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SUFFICIENT CONDITIONS FOR THE HÖLDER SMOOTHNESS OF A FUNCTION

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To dear Boris Mikhaĭlovich Makarov, with great respect

ABSTRACT. An outer function in the unit disk satisfies the Hölder condition of order $ps/(p+1)$ provided its modulus is s-Hölder on the unit circle and has logarithm in L^p.

Suppose that a nonnegative 2π-periodic function $\rho(\theta)$; belongs to the Hölder class H^α (recall that if $0 < \alpha \le 1$, then $\rho \in H^\alpha \Leftrightarrow |\rho(\theta_2) - \rho(\theta_1)| \le c|\theta_2 - \theta_1|^\alpha$ for every θ_1, θ_2; if $n < \alpha \le n+1$ with a natural n, then $\rho \in H^\alpha \Leftrightarrow \rho^{(n)} \in H^{\alpha-n}$).

If, moreover, ρ satisfies the condition

$$\int_0^{2\pi} \log \rho(\theta)\, d\theta > -\infty, \tag{1}$$

then for the function $f_\rho(z)$ constructed by the formula

$$f_\rho(z) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log\rho(\theta) \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\theta\right), \quad |z| < 1, \tag{2}$$

we have $|f_\rho(e^{i\theta})| = \rho(\theta)$. The function f_ρ is said to be outer, see [1, 2]. Should the function $\rho(\theta)$ satisfy $\rho(\theta) \ge c_0 > 0$, $\theta \in \mathbb{R}$, the classical Zygmund–Privalov theorem [3] would imply that $\log f(z) \in H^\alpha(\bar{\mathbb{D}})$ in the closed unit disk $\bar{\mathbb{D}}$ for $0 < \alpha < 1$, so that $F(z) \in H^\alpha(\bar{\mathbb{D}})$.

If we only know that $\rho(\theta) \ge 0$ and (1) is true, the situation changes. Havin and Shamoyan [4] and, independently, L. Carleson and I. Yakobs proved that, for $0 < \alpha \le 1$, the nonrefinable relation $f \in H^{\alpha/2}(\mathbb{D})$ holds true. Later, Brennan [5] extended this theorem to the case where $1 < \alpha < 2$, and the present author [6] to the case where $\alpha > 0$ is arbitrary. If $\alpha/2 = n$ with $n \in \mathbb{N}$, then the relation $f \in H^n(\mathbb{D})$ means that $f^{(n-1)}$ belongs to the Zigmund class $Z(\mathbb{D})$ defined as follows: a continuous function φ on $\mathbb{D}$ lies in $Z(\mathbb{D})$ if

$$\left|\varphi(z_2) - 2\varphi\Big(\frac{z_1+z_2}{2}\Big) + \varphi(z_1)\right| \le c|z_2 - z_1|$$

for any z_1, $z_2 \in \bar{\mathbb{D}}$.

In the present paper we prove that, under stronger assumptions imposed on $\log\rho(\theta)$, the corresponding outer function will be more smooth.

Theorem. *Suppose that a 2π-periodic nonnegative function ρ satisfies condition* (1) *and belongs to H^α, $\alpha > 0$. Suppose also that* (a) $\log\rho \in L^p(0, 2\pi)$, $1 < p < \infty$. *Then for the function f_ρ constructed as in* (2) *we have $f_\rho \in H^\beta(\bar{\mathbb{D}})$ with $\beta = \frac{p}{p+1}\alpha$, and the exponent β cannot be improved. If* (b) $\log\rho \in \mathrm{BMO}$, *then $f_\rho \in H^\alpha(\bar{\mathbb{D}})$.*

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In case (b), for $0 < \alpha < 1$, the above claim was obtained earlier by Bomash in [7].

Recall that a function $h \in L^1_{\mathrm{loc}}(\mathbb{R})$ belongs to the class BMO if there exists a constant C such that for any interval $I \subset \mathbb{R}$ we have

$$\frac{1}{|I|}\int_I |h(x) - h_I|\,dm \le C,$$

where m denotes the Lebesgue measure on $\mathbb{R}$, $|I|$ is the length of I, and $h_I = \frac{1}{|I|}\int_I h\,dm$.

For the proof of the theorem we apply the criterion, proved in [6, Chapter 3], for an outer function to belong to $H^{\alpha_0}(\bar{\mathbb{D}})$.

For a point $z = re^{i\theta_0} \in \mathbb{D}$, we denote by $\gamma(z)$ the segment $\left[\theta_0 - \frac{1-r}{2},\, \theta_0 + \frac{1-r}{2}\right]$ and put $M_\rho(z) = \max_{\theta\in\gamma(z)} \rho(\theta)$.

Theorem A [6, Chapter 3]**.** *Suppose that $\rho \ge 0$, ρ is 2π-periodic, $\rho \in H^{\alpha_0}$, $\log\rho \in L^1_{\mathrm{loc}}(\mathbb{R})$, and there exist constants $A_{\alpha_0} > 0$ and $C_{\alpha_0} > 0$ such that for any point $z \in \mathbb{D}$ satisfying $M_\rho(z) \ge A_{\alpha_0}(1-r)^{\alpha_0}$ we have*

$$\int_0^{2\pi} \left|\log\frac{M_\rho(z)}{\rho(\theta)}\right| \frac{1-r^2}{|e^{i\theta}-z|^2}\,dm(\theta) \le C_{\alpha_0}. \tag{3}$$

Then $f_\rho \in H^{\alpha_0}(\bar{\mathbb{D}})$.

In [6] it was checked that the existence of a pair of constants $A^0_{\alpha_0}$, $C^0_{\alpha_0}$ satisfying (3) implies that for any constant A_{α_0} we can find a constant C_{α_0} so that (3) is fulfilled. Therefore, in the proof of the theorem we can use sufficiently large constants A_{α_0}, and $\alpha_0 = \beta$ in case (a) and $\alpha_0 = \alpha$ in case (b).

The following technical lemma (see [6, Chapter 2]) plays an important role in the proof.

Lemma. *Suppose $\rho \ge 0$, ρ is 2π-periodic, $\rho \in H^{\alpha_0}$, and $\log\rho \in L^1_{\mathrm{loc}}(\mathbb{R})$. For any closed segment $I \subset \mathbb{R}$, put $M_I(\rho) = \max_{\theta\in I}\rho(\theta)$. There exist constants A, C, independent of I and such that if $|I| \le 2\pi$ and $M_I(\rho) \ge A|I|^{\alpha_0}$, then*

$$\int_I \log\frac{M_I(\rho)}{\rho(\theta)}\,dm(\theta) \le C|I|. \tag{4}$$

For the reader's convenience, we present the proof of this claim. Let $n < \alpha_0 \le n+1$, where n is a nonnegative integer. Since $\frac{M_i(\rho)}{\rho(\theta)} = \frac{M_I(b\rho)}{b\rho(\theta)}$, there is no loss of generality in assuming that

$$|\rho^{(n)}(\theta_1) - \rho^{(n)}(\theta_2)| \le |\theta_1 - \theta_2|^{\alpha_0 - n}.$$

If $n = 0$, then, taking $A = 2$ and assuming that $M_I(\rho) = \rho(\theta_0)$, we have

$$\rho(\theta) \ge \rho(\theta_0) - |\theta - \theta_0|^{\alpha_0} \ge M_I(\rho) - \frac{1}{2}M_I(\rho) = \frac{1}{2}M_I(\rho),$$

which yields (4). Let $n \ge 1$; we put

$$E(h) = \{\theta \in I : \rho(\theta) \le h\}, \quad \nu(h) = mE(h). \tag{5}$$

Then

$$\int_I \log\frac{M_I(\rho)}{\rho(\theta)}\,dm = \int_0^{M_I(\rho)} \frac{\nu(h)}{h}\,dh. \tag{6}$$

Let $P(\theta)$ be the Lagrange interpolation polynomial for the function $\rho(\theta)$, with $P(\theta_j) = \rho(\theta_j)$, $\theta_j \in I$, $i \le j \le n$. By [8, Chapter 1], there exists a constant $\widetilde{A}_{\alpha_0}$ such that

$$|\rho(\theta) - P(\theta)| \le \widetilde{A}_{\alpha_0}|I|^{\alpha_0}. \tag{7}$$

For any fixed h, we can find $n+1$ points $\theta_0 < \theta_1 < \dots < \theta_n$, $\theta_j \in E(h)$, $0 \le j \le n$, such that $\theta_{j+1} - \theta_j \ge \frac{1}{n}(h)$, $0 \le j \le n-1$.

Let $P(\theta)$ be the Lagrange interpolation polynomial with nodes at these points θ_j; then

$$P(\theta)=\sum_{j=0}^{n}\rho(\theta_j)\frac{(\theta-\theta_0)\dots\widehat{(\theta-\theta_j)}\dots(\theta-\theta_n)}{(\theta_j-\theta_0)\dots(\theta_j-\theta_n)},$$

and for some constant A'_n we have

$$|P(\theta)|\le\sum_{j=0}^{n}h\frac{(\theta-\theta_0)\dots\widehat{(\theta-\theta_j)}\dots(\theta-\theta_n)}{(\theta_j-\theta_0)\dots(\theta_j-\theta_n)}\le A'_nh\frac{|I|^n}{(\nu(h))^n}. \tag{8}$$

Now we choose $A_{\alpha_0}=2\widetilde{A}_{\alpha_0}$; let $\rho(\theta_*)=M_I(\rho)$, $\theta_*\in I$. Using (7) and (8), we obtain

$$M_I(\rho)\rho(\theta_*)\le|\rho(\theta_*)-P(\theta_*)|+P(\theta_*)|\le\widetilde{A}_{\alpha_0}|I|^{\alpha_0}+A'_nh\frac{|I|^n}{(\nu(h))^n}\le\frac{1}{2}M_i(p)+A'_nh\frac{|I|^n}{(\nu(h))^n},$$

which implies that

$$\nu(h)\le A''_n(I)\cdot\Big(\frac{h}{M_I(\rho)}\Big)^{\frac{1}{n}}. \tag{9}$$

Relations (6) and (9) yield the estimate

$$\int_I\log\frac{M_i(\rho)}{\rho(\theta)}\,dm\le A''_n(I)\int_0^{M_I(\rho)}\Big(\frac{h}{M_I(\rho)}\Big)^{\frac{1}{n}}\frac{dh}{h}=C_n|I|,$$

which proves the lemma.

We pass to the proof of part (a) of the theorem. Put $\alpha_0=\beta=\frac{p}{p+1}\alpha$. Suppose that a point $z=re^{i\theta_0}\in\mathbb{D}$, $r\ge\frac{1}{2}$, satisfies the inequality $M_I(\rho)\ge\widetilde{A}_\beta|I|^\beta$, $I=\gamma(z)$, with a constant $\widetilde{A}_\beta$ to be defined below. Let J denote the segment $[\theta_0-\frac{1}{2}(1-r)^{\frac{p}{p+1}}$, $\theta_0+\frac{1}{2}(1-r)^{\frac{p}{p+1}}]$. Then $|J|=|I|^{\frac{p}{p+1}}$ and $M_I(\rho)\ge\widetilde{A}_\beta|J|^\alpha$. Let A_α be as in the lemma, and put $\widetilde{A}_\beta=4^\alpha A_\alpha$.

We find n_0 satisfying

$$\frac{1}{2}(1-r)^{\frac{p}{p+1}}\le 2^{n_0}(1-r)<(1-r)^{\frac{p}{p+1}},$$

and introduce the following segments I_n, $0\le n\le n_0$:

$$I_n=\Big[\theta_0-\frac{1}{2}\cdot 2^n(1-r),\theta_0+\frac{1}{2}\cdot 2^n(1-r)\Big].$$

Let $M_{I_n}=\max_{\theta\in I_n}\rho(\theta)$. Clearly, $M_{I_{n-1}}\le M_{I_n}$, $n\ge1$. Using the Lagrange interpolation polynomial as in the proof of the lemma, from the inequality $M_I\ge\widetilde{A}_\beta|J|^\alpha$ we deduce the estimate

$$M_{I_n}\le M_I+a\Big(\frac{I_n}{I_0}\Big)^\alpha M_I\le(1+2^na)M_I$$

with some constant $a=a(f,\alpha)$, whence

$$\frac{M_{I_n}}{M_I}\le C2^{n\alpha}. \tag{10}$$

Now we apply (10) and relation (6) from the lemma, which is possible by the choice of $\widetilde{A}_\beta$.

We find

$$\int_{I_0} \log \frac{M_I}{\rho(\theta)} \frac{1-r^2}{|e^{i\theta}-z|^2}\, dm(\theta) \le \frac{2}{1-r} \int_I \log \frac{M_I}{\rho(\theta)}\, d\theta \le \frac{2}{1-r} C_\alpha |I| = C'_\alpha, \tag{11}$$

$$\begin{aligned}
&\int_{I_n \setminus I_{n-1}} \left| \log \frac{M_I}{\rho(\theta)} \right| \frac{1-r^2}{|e^{i\theta}-z|^2}\, dm(\theta) \\
&\quad \le \int_{I_n \setminus I_{n-1}} \log \frac{M_{I_n}}{\rho(\theta)} \cdot \frac{1-r^2}{|e^{i\theta}-z|^2}\, dm + \log \frac{M_{I_n}}{M_I} \int_{I_n \setminus I_{n-1}} \log \frac{1-r^2}{|e^{i\theta}-z|^2}\, dm \qquad (12) \\
&\quad \le \frac{10}{1-r} \cdot 2^{-2n} \int_{I_n \setminus I_{n-1}} \log \frac{M_{I_n}}{\rho(\theta)}\, dm + \frac{10}{1-r} \cdot 2^{-2n} \int_{I_n \setminus I_{n-1}} \log(c 2^{n\alpha})\, dm \le c''_\alpha \frac{n}{2^n}.
\end{aligned}$$

Now (11) and (12) imply the estimate

$$\int_{I_{n_0}} \left| \log \frac{M_I}{\rho(\theta)} \right| \frac{1-r^2}{|e^{i\theta}-z|^2}\, dm = \int_{I_0} + \int_{I_1 \setminus I_0} + \dots + \int_{I_{n_0} \setminus I_{n_0 - 1}} \le c'_\alpha + c''_\alpha \sum_{n=1}^{n_0} \frac{n}{2^n} = \widetilde{c}_\alpha. \tag{13}$$

Next, the inequality $M_i \ge \widetilde{A}_\beta (1-r)^\beta$ yields

$$(1-r)^{\frac{1}{p+1}} \log M_I \le A'', \quad \frac{1}{2} \le r < 1.$$

Therefore, by the Hölder inequality, we have

$$\begin{aligned}
&\int_{[\theta_0-\pi, \theta_0+\pi] \setminus I_{n_0}} \left| \log \frac{M_I}{\rho(\theta)} \right| \frac{1-r^2}{|e^{i\theta}-z|^2}\, dm \\
&\quad \le \int_{[\theta_0-\pi, \theta_0+\pi] \setminus I_{n_0}} |\log M_I| \cdot \frac{1-r^2}{|e^{i\theta}-z|^2}\, d\theta + \int_{[\theta_0-\pi, \theta_0+\pi]} \left| \log \rho(\theta) \right| \frac{1-r^2}{|e^{i\theta}-z|^2}\, d\theta \\
&\quad \le c |\log M_I| \cdot (1-r)^{\frac{1}{p+1}} \qquad (14) \\
&\qquad + 2(1-r) \left(\int_{\theta_0-\pi}^{\theta_0+\pi} |\log \rho(\theta)|^p\, dm \right)^{\frac{1}{p}} \left(\int_{[\theta_0-\pi, \theta_0+\pi] \setminus I_{n_0}} \frac{dm}{|e^{i\theta}-z|^{2p'}} \right)^{\frac{1}{p'}} \\
&\quad \le cA' + c(1-r) \cdot (1-r)^{-1} = c'',
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$, because

$$\left(\int_{[\theta_0-\pi, \theta_0+\pi] \setminus I_{n_0}} \frac{dm}{|e^{i\theta}-z|^{2p'}} \right)^{\frac{1}{p'}} \le c \Big[\big((1-r)^{\frac{p}{p+1}} \big)^{-2p'+1} \Big]^{\frac{1}{p'}} = c(1-r)^{-1}.$$

To prove part (b), we put $\alpha_0 = \alpha$ and take the constant A_α as in the lemma. Let N_I be such that

$$\log N_I = \frac{1}{|I|} \int_I \log \rho\, dm, \tag{15}$$

where $I = \gamma(z)$; suppose that for the point Z we have $M_I = M_I(\rho) \ge A_\alpha |I|^\alpha$. Then we can apply the lemma to obtain

$$|\log N_I - \log M_I| = \frac{1}{|I|} \left| \int_I \log \frac{\rho(\theta)}{M_I}\, d\theta \right| \le C_\alpha.$$

Therefore,

$$\left| \log \frac{M_I}{\rho(\theta)} \right| \le \left| \log \frac{N_I}{\rho(\theta)} \right| + \left| \log \frac{M_I}{N_I} \right| \le \left| \log \frac{N_I}{\rho(\theta)} \right| + c_\alpha,$$

whence

$$\int \left| \log \frac{M_I}{\rho(\theta)} \right| \frac{1-r^2}{|e^{i\theta}-z|^2}\, d\theta \le \int_{[\theta_0-\pi, \theta_0+\pi]} |\log N_I - \log \rho(\theta)| \frac{1-r^2}{|e^{i\theta}-z|^2}\, dm + 2\pi c_\alpha.$$

Applying a property of the functions in BMO (see [9, Chapter 6]) to the function $\log \rho \in \mathrm{BMO}$, we get

$$\int_{[\theta_0-\pi,\theta_0+\pi]} |\log N_I - \log \rho(\theta)| \frac{1-r^2}{|e^{i\theta}-r|^2}\, dm \le \widetilde{c}, \tag{16}$$

with $\widetilde{c}$ independent of I. To complete the proof of part (b) of the theorem, now it suffices to combine relations (15), (16) and Theorem A.

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