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# A THIRD ORDER OPERATOR WITH PERIODIC COEFFICIENTS ON THE REAL LINE

A. V. BADANIN AND E. L. KOROTYAEV

Abstract. The operator $i\partial^3+i\partial p+ip\partial+q$ with 1-periodic coefficients $p,q\in L^1_{\mathrm{loc}}(\mathbb{R})$ is considered on the real line. The following results are obtained: 1) the spectrum of this operator is absolutely continuous, covers the entire real line, and has multiplicity one or three; 2) the spectrum of multiplicity three is bounded and expressed in terms of real zeros of a certain entire function; 3) the Lyapunov function, analytic on a 3-sheeted Riemann surface, is constructed and investigated.

## §1. Introduction and main results

**1.1. The operator $H$.** In the Hilbert space $L^2(\mathbb{R})$, we consider a third order differential operator

$$H=i\partial^3+i\partial p+ip\partial+q, \tag{1.1}$$

where the real 1-periodic coefficients $p$ and $q$ belong to $L^1(\mathbb{T})$, $\mathbb{T}=\mathbb{R}/\mathbb{Z}$. The domain of $H$ is the set

$$\mathscr{D}(H)=\big\{f\in L^2(\mathbb{R}):i(f''+pf)'+ipf'+qf\in L^2(\mathbb{R}),f'',(f''+pf)'\in L^1_{\mathrm{loc}}(\mathbb{R})\big\}. \tag{1.2}$$

This operator is selfadjoint (see Lemma 5.1, iii). The operator $H$ occurs in the inverse problem method of integration for the nonlinear evolution Boussinesq equation on the circle (“bad Boussinesq”, see [29]):

$$\ddot p=\partial^2\Big(\frac43p^2+\frac13\partial^2p\Big),\quad \dot p=\partial q. \tag{1.3}$$

Namely, for the Boussinesq equation (1.3) there exists an $L$-$A$ Lax pair, where the $L$-operator in this pair is equal to $H$.

Note that the “good Boussinesq” equation (with the sign “$-$” on the right-hand side in (1.3)) leads to the nonselfadjoint operator $-\partial^3+\partial p+p\partial+q$. Such an operator was considered in the paper [29] by McKean in the case of smooth periodic $p$ and $q$. In [29], in the case of sufficiently small $p$ and $q$ and a nonselfadjoint operator, deep results concerning the inverse spectral problem were obtained among other things. At the same time, in McKean's opinion, the spectral analysis for the selfadjoint operator is, in a sense, much more complicated than that in the nonselfadjoint case, and it was not considered.

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**1.2. The main results of the paper.** Our prime goal in the present paper is to describe the spectrum in terms of the Lyapunov function given on a 3-sheeted Riemann surface. Moreover, it turns out that, as in the case of the Hill operator, to describe the spectrum it suffices to know one entire function $\rho(\lambda)$, equal to the discriminant of the characteristic polynomial of the monodromy matrix (see (1.9)). We obtain the following results.

1) The values of the three branches of the Lyapunov function on the real axis determine the spectrum in the same way as in the case of the fourth order operator and periodic systems (Theorem 5.3).

2) The spectrum covers the real line and has multiplicity 1 or 3. The spectrum of multiplicity 3 consists of a finite number of nondegenerate intervals and coincides with the subset of the real line on which the function $\rho$ is nonpositive.

3) The endpoints of the spectral intervals with spectrum of multiplicity 3 are branch points of the Lyapunov function and, simultaneously, are zeros of the function $\rho$.

4) We find the asymptotics of the Lyapunov function as $|\lambda| \to \infty$ (Proposition 3.4).

**1.3. The main theorem.** Consider the equation

$$i(y'' + py)' + ipy' + qy = \lambda y, \qquad \lambda \in \mathbb{C}. \tag{1.4}$$

Generally speaking, under the condition $p, q \in L^1_{\mathrm{loc}}(\mathbb{R})$ the second derivative $y''$ of a solution $y$ of this equation may fail to be well defined. For this reason, we cannot introduce the fundamental matrix in the usual way. We introduce the modified fundamental matrix by the formula

$$M(t,\lambda) = \begin{pmatrix} \varphi_1 & \varphi_2 & \varphi_3 \\ \varphi_1' & \varphi_2' & \varphi_3' \\ \varphi_1'' + p\varphi_1 & \varphi_2'' + p\varphi_2 & \varphi_3'' + p\varphi_3 \end{pmatrix}(t,\lambda), \qquad (t,\lambda) \in \mathbb{R} \times \mathbb{C}. \tag{1.5}$$

Here, $\varphi_1$, $\varphi_2$, and $\varphi_3$ are fundamental solutions of equation (1.4) satisfying the condition

$$M(0,\lambda) = \mathbb{1}_3, \qquad \lambda \in \mathbb{C}, \tag{1.6}$$

where $\mathbb{1}_3$ is the identity $(3 \times 3)$-matrix. We call the matrix $M(1,\lambda)$ the *modified monodromy matrix*. Its characteristic polynomial has the form

$$D(\tau,\lambda) = \det(M(1,\lambda) - \tau \mathbb{1}_3), \qquad (\tau,\lambda) \in \mathbb{C}^2. \tag{1.7}$$

An eigenvalue of the matrix $M(1,\lambda)$ is called a *multiplier*. Each $(3\times 3)$-matrix $M(1,\lambda)$, $\lambda \in \mathbb{C}$, has precisely 3 (counted with multiplicities) mutipliers $\tau_1(\lambda)$, $\tau_2(\lambda)$, and $\tau_3(\lambda)$. In Proposition 3.4, we show that the $\tau_j$ are three branches of one and the same function $\tau$ analytic on a certain connected 3-sheeted Riemann surface.

Introduce the Lyapunov functions

$$\Delta_j = \frac{1}{2}(\tau_j + \tau_j^{-1}), \qquad j = 1, 2, 3. \tag{1.8}$$

These functions are branches of a function $\Delta$ analytic on a certain connected 3-sheeted Riemann surface $\mathscr{R}$. Note that in our case, contrary to the case of periodic systems (see [7]), the surface $\mathscr{R}$ does not split into connected components.

We define the *discriminant* $\rho(\lambda)$, $\lambda \in \mathbb{C}$, of the polynomial $D(\,\cdot\,,\lambda)$ by the relation

$$\rho = (\tau_1 - \tau_2)^2 (\tau_1 - \tau_3)^2 (\tau_2 - \tau_3)^2. \tag{1.9}$$

The zeros of the function $\rho$ determine the branch points of the function $\tau$ and, thus, the branch points of the Lyapunov function (see the remark to Proposition 3.4).

**Theorem 1.1.** i) *The function $\rho$ is entire, $\rho$ is real on $\mathbb{R}$, and for all $\lambda \in \mathbb{R}$ we have*

$$\rho(\lambda) = |T(\lambda)|^4 - 8 \operatorname{Re} T^3(\lambda) + 18|T(\lambda)|^2 - 27, \tag{1.10}$$

*where*

$$T(\lambda) = \operatorname{Tr} M(1, \lambda), \qquad \lambda \in \mathbb{C}.$$

ii) *The spectrum $\sigma(H)$ of the operator $H$ is absolutely continuous and covers the entire real line.*

iii) *The spectrum $\sigma(H)$ has multiplicity* 1 *or* 3; *the spectrum $\mathfrak{S}_3$ of multiplicity* 3 *is bounded, consists of a finite number of nondegenerate intervals, and is equal to*

$$\mathfrak{S}_3 = \{\lambda \in \mathbb{R} : \rho(\lambda) \leq 0\}. \tag{1.11}$$

*Remark* 1.2. 1) The proof of the absolute continuity of the spectrum is based on the decomposition of the operator $H$ into the direct integral of operators $H(k)$ and on the analyticity of the eigenvalues of the operators $H(k)$ with respect to $k$. Largely, our proof follows a standard line of arguments (see [12, 30, 34]), but demands considerable modifications caused by the nonsmoothness of the coefficients (see Remarks 3.1 and 4.1).

2) Relation (1.11) shows that the multiplicity of the spectrum of $H$ is fully determined by the values of the entire function $\rho$. For the fourth order differential operator, we can also introduce the discriminant of the characteristic polynomial for the monodromy matrix. But this does not provide complete information about the spectrum. The discriminant describes well the spectrum of multiplicity 2, but it does not distinguish the spectrum of multiplicity 4 from lacunas.

3) The endpoints of the spectral intervals with spectrum of multiplicity 3 are zeros of the function $\rho$, and $\rho$ changes its sign at these points. For this reason, they are branch points of the Lyapunov function. The periodic and antiperiodic eigenvalues are zeros of one of the functions $\Delta_j^2 - 1$ (see the remark to Lemma 4.3) and, generally speaking, are in no way related to the multiplicity of the spectrum.

4) All the zeros of the function $\rho$ except finitely many are nonreal and have the asymptotics $i\big(\frac{2\pi n}{\sqrt{3}}\big)^3(1 + O(n^{-2}))$ as $n \to \pm\infty$, and the eigenvalues of the periodic and antiperiodic problems are all real and have the asymptotics $(\pi n)^3(1+O(n^{-2}))$ as $n \to \pm\infty$ (see [4]).

5) For a nonselfadjoint differential third-order operator with smooth coefficients, the function $\rho$ was analyzed by McKean in [29]. In this case, all the zeros of $\rho$, except for a finite number, are real and are the spectral data for the inverse problem. This simplifies the analysis of the problem. In his paper, McKean considered the inverse problem only in the case of small coefficients, because in this case the function $\rho$ has at most 2 nonreal zeros.

6) The Lyapunov function, the function $\rho$, and the spectrum $\mathfrak{S}_3$ of multiplicity 3 are sketched in Figure 1.

**1.4. Review of the literature.** Many papers were devoted to direct and inverse spectral problems for the Schrödinger operator $-y'' + qy$ on $L^2(\mathbb{R})$ with a periodic potential $q$: J. Garnett and E. Trubowitz [16], B. A. Dubrovin [8], A. P. Its and N. B. Matveev [18], P. Kargaev and E. Korotyaev [19], V. A. Marchenko and I. V. Ostrovskiĭ [28], etc. Korotyaev [23] extended the results of [28, 16, 19] to the case of periodic distributions, i.e., to the equation $-y'' + q'y$ with $q \in L^2_{\mathrm{loc}}(\mathbb{R})$. The spectral problem for periodic systems is much more complicated than that for the scalar operator, and there are many papers devoted only to the direct spectral problem for periodic systems of the first and second order: I. M. Gelfand and V. B. Lidskiĭ [13], F. Gesztesy et al. [6], R. Carlson [5], E. Kortyaev and D. Chelkak [7], M. G. Krein [26], etc. In particular, the following results

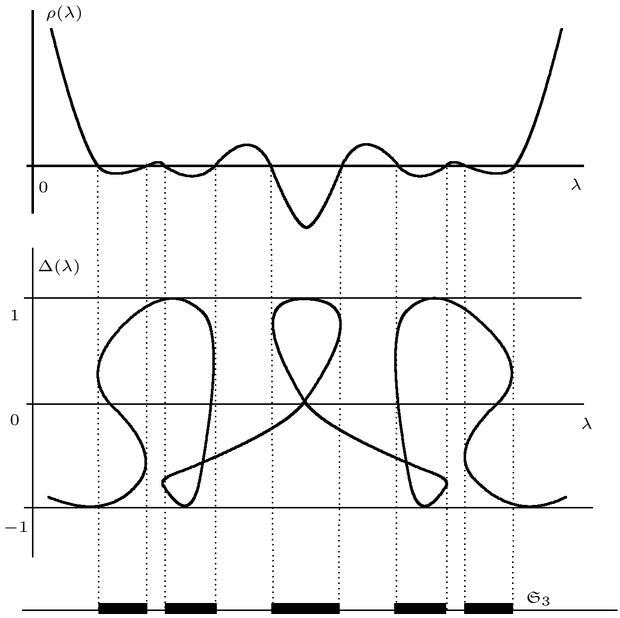


FIGURE 1. The function $\rho$, the Lyapunov function $\Delta$, and the spectrum $\mathfrak{S}_3$ of multiplicity 3.

were obtained in [7, 21, 22] for operators of the first and second order with periodic matrix-valued potentials:

1) the many-sheeted Lyapunov function was constructed and studied;

2) a conformal mapping generalizing the Marchenko–Ostrovskiĭ mapping [28], known in the theory of the Hill operator, was constructed, and the main properties of it were studied;

3) trace formulas (similar to those in the scalar case) were obtained;

4) estimates for the lengths of the spectral gaps were found in terms of the norm of the potential;

5) the asymptotics of eigenvalues of the periodic and antiperiodic problems and of branch points of the Lyapunov function were derived;

6) it was shown that the endpoints of the gaps are periodic or antiperiodic eigenvalues or are branch points of the Lyapunov function.

A similar analysis for periodic systems of difference equations was performed in the papers [24, 25] by Korotyaev and Kutsenko.

The spectral analysis of operators of order $\nu \geq 3$ with periodic coefficients is considerably more complicated than that for operators of the second order. This is caused by the fact that the monodromy matrix contains both elements like $\cos \lambda^{\frac{1}{\nu}}$, which are bounded for large real values of the spectral parameter, and increasing elements like $\cosh \lambda^{\frac{1}{\nu}}$. The conformal Marchenko–Ostrovskiĭ mapping has not yet been constructed for higher order operators. Operators of even ($\geq 4$) order with periodic coefficients were considered in recent papers: Badanin and Korotyaev [1, 2, 3], Papanicolaou [31, 32], Tkachenko [35] (see also the references in those papers).

The selfadjoint operator of odd order at least 3 with periodic coefficients is far less investigated. In the case of smooth coefficients, this operator was considered in the book of Dunford and Schwartz [9, Chapter XIII.7] and in the paper [30] by McGarvey, where it was shown that the spectrum is absolutely continuous and covers the real line. Most likely, no operators with nonsmooth coefficients have been considered so far.

In [4], where we substantially used the results obtained in the present paper, we found the asymptotics for the branch points of the Riemann surface and for eigenvalues of the periodic and antiperiodic problems.

In particular, in [4] it was shown that in the case of generic coefficients $p$ and $q$, the Riemann surface of the Lyapunov function has infinite genus. Moreover, the case of small $p$, $q \to 0$ was studied. We show that in this case the entire spectrum of the operator $H$ has multiplicity 0, with the possible exception of a small interval in a neighborhood of zero where the spectrum has multiplicity 3. We obtain explicit conditions on $p$ and $q$ under which such an interval is absent, as well as conditions under which it exists, and find its asymptotics.

**1.5. Description of the paper.** Since we are not aware of any results concerning the operators of odd order with nonsmooth periodic coefficients, in our paper we pay attention to the proof of the selfadjointness of the operator $H$ and the absolute continuity of the spectrum under the condition $p, q \in L^1_{\rm loc}(\mathbb{R})$.

Note that, in the case of smooth coefficients, the questions of selfadjointness and the absolute continuity of the spectrum do not arise. At the same time, a direct application of the methods of [9] and [30] to the case of nonsmooth coefficients meets some difficulties. In particular, we cannot use the monodromy matrix in the form given in those papers and in our paper in (3.1), because, generally speaking, the functions in the domain of the operator may fail to be twice differentiable. For this reason, we define a modified monodromy matrix. In the case of smooth coefficients, the monodromy matrix and the modified monodromy matrix are related to each other by a simple similarity transformation (see Theorem 3.2, iii), so that their eigenvalues and, thus, the Lyapunov functions coincide.

In §2, we study the properties of the monodromy matrix. In §3, we study the Lyapunov function and the discriminant of the characteristic polynomial of the monodromy matrix. §4 is devoted to the description of the decomposition of the operator $H$ into a direct integral of the operators $H(k)$ in a layer. In §5, we describe the spectrum of the operator $H$ and prove Theorem 1.1.

## §2. Estimates for the monodromy matrix

In this section we obtain estimates for the monodromy matrix $M(1,\lambda)$. We write equation (1.4) in the vector form:

$$Y' - P(\lambda)Y = Q(t)Y, \qquad (t,\lambda) \in \mathbb{R}\times\mathbb{C}, \tag{2.1}$$

where the vector-valued function $Y$ and the $(3\times 3)$-matrix-valued functions $P$ and $Q$ have the form

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} y \\ y' \\ y'' + py \end{pmatrix}, \qquad P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -i\lambda & 0 & 0 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & 0 & 0 \\ -p & 0 & 0 \\ iq & -p & 0 \end{pmatrix}. \tag{2.2}$$

Note that the $(3\times 3)$-matrix-valued function $M(t,\lambda)$ given by (1.5) is a solution of the initial problem

$$M' - P(\lambda)M = Q(t)M, \qquad M(0,\lambda) = \mathbb{1}_3. \tag{2.3}$$

In the nonperturbed case where $p = q = 0$, the solution $M_0$ of problem (2.3) is of the form $M_0 = e^{tP(\lambda)}$. The function $M_0(t,\lambda)$ is entire in $\lambda$ for each $t \in \mathbb{R}$. The eigenvalues of the matrix $P(\lambda)$ are equal to $iz\omega^j$, $j=0,1,2$, where

$$\omega = e^{i\frac{2\pi}{3}}, \qquad z = x+iy = \lambda^{\frac13}, \qquad \arg\lambda \in \Big(-\frac{\pi}{2}, 3\frac{\pi}{2}\Big], \qquad \arg z \in \Big(-\frac{\pi}{6}, \frac{\pi}{2}\Big]. \tag{2.4}$$

The matrix $M_0(t,\lambda)$ has eigenvalues $e^{iz\omega^j t}$, $j=0,1,2$. From condition (2.4) we get $x \ge \max\{0, -y\sqrt{3}\}$. Then the relations

$$\operatorname{Re}(iz) = -y, \qquad \operatorname{Re}(iz\omega) = \frac{y-\sqrt{3}x}{2}, \qquad \operatorname{Re}(iz\omega^2) = \frac{y+\sqrt{3}x}{2}$$

imply that, for all $\lambda \in \mathbb{C}$,

$$\max\{\operatorname{Re}(iz), \operatorname{Re}(iz\omega)\} \le z_0 = \operatorname{Re}(iz\omega^2). \tag{2.5}$$

The estimates $|e^{iz\omega^j t}| \le e^{z_0|t|}$ yield

$$|M_0(t,\lambda)| \le e^{z_0|t|} \qquad \text{for all} \qquad (t,\lambda) \in \mathbb{R}\times\mathbb{C}. \tag{2.6}$$

Henceforth we use the following *norm* for a matrix $A$:

$$|A| = \max\{\sqrt{h} \, : \, h \ \text{ is an eigenvalue of the matrix } \ A^*A\}.$$

For $\lambda \ne 0$, the relation

$$P = (\mathcal{Z}U)(izB)(\mathcal{Z}U)^{-1} \tag{2.7}$$

is valid, where

$$U = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ \mathbb{1} & \omega & \omega^2 \\ \mathbb{1} & \omega^2 & \omega \end{pmatrix} = (U^*)^{-1}, \ \ B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \ \ \mathcal{Z} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & iz & 0 \\ 0 & 0 & (iz)^2 \end{pmatrix}. \tag{2.8}$$

Applying the similarity transformation (2.7) to the two sides of (2.3), we get

$$\mathcal{M}' - izB\mathcal{M} = \mathcal{Q}(t,\lambda)\mathcal{M}, \qquad \mathcal{M}(0,\lambda) = \mathbb{1}_3, \tag{2.9}$$

where

$$\mathcal{M} = (\mathcal{Z}U)^{-1}M(\mathcal{Z}U), \qquad \mathcal{Q} = (\mathcal{Z}U)^{-1}Q(\mathcal{Z}U) = \frac{1}{iz}\, U^{-1}\begin{pmatrix} 0 & 0 & 0 \\ -p & 0 & 0 \\ \frac{q}{z} & -p & 0 \end{pmatrix} U. \tag{2.10}$$

Equation (2.9) is convenient for estimating of the monodromy matrix for large $|\lambda|$, because the coefficient on the left-hand side is a diagonal matrix constant in $t$, and the coefficient on the right-hand side decays as $|\lambda| \to \infty$.

Recall that $T = \operatorname{Tr}\mathcal{M}(1,\cdot)$, and put $T_0 = \operatorname{Tr}\mathcal{M}_0(1,\cdot)$ in the nonpertubed case.

**Lemma 2.1.** *The matrix-valued function $M(1,\cdot)$ is entire. We have*

$$|T(\lambda)| \le 3e^{z_0+\varkappa} \quad \text{for all} \quad \lambda \in \mathbb{C}, \tag{2.11}$$

$$|\mathcal{M}(1,\lambda) - e^{izB}| \le \frac{\varkappa}{|z|}e^{z_0+\varkappa}, \tag{2.12}$$

$$|T(\lambda) - T_0(\lambda)| \le \frac{3\varkappa}{|z|}e^{z_0+\varkappa} \tag{2.13}$$

*for all $|\lambda| \ge 1$, where $\varkappa = \int_0^1(|p(t)|+|q(t)|)\,dt$ and $z_0 = \operatorname{Re}(iz\omega^2)$.*

*Proof.* From (2.3) it follows that the function $M(t,\lambda)$ satisfies the integral equation

$$M(t,\lambda) = M_0(t,\lambda) + \int_0^t M_0(t-s,\lambda)Q(s)M(s,\lambda)\,ds. \tag{2.14}$$

Iterating, we get

$$(2.15)\qquad M(t,\lambda)=\sum_{n\geq 0}M_n(t,\lambda),\qquad M_n(t,\lambda)=\int_0^t M_0(t-s,\lambda)Q(s)M_{n-1}(s,\lambda)\,ds.$$

From (2.15) it follows that
(2.16)

$$M_n(t,\lambda)=\int_{0<t_1<\dots<t_n<t_{n+1}=t}\prod_{k=1}^{n}\big(M_0(t_{k+1}-t_k,\lambda)Q(t_k)\big)M_0(t_1,\lambda)\,dt_1\,dt_2\dots dt_n,$$

$(t,\lambda)\in\mathbb{R}_+\times\mathbb{C}$, the factors in the product are ordered from right to left. Substituting estimates (2.6) in (2.16), we see that

$$(2.17)\qquad |M_n(t,\lambda)|\leq\frac{e^{z_0t}}{n!}\left(\int_0^t|Q(s)|\,ds\right)^n\qquad\text{for all}\qquad(n,t,\lambda)\in\mathbb{N}\times\mathbb{R}_+\times\mathbb{C}.$$

These estimates show that the formal series (2.15) converges absolutely and uniformly on any bounded subset in $\mathbb{R}_+\times\mathbb{C}$. Each term of this series is an entire function of the variable $\lambda$. The sum also possesses this property. Inequalities (2.17) and $|Q|\leq|p|+|q|$ provide the estimates

$$|\operatorname{Tr}M_n(1,\lambda)|\leq 3|M_n(1,\lambda)|\leq\frac{3\varkappa^n}{n!}e^{z_0}\quad\text{for all}\quad(n,t,\lambda)\in\mathbb{N}\times\mathbb{R}_+\times\mathbb{C}.$$

This implies that

$$|\operatorname{Tr}M(1,\lambda)|=\left|\sum_0^\infty\operatorname{Tr}M_n(1,\lambda)\right|\leq 3e^{z_0}\sum_0^\infty\frac{\varkappa^n}{n!},$$

which yields (2.11).

The solutions $\mathcal{M}(t,\lambda)$ of problem (2.9) satisfy the integral equation

$$\mathcal{M}(t,\lambda)=e^{iztB}+\int_0^t e^{iz(t-s)B}\mathcal{Q}(s)\mathcal{M}(s,\lambda)\,ds,$$

whence

$$(2.18)\qquad \mathcal{M}(t,\lambda)=\sum_{n\geq 0}\mathcal{M}_n(t,\lambda),\qquad \mathcal{M}_n(t,\lambda)=\int_0^t e^{iz(t-s)B}\mathcal{Q}(s)\mathcal{M}_{n-1}(s,\lambda)\,ds.$$

From (2.18) we get

$$\mathcal{M}_n(t,\lambda)=\int_{0<t_1<\dots<t_n<t_{n+1}=t}\prod_{k=1}^{n}\big(e^{iz(t_{k+1}-t_k)B}\mathcal{Q}(t_k)\big)e^{izt_1B}\,dt_1\,dt_2\dots dt_n,$$

whence

$$|\mathcal{M}_n(t,\lambda)|\leq\frac{e^{z_0t}}{n!}\left(\int_0^t|\mathcal{Q}(s)|\,ds\right)^n\qquad\text{for all}\qquad(n,t,\lambda)\in\mathbb{N}\times\mathbb{R}_+\times\mathbb{C}.$$

These estimates show that the formal series (2.18) converges absolutely and uniformly on any bounded subset of $\mathbb{R}_+\times\mathbb{C}$. By (2.10), the estimate $\int_0^1|\mathcal{Q}|\,ds\leq\frac{\varkappa}{|z|}$ is true for all $|z|\geq 1$. Summing the majorants and using this estimate, we arrive at inequalities (2.12) and (2.13). $\square$

## §3. Analysis of the monodromy matrix

In this section we analyze the monodromy matrix and functions associated with it, namely, multipliers, the Lyapunov function, and the discriminant.

We begin with the following remark.

*Remark* 3.1. In the case where the coefficients $p$, $p'$, and $q$ belong to $L^1(\mathbb{T})$, we define the *fundamental matrix* for equation (1.4):

$$\widetilde{\mathcal{M}}(t,\lambda) = (\widetilde{\varphi}_j^{(k-1)}(t,\lambda))_{k,j=1}^3, \qquad (t,\lambda) \in \mathbb{R}\times\mathbb{C},$$

where the $\widetilde{\varphi}_j$ are solutions of equation (1.4) satisfying the initial conditions $\widetilde{\mathcal{M}}(0,\lambda) = \mathbb{1}_3$, and we introduce the *monodromy matrix*

$$\widetilde{M}(\lambda) = \widetilde{\mathcal{M}}(1,\lambda) \tag{3.1}$$

(for example, see [9, Chapter XIII.7]). If $p \in L^1(\mathbb{T})$ and $p' \notin L^1(\mathbb{T})$, then the monodromy matrix is not well defined, because the derivative $y''$ may fail to exist. For this reason, a modification is needed, and we define the modified monodromy matrix $M(1,\lambda)$ by formula (1.5).

**Theorem 3.2.** i) *The matrix-valued function* $M(1,\cdot)$ *defined by formula* (1.5) *is entire, and for all* $\tau$, $\lambda \in \mathbb{C}$ *we have*:

$$M^*(1,\bar\lambda)JM(1,\lambda) = J, \quad \textit{where} \quad J = \begin{pmatrix} 0 & 0 & i \\ 0 & -i & 0 \\ i & 0 & 0 \end{pmatrix}, \tag{3.2}$$

$$D(\tau,\lambda) = \det(M(1,\lambda) - \tau\mathbb{1}_3) = -\tau^3 + \tau^2 T(\lambda) - \tau\bar{T}(\bar\lambda) + 1, \tag{3.3}$$

$$\det M(1,\lambda) = 1. \tag{3.4}$$

ii) *Let* $\lambda \in \mathbb{R}$. *If* $\tau(\lambda)$ *is a multiplier, then* $\bar\tau^{-1}(\lambda)$ *is also a multiplier. Only two cases may occur*:

a) *all three multipliers lie on the unit circle*;

b) *exactly one* (*simple*) *multiplier lies on the unit circle.*

*Moreover, in case* b) *the multipliers take the form*

$$e^{ik}, \quad e^{i\bar k}, \quad e^{-i2\operatorname{Re} k} \quad \textit{for some} \quad k \in \mathbb{C} \quad \textit{with} \quad \operatorname{Im} k \neq 0. \tag{3.5}$$

iii) *Let* $p$, $p'$, $q \in L^1(\mathbb{T})$. *Then the modified monodromy matrix* $M$ *and the standard monodromy matrix* $\widetilde{M}$ *are related by the formula*

$$M(1,\cdot) = \mathcal{S}^{-1}\widetilde{M}(\cdot)\mathcal{S}, \qquad \textit{where} \qquad \mathcal{S} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -p(0) & 0 & 1 \end{pmatrix}. \tag{3.6}$$

*Proof.* i) From (2.3) it follows that $JM' = VM$, where

$$V = J(P+Q) = i\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ \mathbb{1} & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ -p & 0 & 1 \\ -i\lambda + iq & -p & 0 \end{pmatrix} = \begin{pmatrix} \lambda - q & -ip & 0 \\ ip & 0 & -i \\ 0 & i & 0 \end{pmatrix},$$

and $J$ is defined by (3.2). Then $-(M^*)'J = M^*V$ for $\lambda \in \mathbb{R}$ and

$$(M^*JM)' = (M^*)'JM + M^*JM' = -M^*VM + M^*VM = 0,$$

which implies $(M^*JM)(t,\lambda) = (M^*JM)(0,\lambda) = J$ for all $(t,\lambda) \in \mathbb{R}\times\mathbb{C}$, so that (3.2) is true. Relation (3.4) follows from the Liouville formula. Next, immediate calculations show that

$$D(\tau,\lambda) = \det(M(1,\lambda) - \tau\mathbb{1}_3) = -\tau^3 + \tau^2 \operatorname{Tr} M(1,\lambda) + B(\lambda)\tau - 1 \text{ for all } (\tau,\lambda) \in \mathbb{C}^2,$$

where $B(\lambda) = \partial_\tau D(0, \lambda)$. A formula known from the matrix theory (see, e.g., [15, IV.1.3]) shows that

$$\partial_\tau D(\tau, \lambda) = -D(\tau, \lambda) \operatorname{Tr}(M(1, \lambda) - \tau \mathbb{1}_3)^{-1}.$$

Using relation $D(0, \lambda) = 1$, we obtain $B(\lambda) = -\operatorname{Tr} M^{-1}(1, \lambda)$. Relation (3.2) gives $M^{-1}(1, \lambda) = -JM^*(1, \bar{\lambda})J$, which implies that $\operatorname{Tr} M^{-1}(1, \lambda) = \operatorname{Tr} M^*(1, \bar{\lambda})$ for all $\lambda \in \mathbb{C}$. Then $B(\lambda) = -\operatorname{Tr} M^*(1, \bar{\lambda})$, and (3.3) follows.

ii) From relation (3.3) we have

$$D(\tau, \lambda) = -\tau^3 \bar{D}(\bar{\tau}^{-1}, \lambda) \qquad \text{for all} \qquad (\tau, \lambda) \in \mathbb{C} \times \mathbb{R}, \qquad \tau \neq 0. \tag{3.7}$$

Thus, if $\tau(\lambda)$ is a root of $D(\tau, \lambda)$ for some $\lambda \in \mathbb{R}$, then $\bar{\tau}^{-1}(\lambda)$ is also a root. Since $\tau_1 \tau_2 \tau_3 = \det M(1, \cdot) = 1$, we obtain the required statements.

iii) Since the $\widetilde{\varphi}_j$, $j = 1, 2, 3$, are fundamental solutions of equation (1.4) and the $\varphi_j$ are also its solutions, all the $\varphi_j$ are linear combinations of $\widetilde{\varphi}_j$. The initial conditions (1.6) yield

$$\varphi_1(t, \lambda) = \widetilde{\varphi}_1(t, \lambda) - p(0)\widetilde{\varphi}_3(t, \lambda), \qquad \varphi_2(t, \lambda) = \widetilde{\varphi}_2(t, \lambda), \qquad \varphi_3(t, \lambda) = \widetilde{\varphi}_3(t, \lambda),$$

for all $(t, \lambda) \in \mathbb{R} \times \mathbb{C}$, whence we obtain

$$\begin{pmatrix} \varphi_1 & \varphi_2 & \varphi_3 \\ \varphi_1' & \varphi_2' & \varphi_3' \\ \varphi_1'' & \varphi_2'' & \varphi_3'' \end{pmatrix} (t, \lambda) = \begin{pmatrix} \widetilde{\varphi}_1 - p(0)\widetilde{\varphi}_3 & \widetilde{\varphi}_2 & \widetilde{\varphi}_3 \\ \widetilde{\varphi}_1' - p(0)\widetilde{\varphi}_3' & \widetilde{\varphi}_2' & \widetilde{\varphi}_3' \\ \widetilde{\varphi}_1'' - p(0)\widetilde{\varphi}_3'' & \widetilde{\varphi}_2'' & \widetilde{\varphi}_3'' \end{pmatrix} (t, \lambda) = \widetilde{\mathcal{M}}(t, \lambda)\mathcal{S}, \quad (t, \lambda) \in \mathbb{R} \times \mathbb{C}. \tag{3.8}$$

Since $\mathcal{S}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \psi(0) & 0 & 1 \end{pmatrix}$, relation (1.5) yields

$$M(t, \lambda) = \mathcal{S}^{-1} \begin{pmatrix} \varphi_1 & \varphi_2 & \varphi_3 \\ \varphi_1' & \varphi_2' & \varphi_3' \\ \varphi_1'' & \varphi_2'' & \varphi_3'' \end{pmatrix} (t, \lambda) \quad \text{for all} \quad (t, \lambda) \in \mathbb{R} \times \mathbb{C}. \tag{3.9}$$

Formulas (3.8) and (3.9) imply (3.6). □

*Remark* 3.3. 1) Relation (3.6) shows that for smooth coefficients $p$ and $q$, the eigenvalues of the matrices $M(1, \lambda)$ and $\widetilde{M}(\lambda)$ coincide, i.e., the multipliers can be determined as eigenvalues of any one of these matrices.

2) In our opinion, even in the case of smooth coefficients the modified matrix $M$ is more convenient for the analysis of the operator $H$ than the matrix $\widetilde{M}$. In particular, the symplecticity relation (3.2) for the matrix $M$ has a simpler form compared to the similar relation for $\widetilde{M}$.

3) A typical location of multipliers on the complex plane in the case of real $\lambda$ is shown in Figure 2.

The coefficients of the polynomial $D$ are entire functions of the variable $\lambda$. It is known (see, e.g., [11, Chapter 8]) that the roots $\tau_j(\lambda)$, $j = 1, 2, 3$, of $D$ form one or several branches of one or several analytic functions having only algebraic singularities in $\mathbb{C}$.

Consider the case where $p = q = 0$. The multipliers are of the form

$$\tau_j^0(\lambda) = e^{i\omega^{j-1} z}, \qquad \lambda \in \mathbb{C}, \qquad j = 1, 2, 3,$$

where

$$\omega = e^{i\frac{2\pi}{3}}, \qquad z = \lambda^{\frac{1}{3}}, \qquad \arg \lambda \in \Big(-\frac{\pi}{2}, \frac{3\pi}{2}\Big], \qquad \arg z \in \Big(-\frac{\pi}{6}, \frac{\pi}{2}\Big].$$

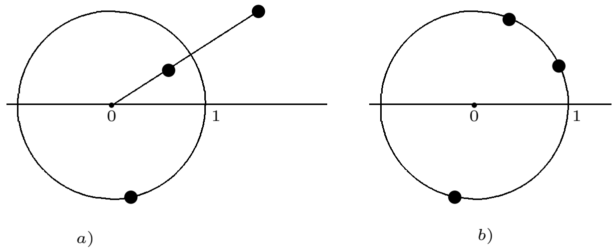


FIGURE 2. Typical location of multipliers on the complex plane in the case of real $\lambda$ (the multipliers are marked with black circles): a) $\lambda \notin \mathfrak{S}_3$; b) $\lambda \in \mathfrak{S}_3$.

The Lyapunov function $\Delta^0$, its branches $\Delta_j^0$, and the discriminant $\rho^0$ are as follows:

$$\Delta^0 = \cos \lambda^{\frac{1}{3}}, \qquad \Delta_j^0 = \cos z\omega^{j-1}, \qquad j = 1, 2, 3,$$
$$\rho^0 = 64 \sinh^2 \frac{\sqrt{3}z}{2} \sinh^2 \frac{\sqrt{3}\omega z}{2} \sinh^2 \frac{\sqrt{3}\omega^2 z}{2}.$$

The asymptotic behavior of the fundamental solutions for $t = 1$ for the second order operator is expressed in terms of the functions $\cos\sqrt{\lambda}$ and $\sin\sqrt{\lambda}$, which are bounded on the real axis. The asymptotics of the fundamental solutions for the operators of order at least 3 is expressed in terms of the functions $e^{i\lambda^{\frac{1}{3}}}, e^{i\omega\lambda^{\frac{1}{3}}}, e^{i\omega^2\lambda^{\frac{1}{3}}}$, unbounded on the real axis. This gives rise to certain difficulties in computing spectral asymptotics. For the third order operator, in computing the asymptotics of multipliers, these difficulties can be overcome as follows. We find the asymptotics of the multiplier $\tau_3(\lambda)$, which has the most rapid growth as $|\lambda| \to \infty$ in all directions except for the imaginary axis. Next, from Theorem 3.2 ii) it follows that, knowing one of the multipliers, we can find the other two. Then from the asymptotics of $\tau_3(\lambda)$ we can deduce the asymptotics of $\tau_1(\lambda)$ and $\tau_2(\lambda)$. Moreover, for large $|\lambda|$, the multiplier $\tau_3(\lambda)$ controls the position of eigenvalues for periodic and antiperiodic problems, as well as the zeros of the function $\rho$, so that the asymptotics of these quantities can also be found (see [4]). Note that, in the case of operators of order at least 4, this method does not work and other approaches are needed (see [1, 2, 3]).

**Proposition 3.4.** i) *The functions $\tau_j$, $j = 1, 2, 3$, are branches of a function $\tau$ analytic on a certain 3-sheeted Riemann surface, and the following asymptotics is valid*:

$$\tau_j(\lambda) = e^{iz\omega^{j-1}}(1 + O(z^{-1})) \qquad \textit{as} \qquad |\lambda| \to \infty, \quad \textit{where} \ \ \omega = e^{i\frac{2\pi}{3}}. \tag{3.10}$$

ii) *The functions $\Delta_j$, $j = 1, 2, 3$, are branches of a function $\Delta$ analytic on a certain connected 3-sheeted Riemann surface, and the following asymptotics is valid*:

$$\Delta_j(\lambda) = \cos(z\omega^{j-1}) + O\left(\frac{e^{|\operatorname{Im}(z\omega^{j-1})|}}{|z|}\right) \qquad \textit{as} \qquad |\lambda| \to \infty. \tag{3.11}$$

*Proof.* i) Denote by $\tau_j^0 = e^{i\omega^{j-1}z}$, $j = 1, 2, 3$, the eigenvalues of the matrix $e^{izB}$. By a well-known theorem of the matrix theory (see, e.g., [17, Corollary 6.3.4]), from (2.12) it follows that, for each $|\lambda| > 1$, the matrix $\mathcal{M}(1, \lambda)$ (and, thus, the matrix $M(1, \lambda)$) has at least one eigenvalue $\tau_j(\lambda)$ in each disk with center $\tau_j^0(\lambda)$, $j = 1, 2, 3$, and radius $\frac{\varkappa}{|z|}e^{z_0 + \varkappa}$. In particular, for $j = 3$ we obtain

$$\left|\tau_3(\lambda) - e^{i\omega^2 z}\right| < \frac{\varkappa}{|z|}e^{z_0+\varkappa} \ \text{ for all } \ |\lambda| > 1, \tag{3.12}$$

whence $|\tau_3(\lambda)e^{-i\omega^2 z} - 1| < \frac{\varkappa}{|z|}e^{\varkappa}$ for all $|\lambda| > 1$. This yields the asymptotics (3.10) for $j = 3$. Using the relations $\tau_2 = \bar{\tau}_3^{-1}$ and $\tau_1 = (\tau_2\tau_3)^{-1}$, we get (3.10) for $j = 1, 2$.

Moreover, the asymptotics (3.10) shows that the functions $\tau_j(\lambda)$, $j = 1, 2, 3$, are distinct. Assume that any one of the functions $\tau_j(\lambda)$ is an entire function of the variable $\lambda$. The asymptotics (3.10) shows that it is an entire function of order $\frac{1}{3}$; thus, it has infinitely many zeros (see, e.g., [27, Chapter I.10]). This contradicts relation (3.4): $\det M(1, \cdot\,) = \tau_1\tau_2\tau_3 = 1$. Thus, none of the functions $\tau_j$ is entire, and then the $\tau_j$ are three branches of one and the same function $\tau$ analytic on a connected 3-sheeted Riemann surface.

ii) The asymptotics (3.10) implies (3.11), which shows that the functions $\Delta_j$, $j = 1, 2, 3$, are distinct. Assume that some of the functions $\Delta_j$ is entire. Then $\tau_j = \Delta_j + \sqrt{\Delta_j^2 - 1}$ is an analytic function on a 2-sheeted Riemann surface, which contradicts statement i). Therefore, none of the functions $\Delta_j$ is entire, and then the $\Delta_j$ are the three branches of one and the same function $\Delta$ analytic on a connected 3-sheeted Riemann surface. $\square$

Concluding this section, we prove Theorem 1.1, i).

*Proof of Theorem* 1.1, i). The function $\rho$ is the discriminant of the cubic polynomial (3.4) with integral coefficients; therefore, $\rho$ is an entire function. The standard formula for the discriminant $d$ of the cubic polynomial $-\tau^3 + a\tau^2 - b\tau + 1$ gives $d = (ab)^2 - 4(a^3 + b^3) + 18ab - 27$, whence (1.10) follows. $\square$

*Remark* 3.5. 1) The simple zeros of the function $\rho$ are branch points of the function $\tau$, and, thus, of the Lyaponov function. A zero of multiplicity at least 2 of the function $\rho$ may fail to be a branch point, but under small perturbations of the coefficients $p$ and $q$ in (1.4), it splits into several simple zeros, thus generating several branch points. For this reason, we call the zeros of the function $\rho$ of any multiplicity the *branch points* of the Lyapunov function.

2) Observe the asymptotics $\rho(\lambda) = \rho^0(\lambda)\big(1 + O(|z|^{-1})\big)$ as $|\lambda| \to \infty$ (see [4]); it shows that $\rho$ is not equal to zero identically.

## §4. The operator in a layer

It is known that if $p$, $q \in C^\infty(\mathbb{T})$, then the selfadjoint operator $H$ can be defined as the closure of the corresponding minimal operator. Moreover, the spectrum $\sigma(H)$ of the operator $H$ is absolutely continuous and covers the entire axis (see, e.g., [9, Chapter XIII], [30]). Our aim is to define the selfadjoint operator $H$ for a wider class of coefficients $p$, $q \in L^1(\mathbb{T})$ and to describe the spectrum in terms of the Lyapunov function.

To obtain the direct integral expansion of the operator $H$, we introduce the Hilbert spaces

$$\mathscr{H}' = L^2([0,1], dt), \qquad \mathscr{H} = \int_{[0,2\pi)}^{\oplus} \mathscr{H}'\,\frac{dk}{2\pi} \tag{4.1}$$

and the operators

$$H(k) = i\partial^3 + i\partial p + ip\partial + q, \qquad k \in [0, 2\pi),$$

which act in $\mathscr{H}' = L^2(0,1)$ and are selfadjoint on the domain

$$\begin{aligned} \mathscr{D}(H(k)) = \Big\{ & f \in L^2(0,1) \,:\, i(f'' + pf)' + ipf' + qf \in L^2(0,1), \\ & f'', (f''+pf)' \in L^1(0,1),\ f_j(1) = e^{ik} f_j(0) \\ & \text{for all } \ j = 1,2,3, \ \text{ where } \ f_1 = f,\ f_2 = f',\ f_3 = f'' + pf \Big\} \end{aligned} \tag{4.2}$$

(see Lemma 4.3). In the expression $i(f''+pf)'+ipf'+qf$ in (4.2), each term belongs to $L^1(0,1)$, but the sum lies in $L^2(0,1)$.

*Remark* 4.1. In analyzing the direct integral expansion, some difficulties with nonsmooth coefficients emerge. Namely, applying the standard unitary transformation $\mathcal{U}^{-1}H(k)\mathcal{U}$, where $\mathcal{U}$ is the operator of multiplication by $e^{ikt}$, we cannot make the domain (4.2) of the operator $H(k)$ independent of $k$. Nevertheless (see Lemma 4.5), the operator function $H(k)$ is analytic in Kato's sense (in terms of the resolvent), and this turns out to be sufficient for the proof of the absolute continuity of the spectrum.

If $p=q=0$, then each operator $H_0(k)=i\partial^3, k\in[0,2\pi)$, acting in $L^2(0,1)$, is selfadjoint on the domain

$$\mathscr{D}(H_0(k))=\big\{f,f'''\in L^2(0,1)\,:\,f^{(j)}(1)=e^{ik}f^{(j)}(0)\ \text{ for all }\ j=0,1,2\big\}.$$

The eigenvalues of the operator $H_0(k)$, $k\in[0,2\pi)$, are all simple and are equal to $\lambda_n^0(k)=(2\pi n+k)^3$, $n\in\mathbb{Z}$. The corresponding eigenfunctions $\psi_{n,k}^0=e^{i(2\pi n+k)t}$ form an orthonormal basis in $L^2(0,1)$.

We need the following properties of the entire functions $D(e^{ik},\,\cdot\,)$, $k\in[0,2\pi)$.

**Lemma 4.2.** i) *Each of the functions $D(e^{ik},\lambda)$, $k\in[0,2\pi)$, has the asymptotics*

$$D(e^{ik},\lambda)=D_0(e^{ik},\lambda)\big(1+O(|z|^{-1})\big)\quad \text{as}\quad |\lambda|\to\infty \tag{4.3}$$

*if $|z-k-2\pi n|\ge\frac{\pi}{2}, |z\omega-k+2\pi n|\ge\frac{\pi}{2}$ for all $n\in\mathbb{N}$, where*

$$D_0(\tau,\lambda)=-\tau^3+\tau^2T_0(\lambda)-\tau\bar{T}_0(\bar\lambda)+1. \tag{4.4}$$

ii) *There exists $n_0\ge1$ such that for every integer $N>n_0$ the following is true*:

a) *for all $k\in[0,\frac{\pi}{2})\cup(\frac{3\pi}{2},2\pi)$, the function $D(e^{ik},\,\cdot\,)$ has precisely $2N+1$ zeros, counted with multiplicities, in the disk $\{\lambda:|\lambda|<(\pi(2N+1))^3\}$;*

b) *for all $k\in[\frac{\pi}{2},\frac{3\pi}{2}]$, the function $D(e^{ik},\,\cdot\,)$ has precisely $2N$ zeros, counted with multiplicities, in the disk $\{\lambda:|\lambda|<(2\pi N)^3\}$.*

*Moreover, for every $n>N$ and for all $k\in[0,2\pi)$, the function $D(e^{ik},\,\cdot\,)$ has precisely one simple zero in every domain $\{\lambda:|z-k-2\pi n|<\frac{\pi}{2}\}$, $\{\lambda:|z\omega-k+2\pi n|<\frac{\pi}{2}\}$. This function has no other zeros.*

*Proof.* i) Let $k\in[0,2\pi)$. By (3.3) and (4.4), we have

$$|D(e^{ik},\lambda)-D_0(e^{ik},\lambda)|=|e^{ik}(T(\lambda)-T_0(\lambda))-\bar{T}(\bar\lambda)+\bar{T}_0(\bar\lambda)|\ \text{ for all }\ \lambda\in\mathbb{C}.$$

Estimates (2.13) imply $|T(\lambda)-T_0(\lambda)|\le\frac{3\varkappa}{|z|}e^{z_0+\varkappa}$, and then

$$|D(e^{ik},\lambda)-D_0(e^{ik},\lambda)|\le\frac{6\varkappa}{|z|}e^{z_0+\varkappa}\ \text{ for all }\ |\lambda|\ge1. \tag{4.5}$$

Assume that

$$|D_0(1,\lambda)|\ge\frac{e^{z_0}}{8} \tag{4.6}$$

for all $\lambda$ lying in the set

$$\Big\{|\lambda|>R\,:\,|z-k-2\pi n|\ge\frac{\pi}{2},\ |z\omega-k+2\pi n|\ge\frac{\pi}{2}\ \text{ for all }\ n\in\mathbb{N}\Big\}$$

for some sufficiently large $R>0$. Estimates (4.5) and (4.6) yield (4.3).

We prove (4.6). Using the identity

$$D_0(\tau,\lambda)=-(\tau-e^{iz})(\tau-e^{i\omega z})(\tau-e^{i\omega^2z}),$$

we obtain

$$
\begin{aligned}
D_0(e^{ik},\lambda) &= -(e^{ik}-e^{iz})(e^{ik}-e^{i\omega z})(e^{ik}-e^{i\omega^2 z})\\
&= -i8e^{i\frac{3k}{2}}\sin\frac{z-k}{2}\sin\frac{z\omega-k}{2}\sin\frac{z\omega^2-k}{2}
\end{aligned}
$$

for all $\lambda\in\mathbb{C}$. With the help of the standard estimate $|\sin z| > \frac14 e^{|\operatorname{Im} z|}$ for $|z-\pi n|\geq\frac{\pi}{4}$ for all $n\in\mathbb{Z}$ (see [33, Lemma 2.1]), we get

$$
|D_0(e^{ik},\lambda)| > \frac18 e^{\frac12(|\operatorname{Im} z|+|\operatorname{Im} z\omega|+|\operatorname{Im} z\omega^2|)} \geq \frac{e^{z_0}}{8}
$$

for all $\lambda\in\mathbb{C}$ such that $|z\omega^j-k-2\pi n|>1, j=0,1,2,\ n\in\mathbb{N}$. Since the inequalities $|z\omega^2-k\pm 2\pi n|>1, |z-k+2\pi n|\geq\frac{\pi}{2}, |z\omega-k-2\pi n|\geq\frac{\pi}{2}, n\in\mathbb{N}$, are fulfilled for all sufficiently large $|\lambda|$, we get (4.6).

ii) Consider $k\in[0,\frac{\pi}{2})$. The proof for other values of $k$ is similar. Let $N\geq 1$ be sufficiently large, and let $N'>N$ be any integer. Consider the contours $C_\alpha(r)=\{\lambda : |z-\alpha|=r\}$, $r>0$, $\alpha\geq 0$. Let $\lambda$ belong to one of the contours

$$
C_0(\pi(2N+1)),\quad C_0(\pi(2N'+1)),\quad C_{k+2\pi n}\Big(\frac{\pi}{2}\Big),\quad C_{(k-2\pi n)\omega^2}\Big(\frac{\pi}{2}\Big),\quad n>N.
$$

The asymptotics (4.3) gives

$$
\begin{aligned}
|D(e^{ik},\lambda)-D_0(e^{ik},\lambda)| &= |D_0(e^{ik},\lambda)|\Big|\frac{D(e^{ik},\lambda)}{D_0(e^{ik},\lambda)}-1\Big|\\
&= |D_0(e^{ik},\lambda)|O(|z|^{-1}) < |D_0(e^{ik},\lambda)|
\end{aligned}
$$

on each contour. By the Rouché theorem, $D(e^{ik},\cdot)$ has as many zeros as $D_0(e^{ik},\cdot)$ in each of the bounded domains and in the remaining unbounded domain. Since $D_0(e^{ik},\cdot)$ has one simple zero at every point $(2\pi n+k)^3$, $n\in\mathbb{Z}$, and $N'>N$ can be taken arbitrarily large, we arrive at the required result. ☐

Now we study properties of the operator $H(k)$ in a layer. In particular, we prove that it is selfadjoint, compute its resolvent, and show that it depends on the variable $k$ analytically.

**Lemma 4.3.** i) *Each operator $H(k)$, $k\in[0,2\pi)$, is selfadjoint.*

ii) *The operator $H(k)$, $k\in[0,2\pi)$, has discrete spectrum*

$$
\sigma(H(k)) = \{\lambda\in\mathbb{R} : e^{ik} \text{ is an eigenvalue of } M(1,\lambda)\}. \tag{4.7}
$$

*Its resolvent $(H(k)-\lambda)^{-1}$, $\lambda\in\mathbb{C}\setminus\sigma(H(k))$, is a Hilbert–Schmidt operator and has the form*

$$
\big((H(k)-\lambda)^{-1}f\big)(t) = -i\int_0^1 R_{k,13}(t,s,\lambda)f(s)\,ds,\qquad t\in[0,1], \tag{4.8}
$$

*where*

$$
\begin{aligned}
R_k(t,s,\lambda) &= \big(R_{k,jk}(t,s,\lambda)\big)_{j,k=1}^3\\
&= M(t,\lambda)\big(\chi(t-s)\mathbb{1}_3 - (M(1,\lambda)-e^{ik}\mathbb{1}_3)^{-1}M(1,\lambda)\big)M^{-1}(s,\lambda),\\
\chi(t) &= \begin{cases}1, & t\geq 0\\ 0, & t<0.\end{cases}
\end{aligned} \tag{4.9}
$$

*Moreover, for sufficiently large $\alpha>0$, the resolvent $(H(k)-i\alpha)^{-1}$ is an analytic operator-valued function of the variable $k$ in a neighborhood of the interval $[0,2\pi]$ in $\mathbb{C}$.*

iii) *The eigenfunctions $\psi_{n,k}, n\in\mathbb{Z}$, of the operator $H(k)$, $k\in[0,2\pi)$, with eigenvalues $\lambda_n(k)$ form an orthonormal basis in $L^2(0,1)$. The eigenvalues $\lambda_n(k)$ that are large in* *modulus are simple. We enumerate the* $\lambda_n(k)$ *in the increasing order:* $\ldots \le \lambda_{-1}(k) \le \lambda_0(k) \le \lambda_1(k) \le \lambda_2(k) \le \ldots$ *with regard to multiplicities. The following asymptotics is valid:*

$$\lambda_n(k) = \big(2\pi n + k\big)^3\big(1 + O(n^{-1})\big) \quad as \quad n \to \pm\infty \tag{4.10}$$

*uniformly in* $k$.

*Proof.* i) Introduce the following operator in $L^2(0,1)$:

$$H_+ = i\partial^3 + i\partial p + ip\partial + q,$$

$$\mathscr{D}(H_+) = \big\{y \in L^2(0,1) \,:\, i(y'' + py)' + ipy' + qy \in L^2(0,1), y'', (y'' + py)' \in L^1([0,1])\big\},$$

and let $H_-$ be the restriction of $H_+$ to the domain

$$\mathscr{D}(H_-) = \big\{y \in \mathscr{D}(H_+) \,:\, y(0) = y(1) = y'(0) = y'(1) = (y''+py)(0) = (y''+py)(1) = 0\big\}.$$

The operator $H_-$ is a densely defined symmetric operator with defect index $(3,3)$, and $H_+ = H_-^*$, $H_- = H_+^*$ (see [14]). We introduce linear maps $\Gamma_1, \Gamma_2 \,:\, \mathscr{D}(H_+) \to \mathbb{C}^3$ by the relations

$$\Gamma_1 y = \begin{pmatrix} i(y'' + py)(1) \\ -i(y'' + py)(0) \\ y'(1) - iy'(0) \end{pmatrix}, \qquad \Gamma_2 y = \begin{pmatrix} y(1) \\ y(0) \\ \frac{2+i}{2}y'(1) - \frac{1+2i}{2}y'(0) \end{pmatrix}. \tag{4.11}$$

A straightforward calculation yields

$$(H_+y, z) - (y, H_+z) = (\Gamma_1 y, \Gamma_2 z) - (\Gamma_2 y, \Gamma_1 z) \quad \text{for all} \quad y, z \in \mathscr{D}(H_+).$$

We shall need the following result from [14].

1) *Let* $A$ *be a unitary operator in* $\mathbb{C}^3$. *Then the restriction of the operator* $H_+$ *to the set of functions* $y$ *in* $\mathscr{D}(H_+)$ *satisfying the condition*

$$(A - \mathbb{1}_3)\Gamma_1 y + i(A + \mathbb{1}_3)\Gamma_2 y = 0, \tag{4.12}$$

*is a selfadjoint extension* $H_A$ *of the operator* $H_-$.

2) *For every selfadjoint extension* $\widetilde{H}$ *of* $H_-$, *there exists a unitary operator* $A$ *in* $\mathbb{C}^3$ *such that* $\widetilde{H} = H_A$.

3) *The correspondence between the set* $\{A\}$ *of unitary operators in* $\mathbb{C}^3$ *and the set* $\{h_A\}$ *of selfadjoint extensions of* $H_-$ *is a bijection.*

We shall choose a unitary operator $A$ so that conditions (4.12) for the operator $H_A$ imply conditions (4.2) for the operator $H(k)$. Let $k \in [0, 2\pi)$, and let $A$ be of the form

$$A = \begin{pmatrix} 0 & -e^{ik} & 0 \\ -e^{-ik} & 0 & 0 \\ 0 & 0 & \frac{ia+b}{ia-b} \end{pmatrix}, \qquad a = ie^{ik} + 1, \qquad b = \frac{2i-1}{2}e^{ik} + \frac{2-i}{2}. \tag{4.13}$$

The relation $\big(\frac{ia+b}{ia-b}\big)^{-1} = \overline{\big(\frac{ia+b}{ia-b}\big)}$ shows that $A$ is a unitary operator. Substituting (4.13) and (4.11) in (4.12), we obtain

$$\begin{pmatrix} -1 & -e^{ik} & 0 \\ -e^{-ik} & -1 & 0 \\ 0 & 0 & \frac{2b}{ia-b} \end{pmatrix}\begin{pmatrix} i(y'' + py)(1) \\ -i(y'' + py)(0) \\ y'(1) - iy'(0) \end{pmatrix} + \begin{pmatrix} 1 & -e^{ik} & 0 \\ -e^{-ik} & 1 & 0 \\ 0 & 0 & \frac{i2a}{ia-b} \end{pmatrix}\begin{pmatrix} iy(1) \\ iy(0) \\ \frac{2i-1}{2}y'(1) + \frac{2-i}{2}y'(0) \end{pmatrix} = 0,$$

which is equivalent to

$$\begin{pmatrix} y \\ y' \\ y'' + py \end{pmatrix}(1) = e^{ik}\begin{pmatrix} y \\ y' \\ y'' + py \end{pmatrix}(0). \tag{4.14}$$

Comparing (4.14) with (4.2), we conclude that $H_A = H(k)$, where $A$ is defined by (4.13). Since $A$ is unitary, $H_A$ and, with it, $H(k)$ are selfadjoint.

ii) The solutions of the equation $iy''' + (py)' + py' + (q - \lambda)y = f$ satisfy the relation

$$
\begin{gathered}
Y(t) = M(t,\lambda)Y(0) + M(t,\lambda)\int_0^t M^{-1}(s,\lambda)F(s)\,ds, \\
Y = \begin{pmatrix} y \\ y' \\ y'' + py \end{pmatrix}, \qquad F = \begin{pmatrix} 0 \\ 0 \\ -if \end{pmatrix},
\end{gathered}
\tag{4.15}
$$

where $(t,\lambda) \in \mathbb{R} \times \mathbb{C}$. Conditions (4.14) yield

$$
M(1,\lambda)Y(0) + M(1,\lambda)\int_0^1 M^{-1}(s,\lambda)F(s)\,ds = e^{ik}Y(0). \tag{4.16}
$$

Suppose the operator $(M(1,\lambda) - e^{ik}\mathbb{1}_3)^{-1}$ is bounded for some $\lambda \in \mathbb{C}$. Then, by (4.16), we have

$$
Y(0) = -\big(M(1,\lambda) - e^{ik}\mathbb{1}_3\big)^{-1} M(1,i\alpha)\int_0^1 M^{-1}(s,\lambda)F(s)\,ds.
$$

Substituting this in (4.15), we get

$$
Y(t) = \int_0^1 R_k(t,s,\lambda)F(s)\,ds \qquad \text{for all} \qquad t \in \mathbb{R}, \tag{4.17}
$$

which implies (4.8).

The matrix-valued function $M$ is absolutely continuous in $t$. Since $\det M(t,\lambda) = 1$, the function $M^{-1}$ is also absolutely continuous. Then relations (4.8), (4.9) show that the resolvent of the operator $H(k)$ is a Hilbert–Schmidt operator. Therefore, the spectrum of $H(k)$ is discrete.

Let $\lambda$ be an eigenvalue of $H(k)$. Then

$$
e^{ik}\begin{pmatrix} y \\ y' \\ y'' + py \end{pmatrix}(0) = \begin{pmatrix} y \\ y' \\ y'' + py \end{pmatrix}(1) = M(1,\lambda)\begin{pmatrix} y \\ y' \\ y'' + py \end{pmatrix}(0).
$$

Consequently, $\lambda$ is a zero of the function $D(e^{ik},\lambda) = \det(M(1,\lambda) - e^{ik}\mathbb{1}_3)$. Conversely, let $\lambda$ be a zero of $D(e^{ik},\lambda)$. Then the matrix $M(1,\lambda)$ has an eigenvector $(x_1,x_2,x_3)^\top$ corresponding to the eigenvalue $e^{ik}$. A solution $y$ of equation (1.4) that satisfies the initial conditions $y^{(j-1)}(0) = x_j$, $j = 1,2,3$, provides an eigenfunction of the operator $H(k)$. Thus, $\lambda$ is an eigenvalue of $H(k)$, which implies (4.7).

Let $\lambda = i\alpha$ with a sufficiently large $\alpha > 0$. Then $z = e^{i\frac{\pi}{6}}\alpha^{\frac13}$, $\alpha^{\frac13} > 0$, whence

$$
\operatorname{Re} iz = \operatorname{Re} i\omega z = -\frac{\alpha^{\frac13}}{2}, \qquad \operatorname{Re} i\omega^2 z = \alpha^{\frac13}.
$$

The asymptotics (3.10) show that

$$
\begin{aligned}
\tau_j(\lambda) &= e^{-\frac{12}{\alpha}^{\frac13}}(1 + O(\alpha^{-\frac13})), \qquad j = 1,2, \\
\tau_3(\lambda) &= e^{\alpha^{\frac13}}(1 + O(\alpha^{-\frac13})) \qquad \text{as} \qquad \alpha \to +\infty.
\end{aligned}
$$

Thus, for $\alpha > 0$ sufficiently large, the eigenvalues $\tau_j(i\alpha)$, $j = 1,2,3$, of the matrix $M(1,i\alpha)$ lie far from the unit circle. Then the operator $(M(1,i\alpha) - e^{ik}\mathbb{1}_3)^{-1}$ depends analytically on $k$ in a neighborhood of the interval $[0,2\pi]$. From (4.8) it follows that the resolvent $(H(k) - i\alpha)^{-1}$ is an analytic operator-valued function of the variable $k$ in this neighborhood.

iii) The resolvent of $H(k)$ is a compact operator, so that its eigenfunctions form an orthonormal basis.

The relation $T = \tau_1 + \tau_2 + \tau_3$ and the asymptotics (3.10) imply the formulas

$$T(\lambda) = e^{i\omega^2 z}\big(1 + O(z^{-1})\big) \quad \text{as} \quad \lambda \to \pm\infty,$$
$$\bar{T}(\lambda) = \big(1 + O(z^{-1})\big) \begin{cases} e^{-i\omega z} & \text{as } \lambda \to +\infty, \\ e^{-iz} & \text{as } \lambda \to -\infty. \end{cases}$$

From (3.3) we deduce that

$$\begin{aligned} D(e^{ik}, \lambda) &= e^{i2k} e^{i\omega^2 z}\big(1 + O(z^{-1})\big) - e^{ik} e^{-i\omega z}\big(1 + O(z^{-1})\big) \\ &= 2ie^{i\frac{3}{2}k + \sqrt{3}z}\Big(\sin \frac{k - z}{2} + O(z^{-1})\Big) \end{aligned} \tag{4.18}$$

as $\lambda \to +\infty$, and

$$\begin{aligned} D(e^{ik}, \lambda) &= e^{i2k} e^{i\omega^2 z}\big(1 + O(z^{-1})\big) - e^{ik} e^{-iz}\big(1 + O(z^{-1})\big) \\ &= 2ie^{i\frac{3}{2}k + \sqrt{3}|z|}\Big(\sin \frac{k + |z|}{2} + O(z^{-1})\Big) \end{aligned} \tag{4.19}$$

as $\lambda \to -\infty$, uniformly with respect to $k \in [0, 2\pi]$. The eigenvalues $\lambda_n(k)$ of the operator $H(k)$ are real and coincide with the set of zeros of the function$D(e^{ik}, \cdot\,)$. Lemma 4.2 ii) implies that the zeros of $D(e^{ik}, \cdot\,)$ that are large in modulus are simple. Moreover, $\lambda_n(k) = (2\pi n + k + \delta_n)^3, |\delta_n| < \frac{\pi}{2}$ for all sufficiently large $|n|$. Substituting $z = \lambda_n(k)^{\frac{1}{3}} = 2\pi n + k + \delta_n$ in (4.18) and (4.19), we see that $\delta_n = O(n^{-1})$ as $n \to \pm\infty$, which provides (4.10). $\square$

*Remark* 4.4. Relation (4.2) shows that the operator $H(0)$ is given on a circle. Consequently, the eigenvalues of this operator are eigenvalues of a *periodic problem*. Similarly, the eigenvalues of the operator $H(\pi)$ are eigenvalues of an antiperiodic problem. From (4.7) it follows that $\lambda$ is an eigenvalue of the periodic problem if and only if one of the multipliers $\tau_j(\lambda) = 1$ is equal to 1 or, what is the same, one of the Lyapunov functions $\Delta_j(\lambda) = 1$ is equal to 1 (for the antiperiodic problem, $\tau_j(\lambda) = -1$ or $\Delta_j(\lambda) = -1$).

In the following lemma we show that the eigenvalues and eigenfunctions of the operator $H(k)$ depend on $k$ analytically and the $\lambda_n(\,\cdot\,), n \in \mathbb{Z}$, cannot be constant. These statements are basic for proving the absolute continuity of the spectrum of the operator $H$.

**Lemma 4.5.** i) *The operator-valued function $H(\,\cdot\,)$ is real-analytic in Kato's sense in a neighborhood of the interval $[0, 2\pi]$ in $\mathbb{C}$. Moreover, $H(k)$ and $H(2\pi - k)$ are antiunitarily equivalent with respect to the usual complex conjugation. In particular, their eigenvalues are equal and the corresponding eigenfunctions are complex conjugate.*

ii) *For $k \in (0, 2\pi)$, the eigenvalues of the operator $H(k)$ have multiplicity* 1 *or* 2. *Moreover, for every $n \in \mathbb{Z}$ there exists a finite number $m_n \geq 0$ of values $k_\ell \in (0, 2\pi)$, $\ell = 1, \ldots, m_n$, such that $\lambda_n(k_\ell)$ is an eigenvalue of multiplicity* 2 *of the operator $H(k_\ell)$. Every function $\lambda_n(\,\cdot\,)$, $n \in \mathbb{Z}$, is continuous on $[0, 2\pi)$ and analytic and nonconstant on each of the intervals $(0, k_1)$, $(k_{m_n}, 2\pi)$, $(k_\ell, k_{\ell+1})$, $\ell = 1, \ldots, m_n - 1$.*

iii) *Let $n \in \mathbb{Z}$. Then the $L^2(0,1)$-valued function $\psi_{n,k}$ is continuous in $k$ on $[0, 2\pi)$ and real-analytic in $k$ on each of the intervals $(0, k_1), (k_{m_n}, 2\pi), (k_\ell, k_{\ell+1}), \ell = 1, \ldots, m_n - 1$.*

*Proof.* i) For sufficiently large $\alpha > 0$, the resolvent $(H(k) - i\alpha)^{-1}$ is an analytic operator-valued function of the variable $k$ in a neighborhood of the interval $[0, 2\pi]$ in $\mathbb{C}$. Consequently, the operators $H(k)$ form a Kato analytic family in this neighborhood (see [34, Chapter XII.2]). The definition (4.2) of the operator $H(k)$ shows that $H(k)$ and $H(2\pi - k)$ are antiunitarily equivalent.

ii) Recall the following known result (see [20, Theorem VII.1.8]).

*If a family of operators $A(k)$ is (Kato) analytic in a neighborhood of zero, then any finite system of eigenvalues of $A(k)$ is represented by branches of one or several analytic functions, which have at most algebraic singularities at zero.*

By Lemma 4.2 ii), there exists a number $N \geq 1$ independent of $k$ and such that all the eigenvalues $\lambda_n(k)$, $k \in [0, 2\pi)$, $|n| > N$, are simple. This implies that every function $\lambda_n(\,\cdot\,)$ with $|n| > N$ is analytic on $[0, 2\pi)$. Applying the above result to the finite system $\lambda_n(k)$, $|n| \leq N$, of eigenvalues of the operator $H(k)$, $k \in [0, 2\pi)$, we deduce that every function $\lambda_n(\,\cdot\,)$, $|n| \leq N$, is continuous and piecewise analytic on $[0, 2\pi)$.

We show that any eigenvalue $\lambda_n(k)$, $n \in \mathbb{N}$, is simple for all $k \in (0, 2\pi)$ except for a finite number of values of $k$. Recall that the number of distinct eigenvalues of $M(1, \lambda)$ is equal to 3 independently of $\lambda$, with the exception of some special points $\lambda \in \mathbb{C}$. Each compact subset in $\mathbb{C}$ can contain only finitely many such exceptional points. If $\lambda$ is a degenerate eigenvalue of $H(k)$, then $e^{ik}$ is a multiple eigenvalue of the matrix $M(1, \lambda)$ and $\lambda$ is an exceptional point.

Suppose there are infinitely many values $k_\ell \in (0, 2\pi)$ such that $\lambda_n(k_\ell)$ (with a fixed $n \in \mathbb{Z}$) is a degenerate eigenvalue. Then the set $\{k_\ell, \ell \in \mathbb{N}\}$ has at least one limit point $k = \lim_{\ell \to \infty} k_\ell \in [0, 2\pi]$, and either a) $\lim_{\ell \to \infty} \lambda_n(k_\ell) = \infty$, or b) the set $\{\lambda_n(k_\ell), \ell \in \mathbb{N}\}$ has a finite limit. The asymptotics (4.10) shows that a) is impossible. Since in every compact subset in $\mathbb{C}$ there are only finitely many exceptional values, b) is also impossible. Thus, there are a finite number $m_n \geq 0$ of values $k_\ell \in (0, 2\pi)$ such that $\lambda_n(k_\ell)$ is a degenerate eigenvalue of the operator $H(k_\ell)$. The function $\lambda_n(\,\cdot\,)$ is continuous on $[0, 2\pi)$ and analytic on each of the intervals $(0, k_1)$, $(k_\ell, k_{\ell+1})$, $(k_{m_n}, 2\pi)$.

If $\lambda$ is an eigenvalue of $H(k)$ of multiplicity 3, then the multiplier $\tau = e^{ik}$ has multiplicity 3. Relation (3.4) shows that in this case we have $\tau = 1$, which implies that $k = 0$.

Assume that, for some $n \in \mathbb{Z}$, $\lambda_n(k) = c = \text{const}$ on a nonempty interval $(\alpha, \beta) \subset [0, 2\pi)$, and let $A = T(c)$. Then, (3.3) implies the relation

$$-e^{i3k} + e^{i2k} A - e^{ik} \overline{A} + 1 = 0 \quad \text{for all} \quad k \in (\alpha, \beta). \tag{4.20}$$

Using the Cardano formulas to express $e^{ik}$, we get $k = \text{const}$, i.e., (4.20) cannot be fulfilled for all $k \in (\alpha, \beta)$. This contradiction proves the claim.

iii) The result follows from i), ii), and the Kato–Rellich theorem (see [34, Theorem XII.8]). □

*Remark* 4.6. In our proof of the continuity of the spectrum of $H$, an important role is played by the fact that the zeros of the function $D(e^{ik}, \,\cdot\,)$ cannot be constant with respect to $k \in [0, 2\pi)$. This immediately implies the absence of the point spectrum. However, the absence of the point spectrum can be proved with the help of the approach used in [9]. The idea of such a proof is that, by the periodicity of the operator, the spectrum contains no eigenvalues of finite multiplicity; moreover, it contains no eigenvalues of infinite multiplicity, because the number of linearly independent solutions of equation (1.4) is finite for each fixed $\lambda \in \mathbb{R}$.

## §5. The spectrum of $H$

Denote by $C_0^\infty(\mathbb{R})$ the set of smooth functions on $\mathbb{R}$ with bounded support; this set is dense in $L^2(\mathbb{R})$. Define a unitary operator $U : L^2(\mathbb{R}) \to \mathscr{H}$ by the relation

$$(Uf)_k(t) = \sum_{n \in \mathbb{Z}} e^{-ink} f(t + n), \qquad (k, t) \in [0, 2\pi) \times [0, 1]. \tag{5.1}$$

Here, $\mathscr{H} = \int_{[0,2\pi)}^{\oplus} \mathscr{H}' \, \frac{dk}{2\pi}$, product $(\,\cdot\,, \,\cdot\,)_0$ and the norm $\|\cdot\|_0$ in the Hilbert space $\mathscr{H}'$.

**Lemma 5.1.** i) *We extend* $\psi_{n,k}, (n,k) \in \mathbb{Z} \times [0, 2\pi)$ *from* $[0,1]$ *to* $\mathbb{R}$ *by the formula* $\psi_{n,k}(t+1) = e^{ik}\psi_{n,k}(t)$. *For* $f \in C_0^\infty(\mathbb{R})$, *consider*

$$\widetilde{f}_n(k) = \int_{\mathbb{R}} f(t)\overline{\psi_{n,k}(t)}\,dt. \tag{5.2}$$

*Then*

$$\|f\|^2 = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\int_0^{2\pi} |\widetilde{f}_n(k)|^2\,dk, \qquad f(t) = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\int_0^{2\pi} \widetilde{f}_n(k)\psi_{n,k}(t)\,dk, \qquad t\in\mathbb{R}. \tag{5.3}$$

ii) *The operator* $H_1 = U^{-1}\int_{[0,2\pi)}^{\oplus} H(k)\frac{dk}{2\pi}U$ *is selfadjoint on the domain* $\mathscr{D}(H_1) = \mathscr{D}(H)$, *where* $\mathscr{D}(H)$ *is as in* (1.2). *Moreover, the operator* $H_1$ *satisfies*

$$(\widetilde{H_1 f})_n(k) = \lambda_n(k)\widetilde{f}_n(k) \qquad \text{for all} \qquad (n,k)\in\mathbb{Z}\times[0,2\pi), \qquad f\in\mathscr{D}(H_1), \tag{5.4}$$

*where the operation* $\widetilde{}$ *is assumed to be extended to* $L^2(\mathbb{R})$ *by continuity.*

iii) *The operator* $H$ *defined in* (1.1), (1.2) *is selfadjoint and satisfies*

$$UHU^{-1} = \int_{[0,2\pi)}^{\oplus} H(k)\frac{dk}{2\pi}. \tag{5.5}$$

*Proof.* i) For $f \in C_0^\infty(\mathbb{R})$, the sum in (5.1) is finite, and $u_k$ satisfies

$$u_k(t+1) = e^{-ik}u_k(t), \qquad f(t) = \frac{1}{2\pi}\int_0^{2\pi} u_k(t)\,dk \ \text{ for all } \ t\in\mathbb{R}. \tag{5.6}$$

Moreover,

$$\begin{aligned}\widetilde{f}_n(k) &= \int_0^1 \sum_{\ell\in\mathbb{Z}} f(t+\ell)\overline{\psi_{n,k}(t+\ell)}\,dt \\ &= \int_0^1 \sum_{\ell\in\mathbb{Z}} e^{-ik\ell} f(t+\ell)\overline{\psi_{n,k}(t)}\,dt = \int_0^1 u_k(t)\overline{\psi_{n,k}(t)}\,dt.\end{aligned}$$

Since the $\psi_{n,k}$ form an orthonormal basis in $L^2(0,1)$, we have

$$\sum_{n\in\mathbb{Z}}|\widetilde{f}_n(k)|^2 = \int_0^1 |u_k(t)|^2\,dt, \qquad u_k(t) = \sum_{n\in\mathbb{Z}}\widetilde{f}_n(k)\psi_{n,k}(t) \ \text{ for all } \ t\in[0,1]. \tag{5.7}$$

Since $U$ is unitary, we obtain $\|f\|^2 = \|u\|^2_{\mathscr{H}} = \frac{1}{2\pi}\int_0^{2\pi} dk \int_0^1 |u_k(t)|^2\,dt$. This and the first relation in (5.7) imply the first identity in (5.3). Substituting the second formula in (5.7) in the second relation of (5.6), we obtain the second relation of (5.3).

ii) The operator $A = \int_{[0,2\pi)}^{\oplus} H(k)\frac{dk}{2\pi}$ on the domain

$$\mathscr{D}(A) = \left\{u\in\mathscr{H} \,:\, u_k\in\mathscr{D}(H(k)) \ \text{ for all } \ k\in[0,2\pi), \ \int_0^{2\pi} \|H(k)u_k\|_0^2\,dk < \infty\right\}$$

is selfadjoint (see [34, Theorem XIII.85]). Then $H_1 = U^{-1}AU$ is selfadjoint on the domain $\mathscr{D}(H_1) = U^{-1}\mathscr{D}(A)$.

We show that $\mathscr{D}(H_1) = \mathscr{D}(H)$, where $\mathscr{D}(H)$ is the domain given by (1.2). Let $f\in\mathscr{D}(H)$. Since (5.1), it follows that $u_k = (Uf)(k) \in \mathscr{D}(H(k))$ for all $k\in[0,2\pi)$. Moreover, we have $hu_k = (U(hf))(k)$, where $h = i(\partial^2+p)\partial + i\partial p + q$. Then $\frac{1}{2\pi}\int_0^{2\pi}\|H(k)u_k\|_0^2\,dk = \|hf\|^2$. From $hf\in L^2(\mathbb{R})$ it follows that $\int_0^{2\pi}\|H(k)u_k\|_0^2\,dk < \infty$. Thus, $u = Uf \in \mathscr{D}(A)$, implying that $U\mathscr{D}(H) \subset \mathscr{D}(A)$, whence $\mathscr{D}(H) \subset U^{-1}\mathscr{D}(A) = \mathscr{D}(H_1)$. Conversely, let $u\in\mathscr{D}(A)$. Then $u_k\in\mathscr{D}(H(k))$ for all $k\in[0,2\pi)$, and the relation $f(t) = \frac{1}{2\pi}\int_0^{2\pi}u_k(t)\,dk$ shows that $f', f''+pf \in AC(\mathbb{R})$. Moreover, $\int_0^{2\pi}\|H(k)u_k\|_0^2\,dk<\infty$. Since $U$ is unitary, we have $\|hf\|^2 = \frac{1}{2\pi}\int_0^{2\pi}\|H(k)u_k\|_0^2\,dk < \infty$, which means that $hf \in L^2(\mathbb{R})$. Thus, $f = U^{-1}A \in \mathscr{D}(H)$, and $U^{-1}\mathscr{D}(A) = \mathscr{D}(H_1) \subset \mathscr{D}(H)$. Consequently, $\mathscr{D}(H_1) = \mathscr{D}(H)$.

We prove (5.4). Let $f \in \mathscr{D}(H_1) \cap C_0^\infty(\mathbb{R})$. It is known that the set $\mathscr{D}(H_1) \cap C_0^\infty(\mathbb{R})$ is dense in $L^2(\mathbb{R})$ (see, e.g., [10, Appendix A]). Relations (5.2) and (5.1) yield

$$(5.8)\qquad \begin{aligned}(\widetilde{H_1 f})_n(k) &= \int_{\mathbb{R}} (H_1 f)(t)\overline{\psi_{n,k}(t)}\,dt\\ &= \sum_{\ell\in\mathbb{Z}} \int_0^1 (H_1 f)(t+\ell)\overline{\psi_{n,k}(t)}e^{-i\ell k}\,dt = ((UH_1 f)(k), \psi_{n,k})_0,\end{aligned}$$

and this identity extends by continuity to the set of all $f \in \mathscr{D}(H_1)$. From the definition of $H_1$ it follows that $(UH_1 f)(k) = (AUf)(k) = H(k)u_k$. Substituting this in (5.8), we obtain

$$(\widetilde{H_1 f})_n(k) = \big(H(k)u_k, \psi_{n,k}\big)_0 = \lambda_n(k)(u_k, \psi_{n,k})_0, \qquad f \in \mathscr{D}(H_1).$$

The second formula in (5.7) implies $(u_k, \psi_{n,k})_0 = \widetilde{f}_n(k)$, which gives (5.4).

iii) Let $f \in \mathscr{D}(H_1) = \mathscr{D}(H)$. Integration by parts yields

$$(5.9)\qquad \begin{aligned}(\widetilde{Hf})_n(k) &= \int_{\mathbb{R}} (Hf)(t)\overline{\psi_{n,k}(t)}\,dt\\ &= \int_{\mathbb{R}} f(t)\big((i\partial^3 + ip\partial + i\partial p + q)\bar{\psi}_{n,k}\big)(t)\,dt = \lambda_n(k)\widetilde{f}_n(k)\end{aligned}$$

for all $(n,k) \in \mathbb{Z} \times [0, 2\pi)$. Relations (5.4) and (5.9) show that $((\widetilde{H-H_1})f)_n(k) = 0$ for all $(n,k) \in \mathbb{Z}\times[0,2\pi)$. Then from (5.3) we have $(H-H_1)f = 0$ for all $f \in \mathscr{D}(H)$, whence $H = H_1$. Lemma 5.1 ii) implies the selfadjointness of $H$ and relation (5.5). ☐

*Remark* 5.2. (1) The functions $\psi_{n,k}$, $(n,k) \in \mathbb{Z} \times [0, 2\pi)$, extended to all $t \in \mathbb{R}$ by the formula $\psi_{n,k}(t+1) = e^{ik}\psi_{n,k}(t)$ are Floquet solutions for the operator $H$. Relations (5.3) provide an expansion of the function $f$ in an integral of Floquet solutions, together with the Parseval relation for this expansion. Relation (5.4) shows the existence of an expansion in eigenfunctions for $H$.

(2) Results similar to Lemma 5.1 are well known for the second order operator (see, e.g., [12] and [34, Chapter XIII.16]). For operators of an arbitrary even order with smooth coefficients, pertinent results can be found in [35]. In our case, the operator has coefficients in $L^1(0,1)$, so that its domain is more complicated than that of an operator with smooth coefficients. Moreover, our operator has an odd order and is not semibounded from below. This case requires additional analysis.

In the following theorem we describe the spectrum of $H$ is terms of multipliers and the Lyapunov function.

**Theorem 5.3.** *The spectrum $\sigma(H)$ of the operator $H$ is absolutely continuous and is equal to*

$$(5.10)\qquad \begin{aligned}\sigma(H) &= \bigcup_{n\in\mathbb{Z}} \lambda_n([0,2\pi]) = \big\{\lambda \in \mathbb{R} : |\tau_j(\lambda)| = 1 \ \ \textit{for} \ \ j = 1, 2, \ \ \textit{or} \ \ 3\big\}\\ &= \big\{\lambda \in \mathbb{R} : \Delta_j(\lambda) \in [-1, 1] \ \ \textit{for} \ \ j = 1, 2, \ \textit{or}\ 3\big\}.\end{aligned}$$

*Moreover, the multiplicity of the spectrum is equal to the number of branches of the function $\tau(\lambda)$ (or $\Delta(\lambda)$) that satisfy* (5.10).

*Proof.* We use the following known results (see [34, Theorem XIII.85,86]).

*Let $A = \int_{[0,2\pi)} H(k)\frac{dk}{2\pi}$, and let $H(k)$ be a selfadjoint operator in $L^2(0,1)$ for each $k \in [0, 2\pi)$. Assume we are given $L^2(0,1)$-valued functions $\{\psi_n(\,\cdot\,)\}_{n\in\mathbb{Z}}$ on $[0, 2\pi]$ that are* *continuous on* $[0, 2\pi)$ *and piecewise real-analytic on* $(0, 2\pi)$, *and complex-valued functions* $\lambda_n(\,\cdot\,)$ *continuous on* $[0, 2\pi)$ *and piecewise analytic on* $[0, 2\pi)$ *such that*

a) *every function* $\lambda_n(\,\cdot\,)$ *is nonconstant on any subinterval in* $[0, 2\pi)$;

b) $H(k)\psi_n(k) = \lambda_n(k)\psi_n(k)$ *for all* $(n, k) \in \mathbb{Z} \times [0, 2\pi)$;

c) *for every* $k \in [0, 2\pi)$, *the collection* $\{\psi_n(k)\}_{n\in\mathbb{Z}}$ *forms an orthonormal basis in the space* $L^2(0, 1)$.

*Then* $A$ *has a purely absolutely continuous spectrum, and* $\sigma(A) = \overline{\bigcup_{k\in[0,2\pi)} \sigma(H(k))}$.

From these results and Lemmas 4.3 and 4.5, it follows that the spectrum of the operator $H$ is purely absolutely continuous and satisfies the first and the second identity in (5.10). Since $\Delta_j = \frac{1}{2}(\tau_j + \tau_j^{-1})$, $j = 1, 2, 3$, we have the third identity in (5.10). □

*Remark* 5.4. The asymptotics (3.11) and identity (5.10) show that, at high energy, exactly one branch of the Lyapunov function contributes to the spectrum, and the other two branches take nonreal values.

*Proof of Theorem* 1.1 ii), iii). Theorem 3.2 ii) shows that $\sigma(H) = \mathbb{R}$ and for any $\lambda \in \sigma(H)$ only two possibilities occur: a) precisely one multiplier $\tau(\lambda)$ lies on the unit circle; b) all three multipliers lie on the unit circle.

Consider case a): precisely one multiplier $\tau(\lambda)$ lies on the unit circle for all $\lambda \in (\alpha, \beta)$ with some $\alpha < \beta$. Then $(\alpha, \beta) \in \sigma(H)$ and (5.4) shows that the spectral projection $\chi_{(\alpha,\beta)}(H)$ is unitarily equivalent to the operator of multiplication by $\lambda \in (\alpha, \beta)$. Therefore, the spectrum of $H$ in the interval $(\alpha, \beta)$ has multiplicity 1.

Consider case b): all three multipliers lie on the unit circle for all $\lambda \in (\alpha, \beta)$ with some $\alpha < \beta$. Then the spectral projection $\chi_{(\alpha,\beta)}(H)$ is unitarily equivalent to the operator of multiplication by $\lambda \mathbb{1}_3, \lambda \in (\alpha, \beta)$. Therefore, the spectrum of $H$ in the interval $(\alpha, \beta)$ has multiplicity 3.

The asymptotics (3.10) shows that precisely one multiplier satisfies (5.10) for $\lambda \in \mathbb{R} \setminus [-R, R]$ if $R > 0$ is sufficiently large; therefore, the spectrum has multiplicity 1 for such $\lambda$.

Let $\tau_j = e^{ik_j}$ for all $j = 1, 2, 3$. By Proposition 3.4 i), all $k_j(\lambda)$, $j = 1, 2, 3$, are distinct for all $\lambda \in \mathbb{C}$ apart from some exceptional values of $\lambda$, and the number of such exceptional values is finite in each finite domain. Since $e^{i(k_1+k_2+k_3)} = 1$, formula (1.9) yields

$$\begin{aligned} \rho &= (e^{ik_1} - e^{ik_2})^2 (e^{ik_1} - e^{ik_3})^2 (e^{ik_2} - e^{ik_3})^2 \\ &= -64 \sin^2 \frac{k_1 - k_2}{2} \sin^2 \frac{k_1 - k_3}{2} \sin^2 \frac{k_2 - k_3}{2}. \end{aligned} \tag{5.11}$$

If the spectrum has multiplicity 3 at a point $\lambda$, then, by Theorem 3.2 iii), $k_j(\lambda) \in \mathbb{R}$ for all $j = 1, 2, 3$ and (5.11) shows that $\rho(\lambda) \le 0$. If the spectrum has multiplicity 1 at a point $\lambda$, then precisely one of the $k_j$, say $k_1$, is real and $k_2 = \bar{k}_3$ are not real. From (5.11) it follows that $\rho(\lambda) > 0$, which yields (1.11). The function $\rho$ has finitely many zeros on any bounded interval of the real line; thus, the spectrum of multiplicity 3 consists of a finite number of intervals. □

In conclusion, we prove the following property of the Lyapunov function.

**Proposition 5.5.** *If* $\Delta_j(\lambda) \in (-1, 1)$ *for some* $(j, \lambda) \in \{1, 2, 3\} \times \mathbb{R}$ *and* $\lambda$ *is not a branch point of the function* $\Delta_j$, *then* $\Delta_j'(\lambda) \ne 0$.

*Proof.* The proof is standard, and we present it for completeness. Suppose $\lambda_0 \in \mathbb{R}$ satisfies the following conditions: $\Delta_j(\lambda_0) \in (-1, 1)$ for some $j = 1, 2, 3$, $\lambda_0$ is not a branch point of the function $\Delta_j$, and $\Delta_j'(\lambda_0) = 0$. Then $\Delta_j(\lambda) = \Delta_j(\lambda_0) + \frac{1}{2}\Delta_j''(\lambda_0)(\lambda - \lambda_0)^2 + O((\lambda - \lambda_0)^3)$ as $\lambda - \lambda_0 \to 0$. Consider the mapping $\lambda \to \Delta_j(\lambda)$ in a neighborhood of the point $\lambda_0$. Any angle formed by two lines starting at the point $\lambda_0$ is carried by this mapping to an angle that is two or more times larger. Then the segment $[\Delta_j(\lambda_0)-\delta, \Delta_j(\lambda_0)+\delta] \subset [-1,1]$ for some sufficiently small $\delta > 0$ has a preimage that cannot lie entirely on the real line. From (5.10) it follows that $H$ has a nonreal spectrum, which contradicts the fact that $H$ is selfadjoint. Thus, $\Delta_j'(\lambda_0) \neq 0$, which proves the claim. □

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Department of Physics, St. Petersburg State University, Ul′yanovskaya 2, Staryi Peterhof, St. Petersburg 198904, Russia

*E-mail address*: an.badanin@gmail.com

Department of Physics, St. Petersburg State University, Ul′yanovskaya 2, Staryi Peterhof, St. Petersburg 198904, Russia

*E-mail address*: korotyaev@gmail.com



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