$$\operatorname{Res}_{k=k_j^*}, \Psi_1(x, k) = C_j^* \Psi_4(x, k_j^*), \quad j = 1, 2, \ldots, N,$$

and the function Ψ_2 has singularities on the boundary of Ω_1:

$$(k - k_j)\Psi_4(x, k) = D_j \Psi_2(x, k_j) + o(1), \quad k \to k_j, \quad j = 1, 2, \ldots, M, \quad k_j \in \gamma_1.$$

For all other domains, the structure of the poles and singularities of the functions Ψ_l is in agreement with the symmetry relations in Lemma 2.2 and relation (2.6).

2) $\mathbf{\Psi}^{+m}(x, k) = G_m(k)\mathbf{\Psi}^{-m}(x, k)$.
3) $\mathbf{\Psi}(x, k) = \exp(Jxk)(\mathbf{i}_0 + \mathbf{o}(1))$.

In order to fix the Riemann–Hilbert (**RH**) problem 1)–3), we should have the following data of the inverse problem:

1) $k_j^* \in \Omega_1$, $C_j^* \in \mathbb{C}$, $j = 1, 2, \ldots, N$;
2) $k_j \in \gamma_1$, $D_j \in \mathbb{C}$, $j = 1, 2, \ldots, M$,
3) the coefficients $r_0(k)$ and $r_1(k)$ with the properties described in Lemmas 4.6 and 4.7.

References

[1] R. Beals, P. Deift, and C. Tomei, *Direct and inverse scattering on the line*, Math. Surveys Monogr., vol. 28, Amer. Math. Soc., Providence, RI, 1988. MR954382 (90a:58064)
[2] P. Deift and E. Trubowitz, *Inverse scattering on the line*, Comm. Pure Appl. Math. **32** (1979), no. 2, 121–251. MR512420 (80e:34011)
[3] P. Deift and X. Zhou, *Direct and inverse scattering on the line with arbitrary singularities*, Comm. Pure Appl. Math. **44** (1991), no. 5, 485–533. MR1105873 (92g:34112)
[4] V. V. Sukhanov, *An inverse problem for a selfajoint operator on the axis*, Mat. Sb. (N.S.) **137** (1988), no. 2, 242–259; English transl., Math. USSR-Sb. **65** (1990), no. 1, 249–266. MR971696 (89k:34036)
[5] J. Östensson, *Trace formulae and spectral properties of fourth order differential operators*, KTH, PhD-thesis.
[6] A. Laptev, R. Shterenberg, V. Sukhanov, and J. Östensson, *Reflectionless potentials for an ordinary differential operator of order four*, Inverse Problems **22** (2006), no. 1, 135–153. MR2194188 (2006k:34230)

Department of Mathematics, University of Alabama at Birmingham, 1300 University Blvd, Birmingham, Alabama 35294
E-mail address: rshterenberg@gmail.com

Department of Physics, St. Petersburg State University, Ul′yanovskaya 3, Petrodvorets, St. Petersburg 185504, Russia
E-mail address: vvsukhanov@mail.ru

Received 5/NOV/2012

Translated by THE AUTHOR