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# ON SPECTRAL ESTIMATES FOR THE SCHRÖDINGER OPERATORS IN GLOBAL DIMENSION 2

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*To Boris Mikhaĭlovich Makarov,*
*on the occasion of his 80th birthday*

Abstract. The problem of finding eigenvalue estimates for the Schrödinger operator turns out to be most complicated for the dimension 2. Some important results for this case have been obtained recently. In the paper, these results are discussed, and their counterparts are established for the operator on the combinatorial and metric graphs corresponding to the lattice $\mathbb{Z}^2$.

## §1. Introduction

For a selfadjoint operator $\mathbf{H}$ acting in the Hilbert space $\mathfrak{H}$ and having a discrete negative spectrum, we denote by $N_-(\mathbf{H})$ the dimension of its spectral projection that corresponds to the negative semiaxis $(-\infty, 0)$. In other words, $N_-(\mathbf{H})$ is the total number of all negative eigenvalues of $\mathbf{H}$, counted with their multiplicities. For the Schrödinger operator $\mathbf{H} = \mathbf{H}_V = -\Delta - \alpha V$ (where $\alpha$ is a large parameter, the *coupling constant,* and $V \geq 0$) and its analogs, the problem of obtaining estimates for $N_-(\mathbf{H})$ has been attracting the interest of researchers for several last decades. In the case of the standard Schrödinger operator in $\mathbb{R}^d$ with $d \geq 3$, the CLR estimate

$$N_-(-\Delta - \alpha V) \leq C\alpha^{d/2} \int_{\mathbb{R}^d} V^{d/2}\, dx, \quad d > 2, \tag{1.1}$$

is sharp in order in $\alpha$ and in the function class for the potentials. It was obtained about 40 years ago, and numerous generalizations have been found since then. A possible direction for such generalizations concerns Schrödinger-like operators on structures that look globally as $\mathbb{R}^d$, but locally have a different dimension $\delta$. The leading example here is the lattice $\mathbb{Z}^d$, which locally has dimension $\delta = 0$ but globally looks like $\mathbb{R}^d$. An exact explanation of the terms "locally" and "globally" in this context, as well as the corresponding results, can be found in [12, 13, 14, 15] and in [9].

The case where $d = 2$ proves to be the most complicated one and it is not completely understood up to now, both for the classical Schrödinger operator and its generalizations.

Recently, several important results for operators in $\mathbb{R}^2$ were obtained. In the present paper we discuss some of these results and their counterparts for operators with local dimension 0 and 1. To stress the close relationship between three cases under consideration, we denote the operators, functions, etc. by the same symbols in all cases, marking the local dimension by the overset numeral, like in $\overset{2}{\Delta}$, or $\overset{1}{V}$, when this is not clear from the context. We concentrate ourselves on estimates having semiclassical order with respect to $\alpha$. In the case where the global dimension is $d = 2$, this means that we are

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interested in estimates of the type

$$N_-(\mathbf{H}_V) \le 1 + \Phi(V), \tag{1.2}$$

where the functional $\Phi(V)$ is homogeneous of order 1 with respect to $V$, so that (1.2) automatically implies

$$N_-(\mathbf{H}_{\alpha V}) \le 1 + \Phi(V)\alpha, \quad \alpha > 0. \tag{1.3}$$

Note that the term 1 in (1.2) and in (1.3) reflects the well-known fact that for all the cases under study the operator $\mathbf{H}_{\alpha V}$ has at least one negative eigenvalue for any $\alpha > 0$.

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## §2. The setting

We are interested in estimates for $N_-(\mathbf{H})$ (in particular, in the conditions guaranteeing $N_-(\mathbf{H}) < \infty$) for three cases.

**1.** $\mathfrak{H} = L^2(\mathbb{R}^2)$ and $\mathbf{H} = -\overset{2}{\Delta} - \overset{2}{V}$, i.e., it is the standard Schrödinger operator with a real-valued, nonpositive potential $-\overset{2}{V}$. Here $\delta = 2$.

**2.** $\mathfrak{H} = \ell^2(\mathbb{Z}^2)$ and $\mathbf{H} = -\overset{0}{\Delta} - \overset{0}{V}$ is the discrete Schrödinger operator with a real-valued, nonpositive potential $-\overset{0}{V}$ defined on the lattice $\mathbb{Z}^2$. Here $\delta = 0$.

**3.** The third case is intermediate between the previous two ones. Here we are dealing with the metric graph $\Gamma_{\mathrm{ch}}$ that can be visualized as the union of straight lines on $\mathbb{R}^2$ dividing the plane into the union of unit squares. We will call it "the *chessboard mesh*". We give all the necessary details about the graph $\Gamma_{\mathrm{ch}}$ and about the corresponding Schrödinger operator $\mathbf{H} = \overset{1}{\mathbf{H}}_{\overset{1}{V}} = -\overset{1}{\Delta} - \overset{1}{V}$ in Subsection 6.1. Here we only note that $\delta = 1$ for this graph. The operator $\overset{1}{\mathbf{H}}_{\overset{1}{V}}$ acts as $-u'' - \overset{1}{V}u$ on each edge, and the functions from its domain meet some matching conditions at each vertex. The potential $\overset{1}{V} \ge 0$ is a function defined on the union of edges.

The spectral nature of the Laplacians $\overset{2}{\Delta}$, $\overset{0}{\Delta}$, and $\overset{1}{\Delta}$ is quite different. In particular, the operator $-\overset{0}{\Delta}$ is bounded in $\ell^2(\mathbb{Z}^2)$, its spectrum is purely a.c. and fills $[0, 4]$. The classical Laplacian $-\overset{2}{\Delta}$ is unbounded in $L^2(\mathbb{R}^2)$, its spectrum is purely a.c. and fills $[0, \infty)$. The operator $-\overset{1}{\Delta}$ is also unbounded. Its spectrum also fills $[0, \infty)$; however, along with the a.c. component, it contains embedded eigenvalues at the points $\pi^2 l^2$, $l = 1, 2, \dots$, and these eigenvalues have infinite multiplicity (see [4] for details).

The Schrödinger operators $\overset{\delta}{\mathbf{H}}_{\overset{\delta}{V}}$ are defined by means of quadratic forms. For $\overset{2}{\mathbf{H}}$ this procedure is standard; for operators on quantum and combinatorial graphs the details can be found, e.g., in [14]. The perturbation $\overset{\delta}{V}$ generates a quadratic form $\overset{\delta}{\mathbf{b}}$. For $\delta = 0$, $\overset{0}{\mathbf{b}}[u]$ is defined as $\sum_{\mathbf{x} \in \mathbb{Z}^2} V(\mathbf{x})|u(\mathbf{x})|^2$; in dimensions 1 and 2 the sum is replaced by the usual integrals, see [14]. At the moment, we do not specify the conditions imposed on the potential $\overset{\delta}{V}$; the assumptions made later, when studying the eigenvalue bounds, guarantee that the operators are well defined.

*Remark* 2.1. Alternatively to $N_-(\mathbf{H}_V)$, one can look for estimates for the number of nonpositive eigenvalues, $N_{\le 0}(\mathbf{H}_V)$, like in [1, 9, 10]. Since there are no zero energy eigenfunctions of the unperturbed Laplacian, the quadratic form $-(Vu, u)$ is negative definite on the null space of the operator $\mathbf{H}_V$, and therefore the quadratic form $(\mathbf{H}_V u, u) - (Vu, u) = (\mathbf{H}_{2V} u, u)$ is negative definite on the spectral subspace of $\mathbf{H}_V$ corresponding to the nonpositive part of the spectrum (cf. the observation in [9]). Thus,

$$N_{\le 0}(\mathbf{H}_V) \le N_-(\mathbf{H}_{2V}).$$

Therefore, any estimate for the number of negative eigenvalues carries over automatically to a similar estimate for the number of nonpositive eigenvalues (with merely a constant changed), so the former estimate is only formally weaker than the latter, provided we do not care for sharp constants. Keeping this in mind and following the tradition, we discuss estimates for $N_-(\mathbf{H}_V)$ only.

## §3. The $\mathbb{R}^2$-case: Shargorodsky estimate

In spite of important differences, the estimates for the quantities $N_-\big(\overset{\boldsymbol{\delta}}{\mathbf{H}}_V\big)$ for $\delta = 0, 1, 2$ have much in common. For $d \geq 3$ this was discovered in [12, 13, 14]. The situation in $\mathbb{R}^d$, $d > 2$, is governed by the CLR-inequality (1.1). A similar inequality is valid for the discrete Schrödinger operator on $\mathbb{Z}^d$. It can easily be derived from (1.1), see [13], but it can also be obtained independently [8, 12]. A CLR inequality for the $d$-dimensional analog of the chessboard mesh has a similar form, with $V$ replaced by a certain effective discrete potential, see [14] for the exact definition and statements.

On the contrary, the case where $d = 2$ is not completely understood up to now. Several upper bounds for $N_-\big(\overset{\mathbf{2}}{\mathbf{H}}\big)$ are known. The estimate formulated below, the sharpest one known up to now, was obtained recently by Shargorodsky [16], and is a refinement of earlier estimates in [17, 5, 7].

The estimate concerns the $\mathbb{R}^2$-case, and the overscript will be suppressed till the end of this section. The formulation of the estimate is rather complicated, because it makes use of function spaces that appear in the spectral theory not frequently. Let us present the necessary auxiliary material.

Below, $(r, \vartheta)$ stand for the polar coordinates in $\mathbb{R}^2$, and $\mathbb{S}$ denotes the unit circle $r = 1$. Given a function $V$ such that $V(r, \cdot) \in L^1(\mathbb{S})$ for almost all $r > 0$, we introduce its radial and nonradial parts

$$V_{\mathrm{rad}}(r) = \frac{1}{2\pi} \int_{\mathbb{S}} V(r, \vartheta)\, d\vartheta; \quad V_{\mathrm{nrad}}(r, \vartheta) = V(r, \vartheta) - V_{\mathrm{rad}}(r).$$

The conditions will be imposed on $V_{\mathrm{rad}}$ and on $V_{\mathrm{nrad}}$ separately. For handling the radial part, we need a certain auxiliary operator on the real line,

$$(\mathbf{M}_G\varphi)(t) = -\varphi''(t) - G(t)\varphi(t), \quad \varphi(0) = 0, \tag{3.1}$$

with the "effective potential"

$$G(t) = G_V(t) = e^{2|t|} V_{\mathrm{rad}}(e^t). \tag{3.2}$$

Due to the condition $\varphi(0) = 0$ in (3.1), the operator $\mathbf{M}_G$ is the direct orthogonal sum of two operators, each acting on the half-line. The sharp spectral estimates for $\mathbf{M}_G$ can be given in terms of the number sequence (see [7, (1.13)])

$$\widehat{\boldsymbol{\zeta}}(G) = \big\{\widehat{\zeta}_j(G)\big\}_{j\geq 0} : \widehat{\zeta}_0(G) = \int_{D_0} G(t)\, dt, \ \widehat{\zeta}_j(G) = \int_{|t|\in D_j} |t| G(t)\, dt \ \ (j \in \mathbb{N}), \tag{3.3}$$

where $D_0 = (-1, 1)$ and $D_j = (e^{j-1}, e^j)$ for $j \in \mathbb{N}$. The estimate is

$$N_-(\mathbf{M}_G) \leq 1 + C \sup_{s>0} \big(s \#\{j : \widehat{\zeta}_j(G) > s\}\big), \tag{3.4}$$

see Theorem 5.4, **2** in [18].

Note that the functional appearing on the right-hand side in (3.4) is nothing else than the quasinorm of the sequence $\big\{\widehat{\zeta}_j(G)\big\}$ in the "weak $\ell_1$-space" $\ell_{1,\infty}$.

The condition on $V_{\mathrm{nrad}}$ is given in terms of the space $L_1(\mathbb{R}_+, L_{\mathfrak{B}}(\mathbb{S}))$, i.e., the space of functions on the half-line with values in the space $L_{\mathfrak{B}}(\mathbb{S})$. The latter is the Orlicz space of functions on the unit circle determined by the $\mathcal{N}$-function

$$\mathfrak{B}(t) = (1+t)\ln(1+t) - t. \tag{3.5}$$

See [2], or [11] for the basics in Orlicz spaces, and in particular, for the definition of the norm in them. Some additional details are presented in §5 below. There we need also the $\mathcal{N}$-function

$$\mathfrak{A}(t) = e^t - t - 1, \tag{3.6}$$

complementary to $\mathfrak{B}(t)$. In what follows, for brevity, we write $\mathfrak{B}(\mathbb{S}) = L_{\mathfrak{B}}(\mathbb{S})$.

We shall suppose that $V_{\mathrm{nrad}} \in \mathcal{X} = L_1(\mathbb{R}_+, \mathfrak{B}(\mathbb{S}))$. The last space is determined by the norm

$$\|f\|_{\mathcal{X}} = \int_{\mathbb{R}_+} \|f(r,\cdot)\|_{\mathfrak{B}(\mathbb{S})} r\, dr.$$

The resulting estimate, see [16], is given in the following statement.

**Theorem 3.1.** *Let $V \geq 0$ be a potential on $\mathbb{R}^2$, and let $G = G_V$ be the corresponding effective potential as in* (3.2). *Suppose that the sequence $\{\widehat{\zeta}_j(G)\}$ belongs to the space $\ell_{1,\infty}$ and that $V_{\mathrm{nrad}} \in L_1(\mathbb{R}_+, \mathfrak{B}(\mathbb{S}))$. Then*

$$N_-(\mathbf{H}_V) \leq 1 + C\Big(\sup_{s>0}\,(s\#\{j : \zeta_j(G_V) > s\}) + \|V_{\mathrm{nrad}}\|_{L_1(\mathbb{R}_+,\ L\log L(\mathbb{S}))}\Big). \tag{3.7}$$

Still, this theorem gives only a sufficient condition for the semiclassical behavior $N_-(\mathbf{H}_{\alpha V}) = O(\alpha)$ in the large coupling constant regime. A number of estimates, non-linear in $\alpha$, are also known, see [10].

For the radial potentials $V(x) = F(|x|)$, estimate (3.7) simplifies because the second term in brackets disappears. It was established in [6] that for such potentials this estimate gives not only sufficient, but also necessary condition for the semiclassical behavior. For arbitrary (i.e., not necessarily radial) potentials the following important result of "negative" nature was recently established in [1, Sec. 2.7].

**Proposition 3.2.** *For the operator $\mathbf{H}_V$ on $\mathbb{R}^2$, no estimate of the type*

$$N_-(\mathbf{H}_V) \leq \mathrm{const} + \int_{\mathbb{R}^2} VW\, dx \tag{3.8}$$

*can occur provided the weight function $W$ is bounded in a neighborhood of at least one point.*

## §4. The $\mathbb{Z}^2$-case: Reduction to the $\mathbb{R}^2$-case

In this section we present the machinery for transferring spectral estimates from the operator $\overset{2}{\mathbf{H}}$ to $\overset{0}{\mathbf{H}}$. This simple approach was already used in [13] for the study of the Schrödinger operator on $\mathbb{Z}^d$, $d \geq 3$, however certain modifications are needed in the two-dimensional case.

**4.1. Interpolation. The lattice Sobolev space.** Below $e_1 = (1,0)$, $e_2 = (0,1)$ denote the standard basis in $\mathbb{Z}^2$. Consider the seminorm

$$\|u\|^2_{\overset{0}{\mathcal{H}}} = \sum_{\mathbf{x}\in\mathbb{Z}^2} \big(|u(\mathbf{x}+e_1) - u(\mathbf{x})|^2 + |u(\mathbf{x}+e_2) - u(\mathbf{x})|^2\big) \tag{4.1}$$

on the space of functions on $\mathbb{Z}^2$ for which the sum in (4.1) is finite. It is the quadratic form of the discrete Laplacian on $\mathbb{Z}^2$ and is an analog of the Dirichlet integral on $\mathbb{R}^2$.

With any function $u(\mathbf{x})$, $\mathbf{x} \in \mathbb{Z}^2$, we associate the function $U_0(x) = (\mathfrak{J}_0 u)(x)$, $x \in \mathbb{R}^2$, in the following way: first, in each square with the vertices $\mathbf{x}$, $\mathbf{x}+e_1$, $\mathbf{x}+e_2$, $\mathbf{x}+e_1+e_2$ ($\mathbf{x} \in \mathbb{Z}^2$), we take the function $U_0$ that is bilinear, that is, linear in $x_1$ and in $x_2$ (separately), and coincides with $u$ at the vertices. Such a function is unique. The following simple fact was used in [13].

**Proposition 4.1.** *For any finitely supported function $u$ on $\mathbb{Z}^2$, the function $U_0 = \mathfrak{J}_0 u$ belongs to the usual Sobolev space $H^1(\mathbb{R}^2)$ and*

$$\int_{\mathbb{R}^2} |\nabla U_0|^2\, dx \le C\|u\|^2_{\overset{\circ}{\mathcal{H}}}, \quad U_0 = \mathfrak{J}_0 u. \tag{4.2}$$

We borrow the proof from [13].

*Proof.* Consider the space $\mathcal{L}(\mathcal{C})$ of bilinear functions on the unit cell $\mathcal{C} = [0,1]^2$. Clearly, $\dim \mathcal{L}(\mathcal{C}) = 4$. On $\mathcal{L}(\mathcal{C})$ we consider the quadratic forms

$$\widetilde{Q}[U_0;\mathcal{C}] = \sum_{\substack{\mathbf{x},\mathbf{y}\in\{0,1\}^2\\ \mathbf{x}\sim\mathbf{y}}} |U_0(\mathbf{x}) - U_0(\mathbf{y})|^2; \quad \widetilde{D}[U_0;\mathcal{C}] = \int_{\mathcal{C}} |\nabla U(x)|^2\, dx.$$

These two quadratic forms vanish on the same subspace in $\mathcal{L}(\mathcal{C})$, the one consisting of constant functions. Therefore, they are equivalent, i.e., with some $c, c' > 0$ we have

$$c\widetilde{Q}[U_0;\mathcal{C}] \le \widetilde{D}[U_0;\mathcal{C}] \le c'\widetilde{Q}[U_0;\mathcal{C}]. \tag{4.3}$$

The function $U_0$ is compactly supported, continuous on the whole of $\mathbb{R}^2$, and smooth inside each cell. Hence, $U_0 \in H^1(\mathbb{R}^2)$. Summing inequalities of the form (4.3) for all cells $\mathcal{C} + \mathbf{x},\ \mathbf{x} \in \mathbb{Z}^2$, we arrive at (4.2). ☐

Now we fix a smooth cut-off function $\psi(x),\ x \in \mathbb{R}^2$, that vanishes in a neighborhood of $x = (0,0)$ and equals 1 for $|x| > \frac{1}{2}$. We define the interpolation operator $\mathfrak{J}$ by setting $U = \mathfrak{J}u = \psi\mathfrak{J}_0 u = \psi U_0$.

Since the function $U$ differs from the corresponding function $U_0$ on four squares only, from (4.2) it follows that for all finitely supported functions $u$ on $\mathbb{Z}^2$ such that $u(0,0) = 0$, an inequality similar to (4.2) holds true, with $U$ replacing $U_0$:

$$\int_{\mathbb{R}^2} |\nabla U|^2\, dx \le C\|u\|^2_{\overset{\circ}{\mathcal{H}}}, \quad U = \mathfrak{J}u. \tag{4.4}$$

The next step is to obtain a weighted estimate for functions on the lattice.

**Proposition 4.2.** *For all functions $u(\mathbf{x}),\ \mathbf{x} \in \mathbb{Z}^2$, with finite support and satisfying $u(0,0) = 0$, the following discrete Hardy inequality is valid*:

$$\|u\|^2_{\overset{\circ}{\mathcal{H}}} \ge C \sum_{\substack{\mathbf{x}\in\mathbb{Z}^2,\\ \mathbf{x}\ne(0,0)}} |u(\mathbf{x})|^2 |\mathbf{x}|^{-2} (\log(|\mathbf{x}| + 2))^{-2}. \tag{4.5}$$

*Proof.* From (4.2) and the logarithmic Hardy inequality for functions on $\mathbb{R}^2$ vanishing near the origin, it follows that for $U = \mathfrak{J}u$ we have

$$\|u\|^2_{\overset{\circ}{\mathcal{H}}} \ge C \int |\nabla U|^2\, dx \ge C' \int |U(x)|^2 |x|^{-2} (\log^2(|x| + 2))^{-1}\, dx. \tag{4.6}$$

Recalling that $U(0,0) = u(0,0) = 0$ and that $U$ is piecewise bilinear (with exception of 4 central squares), we can estimate the last integral in (4.6) from below by the sum of the values of the integrand at the lattice points, which produces the Hardy type inequality in question. ☐

**4.2. Main result for the $\mathbb{Z}^2$-case.** Given a discrete potential $V = \overset{\circ}{V} \ge 0$, we associate with it the piecewise-constant potential $\overset{2}{V} = \mathcal{I}\overset{\circ}{V}$, assigning at each point $(x_1, x_2) \in \mathbb{R}^2$ the value $\overset{2}{V}(x_1, x_2) = \overset{\circ}{V}([x_1], [x_2])$ (as usual, $[\cdot]$ denotes the integral part of the number in brackets).

**Proposition 4.3.** *Let $\overset{\circ}{V} \ge 0$ be a discrete potential on $\mathbb{Z}^2$, and let $\overset{2}{V} = \mathcal{I}\big(\overset{\circ}{V}\big)$. Then*

$$N_-\big( - \overset{\circ}{\Delta} - \overset{\circ}{V}\big) \le N_-\big( - \overset{2}{\Delta} - \gamma\overset{2}{V}\big), \tag{4.7}$$

*with some constant $\gamma > 0$ independent of $\overset{\circ}{V}$.*

*Proof.* First, let $N_-\big( - \overset{\circ}{\Delta} - \overset{\circ}{V}\big) = \mathbf{m} < \infty$. This means that there exists a subspace $\mathcal{L} \subset \ell^2(\mathbb{Z}^2)$ of dimension $\mathbf{m}$ such that $\|u\|^2_{\overset{\circ}{\mathcal{H}}} < \mathbf{b}[u]$ for all $u \in \mathcal{L}$, $u \ne 0$. By compactness, it may be assumed that all functions in $\mathcal{L}$ have support in a common compact set $\mathfrak{K} \subset \mathbb{Z}^2$.

Consider the set $\boldsymbol{\mathcal{L}} = \mathfrak{J}_0\mathcal{L}$ consisting of the interpolants $U = \mathfrak{J}_0 u,\ u \in \mathcal{L}$, where $\mathfrak{J}_0$ is the interpolation operator described in Subsection 4.1. This is a space of functions on $\mathbb{R}^2$, contained in $\overset{2}{\mathcal{H}}(\mathbb{R}^2)$ and having the same dimension $\mathbf{m}$. It is readily seen that $\|\nabla U\|^2_{L^2} \le C_1\|u\|^2_{\overset{\circ}{\mathcal{H}}}$ and $\mathbf{b}_{\overset{2}{V}}[U] \ge C_2\mathbf{b}_{\overset{\circ}{V}}[u]$. Therefore, for $U \in \boldsymbol{\mathcal{L}},\ U \ne 0$, we have

$$\|\nabla U\|^2_{L^2} - \gamma\alpha\int V|U|^2\,dx < C_1\big(\|u\|^2_{\overset{\circ}{\mathcal{H}}}\big) - \mathbf{b}[u]) < 0, \quad \gamma = C_1C_2^{-1}. \tag{4.8}$$

The last inequality means that $N_-\big(\overset{2}{\Delta} - \gamma\overset{2}{V}\big) \ge \mathbf{m}$, which proves (4.7) for $\mathbf{m} < \infty$. In the case where $\mathbf{m} = \infty$, we can repeat the above argument for any finite-dimensional subspace $\mathcal{L}$, concluding that $N_-\big(\overset{2}{\Delta} - \gamma\overset{2}{V}\big)$ exceeds any given natural number; in other words, $N_-\big(\overset{2}{\Delta} - \gamma\overset{2}{V}\big) = \infty$. $\square$

Note that we did not use Proposition 4.2 in the proof.

## §5. The weighted estimate for $\mathbb{Z}^2$

As it follows from Proposition 4.3, any eigenvalue estimate for the usual Schrödinger operator has its counterpart for the discrete one. The weak side of this approach is that it gives an estimate for the $\mathbb{Z}^2$-case in terms of the associated potential $\overset{2}{V} = \mathcal{I}\overset{\circ}{V}$, rather than in terms of the original discrete potential $\overset{\circ}{V}$, and some features, in particular related to the circle symmetry, are hopelessly lost. In this connection, a recent result by Molchanov and Vainberg (see [10, Theorem 6.1]) deserves a special attention, because it is free from this defect.

**Theorem 5.1.** *For any discrete potential $\overset{\circ}{V} \ge 0$ on $\mathbb{Z}^2$, we have*

$$N_-\big(\overset{\circ}{\mathbf{H}}_{\overset{\circ}{V}}\big) \le 1 + C\sum_{\mathbf{x}\in\mathbb{Z}^2} \overset{\circ}{V}(\mathbf{x})\log(2 + |\mathbf{x}|). \tag{5.1}$$

By Proposition 3.2 no corresponding $\mathbb{R}^2$-estimate of a similar form can exist. Our aim in this section is to show that, nevertheless, (5.1) can be derived by the interpolation procedure from an existing $\mathbb{R}^2$-estimate, namely, from the result of [17].

To formulate the last-mentioned result, we use the pair (3.5) and (3.6) of mutually complementary $\mathcal{N}$-functions. For any measurable set $E \subset \mathbb{R}^2$ of finite Lebesgue measure, the Orlicz space $L_{\mathfrak{B}}(E)$ is defined as the space of measurable functions $v$ on $E$ such that $\int_E \mathfrak{B}(|v(x)|)\,dx < \infty$. Since the $\mathcal{N}$-function $\mathfrak{B}(t)$ meets the so-called $\Delta_2$-condition, see [2, 11], such functions form a Banach space, and one of the equivalent norms in it is the *averaged norm* (introduced in [17]):

$$\|v\|^{(av)}_{\mathfrak{B},E} = \sup\left\{\left|\int_E vg\,dx\right| \,:\, \int_E \mathfrak{A}(|g(x)|)\,dx \le |E|\right\}.$$

Now, to formulate the estimate from [17], we consider the following partition of $\mathbb{R}^2$:

$$\Omega_0 = \{x \,:\, |x| \le 1\}, \quad \Omega_k = \{x \,:\, 2^{k-1} \le |x| \le 2^k\}.$$

With a given potential $\overset{\mathbf{2}}{V} \ge 0$, we associate the number sequence $\boldsymbol{\mu}(\overset{\mathbf{2}}{V}) = \{\mu_k(\overset{\mathbf{2}}{V})\}$, where

$$\mu_k(\overset{\mathbf{2}}{V}) = \|\overset{\mathbf{2}}{V}\|^{(av)}_{\mathfrak{B},\Omega_k}.$$

With this notation, by Theorem 3 in [17] (more precisely, by its simpler version, see (32) there), we have

$$N_-(\overset{\mathbf{2}}{\Delta} - \alpha\overset{\mathbf{2}}{V}) \le 1 + C\|\boldsymbol{\mu}(\overset{\mathbf{2}}{V})\|_{\ell^1} + \int \overset{\mathbf{2}}{V}(x)|\log|x||\,dx. \tag{5.2}$$

Now, using Proposition 4.1, we derive estimate (5.1) from (5.2). To this end, for a given $\overset{\mathbf{0}}{V}$, we consider the piecewise constant potential $\overset{\mathbf{2}}{V} = \mathcal{I}(\overset{\mathbf{0}}{V})$. Estimate (5.1) follows from (5.2) immediately as soon as we prove that

$$\|\boldsymbol{\mu}(\overset{\mathbf{2}}{V})\|_{\ell^1} \le C\sum_{\mathbf{x}\in\mathbb{Z}^2} \overset{\mathbf{0}}{V}(\mathbf{x})\log(2+|\mathbf{x}|). \tag{5.3}$$

For $k$ fixed, we consider the set $E_k$ that is the union of all closed unit lattice squares having nonempty intersection with $\Omega_k$. Such sets form a covering of $\mathbb{R}^2$ with multiplicity 2, moreover, $|E_k| \asymp |\Omega_k|$. So, since $\log(|\mathbf{x}|+2) \asymp k+1$ for $\mathbf{x} \in E_k$, to prove (5.3) it suffices to establish the inequality

$$\|\overset{\mathbf{2}}{V}\|^{(av)}_{\mathfrak{B},E_k} \le C(k+1)\sum_{\mathbf{x}\in E_k\cap\mathbb{Z}^2} \overset{\mathbf{0}}{V}(\mathbf{x}). \tag{5.4}$$

Further on, in this proof, all summations are performed over $\mathbf{x} \in E_k \cap \mathbb{Z}^2$. For any unit square $Q$ (with vertices in $\mathbb{Z}^2$) and a nonnegative function $g$ we denote by $g_Q(x)$ the constant function on $Q$ equal to the mean value of $g$ over $Q$, so that $\int_Q g\overset{\mathbf{2}}{V}\,dx = \int_Q g_Q\overset{\mathbf{2}}{V}\,dx$. By Jensen's inequality,

$$\int_Q \mathfrak{A}(g(x))\,dx \ge \mathfrak{A}\left(\int_Q g(x)\,dx\right) = \int_Q \mathfrak{A}(g_Q(x))\,dx.$$

Therefore, to obtain an upper estimate for the averaged norm of $\overset{\mathbf{2}}{V}$, it suffices to maximize over the piecewise constant functions $g$:

$$\|\overset{\mathbf{2}}{V}\|^{(av)}_{\mathfrak{B},E_k} \le \sup\left\{\sum \overset{\mathbf{0}}{V}(\mathbf{x})f(\mathbf{x}) \,:\, \sum \mathfrak{A}(f(\mathbf{x})) = |E_k|\right\}. \tag{5.5}$$

By linearity, it suffices to estimate the quantity occurring in (5.5) under the normalization $\sum \overset{\mathbf{0}}{V}(\mathbf{x}) = 1$; so, we aim at estimating the quantity

$$\sup\left\{\sum \overset{\mathbf{0}}{V}(\mathbf{x})g(\mathbf{x}) \,:\, \sum \mathfrak{A}(g(\mathbf{x})) = |E_k| \asymp C2^k\right\}. \tag{5.6}$$

Instead, we can consider the expression $S(\overset{\mathbf{0}}{V}, g) = \sum \overset{\mathbf{0}}{V}(\mathbf{x})g(\mathbf{x})$ and maximize it over all collections $\{\overset{\mathbf{0}}{V}(\mathbf{x}) \ge 0\}, \{g(\mathbf{x}) \ge 0\}$ under the conditions $\sum \overset{\mathbf{0}}{V}(\mathbf{x}) = 1$, $\sum \mathfrak{A}(g(\mathbf{x})) = |E_k|$. By compactness and continuity, a point of maximum must exist. We claim that at the point of maximum, only one of $\overset{\mathbf{0}}{V}(\mathbf{x})$ is not zero. Indeed, if, say, $\overset{\mathbf{0}}{V}(\mathbf{x}_1), \overset{\mathbf{0}}{V}(\mathbf{x}_2) \ne 0$ and

$g(\mathbf{x}_1)$ (or $g(\mathbf{x}_2)$) is 0, then we can change $\overset{\mathbf{o}}{V}(\mathbf{x}_1)$ to 0 and add this $\overset{\mathbf{o}}{V}(\mathbf{x}_1)$ to $\overset{\mathbf{o}}{V}(\mathbf{x})$ for which $g(\mathbf{x}) > 0$, thus keeping $\sum \overset{\mathbf{o}}{V}(\mathbf{x}) = 1$ and increasing $S(\overset{\mathbf{o}}{V}, g)$. If both $g(\mathbf{x}_1)$, $g(\mathbf{x}_2)$ are positive and, say, $g(\mathbf{x}_1) \geq g(\mathbf{x}_2)$, we change $\overset{\mathbf{o}}{V}(\mathbf{x}_1) \mapsto \overset{\mathbf{o}}{V}(\mathbf{x}_1) + \overset{\mathbf{o}}{V}(\mathbf{x}_2)$, $\overset{\mathbf{o}}{V}(\mathbf{x}_2) \mapsto 0$ and thus increase $S(\overset{\mathbf{o}}{V}, g)$, keeping $\sum \overset{\mathbf{o}}{V}(\mathbf{x}) = 1$.

So, the function $\overset{\mathbf{2}}{V}$ maximizing the norm $\|\overset{\mathbf{2}}{V}\|^{av}_{\mathcal{B},E_k}$ among the piecewise constant nonnegative functions on $E_k$ with $\int_{E_k} \overset{\mathbf{2}}{V}\, dx = \sum_{\mathbf{x}\in E_k\cap\mathbb{Z}^2} \overset{\mathbf{o}}{V}(\mathbf{x}) = 1$, is the function supported on one square and having value 1 there. For such a function, the averaged Orlicz norm over the domain $E_k$ equals

$$\|\overset{\mathbf{2}}{V}\|^{av}_{\mathfrak{B},E_k} = \overset{\mathbf{2}}{V}\mathfrak{A}^{-1}(|E_k|) \asymp \log|E_k| \asymp k+1.$$

Summing up such inequalities over all $k$, we arrive at (5.4), thus finishing the (alternative) proof of (5.1).

*Remark* 5.2. Along with estimate (5.1), we have

$$N_-\big(\overset{\mathbf{o}}{\Delta} - \alpha\overset{\mathbf{o}}{V}\big) = o(\alpha), \quad \alpha \to \infty.$$

Indeed, this is certainly true for the finitely supported potentials. Since such potentials form a dense set in the cone of nonnegative elements in the weighted space $\ell^1(\mathbb{Z}^2, \log(2+|x|)$, the result extends by continuity to all such weights.

The details can easily be restored by analogy with [14], see the proof of Theorem 2.3 there.

## §6. The chessboard mesh

**6.1. Basic definitions.** First, we present a detailed description of the metric graph $\Gamma_{\mathrm{ch}}$ that we call the chessboard mesh, see §2. Its set of vertices is $\mathbb{Z}^2$ (viewed as naturally imbedded in $\mathbb{R}^2$) and the set $\mathcal{E}$ of edges consists of the intervals $\mathbf{e}_\nu$ of length one connecting the neighboring vertices of $\mathbb{Z}^2$. Here $\nu$ in the subscript standing for the pair $\nu = (\mathbf{x}, \mathbf{y}) = (\mathbf{y}, \mathbf{x})$ of vertices that are the endpoints of an edge, so $\mathbf{x} - \mathbf{y} = \pm e_1$ or $\mathbf{x} - \mathbf{y} = \pm e_2$. The measure $dz$ on $\Gamma_{\mathrm{ch}}$ is induced by the Lebesgue measure on the edges, and our main Hilbert space is $L^2(\Gamma_{\mathrm{ch}}, dz) = \bigoplus_{\mathbf{e}\in\mathcal{E}} L^2(\mathbf{e}, dz)$. A function $u$ on $\Gamma_{\mathrm{ch}}$ belongs to the Sobolev space $H^1(\Gamma_{\mathrm{ch}})$ if it is continuous on $\Gamma_{\mathrm{ch}}$, lies in $H^1(\mathbf{e})$ for each $\mathbf{e} \in \mathcal{E}$, and satisfies

$$\|u\|^2_{H^1(\Gamma_{\mathrm{ch}})} = \int_{\Gamma_{\mathrm{ch}}} (|u'(z)|^2 + |u(z)|^2)\, dz = \sum_{\mathbf{e}\in\mathcal{E}} \int_{\mathbf{e}} (|u'(z)|^2 + |u(z)|^2)\, dz < \infty.$$

The (minus) Laplacian on $\Gamma_{\mathrm{ch}}$ is defined as the selfadjoint operator in $L^2(\Gamma_{\mathrm{ch}})$ associated with the quadratic form $\mathbf{a}_{\Gamma_{\mathrm{ch}}}[u] := \int_{\Gamma_{\mathrm{ch}}} |u'(z)|^2\, dz$ considered on the form-domain $H^1(\Gamma_{\mathrm{ch}})$. Its operator domain $\mathcal{D}(\Delta)$ can be described explicitly as follows: $u \in \mathcal{D}(\Delta)$ if and only if $u$ is continuous on $\Gamma_{\mathrm{ch}}$, $u''(z) \in L^2(\mathbf{e})$ on each edge $\mathbf{e} \in \mathcal{E}$, the Kirchhoff matching condition is fulfilled at each vertex $\mathbf{x} \in \mathbb{Z}^2$ (that is, the sum of the outgoing derivatives of $u$ at the point $\mathbf{x}$ equals zero), and finally,

$$\sum_{\mathbf{e}\in\mathcal{E}} \int_{\mathbf{e}} (|u''|^2 + |u|^2)\, dz < \infty.$$

On this domain the Laplacian acts as $\Delta u = u''$ on each edge; see [3] for more detail on the Laplacian on metric trees.

The spectrum of $-\Delta$ coincides with $[0, \infty)$. It consists of the a.c. component filling the same interval, and the embedded eigenvalues $\lambda_l = \pi^2 l^2$, each of infinite multiplicity (see [4], where, in particular, the eigenfunctions are described explicitly).

The Schrödinger operator $-\Delta - V$ on $\Gamma_{\mathrm{ch}}$ is defined standardly as the form-sum, the potential $V \geq 0$ being a measurable function on $\Gamma_{\mathrm{ch}}$. The assumptions we impose on $V$ further on, in Theorem 6.2, guarantee that the quadratic form $\mathbf{a}_{\Gamma_{\mathrm{ch}}}[u] - \int_{\Gamma_{\mathrm{ch}}} V|u|^2\,dz$ is lower bounded and closed on $H^1(\Gamma_{\mathrm{ch}})$.

For the spectral analysis of the Schrödinger operator on $\Gamma_{\mathrm{ch}}$, it is convenient to consider two pre-Hilbert spaces that are linear subspaces in the space $H^1_{\mathrm{comp}}(\Gamma_{\mathrm{ch}})$ of all compactly supported functions in $H^1(\Gamma_{\mathrm{ch}})$. One of them, $H^1_{\mathrm{comp,pl}}(\Gamma_{\mathrm{ch}})$, is formed by functions linear on each edge $\mathbf{e} \in \mathcal{E}$; the subscript pl stands for "piecewise-linear". Any function $\varphi \in H^1_{\mathrm{comp,pl}}(\Gamma_{\mathrm{ch}})$ is determined by its values $\varphi(\mathbf{x})$ at the vertices. Given a lattice function $u = \{u(\mathbf{x})\}$, $\mathbf{x} \in \mathbb{Z}^2$, with finite support, we denote by $Ju$ the piecewise-linear extension of $u$ to $\Gamma_{\mathrm{ch}}$, i.e., a unique function in $H^1_{\mathrm{comp,pl}}(\Gamma_{\mathrm{ch}})$ such that $(Ju)(\mathbf{x}) = u(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{Z}^2$. The mapping $J$ gives rise to an isometry between the pre-Hilbert space $H^1_{\mathrm{comp,pl}}(\Gamma_{\mathrm{ch}})$, equipped with the metric $\mathbf{a}_{\Gamma_{\mathrm{ch}}}$, and the space $\mathcal{H}_{\mathrm{fin}}(\mathbb{Z}^2)$ of finitely supported lattice functions equipped with the metric (4.1). This isometry allows us to identify these pre-Hilbert spaces.

Another subspace is $H^1_{\mathrm{comp},\mathcal{D}}$ ($\mathcal{D}$ hints for *Dirichlet*) consists of all functions $\varphi \in H^1_{\mathrm{comp}}(\Gamma_{\mathrm{ch}})$ such that $\varphi(\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathbb{Z}^2$. It is clear that

$$H^1_{\mathrm{comp}} = H^1_{\mathrm{comp,pl}} \oplus H^1_{\mathrm{comp},\mathcal{D}} \tag{6.1}$$

(orthogonal decomposition is in the metric $\mathbf{a}_{\Gamma_{\mathrm{ch}}}$). We denote by $\varphi_{\mathrm{pl}}$ and $\varphi_{\mathcal{D}}$ the components of a given element $\varphi \in H^1_{\mathrm{comp}}(\Gamma_{\mathrm{ch}})$ with respect to this decomposition.

**Proposition 6.1.** *For any function $\varphi \in H^1_{\mathrm{comp}}$ such that $\varphi(0,0) = 0$, the following Hardy type inequality holds true*:

$$\mathbf{a}_{\Gamma_{\mathrm{ch}}}[\varphi] \geq C \int_{\Gamma_{\mathrm{ch}}} |\varphi(z)|^2 (|z|^2 (\log^2(|z|+2)))^{-1}\,dz.$$

*Proof.* It suffices to prove this inequality for $\varphi_{\mathrm{pl}}$ and $\varphi_{\mathcal{D}}$ separately. For $\varphi_{\mathrm{pl}}$, by the isometry $J$, it obviously follows from (4.5), and for $\varphi_{\mathcal{D}}$ from the usual Hardy inequality applied to each of the intervals $\mathbf{e} \in \mathcal{E}$. $\square$

Now we can consider the space $\overset{1}{\mathcal{H}}$ of functions on $\Gamma_{\mathrm{ch}}$ that is the closure of $H^1_{\mathrm{comp}}$ in the metric $\mathbf{a}_\Gamma$. By Proposition 6.1, this space of functions is embedded in $L^2(\Gamma_{\mathrm{ch}})$ with the weight $|z|^2(\log^2(|z|+2))^{-1}$. The closures of the terms in (6.1) are Hilbert spaces $\overset{1}{\mathcal{H}}_{\mathrm{pl}}$ and $\overset{1}{\mathcal{H}}_{\mathcal{D}}$, respectively, giving an orthogonal decomposition of $\overset{1}{\mathcal{H}}$,

$$\overset{1}{\mathcal{H}} = \overset{1}{\mathcal{H}}_{\mathrm{pl}} \oplus \overset{1}{\mathcal{H}}_{\mathcal{D}}. \tag{6.2}$$

**6.2. Spectral estimates on $\Gamma_{\mathrm{ch}}$.** The passage from eigenvalue estimates for $\overset{0}{\mathbf{H}}$ to those for $\overset{1}{\mathbf{H}}$ was elaborated in detail in [14] for general graphs. Here we explain how it works in our particular case. Let a potential $\overset{1}{V}$ on $\Gamma_{\mathrm{ch}}$ be given. For each edge $\mathbf{e}$ in $\Gamma_{\mathrm{ch}}$, we consider the Sturm–Liouville operator in $L^2(\mathbf{e})$, $\overset{1}{\mathbf{H}} = \mathbf{H}_{\mathbf{e},\mathcal{D}}(\overset{1}{V}) = -\frac{d^2}{dz^2} - \overset{1}{V}(z)$, with Dirichlet boundary condition at the endpoints of $\mathbf{e}$. The direct sum of these operators is denoted by $\overset{\mathcal{D}}{\mathbf{H}}$; this operator corresponds to the second term in the decomposition (6.2). Another operator, stemming from the first term in the decomposition (6.2), is the operator $\overset{0}{\mathbf{H}}$ on the lattice $\mathbb{Z}^2$ with the effective potential

$$\overset{0}{V}(\mathbf{x}) = \sum_{\mathbf{e} \ni \mathbf{x}} \eta(\mathbf{e}),$$

where $\eta\big(\mathbf{e}, \overset{1}{V}\big) = \int_{\mathbf{e}} \overset{1}{V}(z)\, dz$. We denote by $\boldsymbol{\eta}\big(\overset{1}{V}\big)$ the sequence $\big\{\eta\big(\mathbf{e}, \overset{1}{V}\big)\big\}$. By the general result of [14, § 4.3], we have

$$N_-\big(\overset{1}{\mathbf{H}}_{\frac{1}{2}\overset{1}{V}}\big) \le N_-\big(\overset{\mathcal{D}}{\mathbf{H}}\big) + N_-\big(\overset{0}{\mathbf{H}}\big). \tag{6.3}$$

**Theorem 6.2.** *Denote by $\rho(\mathbf{e})$ the distance from the origin to the point of* $\mathbf{e}$ *nearest to the origin. Suppose that the sum*

$$\mathbf{L}\big(\overset{1}{V}\big) = \sum_{\mathbf{e}} \eta\big(\mathbf{e}, \overset{1}{V}\big) \log(2 + \rho(\mathbf{e})) \tag{6.4}$$

*is finite. Then $N_-\big(\overset{1}{\mathbf{H}}\big) \le 1 + C\mathbf{L}\big(\overset{1}{V}\big)$ with a constant $C$ independent of $V$.*

*Proof.* We need to estimate the two terms in (6.3). For the second term, the result of Theorem 5.1 applies, because the sum $\sum \overset{0}{V}(\mathbf{x}) \log(2 + |\mathbf{x}|)$ is estimated from both sides by the sum in (6.4). For the first term in (6.3), we can apply [14, Lemma 4.2, 2°] for the particular value $q = 1$, which gives

$$N_-\big(\overset{0}{\mathbf{H}}\big) \le C\big\|\boldsymbol{\eta}\big(\overset{1}{V}\big)\big\|_{\ell^1_w}.$$

The last expression is dominated by the right-hand side of (6.4), and we are done. □

The sum in (6.4) can be estimated from above by the integral

$$\int_\Gamma \overset{1}{V}(z) \log(2 + |z|)\, dz,$$

and this leads us to the final result.

**Theorem 6.3.** *Suppose that the integral $\mathbf{M}\big(\overset{1}{V}\big) = \int_\Gamma \overset{1}{V}(z) \log(2+|z|)\, dz$ converges. Then*

$$N_-\big(\overset{1}{\mathbf{H}}\big) \le 1 + C\mathbf{M}\big(\overset{1}{V}\big).$$

Here we note that Theorems 5.1 and 6.3 demonstrate that, in local dimension 0 and 1, no forbidding result similar to Proposition 3.2 can exist.

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