Алгебра и анализ
Том 25 (2013), № 6

St. Petersburg Math. J.
Vol. 25 (2014), No. 6, Pages 929–963
S 1061-0022(2014)01324-6
Article electronically published on September 8, 2014

TOWARD THE THEORY OF ORLICZ–SOBOLEV CLASSES

D. A. KOVTONYUK, V. I. RYAZANOV, R. R. SALIMOV, AND E. A. SEVOST′YANOV

Abstract. It is shown that, under a Calderón type condition on the function φ, the continuous open mappings that belong to the Orlicz–Sobolev classes $W^{1,\varphi}_{\mathrm{loc}}$ have total differential almost everywhere; this generalizes the well-known theorems of Gehring–Lehto–Menchoff in the case of $\mathbb{R}^2$ and of Väisälä in $\mathbb{R}^n$, $n \geq 3$. Appropriate examples show that the Calderón type condition is not only sufficient but also necessary. Moreover, under the same condition on φ, it is also proved that the continuous mappings of class $W^{1,\varphi}_{\mathrm{loc}}$ and, in particular, of class $W^{1,p}_{\mathrm{loc}}$ for $p > n-1$ have Lusin's (N)-property on a.e. hyperplane. On that basis, it is shown that, under the same condition on φ, the homeomorphisms f with finite distortion of class $W^{1,\varphi}_{\mathrm{loc}}$ and, in particular, those belonging to $W^{1,p}_{\mathrm{loc}}$ for $p > n-1$, are what is called lower Q-homeomorphisms, where Q is equal to their outer dilatation K_f; also, they are so-called ring Q_*-homeomorphisms with $Q_* = K_f^{n-1}$. The latter fact makes it possible to fully apply the theory of the boundary and local behavior of the ring and lower Q-homeomorphisms, as developed earlier by the authors, to the study of mappings in the Orlicz–Sobolev classes.

Part 1. Differentiability and behavior of Hausdorff's measures in the Orlicz–Sobolev classes

§1. Introduction

The theory of mappings with bounded distortion, as developed in the work of academician Yu. G. Reshetnyak and his school (S. K. Vodop′yanov, V. M. Goldstein and others), became classics of the mapping theory long ago, see, e.g., the monographs [146, 27] and [60], and also [28], as well as the relatively recent papers [25, 26]. Recall that a continuous mapping $f\colon U \to \mathbb{R}^n$ of an open set U in $\mathbb{R}^n$, $n \geq 2$, is called a *mapping with bounded distortion* if $f \in W^{1,n}_{\mathrm{loc}}$, its Jacobian $J_f(x) = \det f'(x)$ keeps its sign in U, and

$$\|f'(x)\|^n \leq K\,|J_f(x)| \qquad \text{a.e.} \tag{1}$$

for some number $K \in [1,\infty)$, where $f'(x)$ is the Jacobi matrix of f, and $\|f'(x)\|$ is its operator norm: $\|f'(x)\| = \sup |f'(x)\cdot h|$ with the supremum taken over all vector columns h in $\mathbb{R}^n$ of unit length. Such mappings are said to be *quasiregular* in the foreign literature, see, e.g., [71, 119] and [150].

During the last decades, a new theory of mappings with finite distortion has been developed. Recall that a mapping $f\colon U \to \mathbb{R}^n$ of an open set U in $\mathbb{R}^n$, $n \geq 2$, is said to have *finite distortion* if $f \in W^{1,1}_{\mathrm{loc}}$, $J_f \in L^1_{\mathrm{loc}}$, and

$$\|f'(x)\|^n \leq K(x)\cdot J_f(x) \tag{2}$$

2010 *Mathematics Subject Classification.* Primary 46E35.

Key words and phrases. Moduli of families of curves and surfaces, mappings with bounded and finite distortion, differentiability, Lusin and Sard properties, Sobolev classes, Orlicz–Sobolev classes, boundary and local behavior.

with an a.e. finite function K. In what follows, we use the notation $K_f(x)$ for the smallest function $K(x) \geq 1$ in (2), i.e., we set $K_f(x) = \|f'(x)\|^n / J_f(x)$ if $J_f(x) \neq 0$, $K_f(x) = 1$ if $f'(x) = 0$ and $K_f(x) = \infty$ at the other points.

This notion was introduced in [82] in the plane case for $f \in W^{1,2}_{\mathrm{loc}}$. Afterward, this condition was replaced by the requirement $f \in W^{1,1}_{\mathrm{loc}}$ implying in addition that $J_f \in L^1_{\mathrm{loc}}$, see, e.g., the book [80]. In fact, similar mappings were studied long ago in the framework of the theory of mappings with bounded Dirichlet integral, see, e.g., [112] and [174, 175, 176], and of mappings quasiconformal in the mean, see [1, 18, 60, 65, 99, 100, 101, 102, 103, 104, 108, 136, 137, 138] and [152].

Note that the above additional condition $J_f \in L^1_{\mathrm{loc}}$ in the definition of the mappings with finite distortion can be omitted for homeomorphisms. Indeed, for each homeomorphism f between domains D and D' in $\mathbb{R}^n$ having the first partial derivatives a.e. in D, there is a set E of Lebesgue measure zero such that f satisfies Lusin's (N)-property on $D \setminus E$, and

$$\int_A J_f(x)\, dm(x) = |f(A)| \tag{3}$$

for every Borel set $A \subset D \setminus E$, see, e.g., 3.1.4, 3.1.8 and 3.2.5 in [50].

Moduli of families of curves and surfaces are the main geometric tool in the mapping theory. The recent development of the moduli method is closely linked with the modern classes of mappings, see, e.g., [123], and partial differential equations, see, e.g., [21] and [64] and also the recent books [13, 47] and [183] on the moduli and capacity theory, as well as the following papers and monographs: [5, 6, 7, 8, 9, 10, 32, 39, 40, 41, 42, 43, 44, 66, 58, 56, 57, 59, 87, 107, 171, 177, 179, 182, 185], and further references therein.

In the present paper, we show that the theories of mappings satisfying certain modulus conditions (the so-called lower and ring Q-homeomorphisms), recently developed by us and published, e.g., in [123], can be applied to a wide range of Orlicz–Sobolev mappings and, in particular, to the Sobolev classes $W^{1,p}_{\mathrm{loc}}$ for $p > n-1$ in $\mathbb{R}^n$, $n \geq 3$. Note that the corresponding plane case was studied by us earlier, see, e.g., the papers [88, 92, 90] and [114] where it was established that any homeomorphism f with finite distortion is a lower and ring Q-homeomorphism with $Q(x) = K_f(x)$.

In what follows, D is a domain in $\mathbb{R}^n$, $n \geq 2$, and m is the Lebesgue measure in $\mathbb{R}^n$. Following Orlicz, see [133] and [134], given a convex monotone increasing function $\varphi\colon \mathbb{R}^+ \to \mathbb{R}^+$, $\varphi(0) = 0$, we denote by L_φ the space of all functions $f\colon D \to \mathbb{R}$ such that

$$\int_D \varphi\left(\frac{|f(x)|}{\lambda}\right) dm(x) < \infty \tag{4}$$

for some $\lambda > 0$, see also [98]. L_φ is called the *Orlicz space*. In other words, L_φ is the cone over the class of all functions $g\colon D \to \mathbb{R}$ such that

$$\int_D \varphi(|g(x)|)\, dm(x) < \infty, \tag{5}$$

which is called the *Orlicz class*, see [19].

The *Orlicz–Sobolev class* $W^{1,\varphi}(D)$ is the class of all functions $f \in L^1(D)$ having the first distributional derivatives and with gradient ∇f belonging to the Orlicz class in D. We write $f \in W^{1,\varphi}_{\mathrm{loc}}(D)$ if $f \in W^{1,\varphi}(D_*)$ for every domain D_* with compact closure in D. Note that, by definition, $W^{1,\varphi}_{\mathrm{loc}} \subseteq W^{1,1}_{\mathrm{loc}}$. As usual, we write $f \in W^{1,p}_{\mathrm{loc}}$ if $\varphi(t) = t^p$, $p \geq 1$. It is known that a continuous function f belongs to $W^{1,p}_{\mathrm{loc}}$ if and only if $f \in ACL^p$, i.e., if f is locally absolutely continuous on a.e. straight line parallel to a coordinate axis and the first partial derivatives of f are locally integrable with the power p, see, e.g., 1.1.3

in [128]. The concept of the distributional (generalized) derivative was introduced by Sobolev [170] in $\mathbb{R}^n$, $n \geq 2$, and at present it is developed under wider settings by many authors, see, e.g., [4, 29, 30, 67, 70, 72, 73, 118, 123] and [148].

In what follows, we also write $f \in W^{1,\varphi}_{\mathrm{loc}}$ for a locally integrable vector-valued function $f = (f_1, \dots, f_m)$ of n real variables $x_1, \dots, x_n$ if $f_i \in W^{1,1}_{\mathrm{loc}}$ and

$$\int_D \varphi(|\nabla f(x)|)\, dm(x) < \infty, \tag{6}$$

where

$$|\nabla f(x)| = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} \left(\frac{\partial f_i}{\partial x_j}\right)^2}.$$

We also use the notation $W^{1,\varphi}_{\mathrm{loc}}$ in the case of functions φ more general than in the Orlicz classes, always assuming the convexity of φ. Note that the Orlicz–Sobolev classes have been studied intensively in various aspects by many authors, both recently and in the past, see, e.g., [2, 12, 31, 34, 46, 61, 76, 79, 85, 86, 95, 96, 97, 109, 110, 132, 154, 157, 155, 178] and [184].

§2. Preliminaries

In this paper, we denote by H^k, $k \geq 0$, the *k-dimensional Hausdorff measure* in $\mathbb{R}^n$, $n \geq 1$. More precisely, if A is a set in $\mathbb{R}^n$, then

$$H^k(A) = \sup_{\varepsilon > 0} H^k_\varepsilon(A), \tag{7}$$

$$H^k_\varepsilon(A) = \inf \sum_{i=1}^{\infty} (\mathrm{diam}\ A_i)^k\,, \tag{8}$$

where the infimum in (8) is taken over all coverings of A by sets A_i with diam $A_i < \varepsilon$, see, e.g., [77] and [127]. It is known that the outer Lebesgue measure $m(A)$ is equal to $\Omega_n \cdot 2^{-n} H^n(A)$ for sets A in $\mathbb{R}^n$, where Ω_n denotes the volume of the unit ball in $\mathbb{R}^n$, see [166]. Note that H^k is an *outer measure in the sense of Caratheodory*, i.e.,

(1) $H^k(X) \leq H^k(Y)$ whenever $X \subseteq Y$,

(2) $H^k\left(\bigcup_i X_i\right) \leq \sum_i H^k(X_i)$ for each sequence of sets X_i,

(3) $H^k(X \cup Y) = H^k(X) + H^k(Y)$ whenever $\mathrm{dist}(X, Y) > 0$.

A set $E \subset \mathbb{R}^n$ is said to be *measurable* with respect to H^k if $H^k(X) = H^k(X \cap E) + H^k(X \setminus E)$ for every set $X \subset \mathbb{R}^n$. It is well known that every Borel set is measurable with respect to any outer measure in the sense of Caratheodory, see, e.g., [163, Theorem II (7.4)]. Moreover, H^k is Borel regular, i.e., for every set $X \subset \mathbb{R}^n$, there is a Borel set $B \subset \mathbb{R}^n$ such that $X \subset B$ and $H^k(X) = H^k(B)$, see, e.g., [163, Theorem II (8.1)] and [50, Section 2.10.1]. This implies that, for every measurable set $E \subset \mathbb{R}^n$, there exist Borel sets B_* and $B^* \subset \mathbb{R}^n$ such that $B_* \subset E \subset B^*$ and $H^k(B^* \setminus B_*) = 0$, see, e.g., [50, Section 2.2.3]. In particular, $H^k(B^*) = H^k(E) = H^k(B_*)$.

If $H^{k_1}(A) < \infty$, then $H^{k_2}(A) = 0$ for every $k_2 > k_1$, see, e.g., [77, VII.1.B]. The quantity

$$\dim_H A = \sup_{H^k(A) > 0} k$$

is called the *Hausdorff dimension* of A. It was shown in [55] that any set A with $\dim_H A = p$ can be transformed into a set $B = f(A)$ with $\dim_H B = q$ for each pair of numbers p and $q \in (0, n)$ by a quasiconformal mapping f of $\mathbb{R}^n$ onto itself, see also [14] and [20].

Recall also that a *k-dimensional direction* Γ in $\mathbb{R}^n$ is the equivalence class of all k-dimensional planes in $\mathbb{R}^n$ that can be obtained each from each other by a parallel shift. Note that each $(n-k)$-dimensional plane $\mathcal{T}$ orthogonal to a k-dimensional plane $\mathcal{P}$ in Γ intersects $\mathcal{P}$ at a single point $X(\mathcal{P})$. If E is a subset of Γ, then $X(E)$ denotes the collection of all points $X(\mathcal{P})$, $\mathcal{P} \in E$. It is clear that the $(n-k)$-dimensional measure $\mu_{n-k}(E)$ of the set $X(E)$ does not depend of the choice of the plane $\mathcal{T}$. A property is said to hold for almost every plane in Γ if $\mu_{n-k}(E) = 0$ for the set E of all planes $\mathcal{P}$ for which the property fails.

The following remarkable property of functions f in the Sobolev classes $W^{1,p}_{\mathrm{loc}}$ was proved in the monograph [60], see Theorem 5.5 therein, and can be extended to the Orlicz–Sobolev classes. This statement follows directly from the Fubini theorem and the known characterization of functions in Sobolev's class $W^{1,1}_{\mathrm{loc}}$ in terms of ACL (absolute continuity on lines), see, e.g., [128, 1.1.3], and the comments in the Introduction.

Proposition 1. *Let U be an open set in $\mathbb{R}^n$, and let $f\colon U \to \mathbb{R}^m$, $m \geq 1$, be a mapping in the Orlicz–Sobolev class $W^{1,\varphi}_{\mathrm{loc}}(U)$ with a monotone increasing function $\varphi\colon [0,\infty) \to [0,\infty)$. Then, for every k-dimensional direction Γ and a.e. k-dimensional plane $\mathcal{P} \in \Gamma$, $k = 1, 2, \ldots, n-1$, the restriction of the function f to the set $\mathcal{P} \cap U$ is a function of class $W^{1,\varphi}_{\mathrm{loc}}(\mathcal{P} \cap U)$.*

Here the class $W^{1,\varphi}_{\mathrm{loc}}$ is well defined on an almost every k-dimensional plane, because the partial derivatives are Borel functions and, moreover, Sobolev's classes are invariant with respect to quasiisometric transformations of systems of coordinates, in particular, with respect to rotations, see, e.g., [128, 1.1.7].

Recall the Väisälä–Fadell theorem, see [49] and [181]. With the use of it, the well-known theorems of Gehring–Lehto–Menchoff in the case of the plane and of Väisälä in $\mathbb{R}^n$, $n \geq 3$, see, e.g., [53, 129] and [181], on the differentiability a.e. of open Sobolev classes mapping can be extended to the case of open mappings in Orlicz–Sobolev classes in $\mathbb{R}^n$, $n \geq 3$. Recall that a mapping $f\colon \Omega \to \mathbb{R}^n$ is said to be *open* if the image of every open set in Ω under f is an open set in $\mathbb{R}^n$.

Proposition 2. *Let $f\colon \Omega \to \mathbb{R}^n$ be a continuous open mapping given on an open set Ω in $\mathbb{R}^n$, $n \geq 3$. If f has a total differential a.e. on Ω with respect to each collection of $n-1$ variables, then f has a total differential a.e. on Ω with respect to all n variables.*

Finally, we recall the following fundamental Calderón result, see [31, p. 28].

Proposition 3. *Let $\varphi\colon [0,\infty) \to [0,\infty)$ be a monotone increasing function such that $\varphi(0) = 0$ and*

$$A := \int_0^\infty \left[\frac{t}{\varphi(t)}\right]^{\frac{1}{k-1}} dt < \infty \tag{9}$$

for a natural number $k \geq 2$, and let $f\colon D \to \mathbb{R}$ be a continuous function defined on a domain $D \subset \mathbb{R}^k$ and belonging to $W^{1,\varphi}(G)$. Then

$$\operatorname{diam} f(C) \leq \alpha_k A^{\frac{k-1}{k}} \left[\int_C \varphi(|\nabla f|)\, dm(x)\right]^{\frac{1}{k}} \tag{10}$$

for every cube $C \subset G$ whose edges are directed along coordinate axes; here α_k is a constant depending only on k.

Note that Lemma 3.2 in [85] is in fact a reformulation of this Calderón result, with no reference to Calderón's work. Perhaps, the paper [31] had enough time to be forgotten, because it was published long ago in a hardly accessible journal.

Remark 1. The assumption that φ is (strictly!) monotone increasing is not essential. Indeed, let φ be only monotone nondecreasing. Passing if necessary to the new function

$$\widetilde{\varphi}_\varepsilon(t) := \varphi(t) + \sum_i \varphi_i^{(\varepsilon)}(t),$$

where

$$\varphi_i^{(\varepsilon)}(t) := \varepsilon\,\frac{2^{-i}}{(b_i - a_i)} \int_0^t \chi_i(t)\,dt$$

and the χ_i denote the characteristic functions of the constancy intervals (a_i, b_i) of the function φ, we see that $\varphi(t) \le \widetilde{\varphi}_\varepsilon(t) \le \varphi(t) + \varepsilon$, so that condition (6) on the cube C and condition (9) are fulfilled for the (strictly!) monotone increasing function $\widetilde{\varphi}_\varepsilon$. Letting $\varepsilon \to 0$, we obtain estimate (10) with the initial function φ, see, e.g., [163, Theorem I.12.1].

The function $(t/\varphi(t))^{1/(k-1)}$ can have a nonintegrable singularity at zero. However, it is clear that the behavior of φ near zero is not essential for estimate (10). Indeed, we may apply (10) with the replacements $A \mapsto A_*$ and $\varphi \mapsto \varphi_*$, where

$$A_* := t_* \left[\frac{1}{\varphi(t_*)}\right]^{\frac{1}{k-1}} + \int_{t_*}^{\infty} \left[\frac{t}{\varphi(t)}\right]^{\frac{1}{k-1}} dt < \infty \tag{11}$$

and $\varphi_*(0) = 0$, $\varphi_*(t) \equiv \varphi(t_*)$ for $t \in (0, t_*)$ and $\varphi_*(t) = \varphi(t)$ for $t \ge t_*$ if $\varphi(t_*) > 0$. Hence, in particular, the normalization $\varphi(0) = 0$ in Proposition 3 evidently does not matter.

§3. Differentiability of open mappings

We start with the following statement, which is due to Calderón [31], see also [172]. However, here we prefer, in contrast to [31], to prove it on the basis of the classic Stepanov theorem, see [173] and also [115].

Lemma 1. *Let Ω be an open set in $\mathbb{R}^k$, $k \ge 2$, and let $f\colon \Omega \to \mathbb{R}$ be a continuous mapping of class $W^{1,\varphi}_{\mathrm{loc}}(\Omega)$ with a monotone nondecreasing function $\varphi\colon [0,\infty) \to [0,\infty)$ such that, for some $t_* \in (0,\infty)$,*

$$A := \int_{t_*}^{\infty} \left[\frac{t}{\varphi(t)}\right]^{\frac{1}{k-1}} dt < \infty. \tag{12}$$

Then f has a total differential a.e. in Ω.

Proof. Given $x \in \Omega$, we set

$$L(x, f) = \limsup_{y \to x} \frac{|f(y) - f(x)|}{|y - x|}.$$

By the Stepanov theorem, the proof reduces to checking that $L(x, f) < \infty$ a.e. in Ω.

Denote by $C(x, r)$ the oriented cube centered at x such that the ball $B(x, r)$ is inscribed in $C(x, r)$ with $r = |x - y|$. Then

$$L(x, f) = \limsup_{y \to x} \frac{|f(x) - f(y)|}{|x - y|} \le \limsup_{r \to 0} \frac{d(fB(x, r))}{r} \le \limsup_{r \to 0} \frac{d(fC(x, r))}{r}.$$

Using Proposition 3, Remark 1, and the triangle inequality, we get

$$L(x, f) \le m\alpha_k\, A_*^{\frac{k-1}{k}} \limsup_{r \to 0} \left[\frac{1}{r^k} \int_{C(x,r)} \varphi_*(|\nabla f|)\, dm(x)\right]^{\frac{1}{k}} < \infty$$

for a.e. $x \in \Omega$, in view of the Lebesgue theorem on differentiability of indefinite integrals, see, e.g., [163, Theorem IV.5.4]. The proof is complete. □

Combining Lemma 1 and Proposition 1, we obtain the following statement.

Corollary 1. *Let Ω be an open set in $\mathbb{R}^n$, $n \geq 3$, and let $f\colon \Omega \to \mathbb{R}$ be a continuous mapping of class $W^{1,\varphi}_{\mathrm{loc}}(\Omega)$ with a monotone nondecreasing function $\varphi\colon (0,\infty) \to (0,\infty)$ such that, for some $t_* \in (0,\infty)$,*

$$\int_{t_*}^{\infty} \left[\frac{t}{\varphi(t)}\right]^{\frac{1}{n-2}} dt < \infty. \tag{13}$$

Then on a.e. hyperplane $\mathcal{P}$ parallel to a fixed hyperplane $\mathcal{P}_0$, the mapping $f\big|_{\mathcal{P}}$ has a total differential a.e..

Combining Corollary 1 and the Väisälä–Fadell result, see Proposition 2 above, we obtain the main result of this section.

Theorem 1. *Let Ω be an open set in $\mathbb{R}^n$, $n \geq 3$, and let $f\colon \Omega \to \mathbb{R}^n$ be a continuous open mapping of class $W^{1,\varphi}_{\mathrm{loc}}(\Omega)$ with a monotone nondecreasing function $\varphi\colon (0,\infty) \to (0,\infty)$ satisfying* (13)*. Then f has a total differential a.e. in Ω.*

Remark 2. In particular, the conclusion of Theorem 1 is valid for continuous open mappings $f \in W^{1,p}_{\mathrm{loc}}$ with $p > n-1$. This statement is the Väisälä result, see [181, Lemma 3]. Theorem 1 is also an extension to high dimensions of the well-known Gegring–Lehto–Menchoff result in the plane, see, e.g., [53, 111] and [129].

The corresponding results for weakly monotone mappings f with derivatives in the Lorentz classes $L^{n-1,1}$ can be found in [132]. It was shown in [85] that the functions on $\mathbb{R}^k$ with generalized derivatives in the Lorentz class $L^{k,1}$ can be described as functions in the Orlicz–Sobolev classes $W^{1,\varphi}_{\mathrm{loc}}$ under the Calderón condition (12) imposed on φ. Thus, for open mappings, the results of [132] follow from the Calderón result on differentiability.

In [31], Calderón proved that condition (12) for the differentiability a.e. of continuous mappings $f \in W^{1,\varphi}_{\mathrm{loc}}$ is sharp in the case of convex φ. However, Theorem 1 shows that we may use the weaker condition (13) to obtain the differentiability a.e. of continuous open mappings.

Condition (13) is not only sufficient but also necessary for continuous open mappings $f \in W^{1,\varphi}_{\mathrm{loc}}$ from $\mathbb{R}^n$ into $\mathbb{R}^n$, $n \geq 3$, to have total differential a.e. Furthermore, if a function $\varphi\colon (0,\infty) \to (0,\infty)$ is monotone increasing, convex, and such that

$$\int_{1}^{\infty} \left[\frac{t}{\varphi(t)}\right]^{\frac{1}{n-2}} dt = \infty\ , \tag{14}$$

then there is a homeomorphism $g\colon \mathbb{R}^n \to \mathbb{R}^n$, $n \geq 3$, of class $W^{1,\varphi}_{\mathrm{loc}}$ that does not have total differential a.e. Indeed, if $f\colon \mathbb{R}^{n-1} \to \mathbb{R}$ is a function in Calderón's construction for $k = n-1$ and $\varphi_n(t) = \varphi(t+n)$, then

$$\int_{1}^{\infty} \left[\frac{t}{\varphi_n(t)}\right]^{\frac{1}{n-2}} dt = \infty, \tag{15}$$

and $g(x,y) = (x, y + f(x))$, $x \in \mathbb{R}^{n-1}$, $y \in \mathbb{R}$, is the desired example, because $|\nabla g| \leq n + |\nabla f|$ and the function φ is monotone. Thus, condition (13) cannot be relaxed even for homeomorphisms.

§4. Lusin and Sard properties on surfaces

Theorem 2. *Let Ω be an open set in $\mathbb{R}^k$, $k \geq 2$, and let $f\colon \Omega \to \mathbb{R}^m$, $m \geq 1$, be a continuous mapping of class $W^{1,\varphi}(\Omega)$ with a monotone nondecreasing $\varphi\colon (0,\infty) \to (0,\infty)$ such that, for some $t_* \in (0,\infty)$, we have*

$$A := \int_{t_*}^{\infty} \left[\frac{t}{\varphi(t)}\right]^{\frac{1}{k-1}} dt < \infty. \tag{16}$$

Then

$$H^k(f(E)) \le \gamma_{k,m}\, A_*^{k-1} \int_E \varphi_*(|\nabla f|)\, dm(x) \tag{17}$$

for every measurable set $E \subset \Omega$ and $\gamma_{k,m} = (m\alpha_k)^k$, where α_k is a constant as in (10) *depending only on k, $A_* = A + 1/[\varphi(t_*)]^{1/(k-1)}$, $\varphi_*(0) = 0$, $\varphi_*(t) \equiv \varphi(t_*)$ for $t \in (0, t_*)$, and $\varphi_*(t) = \varphi(t)$ for $t \ge t_*$.*

The proof of Theorem 2 is based on the following lemma.

Lemma 2. *Let Ω be an open set in $\mathbb{R}^k$, $k \ge 2$, and let $f\colon \Omega \to \mathbb{R}^m$, $m \ge 1$, be a continuous mapping of class $W^{1,\varphi}(\Omega)$ with a monotone nondecreasing $\varphi\colon (0,\infty) \to (0,\infty)$ satisfying* (16). *Then*

$$\operatorname{diam}\, f(C) \le m\alpha_k\, A_*^{\frac{k-1}{k}} \left[\, \int_C \varphi_*(|\nabla f|)\, dm(x)\right]^{\frac{1}{k}} \tag{18}$$

for every cube $C \subset \Omega$ whose edges are directed along the coordinate axes; *here α_k is the constant occurring in* (10) *and depending only on k, whereas A_* and φ_* are as defined in Theorem* 2.

Proof. We prove (18) by induction on $m = 1, 2, \ldots$. Indeed, for $m = 1$ (18) is true by Proposition 3 and Remark 1. Assuming that (18) is valid for some $m = l$, we prove it for $m = l+1$. Consider an arbitrary vector $\overrightarrow{V} = (v_1, v_2, \ldots, v_l, v_{l+1})$ in $\mathbb{R}^{l+1}$ and the vectors $\overrightarrow{V}_1 = (v_1, v_2, \ldots, v_l, 0)$ and $\overrightarrow{V}_2 = (0, \ldots, 0, v_{l+1})$. Then $|\overrightarrow{V}| = |\overrightarrow{V}_1 + \overrightarrow{V}_2| \le |\overrightarrow{V}_1| + |\overrightarrow{V}_2|$. Thus, denoting by $\mathrm{Pr}_1 \overrightarrow{V} = \overrightarrow{V}_1$ and $\mathrm{Pr}_2 \overrightarrow{V} = \overrightarrow{V}_2$ the projections of vectors in $\mathbb{R}^{l+1}$ to the coordinate hyperplane $y_{l+1} = 0$ and to the $(l+1)$st axis in $\mathbb{R}^{l+1}$, respectively, we see that $\operatorname{diam} f(C) \le \operatorname{diam} \mathrm{Pr}_1 f(C) + \operatorname{diam} \mathrm{Pr}_2 f(C)$. Now, applying (18) with $m = l$ and with $m = 1$ and using the monotonicity of φ, we arrive at inequality (18) with $m = l+1$. The proof is complete. □

Proof of Theorem 2. Using the countable additivity of the integral and measure, we may assume with no loss of generality that E is bounded and $\bar{E} \subset \Omega$, i.e., $\bar{E}$ is a compact set in Ω. For each $\varepsilon > 0$, there is an open set $\omega \subset \Omega$ such that $E \subset \omega$ and $|\omega \setminus E| < \varepsilon$, see, e.g., [163, Theorem III (6.6)]. By the above remark, we may assume that $\bar{\omega}$ is compact, so that the mapping f is uniformly continuous on ω. Hence, ω can be covered by a countable collection of closed oriented cubes $C_i \subset \omega$ with mutually disjoint interiors and such that $\operatorname{diam} f(C_i) < \delta$ for any $\delta > 0$ prescribed in advance, and $\left|\bigcup_{i=1}^{\infty} \partial C_i\right| = 0$.

Thus, by Lemma 2 we have

$$H^k_\delta(f(E)) \le H^k_\delta(f(\omega)) \le \sum_{i=1}^{\infty} [\operatorname{diam} f(C_i)]^k \le \gamma_{k,m} A_*^{k-1} \int_\omega \varphi_*(|\nabla f|)\, dm(x).$$

Finally, by the absolute continuity of the indefinite integral and the arbitrariness of ε and $\delta > 0$, we obtain (17). □

Corollary 2. *Under the assumptions of Theorem* 2, *the mapping f has the* (N)*-property of Lusin, and furthermore, f is absolutely continuous with respect to the k-dimensional Hausdorff measure.*

Remark 3. Note that $H^k(\mathbb{R}^m) = 0$ for $m < k$, so that (17) is trivial in this case without condition (16). However, this condition is necessary if $m \ge k$. It is known that each homeomorphism of $\mathbb{R}^k$ onto itself belonging to $W^{1,k}_{\mathrm{loc}}$ has the (N)-property, see [111, Lemma III.6.1] for $k = 2$ and [149] for $k > 2$. The same is true also for continuous open mappings, see [116]. On the other hand, there exist examples of homeomorphisms of class $W^{1,p}_{\mathrm{loc}}$ for all $p < k$ that do not have the (N)-property, see [139]. Moreover, in [33]

Cezari proved that the continuous plane mappings $f\colon D \to \mathbb{R}^2$ of class ACL^p, $p > 2$, have the (N)-property and that there exist examples of such mappings in ACL^2 without the (N)-property. The corresponding examples of continuous mappings $f \in W^{1,k}_{\mathrm{loc}}$ in $\mathbb{R}^k$, $k \geq 3$, can be found in [116]. The fact that the continuous mappings f belonging to the Sobolev classes $W^{1,p}_{\mathrm{loc}}$, $p > k$, in $\mathbb{R}^k$, $k \geq 3$, have the (N)-property was established in [117], see also [23]. The corresponding results for continuous mappings with derivatives in Lorentz classes can be found in [85].

The following Sard type consequence of Theorem 2 is valid for mappings in the Orlicz–Sobolev classes, see also [77, Theorem VII.3].

Corollary 3. *Under the assumptions of Theorem* 2, *we have* $H^k(f(E)) = 0$ *whenever* $|\nabla f| = 0$ *on a measurable set* $E \subset \Omega$; *hence,* $\dim_H f(E) \leq k$ *and also* $\dim f(E) \leq k-1$.

Remark 4. For the first time, such a statement was established by Sard in [165] for the set of *critical points* of f where $J_f(x) = 0$, and then similar problems where studied by many authors for the *critical points of rank* r where $\operatorname{rank} f'(x) \leq r$, and, in particular, for the *supercritical points* where the Jacobian matrix $f'(x)$ is null, see, e.g., [16, 37, 38, 48, 62, 68, 84, 131, 140, 167, 168], and [187]. As a rule, certain smoothness conditions were imposed on f, without which such statements are not true in general.

In this connection, it should be mentioned that our result on supercritical points, Corollary 3, is valid without any assumption on the smoothness of f. For instance, this result holds true for all continuous mappings f of class $W^{1,p}_{\mathrm{loc}}$ with $p > k$, see an excellent survey on Sard type theorems, in particular, for Sobolev mappings in the paper [22].

In what follows, $\nabla_k f$ denotes the k-dimensional gradient of the restriction of the mapping f to the k-dimensional plane P. Combining Proposition 1 and Corollary 2, we obtain the following statement.

Proposition 4. *Let* $k = 2, \ldots, n-1$, *let* U *be an open set in* $\mathbb{R}^n$, $n \geq 3$, *and let* $f\colon U \to \mathbb{R}^m$, $m \geq 1$, *be a continuous mapping of class* $W^{1,\varphi}_{\mathrm{loc}}(U)$ *for some monotone increasing function* $\varphi\colon (0,\infty) \to (0,\infty)$ *such that*

$$\int_{t_*}^{\infty} \left[\frac{t}{\varphi(t)}\right]^{\frac{1}{k-1}} dt < \infty \tag{19}$$

for some $t_* \in (0,\infty)$. *Then, for every* k-*dimensional direction* Γ *and almost every* k-*dimensional plane* $\mathcal{P} \in \Gamma$, *the restriction of* f *to the set* $\mathcal{P} \cap U$ *has the* (N)-*property* (*furthermore, it is locally absolutely continuous*) *with respect to the* k-*dimensional Hausdorff measure. Moreover, for a.e.* $\mathcal{P} \in \Gamma$, *we have* $H^k(f(E)) = 0$ *whenever* $\nabla_k f = 0$ *on a set* $E \subset \mathcal{P}$.

For us, the most important particular case of Proposition 4 is the following statement.

Theorem 3. *Let* U *be an open set in* $\mathbb{R}^n$, $n \geq 3$, *and let* $\varphi\colon (0,\infty) \to (0,\infty)$ *be a monotone nondecreasing function such that*

$$\int_{t_*}^{\infty} \left[\frac{t}{\varphi(t)}\right]^{\frac{1}{n-2}} dt < \infty \tag{20}$$

for some $t_* \in (0,\infty)$. *Then each continuous mapping* $f\colon U \to \mathbb{R}^m$, $m \geq 1$, *of class* $W^{1,\varphi}_{\mathrm{loc}}$ *has the* (N)-*property* (*furthermore, it is locally absolutely continuous*) *with respect to the* $(n-1)$-*dimensional Hausdorff measure on a.e. hyperplane* $\mathcal{P}$ *that is parallel to a fixed hyperplane* $\mathcal{P}_0$. *Moreover,* $H^{n-1}(f(E)) = 0$ *whenever* $|\nabla f| = 0$ *on* $E \subset \mathcal{P}$ *for a.e.* $\mathcal{P}$ *of this type.*

Note that if condition (20) is fulfilled for a monotone nondecreasing function φ, then so it is for the function $\varphi_c = \varphi(ct)$ with $c > 0$. Moreover, the Hausdorff measures are quasiinvariant under quasiisometries.

By the Lindelöf property of $\mathbb{R}^n$, see, e.g., [105, I.5.XI] for the Lindelöf theorem, $U \setminus \{x_0\}$ can be covered by a countable collection of open segments of spherical annuli in $U \setminus \{x_0\}$ centered at x_0, and each such segment can be mapped onto a rectangular oriented segment of $\mathbb{R}^n$ by some quasiisometry. Thus, applying Theorem 3 piecewise, we obtain the following conclusion.

Corollary 4. *Under condition* (20), *each continuous mapping $f \in W^{1,\varphi}_{\mathrm{loc}}$ has the* (N)*-property* (*furthermore, it is locally absolutely continuous*) *on a.e. sphere S centered at a prescribed point $x_0 \in \mathbb{R}^n$. Moreover, $H^{n-1}(f(E)) = 0$ whenever $|\nabla f| = 0$ on $E \subseteq S$ for a.e. sphere S of this type.*

Remark 5. Note that (20) does not imply the (N)-property of $f\colon U \to \mathbb{R}^n$ in U with respect to the Lebesgue measure in $\mathbb{R}^n$. This follows, in particular, from Ponomarev's examples of homeomorphisms $f \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$, for all $p < n$, without the (N)-property, see [139].

In particular, (20) is true for the functions $\varphi(t) = t^p$, $p > n-1$, i.e., the properties given in Theorem 3 hold for $f \in W^{1,p}_{\mathrm{loc}}$, $p > n-1$. However, this is not true for $p < n-1$. Furthermore, this is not true for homeomorphisms $f\colon U \to \mathbb{R}^n$ in $W^{1,p}_{\mathrm{loc}}$ with $p < n-1$, as follows from Ponomarev's examples. Indeed, if $g(x)$ is such an example in $\mathbb{R}^{n-1}$, then the function $f(x,y) = (g(x), y)$, $x \in \mathbb{R}^{n-1}$, $y \in \mathbb{R}$, fails to have the (N)-property on every hyperplane $y = \mathrm{const}$. The case where $p = n-1$ was investigated in [36].

If $m < n-1$, then $H^{n-1}(\mathbb{R}^m) = 0$ and the (N)-property on a.e. hyperplane is obvious without condition (20) for the mapping f in Theorem 3. However, if $m \geq n-1$, then condition (20) is necessary, see Remark 3.

The relationship of estimates of Calderón type (10) with the (N)-property and differentiability was first found in the study of the so-called generalized Lipschitzians in the sense of Rado, see, e.g., [31] and [141, V.3.6], cf. also the recent papers [15, 85, 143].

Part 2. Description of lower Q-homeomorphisms and their relationship with the Orlicz–Sobolev classes

§5. Moduli of families of surfaces

As in [123], in what follows a (continuous) mapping $S\colon \omega \to \mathbb{R}^n$ is called a *k-dimensional surface* S in $\mathbb{R}^n$, where ω be an open set in $\overline{\mathbb{R}^k}$, $k = 1, \dots, n-1$. The number of preimages

$$N(S,y) = \operatorname{card} S^{-1}(y) = \operatorname{card}\{x \in \omega : S(x) = y\}, \quad y \in \mathbb{R}^n, \tag{21}$$

is the *multiplicity function* of the surface S. In other words, $N(S,y)$ denotes the multiplicity of covering of the point y by the surface S. It is known that the multiplicity function is lower semicontinuous, i.e.,

$$N(S,y) \geq \liminf_{m\to\infty} N(S,y_m)$$

for every sequence $y_m \in \mathbb{R}^n$, $m = 1, 2, \dots$, such that $y_m \to y \in \mathbb{R}^n$ as $m \to \infty$; see, e.g., [141, p. 160]. Thus, the function $N(S,y)$ is Borel measurable and hence measurable with respect to every Hausdorff measure H^k; see, e.g., [163, Theorem II(7.6)].

Recall that a k-dimensional Hausdorff area in $\mathbb{R}^n$ (or simply *area*) associated with a surface $S\colon \omega \to \mathbb{R}^n$ is given by

$$\mathcal{A}_S(B) = \mathcal{A}_S^k(B) := \int_B N(S,y)\, dH^k y \tag{22}$$

for every Borel set $B \subseteq \mathbb{R}^n$ and, more generally, for an arbitrary set that is measurable with respect to H^k in $\mathbb{R}^n$, see [50, 3.2.1]. The surface S is said to be *rectifiable* if $\mathcal{A}_S(\mathbb{R}^n) < \infty$, see [123, 9.2].

If $\varrho\colon \mathbb{R}^n \to \overline{\mathbb{R}^+}$ is a Borel function, then its *integral over* S is defined by the formula

$$\int_S \varrho\, d\mathcal{A} := \int_{\mathbb{R}^n} \varrho(y)\, N(S,y)\, dH^k y. \tag{23}$$

Given a family Γ of k-dimensional surfaces S, we say that a Borel function $\varrho\colon \mathbb{R}^n \to \overline{\mathbb{R}^+}$ is *admissible* for Γ and write $\varrho \in \operatorname{adm}\Gamma$ if

$$\int_S \varrho^k\, d\mathcal{A} \geq 1 \tag{24}$$

for every $S \in \Gamma$. For $p \in (0,\infty)$, the *p-modulus* of Γ is the quantity

$$M_p(\Gamma) = \inf_{\varrho \in \operatorname{adm}\Gamma} \int_{\mathbb{R}^n} \varrho^p(x)\, dm(x). \tag{25}$$

We also set

$$M(\Gamma) = M_n(\Gamma) \tag{26}$$

and call the quantity $M(\Gamma)$ the *modulus of the family* Γ. The modulus is an outer measure on the space of all k-dimensional surfaces.

We say that Γ_2 is *minorized* by Γ_1 and write $\Gamma_2 > \Gamma_1$ if every $S \subset \Gamma_2$ has a subsurface that belongs to Γ_1. It is known that $M_p(\Gamma_1) \geq M_p(\Gamma_2)$, see [51, p. 176–178]. We also say that a property P is fulfilled for *p-a.e.* k-dimensional surface S in a family Γ if the subfamily of all surfaces of Γ for which P fails has the p-modulus zero. If $0 < q < p$, then P also occurs for q-a.e. S, see [51, Theorem 3]. In the case where $p = n$, we write simply a.e. (almost every).

Remark 6. The definition of the modulus immediately implies that, for every $p \in (0,\infty)$ and $k = 1, \dots, n-1$,

(1) p-a.e. k-dimensional surface in $\mathbb{R}^n$ is rectifiable;

(2) given a Borel set B in $\mathbb{R}^n$ of (Lebesgue) measure zero, we have

$$\mathcal{A}_S(B) = 0 \tag{27}$$

for p-a.e. k-dimensional surface S in $\mathbb{R}^n$.

The following lemma was first proved in [92, Lemma 2.3], see also [123, Lemma 9.1].

Lemma 3. *Suppose $k = 1, \dots, n-1$, $p \in [k,\infty)$, and C is an open cube in $\mathbb{R}^n$, $n \geq 2$, whose edges are parallel to coordinate axis. If a property P is fulfilled for p-a.e. k-dimensional surface S in C, then P is also fulfilled for a.e. k-dimensional plane in C parallel to a k-dimensional coordinate plane H.*

The last a.e. is relative to the Lebesgue measure in the corresponding $(n-k)$-dimensional coordinate plane $H^\perp$ perpendicular to H.

The following statement, see [93, Theorem 2.11] or [123, Theorem 9.1], is an analog of the Fubini theorem, cf., e.g. [163, Theorem III(8.1)]. It extends Theorem 33.1 in [182], cf. also Theorem 3 in [51], Lemma 2.13 in [120], and Lemma 8.1 in [123].

Theorem 4. *Let $k = 1, \dots, n-1$, let $p \in [k, \infty)$, and let E be a subset in an open set $\Omega \subset \mathbb{R}^n$, $n \geq 2$. Then E is Lebesgue measurable in $\mathbb{R}^n$ if and only if E is measurable with respect to area on p-a.e. k-dimensional surface S in Ω. Moreover, $|E| = 0$ if and only if*

$$\mathcal{A}_S(E) = 0 \tag{28}$$

on p-a.e. k-dimensional surface S in Ω.

Remark 7. The Lusin theorem, see, e.g., [50, Section 2.3.5], implies that, for every measurable function $\varrho \colon \mathbb{R}^n \to \overline{\mathbb{R}^+}$, there is a Borel function $\varrho^* \colon \mathbb{R}^n \to \overline{\mathbb{R}^+}$ with $\varrho^* = \varrho$ a.e. in $\mathbb{R}^n$. Thus, by Remark 6 and Theorem 4, ϱ is measurable on p-a.e. k-dimensional surface S in $\mathbb{R}^n$ for every $p \in (0, \infty)$ and $k = 1, \dots, n-1$.

We say that a Lebesgue measurable function $\varrho \colon \mathbb{R}^n \to \overline{\mathbb{R}^+}$ is *p-extensively admissible* for a family Γ of k-dimensional surfaces S in $\mathbb{R}^n$ and write $\varrho \in \operatorname{ext}_p \operatorname{adm} \Gamma$ if

$$\int_S \varrho^k \, d\mathcal{A} \geq 1 \tag{29}$$

for p-a.e. $S \in \Gamma$. The *p-extensive modulus* $\overline{M}_p(\Gamma)$ of Γ is the quantity

$$\overline{M}_p(\Gamma) = \inf \int_{\mathbb{R}^n} \varrho^p(x) \, dm(x), \tag{30}$$

where the infimum is taken over all $\varrho \in \operatorname{ext}_p \operatorname{adm} \Gamma$. For $p = n$, we use the notation $\overline{M}(\Gamma)$ and write $\varrho \in \operatorname{ext} \operatorname{adm} \Gamma$. For every $p \in (0, \infty)$ and $k = 1, \dots, n-1$, we have

$$\overline{M}_p(\Gamma) = M_p(\Gamma) \tag{31}$$

for every family Γ of k-dimensional surfaces in $\mathbb{R}^n$.

§6. Lower and ring Q-homeomorphisms

The mappings that we start to study in this section are not only of interest themselves but also necessary for us as a tool to derive important statements about the Orlicz–Sobolev classes. In [52, Section 13, F], Gehring defined a K-quasiconformal mapping as a homeomorphism changing the modulus of ring domains by at most K times. The following concept is motivated by the Gehring ring definition of quasiconformal mappings.

Given domains D and D' in $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$, $n \geq 2$, and a measurable function $Q \colon D \to (0, \infty)$, we say that a homeomorphism $f \colon D \to D'$ is a *lower Q-homeomorphism at a point* $x_0 \in \bar{D} \setminus \{\infty\}$ if

$$M(f\Sigma_\varepsilon) \geq \inf_{\varrho \in \operatorname{ext} \operatorname{adm} \Sigma_\varepsilon} \int_{D \cap R_\varepsilon} \frac{\varrho^n(x)}{Q(x)} \, dm(x) \tag{32}$$

for every ring

$$R_\varepsilon = \{x \in \mathbb{R}^n : \varepsilon < |x - x_0| < \varepsilon_0\}, \quad \varepsilon \in (0, \varepsilon_0), \ \ \varepsilon_0 \in (0, d_0),$$

where

$$d_0 = \sup_{x \in D} |x - x_0|, \tag{33}$$

and Σ_ε denotes the family of all intersections of the spheres

$$S(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| = r\}, \qquad r \in (\varepsilon, \varepsilon_0),$$

with D. This notion can be extended to the case of $x_0 = \infty \in \bar{D}$ by applying the inversion T with respect to the unit sphere in $\overline{\mathbb{R}^n}$, $T(x) = x/|x|^2$, $T(\infty) = 0$, $T(0) = \infty$. Namely, a homeomorphism $f \colon D \to D'$ is called a *lower Q-homeomorphism at* $\infty \in \bar{D}$ if $F = f \circ T$ is a lower Q_*-homeomorphism at 0 for $Q_* = Q \circ T$.

Finally, we say that a homeomorphism $f\colon D\to\overline{\mathbb{R}^n}$ is a *lower Q-homeomorphism in D* if f is a lower Q-homeomorphism at every point $x_0\in\bar{D}$.

Recall a criterion for homeomorphisms in $\mathbb{R}^n$ to be lower Q-homeomorphisms, see [92, Theorem 2.1] or [123, Theorem 9.2].

Proposition 5. *Let D and D' be domains in $\overline{\mathbb{R}^n}$, $n\ge 2$, let $x_0\in\bar{D}\setminus\{\infty\}$, and let $Q\colon D\to(0,\infty)$ be a measurable function. A homeomorphism $f\colon D\to D'$ is a lower Q-homeomorphism at x_0 if and only if*

$$M(f\Sigma_\varepsilon)\ge\int_\varepsilon^{\varepsilon_0}\frac{dr}{\|\,Q\|_{n-1}(r)}\quad \text{for all}\quad \varepsilon\in(0,\varepsilon_0)\,,\ \varepsilon_0\in(0,d_0), \tag{34}$$

where

$$\|Q\|_{n-1}(r)=\left(\int_{D(x_0,r)}Q^{n-1}(x)\,d\mathcal{A}\right)^{\frac{1}{n-1}} \tag{35}$$

is the L_{n-1}-norm of Q over $D(x_0,r)=D\cap S(x_0,r)=\{x\in D\,:\,|x-x_0|=r\}$.

Note that the infimum on the right-hand side in (32) is attained at the function

$$\rho_0(x)=\frac{Q(x)}{\|Q\|_{n-1}(|x-x_0|)}\ .$$

In what follows, as usual, for sets A, B, and C in $\overline{\mathbb{R}^n}$, we denote by $\Gamma(A,B,C)$ the family of all paths joining A and B in C.

Now, let $D\subset\mathbb{R}^n$ and $D'\subset\overline{\mathbb{R}^n}$ be two domains, $n\ge 2$, and let $Q\colon D\to[0,\infty]$ be a measurable function. Let $S_i:=S(x_0,r_i)$. A homeomorphism $f\colon D\to D'$ is called a *ring Q-homeomorphism at a point* $x_0\in\bar{D}$ if

$$M(f(\Gamma(S_1,S_2,D)))\le\int_{A\cap D}Q(x)\cdot\eta^n(|x-x_0|)\,dm(x) \tag{36}$$

for every ring $A=A(x_0,r_1,r_2)$, $0<r_1<r_2<d_0=\operatorname{dist}(x_0,\partial D)$, and for every measurable function $\eta\colon(r_1,r_2)\to[0,\infty]$ such that

$$\int_{r_1}^{r_2}\eta(r)\,dr\ge 1. \tag{37}$$

The notion of a ring Q-homeomorphism can be extended to ∞ in a standard way, as in the case of a lower Q-homeomorphism above.

The notion of a ring Q-homeomorphism was first introduced for inner points of a domain in [158] in connection with the study of Beltrami equations on the plane; then it was extended to the space case in [155], see also the book [123]. This notion was extended to boundary points in the papers [113] and [160, 161, 162], see also [64].

Corollary 5. *In $\mathbb{R}^n$ with $n\ge 2$, any lower Q-homeomorphism $f\colon D\to D'$ at a point $x_0\in\bar{D}$ with Q integrable in the power $n-1$ in a neighborhood of x_0, is a ring Q_*-homeomorphism at x_0 with $Q_*=Q^{n-1}$.*

Proof. Indeed, let $0<r_1<r_2<d(x_0,\partial D)$, and let $S_i=S(x_0,r_i)$, $i=1,2$. By the identities of Hesse and Ziemer, see, e.g., [75, 188] and also [123, Supplements A3 and A6], we have

$$M\big(f(\Gamma(S_1,S_2,D))\big)\le M^{1-n}(f(\Sigma)) \tag{38}$$

because $f(\Sigma)\subset\Sigma\big(f(S_1),f(S_2),f(D)\big)$, where Σ denotes the collection of all spheres centered at x_0 and located between the spheres S_1 and S_2, and $\Sigma\big(f(S_1),f(S_2),f(D)\big)$

consists of all $(n-1)$-dimensional surfaces in $f(D)$ separating $f(S_1)$ and $f(S_2)$. By Proposition 5, from (38) we obtain

$$M\left(f\big(\Gamma(S_1,S_2,D)\big)\right) \le I^{1-n}, \tag{39}$$

where the integral I is defined as in (34). However, by [155, Lemma 3.7], see also [123, Lemma 7.4], we have

$$I^{1-n} \le \int_{A\cap D} Q(x)\cdot \eta^n(|x-x_0|)\, dm(x) \tag{40}$$

for every Lebesgue measurable function $\eta\colon (r_1,r_2)\to[0,\infty]$ satisfying condition (37), where A is the ring $A(x_0,r_1,r_2)$. Finally, we deduce the conclusion of Corollary 5 from (39) and (40). □

Corollary 6. *For $n=2$, every lower Q-homeomorphism $f\colon D\to D'$ at a point $x_0\in\bar{D}$, with Q integrable in a neighborhood of x_0, is a ring Q-homeomorphism at x_0.*

Remark 8. Inequality (40) and, consequently, the conclusions of Corollaries 5 and 6 are valid if the function Q is integrable in the power $n-1$ on almost all spheres of sufficiently small radius centered at the point x_0.

Note also that, in the definitions of lower and ring Q-homeomorphisms, it suffices that the function Q be given only in the domain D or be extended by zero outside of D.

In the preprint [88], see also [89, 90, 114], it was proved that every homeomorphism f with finite distortion on the plane is a lower and ring Q-homeomorphism with $Q(x) = K_f(x)$. In the next section we show that a similar statement is true for homeomorphisms f with finite distortion in $\mathbb{R}^n$, $n\ge 3$, belonging to the Orlicz–Sobolev classes $W^{1,\varphi}_{\mathrm{loc}}$ provided the function φ satisfies the Calderón type condition (20).

§7. Lower Q-homeomorphisms and Orlicz–Sobolev classes

Recall first that a map $\varphi\colon X\to Y$ between metric spaces X and Y is said to be *Lipschitz* if $\operatorname{dist}(\varphi(x_1),\varphi(x_2))\le M\cdot\operatorname{dist}(x_1,x_2)$ for some $M<\infty$ and all x_1 and $x_2\in X$. The map φ is *bi-Lipschitz* if it is Lipschitz and $M^*\operatorname{dist}(x_1,x_2)\le\operatorname{dist}(\varphi(x_1),\varphi(x_2))$ for some $M^*>0$ and all x_1, $x_2\in X$.

The following statement plays a key role in our further research.

Theorem 5. *Let D and D' be domains in $\mathbb{R}^n$, $n\ge3$, and let $\varphi\colon(0,\infty)\to(0,\infty)$ be a monotone nondecreasing function such that*

$$\int_{t_*}^{\infty}\left[\frac{t}{\varphi(t)}\right]^{\frac{1}{n-2}} dt<\infty \tag{41}$$

for some $t_*\in(0,\infty)$. Then each finite distortion homeomorphism $f\colon D\to D'$ of class $W^{1,\varphi}_{\mathrm{loc}}$ is a lower Q-homeomorphism at every point $x_0\in\bar{D}$ with $Q(x)=K_f(x)$.

Proof. Let B be the (Borel) set of all points $x\in D$ where f has total differential $f'(x)$ and $J_f(x)\ne0$. Then, applying Kirszbraun's theorem and the uniqueness of the approximate differential, see, e.g., [50, Section 2.10.43 and Theorem 3.1.2], we see that B is the union of a countable collection of Borel sets B_l, $l=1,2,\dots$, such that the $f_l=f|_{B_l}$ are bi-Lipschitz homeomorphisms, see, e.g., [50, Lemma 3.2.2 and Theorems 3.1.4 and 3.1.8]. With no loss of generality, we may assume that the B_l are mutually disjoint. Also we denote by B_* the remaining set of all points $x\in D$ where f has total differential but $f'(x)=0$.

By construction, the set $B_0:=D\backslash(B\cup B_*)$ has Lebesgue measure zero, see Theorem 1. Hence, by Theorem, 4 $\mathcal{A}_S(B_0)=0$ for a.e. hypersurface S in $\mathbb{R}^n$ and, in particular, for a.e. sphere $S_r := S(x_0, r)$ centered at a prescribed point $x_0 \in \bar{D}$. Thus, by Corollary 4, $\mathcal{A}_{S_r^*}(f(B_0)) = 0$ and $\mathcal{A}_{S_r^*}(f(B_*)) = 0$ for a.e. S_r, where $S_r^* = f(S_r)$.

Let Γ be the family of all intersections of the spheres S_r, $r \in (\varepsilon, \varepsilon_0)$, $\varepsilon_0 < d_0 = \sup_{x\in D} |x - x_0|$, with the domain D. Given $\varrho_* \in \operatorname{adm} f(\Gamma)$ with $\varrho_* \equiv 0$ outside $f(D)$, we set $\varrho \equiv 0$ outside D and on B_0,

$$\varrho(x) := \varrho_*(f(x))\|f'(x)\| \quad for \;\; x \in D \setminus B_0.$$

Arguing piecewise on B_l, $l = 1, 2, \ldots$, and using [50, Section 1.7.6 and Lemma 3.2.2] we get

$$\int_{S_r} \varrho^{n-1}\, d\mathcal{A} \geq \int_{S_*^r} \varrho_*^{n-1}\, d\mathcal{A} \geq 1$$

for a.e. S_r, so that $\varrho \in \operatorname{ext}\operatorname{adm}\Gamma$.

A change of variables on each B_l, $l = 1, 2, \ldots$, see, e.g., [50, Theorem 3.2.5], and the countable additivity of integrals give the estimate

$$\int_D \frac{\varrho^n(x)}{K_f(x)}\, dm(x) \leq \int_{f(D)} \varrho_*^n(x)\, dm(x),$$

completing the proof. □

Corollary 7. *Each homeomorphism f of finite distortion in $\mathbb{R}^n$, $n \geq 3$, belonging to the class $W^{1,p}_{\mathrm{loc}}$ for $p > n - 1$ is a lower Q-homeomorphism at every point $x_0 \in \bar{D}$ with $Q(x) = K_f(x)$.*

Corollary 8. *In particular, each homeomorphism $f \in W^{1,1}_{\mathrm{loc}}$ in $\mathbb{R}^n$, $n \geq 3$, such that $K_f \in L^q_{\mathrm{loc}}$ for $q > n - 1$ is a lower Q-homeomorphism at every point $x_0 \in \bar{D}$ with $Q(x) = K_f(x)$.*

Proof. Applying the Hölder inequality and (3), on every compact set $C \subset D$ we obtain the following estimate for the norms of the first partial derivatives:

$$\|\partial_i f\|_p \leq \|K_f^{1/n}\|_s \cdot \|J_f^{1/n}\|_n \leq \|K_f\|_q^{1/n} \cdot |f(C)|^{1/n} < \infty$$

where $\frac{1}{p} = \frac{1}{s} + \frac{1}{n}$ and $s = qn$, i.e., $\frac{1}{p} = \frac{1}{n}\left(\frac{1}{q} + 1\right)$, and if $q > n-1$, then also $p > n-1$. Thus, we have $f \in W^{1,p}_{\mathrm{loc}}$ with $p = nq/(1+q) > n - 1$, and it remains to use Corollary 7. □

Combining Theorem 5 and Corollaries 5 and 8, we also obtain the following conclusion.

Corollary 9. *Each homeomorphism $f \in W^{1,\varphi}_{\mathrm{loc}}$ satisfying condition (41) for the function φ, and, in particular, each homeomorphism $f \in W^{1,p}_{\mathrm{loc}}$ with $p > n - 1$ and with $K_f \in L^{n-1}(D)$, is a ring Q_*-homeomorphism at every point $x_0 \in \bar{D}$ with $Q_*(x) = [K_f(x)]^{n-1}$. In particular, this is true for each homeomorphism $f \in W^{1,1}_{\mathrm{loc}}$ with dilatation $K_f \in L^q(D)$ for $q > n - 1$.*

Remark 9. In view of Remark 8, the condition $K_f \in L^{n-1}(D)$ in Corollary 9 can be replaced by the condition of integrability of K_f in the power $n - 1$ on almost every sphere of sufficiently small radius centered at points $x_0 \in \bar{D}$.

Part 3. On equicontinuous and compact Orlicz–Sobolev classes

§8. On the compactness of Orlicz–Sobolev classes

First, we recall some general facts on normal families of mappings in metric spaces. Let (X, d) and (X', d') be metric spaces with distances d and d', respectively. A family $\mathfrak{F}$ of continuous mappings $f\colon X \to X'$ is said to be *normal* if every sequence of mappings $f_m \in \mathfrak{F}$ has a subsequence f_{m_k} converging uniformly on each compact set $C \subset X$ to a continuous mapping. The family $\mathfrak{F}$ is *compact* if $\mathfrak{F}$ is normal and closed with respect to the locally uniform convergence.

Normality is known to be closely related to the following. A family $\mathfrak{F}$ of mappings $f\colon X \to X'$ is said to be *equicontinuous at a point* $x_0 \in X$ if for every $\varepsilon > 0$ there is $\delta > 0$ such that $d'(f(x), f(x_0)) < \varepsilon$ for all $f \in \mathfrak{F}$ and all $x \in X$ with $d(x, x_0) < \delta$. The family $\mathfrak{F}$ is *equicontinuous* if $\mathfrak{F}$ is equicontinuous at every point $x_0 \in X$.

Given a domain D in $\mathbb{R}^n$, $n \geq 2$, a monotone nondecreasing function $\varphi\colon [0, \infty) \to [0, \infty)$, a number $M \in [0, \infty)$, and a point $x_0 \in D$, we denote by $\mathfrak{F}^{\varphi}_M$ the family of all continuous mappings $f\colon D \to \mathbb{R}^m$, $m \geq 1$, of class $W^{1,1}_{\rm loc}$ such that $f(x_0) = 0$ and

$$\int_D \varphi\big(|\nabla f|\big)\, dm(x) \leq M. \tag{42}$$

We use the notation $\mathfrak{F}^{p}_M$ for the case of the function $\varphi(t) = t^p$, $p \in [1, \infty)$.

Using Proposition 3, Remark 1, and the Arzela–Ascoli theorem, see, e.g., [45, Theorem IV.6.7], we obtain the following statement, cf. [79, Theorem 8.1].

Theorem 6. *Let $\varphi\colon [0, \infty) \to [0, \infty)$ be a nonconstant, continuous, monotone nondecreasing, and convex function such that*

$$\int_{t_*}^{\infty} \left(\frac{t}{\varphi(t)}\right)^{\frac{1}{n-1}} dt < \infty \tag{43}$$

for some $t_* \in (0, \infty)$. Then the class $\mathfrak{F}^{\varphi^\alpha}_M$ with $\alpha > 1$ is equicontinuous and locally bounded, and, consequently, is a normal family of mappings. If, moreover, φ is convex, then the class $\mathfrak{F}^{\varphi^\alpha}_M$ is also closed relative to the locally uniform convergence, i.e., it is compact.

Proof. First, we show that the mappings in $\mathfrak{F}^{\varphi^\alpha}_M$ are equicontinuous. Let z_0 and z be arbitrary points in D such that $z \in C(z_0, \delta)$, $\delta > 0$, where $C(z_0, \delta)$ denotes the n-dimensional open cube centered at the point z_0 with edges of length δ parallel to coordinate axes. Fix $\varepsilon > 0$. Since the function τ^α with $\alpha > 1$ is strictly convex, the integral of $\varphi(|\nabla f|)$ over $\overline{C(z_0, \delta)} \subset D$ is arbitrarily small for sufficiently small $\delta > 0$ for all $f \in \mathfrak{F}^{\varphi^\alpha}_M$, see, e.g., [151, Theorem III.3.1.2]. Thus, by Lemma 2 applied to $\widetilde{\varphi}$, we have $|f(z) - f(z_0)| < \varepsilon$ for all $z \in C(z_0, \delta)$ with some $\delta = \delta(\varepsilon) > 0$.

Now, we show that the family $\mathfrak{F}^{\varphi^\alpha}_M$ is uniformly bounded on compact sets. Indeed, let K be a compact set in D. With no loss of generality we may assume that K is a connected set that contains the point x_0 occurring in the definition of $\mathfrak{F}^{\varphi^\alpha}_M$, see, e.g., [169, Lemma 1]. We cover K by the collection of cubes $C(z, \delta_z)$, $z \in K$, where δ_z corresponds to $\varepsilon := 1$ as in the first part of the proof. Since K is compact, we can find a finite number of cubes $C_i = C(z_i, \delta(z_i))$, $i = 1, 2, \ldots, m$, that cover K. Note that $D_* := \bigcup_{i=1}^N C_i$ is a subdomain of D because K is connected. Consequently, each point $z_* \in K$ can be joined with x_0 in D_* by a polygonal curve with vertices at points x_0, x_1, $\ldots$, x_k, z_* lying, in this order, in the cubes with numbers $i_1, \ldots, i_k$, $k \leq m$, $z \in C(z_{i_1}, \delta(z_{i_1}))$, $x_0 \in C(z_{i_k}, \delta(z_{i_k}))$, and $x_l \in C_{i_l} \cap C_{i_{l+1}}$, $l = 1, \dots, k-1$. By the triangle inequality,

$$|f(z)| \le |f(z) - f(z_{i_1})| + \sum_{l=2}^{k-1} |f(z_{i_l}) - f(z_{i_{l+1}})| + |f(z_{i_k}) - f(x_0)| + |f(x_0)| \le m.$$

Since m depends on the compact set K only, it follows that $\mathfrak{F}_M^{\varphi^\alpha}$ is uniformly bounded on compact sets and, consequently, is normal by the Arzela–Ascoli theorem, see, e.g., [45, IV.6.7].

Finally, we show that the class $\mathfrak{F}_M^{\varphi^\alpha}$ is closed provided φ is convex. Note that φ^α is strictly convex for any $\alpha \in (1, \infty)$. By [151, Theorem III.3.1.2], for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\int_E |\nabla f|\, dm(x) \le \varepsilon$ for all $f \in \mathfrak{F}_M^{\varphi^\alpha}$ whenever $m(E) < \delta$. Suppose $f_j \in \mathfrak{F}_M^{\varphi^\alpha}$ and $f_j \to f$ locally uniformly as $j \to \infty$. Then, by [159, Lemma 2.1] we have $f \in W^{1,1}_{\mathrm{loc}}$. By [146, Theorem 3.3 Chapter III, § 3.4], we have

$$\int_D \varphi^\alpha(|\nabla f|)\, dm(x) \le M, \tag{44}$$

i.e., $\mathfrak{F}_M^{\varphi^\alpha}$ is closed. Thus, the class $\mathfrak{F}_M^{\varphi}$ is compact. □

Corollary 10. *The class $\mathfrak{F}_M^{p}$ is compact with respect to the locally uniform convergence for each $p \in (n, \infty)$.*

Proof. Indeed, $t^p = (t^{p_*})^\alpha$, where $\alpha = p/p_* > 1$, for an arbitrary $p_* \in (n, p)$. □

§9. On functions in the classes BMO, VMO, and FMO

Recall that a real-valued function $\varphi \in L^1_{\mathrm{loc}}(D)$ is said to have *bounded mean oscillation* in $D \subset \mathbb{R}^n$ (we write $\varphi \in \mathrm{BMO}(D)$ or simply $\varphi \in \mathrm{BMO}$) if

$$\|\varphi\|_* = \sup_{B \subset D} ⨍_B |\varphi(z) - \varphi_B|\, dm(z) < \infty, \tag{45}$$

where the supremum is taken over all balls B in D, and

$$\varphi_B = ⨍_B \varphi(z)\, dm(z) = \frac{1}{|B|} \int_B \varphi(z)\, dm(z) \tag{46}$$

is the mean value of the function φ over B.

The space BMO, introduced by John and Nirenberg in [83], is at present one of the most important concepts of harmonic analysis, complex analysis, partial differential equations and relevant areas, see, e.g., [71] and [145].

A function ψ in BMO is said to have *vanishing mean oscillation*, $\psi \in \mathrm{VMO}$, if the supremum in (45) taken over all balls B in D with $|B| < \varepsilon$ converges to 0 as $\varepsilon \to 0$. The space VMO was introduced by Sarason in [164]. There are numerous papers devoted to the study of partial differential equations with coefficients in VMO, see, e.g., [35, 81, 124, 135, 142].

As in [78], we say that a function $\varphi\colon D \to \mathbb{R}$ has *finite mean oscillation at a point* $z_0 \in D$ and write $\varphi \in \mathrm{FMO}(x_0)$ if

$$\limsup_{\varepsilon \to 0} ⨍_{B(z_0, \varepsilon)} |\varphi(z) - \widetilde{\varphi}_\varepsilon(z_0)|\, dm(z) < \infty, \tag{47}$$

where

$$\widetilde{\varphi}_\varepsilon(z_0) = ⨍_{B(z_0, \varepsilon)} \varphi(z)\, dm(z) \tag{48}$$

is the mean value of $\varphi(z)$ over the ball $B(z_0,\varepsilon)$. Condition (47) includes the assumption that φ is integrable in some neighborhood of z_0. We also say that a function φ is of *finite mean oscillation in the domain* D and write $\varphi\in \mathrm{FMO}(D)$ or simply $\varphi\in\mathrm{FMO}$ if this property is fulfilled at every point $x_0\in D$.

Recall that a point $z_0\in D$ is called a *Lebesgue point* of a function $\varphi\colon D\to\mathbb{R}$ if φ is integrable in a neighborhood of z_0 and

$$\lim_{\varepsilon\to 0}\fint_{B(z_0,\varepsilon)}|\varphi(z)-\varphi(z_0)|\,dm(z)=0. \tag{49}$$

It is known that, for every function $\varphi\in L^1(D)$, almost every point in D is its Lebesgue point and, thus, φ has finite mean oscillation.

It is also known that $\mathrm{L}^\infty(D)\subset\mathrm{BMO}(D)\subset\mathrm{L}^p_{\mathrm{loc}}(D)$ for all $1\le p<\infty$, see, e.g., [83] and [145]. However, $\mathrm{FMO}(D)$ is not a subclass of $L^p_{\mathrm{loc}}(D)$ for any $p>1$, although $\mathrm{FMO}(D)\subset L^1_{\mathrm{loc}}(D)$, see the corresponding example in [123, Section 11.2]. Thus, FMO is substantially wider than $\mathrm{BMO}_{\mathrm{loc}}$.

The following facts about the functions of finite mean oscillation are taken from [78].

Proposition 6. *If for some collection of numbers $\varphi_\varepsilon\in\mathbb{R}$, $\varepsilon\in(0,\varepsilon_0]$, we have*

$$\limsup_{\varepsilon\to 0}\fint_{B(z_0,\varepsilon)}|\varphi(z)-\varphi_\varepsilon|\,dm(z)<\infty, \tag{50}$$

then φ has finite mean oscillation at z_0.

Corollary 11. *If for a point $z_0\in D$ we have*

$$\limsup_{\varepsilon\to 0}\fint_{B(z_0,\varepsilon)}|\varphi(z)|\,dm(z)<\infty, \tag{51}$$

then φ has finite mean oscillation at z_0.

The next lemma has a key significance for our further applications.

Lemma 4. *Let D be a domain in $\mathbb{R}^n$, $n\ge 2$, and let $\varphi\colon D\to\mathbb{R}$ be a nonnegative function of finite mean oscillation at the point $0\in D$. Then*

$$\int_{\varepsilon<|x|<\varepsilon_0}\frac{\varphi(x)\,dm(x)}{(|x|\log\frac{1}{|x|})^n}=O\left(\log\log\frac{1}{\varepsilon}\right) \tag{52}$$

as $\varepsilon\to 0$ for a positive number $\varepsilon_0<\operatorname{dist}(0,\partial D)$.

§10. Equicontinuous and normal families

In this section we present certain statements about homeomorphisms in the Orlicz–Sobolev classes $W^{1,\varphi}_{\mathrm{loc}}$ with the Calderón type condition (54), which is, generally speaking, weaker than condition (43). No uniform (even local in the domain) restrictions of the form (42) will be imposed. Here we assume everywhere that the function $\varphi\colon(0,\infty)\to(0,\infty)$ is monotone nondecreasing.

In what follows, we use the *spherical* (*chordal*) *metric* $h(x,y)=|\pi(x)-\pi(y)|$ in $\overline{\mathbb{R}^n}=\mathbb{R}^n\cup\{\infty\}$, where π is the stereographic projection of $\overline{\mathbb{R}^n}$ onto the sphere $S^n\big(\frac12 e_{n+1},\frac12\big)$ in $\mathbb{R}^{n+1}$, i.e.,

$$h(x,\infty)=\frac{1}{\sqrt{1+|x|^2}},\quad h(x,y)=\frac{|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}},\quad x\ne\infty\ne y.$$

It is clear that $\overline{\mathbb{R}^n}$ is homeomorphic to the unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$. The *spherical* (*chordal*) *diameter* of a set $E \subset \overline{\mathbb{R}^n}$ is

$$h(E) = \sup_{x,y\in E} h(x,y). \tag{53}$$

In the sequel we shall use the following Arzela–Ascoli type statement, see, e.g., [123, Corollary 7.5].

Proposition 7. *If (X,d) is a separable metric space and (X',d') is a compact metric space, then a family $\mathfrak{F}$ of mappings $f\colon X \to X'$ is normal if and only if $\mathfrak{F}$ is equicontinuous.*

Combining Corollary 9, see also Remark 9, with the results of [155] on equicontinuous and normal families of ring Q-homeomorphisms, see also [123, Chapter 7], we arrive at the following statements.

Theorem 7. *Let D and D' be domains in $\mathbb{R}^n$, $n \geq 3$, and let $\varphi\colon (0,\infty) \to (0,\infty)$ be a monotone increasing function such that*

$$\int_{t_*}^{\infty} \left[\frac{t}{\varphi(t)}\right]^{\frac{1}{n-2}} dt < \infty \tag{54}$$

for some $t_* \in (0,\infty)$. Let $f\colon D \to D'$ be a finite distortion homeomorphism in the Orlicz–Sobolev class $W^{1,\varphi}_{\rm loc}$ such that $h(\overline{\mathbb{R}^n} \setminus f(D)) \geq \Delta > 0$. Then, for every $x_0 \in D$ and $x \in B(x_0, \varepsilon(x_0))$, $\varepsilon(x_0) < d(x_0) = \operatorname{dist}(x_0, \partial D)$, we have

$$h(f(x), f(x_0)) \leq \frac{\alpha_n}{\Delta} \exp\left\{-\int_{|x-x_0|}^{\varepsilon(x_0)} \frac{dr}{r k_{x_0}^{\frac{1}{n-1}}(r)}\right\}, \tag{55}$$

where α_n is some constant depending only on n, and $k_{x_0}(r)$ is the average of $[K_f(x)]^{n-1}$ over the sphere $S(x_0,r)$.

Remark 10. Estimate (55) can be written in the form

$$h(f(x), f(x_0)) \leq \frac{\alpha_n}{\Delta} \exp\left\{-\omega_{n-1}^{\frac{1}{n-1}} \int_{|x-x_0|}^{\varepsilon(x_0)} \frac{dr}{\|K_f\|_{n-1}(x_0,r)}\right\}, \tag{56}$$

where $\|K_f\|_{n-1}(x_0,r)$ is the norm of K_f in the space L^{n-1} over the sphere $|x-x_0| = r$, and ω_{n-1} is the area of the unit sphere in $\mathbb{R}^n$.

Corollary 12. *In particular, these estimates hold true for finite distortion homeomorphisms f such that $K_f \in L^q_{\rm loc}$ with $q > n-1$.*

Corollary 13. *Let $f\colon D \to D'$ be a homeomorphism of class $W^{1,1}_{\rm loc}$, and let*

$$k_{x_0}(r) \leq \left[\log \frac{1}{r}\right]^{n-1} \tag{57}$$

for $r < \varepsilon(x_0) < \min\{e^{-1}, d(x_0)\}$. Then

$$h(f(x), f(x_0)) \leq \frac{\alpha_n}{\Delta} \frac{\log \frac{1}{\varepsilon(x_0)}}{\log \frac{1}{|x-x_0|}} \tag{58}$$

for all $x \in B(x_0, \varepsilon(x_0))$.

Corollary 14. *In particular, if*

$$K_f(x) \leq \log \frac{1}{|x-x_0|}, \quad x \in B(x_0, \varepsilon(x_0))\ , \tag{59}$$

for some $\varepsilon(x_0) < \min\{e^{-1}, d(x_0)\}$, then (58) is true in the ball $B(x_0, \varepsilon(x_0))$.

Remark 11. If instead of conditions (57) and (59) we require, respectively, the conditions

$$k_{x_0}(r) \le c \cdot \left[\log \frac{1}{r}\right]^{n-1} \tag{60}$$

and

$$K_f(x) \le c \cdot \log \frac{1}{|x-x_0|}\,, \tag{61}$$

then

$$h(f(x), f(x_0)) \le \frac{\alpha_n}{\Delta} \left[\frac{\log \frac{1}{\varepsilon(x_0)}}{\log \frac{1}{|x-x_0|}}\right]^{1/c^{\frac{1}{n-1}}}. \tag{62}$$

Theorem 8. *Let $f\colon \mathbb{B}^n \to \mathbb{B}^n$, $f(0)=0$, be a finite distortion homeomorphism of class $W^{1,\varphi}_{\rm loc}$ satisfying condition* (54) *and such that*

$$\int_{\varepsilon<|x|<1} K_f^{n-1}(x)\, \frac{dm(x)}{|x|^n} \le c \log \frac{1}{\varepsilon}, \qquad \varepsilon \in (0,1). \tag{63}$$

Then

$$|f(x)| \le \gamma_n \cdot |x|^{\beta_n} \tag{64}$$

where the constant γ_n depends only on n, and $\beta_n = (\omega_{n-1}/c)^{\frac{1}{n-1}}$, ω_{n-1} is the area of the unit sphere in $\mathbb{R}^n$.

Theorem 9. *Let D and D' be domains in $\mathbb{R}^n$, $n \ge 3$, and let $\varphi\colon (0,\infty) \to (0,\infty)$ be a monotone increasing function satisfying* (54)*. Suppose $f\colon D \to D'$ is a finite distortion homeomorphism of class $W^{1,\varphi}_{\rm loc}$ such that $h(\overline{\mathbb{R}^n} \setminus f(D)) \ge \Delta > 0$ and $K_f(x) \le Q(x)$, where $Q^{n-1} \in \mathrm{FMO}(x_0)$. Then*

$$h(f(x), f(x_0)) \le \frac{\alpha_n}{\Delta} \left\{\frac{\log \frac{1}{\varepsilon_0}}{\log \frac{1}{|x-x_0|}}\right\}^{\beta} \quad \textit{for all} \quad x \in B(x_0, \varepsilon_0), \tag{65}$$

where $\varepsilon_0 < \operatorname{dist}(x_0, \partial D)$, α_n depends only on n, and β depends on the function Q.

Corollary 15. *In particular, estimate* (65) *is true if*

$$\limsup_{\varepsilon \to 0} \fint_{B(x_0,\varepsilon)} Q^{n-1}(x)\, dm(x) < \infty. \tag{66}$$

Next, let D be a domain in $\mathbb{R}^n$, $n \ge 3$, let $\varphi\colon (0,\infty) \to (0,\infty)$ be a monotone increasing function, and let $Q\colon D \to (0,\infty)$ be a measurable function. Let $\mathcal{O}^{\varphi}_{Q,\Delta}$ be the class of all finite distortion homeomorphisms in the Orlicz–Sobolev class $W^{1,\varphi}_{\rm loc}$ such that $h\big(\overline{\mathbb{R}^n} \setminus f(D)\big) \ge \Delta > 0$ and $K_f(x) \le Q(x)$ a.e. Moreover, let $\mathcal{S}^p_{Q,\Delta}$, $p \ge 1$, denote the classes $\mathcal{O}^{\varphi}_{Q,\Delta}$ with $\varphi(t) = t^p$. Finally, let $\mathcal{K}^p_{Q,\Delta}$ be the class of all finite distortion homeomorphisms such that $K_f \in L^p_{\rm loc}$, $p \ge 1$, $K_f(x) \le Q(x)$ a.e., and $h\big(\overline{\mathbb{R}^n} \setminus f(D)\big) \ge \Delta > 0$.

By Proposition 7, the above estimates for distortion yield the following.

Theorem 10. *Let $\varphi\colon (0,\infty) \to (0,\infty)$ be a monotone increasing function satisfying* (54)*. If $Q^{n-1} \in \mathrm{FMO}$, then $\mathcal{O}^{\varphi}_{Q,\Delta}$ is a normal family.*

Corollary 16. *Under condition* (54)*, the class $\mathcal{O}^{\varphi}_{Q,\Delta}$ is normal if*

$$\limsup_{\varepsilon \to 0} \fint_{B(x_0,\varepsilon)} Q^{n-1}(x)\, dm(x) < \infty \quad \textit{for all} \quad x_0 \in D. \tag{67}$$

Corollary 17. *In particular, the classes* $\mathcal{S}^p_{Q,\Delta}$ *and* $\mathcal{K}^p_{Q,\Delta}$ *are normal for* $p > n-1$ *if either* $Q^{n-1} \in \mathrm{FMO}$ *or* (67) *is true.*

Theorem 11. *Let* $\Delta > 0$*, and let* $Q\colon D \to (0,\infty)$ *be a measurable function such that*

$$\int_0^{\varepsilon(x_0)} \frac{dr}{\|Q\|_{n-1}(x_0,r)} = \infty \quad \text{for all} \quad x_0 \in D, \tag{68}$$

where $\varepsilon(x_0) < \operatorname{dist}(x_0, \partial D)$ *and* $\|Q\|_{n-1}(x_0,r)$ *denotes the norm of* Q *in* L^{n-1} *over the sphere* $|x - x_0| = r$*. Then the classes* $\mathcal{O}^{\varphi}_{Q,\Delta}$*,* $\mathcal{S}^p_{Q,\Delta}$*,* $\mathcal{K}^p_{Q,\Delta}$ *form normal families if* φ *satisfies condition* (54) *or, respectively,* $p > n-1$*.*

Corollary 18. *The classes* $\mathcal{O}^{\varphi}_{Q,\Delta}$*,* $\mathcal{S}^p_{Q,\Delta}$*,* $\mathcal{K}^p_{Q,\Delta}$ *form normal families if* φ *satisfies* (54) *or, respectively,* $p > n-1$*, and* $Q(x)$ *has singularities only of logarithmic type.*

Let D be a fixed domain in the extended space $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$, $n \geq 3$, and let $\varphi\colon (0,\infty) \to (0,\infty)$ be a monotone increasing function. Given a function $\Phi\colon [0,\infty] \to [0,\infty]$ and numbers $M > 0$, $\Delta > 0$, we denote by $\mathcal{O}^{\Phi,\varphi}_{M,\Delta}$ the collection of all finite distortion homeomorphisms in the Orlicz–Sobolev class $W^{1,\varphi}_{\mathrm{loc}}$ such that $h\big(\overline{\mathbb{R}^n} \setminus f(D)\big) \geq \Delta > 0$ and

$$\int_D \Phi\big([K_f(x)]^{n-1}\big) \frac{dm(x)}{(1+|x|^2)^n} \leq M. \tag{69}$$

Similarly, $\mathcal{S}^{\Phi,p}_{M,\Delta}$, $p \geq 1$, denotes the class $\mathcal{O}^{\Phi,\varphi}_{M,\Delta}$ with $\varphi(t) = t^p$. Finally, let $\mathcal{K}^{\Phi,p}_{M,\Delta}$, $p \geq 1$, be the class of all finite distortion homeomorphisms such that $K_f \in L^p_{\mathrm{loc}}$, $p \geq 1$, (69) is true for K_f, and $h\big(\overline{\mathbb{R}^n} \setminus f(D)\big) \geq \Delta > 0$.

Combining Theorem 5, Corollaries 7–9, and also [156, Theorem 4.1], we get the following statement.

Theorem 12. *Let* $\Phi\colon [0,\infty] \to [0,\infty]$ *be a monotone increasing convex function such that*

$$\int_{\delta_0}^{\infty} \frac{d\tau}{\tau\big[\Phi^{-1}(\tau)\big]^{\frac{1}{n-1}}} = \infty \tag{70}$$

for some $\delta_0 > \Phi(0)$*. Then the classes* $\mathcal{O}^{\Phi,\varphi}_{M,\Delta}$ *under condition* (54) *and the classes* $\mathcal{S}^{\Phi,p}_{M,\Delta}$ *and* $\mathcal{K}^{\Phi,p}_{M,\Delta}$ *under the condition* $p > n-1$ *are equicontinuous and, consequently, form normal families of mappings for every* $M \in (0,\infty)$ *and* $\Delta \in (0,1)$*.*

Remark 12. The results of [156] show that condition (70) is not only sufficient but also necessary for these classes to be normal.

Part 4. On the boundary behavior of the Orlicz–Sobolev classes

§11. On domains with regular boundaries

Recall that a domain $D \subset \mathbb{R}^n$, $n \geq 2$, is said to be *locally connected at a point* $x_0 \in \partial D$ if for every neighborhood U of x_0, there is a neighborhood $V \subset U$ of x_0 such that $V \cap D$ is connected. Note that every Jordan domain D in $\mathbb{R}^n$ is locally connected at each point of ∂D, see, e.g., [186, p. 66].

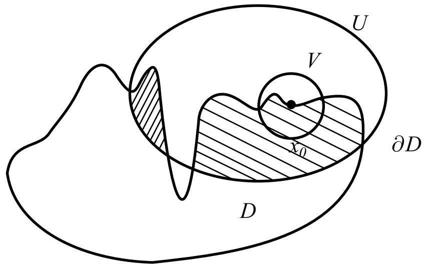

FIGURE 1

We say that ∂D is *weakly flat at a point* $x_0 \in \partial D$ if for every neighborhood U of x_0 and every number $P > 0$ there is a neighborhood $V \subset U$ of x_0 such that

$$M(\Gamma(E, F, D)) \geq P \tag{71}$$

for all continua E and F in D intersecting ∂U and ∂V. We say that the boundary ∂D is *weakly flat* if it is weakly flat at every point in ∂D.

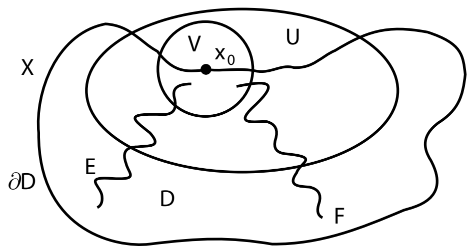

FIGURE 2

We also say that a point $x_0 \in \partial D$ is *strongly accessible* if for every neighborhood U of x_0 there exists a compact set E in D, a neighborhood $V \subset U$ of x_0, and a number $\delta > 0$ such that

$$M(\Gamma(E, F, D)) \geq \delta \tag{72}$$

for all continua F in D intersecting ∂U and ∂V. We say that the boundary ∂D is *strongly accessible* if every point $x_0 \in \partial D$ is strongly accessible.

Here, in the definitions of strongly accessible and weakly flat boundaries, for the role of neighborhoods U and V of x_0 we can take balls (closed or open) centered at x_0 or the sets in any other fundamental system of neighborhoods of x_0. These concepts can also be extended in a natural way to the case of $\overline{\mathbb{R}^n}$ and $x_0 = \infty$. Then we must use the corresponding neighborhoods of ∞.

It is easily seen that if a domain D in $\mathbb{R}^n$ is weakly flat at a point $x_0 \in \partial D$, then the point x_0 is strongly accessible from D. Moreover, in [92, Lemma 5.1] or [123, Lemma 3.15] it was proved that if a domain D in $\mathbb{R}^n$ is weakly flat at a point $x_0 \in \partial D$, then D is locally connected at x_0.

The notions of strong accessibility and weak flatness at boundary points of a domain in $\mathbb{R}^n$, as defined in [91], see also [92, 94, 123, 153], are localizations and generalizations of the corresponding notions introduced in [121, 122], compare with the properties P_1 and P_2 introduced by Väisälä in [182] and also with the quasiconformal accessibility and the quasiconformal flatness introduced by Näkki in [130]. Many theorems on homeomorphic extension to the boundary for quasiconformal mappings and their generalizations are valid under the condition of weak flatness for boundaries. The condition of strong accessibility plays a similar role for continuous extension of mappings to the boundary.

Below we present some significant results obtained by us in this direction, see, e.g., [92, Theorem 10.1 and Lemma 6.1] and also [123, Theorem 9.8 and Lemmas 9.4 and 6.5]. Namely, the following auxiliary result is basic for the proof of the principal statements in the next sections.

Proposition 8. *Let D and D' be bounded domains in $\mathbb{R}^n$, $n \geq 2$, let $Q\colon D \to (0,\infty)$ be a measurable function, and let $f\colon D \to D'$ be a lower Q-homeomorphism on ∂D. Suppose that the domain D is locally connected on ∂D and that the domain D' has a* (*strongly accessible*) *weakly flat boundary. If*

$$\int_0^{\delta(x_0)} \frac{dr}{\|Q\|_{n-1}(x_0,r)} = \infty \quad \text{for all} \quad x_0 \in \partial D \tag{73}$$

with some $\delta(x_0) \in (0, d(x_0))$, where $d(x_0) = \sup_{x\in D} |x - x_0|$ and $\|Q\|_{n-1}(x_0,r)$ is given by (35), *then f admits a* (*continuous*) *homeomorphic extension $\bar{f}$ to $\bar{D}$ that maps $\bar{D}$* (*into*) *onto $\overline{D'}$.*

A domain $D \subset \mathbb{R}^n$ is called a *quasiextremal distance domain* (in short, *QED-domain*), see [54], if

$$M(\Gamma(E,F,\overline{\mathbb{R}^n}) \leq K \cdot M(\Gamma(E,F,D)) \tag{74}$$

for some $K \geq 1$ and all pairs of nonintersecting continua E and F in D.

It is well known, see, e.g., [182, Theorem 10.12], that

$$M(\Gamma(E,F,\mathbb{R}^n)) \geq c_n \log \frac{R}{r} \tag{75}$$

for any sets E and F in $\mathbb{R}^n$, $n \geq 2$, intersecting all the circles $S(x_0,\rho)$, $\rho \in (r,R)$. Consequently, a QED-domain has a weakly flat boundary. An example in [123, Section 3.8] shows that the converse is not true even in the class of simply connected plane domains.

A domain $D \subset \mathbb{R}^n$, $n \geq 2$, is called a *uniform domain* if each pair of points x_1 and $x_2 \in D$ can be joined with a rectifiable curve γ in D such that

$$s(\gamma) \leq a \cdot |x_1 - x_2| \tag{76}$$

and

$$\min_{i=1,2} s(\gamma(x_i,x)) \leq b \cdot d(x,\partial D) \tag{77}$$

for all $x \in \gamma$, where $\gamma(x_i,x)$ is the portion of γ bounded by x_i and x, see [125]. It is known that every uniform domain is a QED-domain, but there exist QED-domains that are not uniform, see, e.g., [54]. Bounded convex domains and bounded domains with smooth boundaries are simple examples of uniform domains and, consequently, QED-domains as well as domains with weakly flat boundaries.

In the mapping theory and in the theory of differential equations, we often meet the so-called Lipschitz boundaries. A domain D in $\mathbb{R}^n$ is said to be *Lipschitz* if every point $x_0 \in \partial D$ has a neighborhood U that can be mapped by a bi-Lipschitz homeomorphism φ onto the unit ball $\mathbb{B}^n \subset \mathbb{R}^n$ in such a way that $\varphi(\partial D \cap U)$ is the intersection of $\mathbb{B}^n$ with a coordinate hyperplane. Note that a bi-Lipschitz homeomorphism is quasiconformal and that the modulus is a quasiinvariant under such mappings. Hence, the Lipschitz domains have weakly flat boundaries.

Recall that a closed set $X \subset \mathbb{R}^n$, $n \geq 2$, is called a *null-set for extremal distances* (in short, a NED-set), if

$$M(\Gamma(E, F, \mathbb{R}^n)) = M(\Gamma(E, F, \mathbb{R}^n \setminus X)) \tag{78}$$

for any two nonintersecting continua E and $F \subset \mathbb{R}^n \setminus X$.

Remark 13. It is known that if $X \subset \mathbb{R}^n$, $n \geq 2$, is a NED-set, then

$$|\, X| = 0 \tag{79}$$

and X does not split $\mathbb{R}^n$ locally, i.e., see [77],

$$\dim X \leq n - 2\,, \tag{80}$$

and, conversely, if a set $X \subset \mathbb{R}^n$ is closed and

$$H^{n-1}(X) = 0, \tag{81}$$

then X is a NED-set, see [181].

Note also that the complement of a NED-set in $\mathbb{R}^n$ is a particular case of a QED-domain.

§12. Continuous extension to boundaries

In this section we always assume that $\varphi\colon (0,\infty) \to (0,\infty)$ is a monotone nondecreasing function.

Using Theorem 5 and [92, Theorem 6.1], see also [123, Lemma 9.4], we get the following statement.

Lemma 5. *Let D and D' be bounded domains in $\mathbb{R}^n$, $n \geq 3$, let $x_0 \in \partial D$, and let*

$$\int_{t_*}^{\infty} \left[\frac{t}{\varphi(t)}\right]^{\frac{1}{n-2}} dt < \infty \tag{82}$$

for some $t_* \in (0,\infty)$. Suppose that the domain D is locally connected at a point $x_0 \in \partial D$ and that the domain D' is strongly accessible. Let $f\colon D \to D'$ be a finite distortion homeomorphism of class $W^{1,\varphi}_{\rm loc}$. If

$$\int_0^{\varepsilon_0} \frac{dr}{\|K_f\|_{n-1}(r)} = \infty, \tag{83}$$

where $0 < \varepsilon_0 < d_0 = \sup_{x\in D}\, |x - x_0|$ and

$$\|K_f\|_{n-1}(r) = \|K_f\|_{n-1}(x_0, r) = \left(\int_{D\cap S(x_0,r)} K_f^{n-1}(x)\, d\mathcal{A}\right)^{\frac{1}{n-1}}, \tag{84}$$

then the mapping f admits extension to the point x_0 by continuity in $\mathbb{R}^n$.

Since the bounded convex, smooth, and Lipschitz domains are particular cases of domains with weakly flat boundaries, Lemma 5 shows that the following is true.

Corollary 19. *Let D and D' be bounded convex, smooth, or Lipschitz domains in $\mathbb{R}^n$, $n \geq 3$, let $x_0 \in \partial D$, and let $f\colon D \to D'$ be a finite distortion homeomorphism of class $W^{1,\varphi}_{\rm loc}$ satisfying* (82) *and* (83)*. Then the mapping f admits extension to the point x_0 by continuity.*

In particular, Lemma 5 implies also the following theorem.

Theorem 13. *Let D and D' be bounded domains in $\mathbb{R}^n$, $n \geq 3$. Suppose that D is locally connected at a point $x_0 \in \partial D$, and that $\partial D'$ is strongly accessible. Let $f\colon D \to D'$ be a finite distortion homeomorphism of class $W^{1,\varphi}_{\rm loc}$ satisfying* (82)*. If*

$$k_{x_0}(r) = O\left(\left[\log \frac{1}{r}\right]^{n-1}\right) \tag{85}$$

as $r \to 0$, where $k_{x_0}(r)$ is the mean value of K_f^{n-1} over the sphere $|x - x_0| = r$, then f admits extension to the point x_0 by continuity in $\mathbb{R}^n$.

Let $\Phi\colon [0,\infty] \to [0,\infty]$ be a monotone nondecreasing convex function, and let $\delta > \Phi(0)$. Since the conditions of the form

$$\int_D \Phi\big(K_f^{n-1}(x)\big)\, dm(x) < \infty \tag{86}$$

and

$$\int_\delta^\infty \frac{d\tau}{\tau[\Phi^{-1}(\tau)]^{\frac{1}{n-1}}} = \infty \tag{87}$$

imply the divergence of the integral in (83), see, e.g., [155, Theorem 3.1], we have the following important consequence of Lemma 5.

Theorem 14. *Let D and D' be bounded domains in $\mathbb{R}^n$, $n \geq 3$. Suppose D is locally connected at a point $x_0 \in \partial D$ and $\partial D'$ is strongly accessible. Let $f\colon D \to D'$ be a finite distortion homeomorphism of class $W^{1,\varphi}_{\rm loc}$ satisfying* (82)*. If conditions* (86) and (87) *are fulfilled for a monotone nondecreasing convex function $\Phi\colon [0,\infty] \to [0,\infty]$ for some $\delta > \Phi(0)$, then f has a continuous extension $\overline{f}\colon \overline{D} \to \overline{D'}$.*

Condition (87) is not only sufficient but also necessary for a mapping f satisfying integral restrictions of the form (86) to admit continuous extension to the boundary; see, e.g., [94, Remark 5.1].

Now, Corollary 9 allows us to obtain the following lemma as a consequence of Lemma 1 in [113] for ring Q-homeomorphisms.

Lemma 6. *Let D and D' be bounded domains in $\mathbb{R}^n$, $n \geq 3$, let D be locally connected at a point $x_0 \in \partial D$, and let $f\colon D \to D'$ be a finite distortion homeomorphism of class $W^{1,\varphi}_{\rm loc}$ satisfying* (82) *and such that $\partial D'$ is strongly accessible at at least at one point of the cluster set $C(x_0, f)$. Suppose that*

$$\int_{D(x_0,\varepsilon)} K_f^{n-1}(x) \cdot \psi^n(|x - x_0|)\, dm(x) = o\big(I^n(\varepsilon, \varepsilon_0)\big) \tag{88}$$

as $\varepsilon \to 0$ for some $\varepsilon_0 = \varepsilon(x_0) > 0$, where $D(x_0, \varepsilon) = \{x \in D : \varepsilon < |x - x_0| < \varepsilon_0\}$ and $\psi(t)$ is a nonnegative measurable function on $(0,\infty)$ such that

$$0 < I(\varepsilon, \varepsilon_0) = \int_\varepsilon^{\varepsilon_0} \psi(t)\, dt < \infty \quad \textit{for all} \quad \varepsilon \in (0, \varepsilon_0).$$

Then f has a continuous extension to the point x_0.

Note that Lemma 6 is also an immediate consequence of Lemma 5 if we use [155, Lemma 3.7], see also [123, Lemma 7.4], and extend K_f by zero outside of D.

Remark 14. Note also that (88) is fulfilled, in particular, if

$$\int_{|x-x_0|<\varepsilon_0} K_f^{n-1}(x) \cdot \psi^n(|x - x_0|)\, dm(x) < \infty \tag{89}$$

for some $\varepsilon_0 > 0$ and $I(\varepsilon, \varepsilon_0) \to \infty$ as $\varepsilon \to 0$. In other words, for f to extend by continuity to the point $x_0 \in \partial D$, it suffices that the integral (89) converge for a nonnegative function $\psi(t)$ that is locally integrable on $(0, \varepsilon_0]$ but has a nonintegrable singularity at zero.

The following statements are special cases of Lemma 6. For example, choosing $\psi(t) := \frac{1}{t \log 1/t}$ in Lemma 6 and using Lemma 4, we obtain the following result.

Theorem 15. *Let D and D' be bounded domains in $\mathbb{R}^n$, $n \geq 3$. Suppose D is locally connected at a point $x_0 \in \partial D$ and $\partial D'$ is strongly accessible. Let $f\colon D \to D'$ be a finite distortion homeomorphism of class $W^{1,\varphi}_{\rm loc}$ satisfying* (82). *If $K_f^{n-1}(x)$ has finite mean oscillation at the point x_0, then f extends to x_0 by continuity in $\mathbb{R}^n$.*

Corollary 20. *In particular, the conclusion of Theorem* 15 *is valid if*

$$\limsup_{\varepsilon \to 0} \fint_{B(x_0,\varepsilon)} K_f^{n-1}(x)\, dm(x) < \infty. \tag{90}$$

Here we assume that K_f is extended by zero outside of D.

The following theorem also follows from Lemma 6 with $\psi(t) = 1/t$.

Theorem 16. *Let D and D' be bounded domains in $\mathbb{R}^n$, $n \geq 3$. Suppose D is locally connected at a point $x_0 \in \partial D$ and $\partial D'$ is strongly accessible. Let $f\colon D \to D'$ be a finite distortion homeomorphism of class $W^{1,\varphi}_{\rm loc}$ satisfying* (82). *If*

$$\int_{\varepsilon<|x-x_0|<\varepsilon_0} Q(x)\, \frac{dm(x)}{|x-x_0|^n} = o\left(\left[\log \frac{1}{\varepsilon}\right]^n\right) \tag{91}$$

as $\varepsilon \to 0$, then f extends to x_0 by continuity.

Remark 15. Choosing in Lemma 6 the function $\psi(t) = 1/(t \log 1/t)$ instead of $\psi(t) = 1/t$, we are able to replace (91) by the weaker condition

$$\int_{\varepsilon<|x-x_0|<\varepsilon_0} \frac{Q(x)\, dm(x)}{\left(|x-x_0|\, \log \frac{1}{|x-x_0|}\right)^n} = o\left(\left[\log\log \frac{1}{\varepsilon}\right]^n\right) \tag{92}$$

and (85) by the condition

$$k_{x_0}(r) = o\left(\left[\log \frac{1}{r} \log\log \frac{1}{r}\right]^{n-1}\right). \tag{93}$$

In general, here we could give a whole scale of the corresponding logarithmic type conditions using the corresponding functions $\psi(t)$.

The following result concerns resolvability of singularities for mappings in the Orlicz–Sobolev classes $W^{1,\varphi}_{\rm loc}$, see, e.g., [92, Theorem 8.1], [123, Theorem 9.5], and also Theorem 5, Proposition 8, and Remark 13.

Theorem 17. *Let D be a domain in $\mathbb{R}^n$, $n \geq 3$, let $X \subset D$, and let $f\colon D \setminus X \to \mathbb{R}^n$ be a finite distortion homeomorphism of class $W^{1,\varphi}_{\rm loc}$ satisfying* (82). *If X and $C(X, f)$ are NED-sets, $x_0 \in X$, and condition* (83) *is fulfilled, then f extends to x_0 by continuity in $\overline{\mathbb{R}^n}$.*

Here and in the sequel we denote by $C(X, f)$ the *cluster set* of the mapping $f\colon D \to \overline{\mathbb{R}^n}$ for a set $X \subset \bar{D}$,

$$C(X, f) := \left\{ y \in \overline{\mathbb{R}^n}\ :\ y = \lim_{k\to\infty} f(x_k),\ x_k \to x_0 \in X,\ x_k \in D \right\}. \tag{94}$$

Note that $C(\partial D, f) \subseteq \partial D'$ for every homeomorphism $f\colon D \to D'$, see, e.g., [123, Proposition 13.5].

§13. Extension of inverse mappings to boundaries

The following lemma on the cluster sets is a base for the proof of the theorem on extension to the boundary of inverse homeomorphisms with finite distortion. This lemma follows from Lemma 9.1 in [92] (see also [123, Lemma 9.5]) and also from Theorem 5.

Lemma 7. *Let D and D' be domains in $\mathbb{R}^n$, $n \geq 2$, let z_1 and z_2 distinct points in ∂D, $z_1 \neq \infty$, and let f be a homeomorphism of the domain D onto D' belonging to $W^{1,\varphi}_{\mathrm{loc}}$ and satisfying* (82)*. Suppose that the function K_f is integrable in the power $n-1$ on the surfaces*

$$D(r) = \{x \in D : |x - z_1| = r\} = D \cap S(z_1, r) \tag{95}$$

for some set E of numbers $r < |z_1 - z_2|$ of positive linear measure. If D is locally connected at z_1 and z_2 and $\partial D'$ is weakly flat, then

$$C(z_1, f) \cap C(z_2, f) = \varnothing. \tag{96}$$

The next statement follows immediately from Lemma 7.

Theorem 18. *Let D and D' be domains in $\mathbb{R}^n$, $n \geq 3$. Suppose that D is locally connected on its boundary and that the boundary of D' is weakly flat. Let $f\colon D \to D'$ be a homeomorphism of class $W^{1,\varphi}_{\mathrm{loc}}$ satisfying* (82) *and such that $K_f \in L^{n-1}(D)$. Then the mapping f^{-1} extends to the closure of the domain $\overline{D'}$ by continuity in $\overline{\mathbb{R}^n}$.*

Proof. Indeed, the Fubini theorem implies that the set

$$E = \left\{r \in \mathbb{R} : K_f\big|_{D(r)} \in L^{n-1}(D(r))\right\} \tag{97}$$

has positive Lebesgue measure because $K_f \in L^{n-1}(D)$. □

Remark 16. The above proof shows that, to apply Lemma 7, it suffices to assume in Theorem 18 that the function K_f is integrable in the power $n-1$ in a neighborhood of the domain D.

Moreover, using Theorem 5 and [92, Theorem 9.2], see also [123, Theorem 9.7], we obtain the following statement.

Theorem 19. *Let D and D' be domains in $\mathbb{R}^n$, $n \geq 3$, let D be locally connected on its boundary, and let the boundary of D' be weakly flat. Suppose that $f\colon D \to D'$ is a finite distortion homeomorphism of class $f \in W^{1,\varphi}_{\mathrm{loc}}$ satisfying* (82) *and that, moreover,*

$$\int_0^{\delta(x_0)} \frac{dr}{\|K_f\|_{n-1}(x_0, r)} = \infty \quad \text{for all} \quad x_0 \in \partial D \tag{98}$$

with some $\delta(x_0) < d(x_0) = \sup_{x \in D} |x - x_0|$, where $\|K_f\|_{n-1}(x_0, r)$ is the quantity defined in (84)*. Then the mapping f^{-1} extends to the closure of the domain $\overline{D'}$ by continuity in $\overline{\mathbb{R}^n}$.*

§14. Homeomorphic extension to the boundary

Combining the results of the last two sections, we arrive at the following statements.

Lemma 8. *Let D and D' be bounded domains in $\mathbb{R}^n$, $n \geq 3$. Suppose D is locally connected on its boundary and $\partial D'$ is weakly flat. Let $f\colon D \to D'$ be a finite distortion homeomorphism of class $W^{1,\varphi}_{\mathrm{loc}}$ satisfying* (82)*. Suppose that*

$$\int_{D(x_0,\varepsilon)} K_f^{n-1}(x) \cdot \psi^n(|x - x_0|)\, dm(x) = o\big(I^n(\varepsilon, \varepsilon_0)\big) \tag{99}$$

as $\varepsilon \to 0$ *for some* $\varepsilon_0 = \varepsilon(x_0) > 0$, *where* $D(x_0, \varepsilon) = \{x \in D : \varepsilon < |x - x_0| < \varepsilon_0\}$ *and* $\psi(t)$ *is a nonnegative measurable function on* $(0, \infty)$ *such that*

$$0 < I(\varepsilon, \varepsilon_0) = \int_{\varepsilon}^{\varepsilon_0} \psi(t)\, dt < \infty \quad \textit{for any} \quad \varepsilon \in (0, \varepsilon_0).$$

Then f *has a homeomorphic extension* $\overline{f}\colon \bar{D} \to \overline{D'}$.

Theorem 20. *Let* D *and* D' *be bounded domains in* $\mathbb{R}^n$, $n \geq 3$. *Suppose* D *is locally connected at* ∂D *and* $\partial D'$ *is weakly flat. Let* $f\colon D \to D'$ *be a finite distortion homeomorphism of class* $W^{1,\varphi}_{\mathrm{loc}}$ *satisfying* (82). *If* $K_f^{n-1}(x) \leq Q(x)$ *a.e., where* $Q \in \mathrm{FMO}(\partial D)$, *then* f *has a homeomorphic extension* $\overline{f}\colon \bar{D} \to \overline{D'}$.

Corollary 21. *In particular, the conclusion of Theorem* 20 *is valid if*

$$\limsup_{\varepsilon \to 0} \fint_{B(x_0, \varepsilon)} K_f^{n-1}(x)\, dm(x) < \infty \quad \textit{for all} \quad x_0 \in \partial D. \tag{100}$$

Theorem 21. *Let* D *and* D' *be bounded domains in* $\mathbb{R}^n$, $n \geq 3$, *with* D *locally connected on its boundary and* $\partial D'$ *weakly flat, and let* $f\colon D \to D'$ *be a finite distortion homeomorphism of class* $W^{1,\varphi}_{\mathrm{loc}}$ *satisfying* (82). *Suppose that*

$$\int_0^{\delta(x_0)} \frac{dr}{\|K_f\|_{n-1}(x_0, r)} = \infty \quad \textit{for all} \quad x_0 \in \partial D \tag{101}$$

with some $\delta(x_0) < d(x_0) = \sup_{x \in D} |x - x_0|$, *where* $\|K_f\|_{n-1}(x_0, r)$ *is defined as in* (84). *Then the mapping* f *has a homeomorphic extension* $\overline{f}\colon \bar{D} \to \overline{D'}$.

As a consequence of Theorem 21, we obtain the following generalization of the well-known theorems of Gehring–Martio and Martio–Vuorinen on homeomorphic extension to the boundary of quasiconformal mappings between QED domains, see [54] and [126].

Corollary 22. *Let* D *and* D' *be bounded domains in* $\mathbb{R}^n$, $n \geq 3$, *with weakly flat boundaries, and let* $f\colon D \to D'$ *be a finite distortion homeomorphism of class* $W^{1,\varphi}_{\mathrm{loc}}$ *satisfying* (82). *If condition* (101) *is fulfilled, then* f *has a homeomorphic extension* $\overline{f}\colon \bar{D} \to \overline{D'}$.

Theorem 5 implies the next result, see, e.g., [92, Theorem 10.3], and also [123, Theorem 9.10].

Theorem 22. *Let* D *be a bounded domain in* $\mathbb{R}^n$, $n \geq 3$, *and let* X *and* $C(X, f)$ *be* NED-*sets, where* $f\colon D \setminus \{X\} \to \overline{\mathbb{R}^n}$ *is a finite distortion homeomorphism of class* $W^{1,\varphi}_{\mathrm{loc}}$ *satisfying* (82) *and such that condition* (101) *is fulfilled at every point* $x_0 \in X$ *with* $\delta(x_0) < \mathrm{dist}(x_0, \partial D)$ *and*

$$\|K_f\|_{n-1}(x_0, r) = \left(\int_{S(x_0, r)} K_f^{n-1}(x)\, dA \right)^{\frac{1}{n-1}}. \tag{102}$$

Then f *has a homeomorphic extension to* D.

Remark 17. In particular, the conclusion of Theorem 22 is valid if X is a closed set and

$$H^{n-1}(X) = 0 = H^{n-1}(C(X, f)). \tag{103}$$

Moreover, condition (101) can be replaced by the condition $K_f^{n-1}(x) \leq Q(x)$ a.e. with a function $Q\colon \mathbb{R}^n \to (0, \infty)$ of class $\mathrm{FMO}(X)$, or by the condition

$$\limsup_{\varepsilon \to 0} \fint_{B(x_0, \varepsilon)} K_f^{n-1}(x)\, dm(x) < \infty \quad \text{for all} \quad x_0 \in X. \tag{104}$$

Theorem 23. *Let D and D' be bounded domains in $\mathbb{R}^n$, $n \geq 3$. Suppose that D is locally connected on ∂D and that D' has weakly flat boundary. Let $f\colon D \setminus \{X\} \to D'$ be a finite distortion homeomorphism of class $W^{1,\varphi}_{\mathrm{loc}}$ satisfying* (82)*. If*

$$\int_D \Phi\big(K_f^{n-1}(x)\big)\, dm(x) < \infty \tag{105}$$

for a monotone nondecreasing convex function $\Phi\colon [0,\infty] \to [0,\infty]$ such that

$$\int_\delta^\infty \frac{d\tau}{\tau[\Phi^{-1}(\tau)]^{\frac{1}{n-1}}} = \infty \tag{106}$$

for some $\delta > \Phi(0)$, then f has a homeomorphic extension $\bar{f}\colon \bar{D} \to \overline{D'}$.

Note that condition (106) is not only sufficient but also necessary for f with integral constraints of the form (105) to admit continuous extension to the boundary, see, e.g., [94, Theorem 5.1 and Remark 5.1]. Note also that all results of this section are valid, in particular, for bounded convex, smooth, and Lipschitz domains.

Acknowledgment

We are grateful to the referee for useful remarks that made it possible to improve the text substantially.

References

[1] L. Ahlfors, *On quasiconformal mappings*, J. Analyse Math. **3** (1954), 1–58. MR0064875 (16:348d)
[2] A. Alberico and A. Cianchi, *Differentiability properties of Orlicz–Sobolev functions*, Ark. Mat. **43** (2005), no. 1, 1–28. MR2134696 (2006a:46038)
[3] P. S. Aleksandrov, *On the dimension of closed sets*, Uspekhi Mat. Nauk **4** (1930), no. 6, 17–88 (Russian). MR0034577 (11:609f)
[4] L. Ambrosio, *Metric space valued functions of bounded variation*, Ann. Scuola. Norm. Sup. Pisa Cl. Sci. (4) **17** (1990), no. 3, 439–478. MR1079985 (92d:26022)
[5] C. Andreian Cazacu, *Foundations of quasiconformal mappings*, Handbook of Complex Analysis: Geometric Function Theory. Vol. 2, Elsevier, Amsterdam, 2005, pp. 687–753. MR2121872 (2005k:30038)
[6] ______, *On the length-area dilatation*, Complex Var. Theory Appl. **50** (2005), no. 7-11, 765–776. MR2155443 (2006m:30032)
[7] ______, *Module inequalities for quasiregular mappings*, Ann. Acad. Sci. Fenn. Math. **2** (1976), 17–28. MR0470213 (57:9971)
[8] ______, *Modules and quasiconformality*, Sympos. Math., vol. 18, Academic Press, London, 1976, pp. 519–534. MR0450548 (56:8842)
[9] ______, *A generalization of the quasiconformality*, Topics in Analysis (Colloq. Math. Anal., Jyvaskyla, 1970), Lecture Notes in Math., vol. 419, Springer-Verlag, Berlin, 1974, pp. 4–17. MR0396940 (53:800)
[10] V. V. Aseev, *Moduli of families of locally quasisymmetric surface*, Sibirsk. Mat. Zh. **30** (1989), no. 3, 9–15; English transl., Sib. Math. J. **30** (1989), no. 3, 353–358. MR1010831 (90i:30030)
[11] K. Astala, T. Iwaniec, P. Koskela, and G. Martin, *Mappings of BMO-bounded distortion*, Math. Ann. **317** (2000), no. 4, 703–726. MR1777116 (2001i:30016)
[12] E. S. Afanas′eva, V. I. Ryazanov, and R. R. Salimov, *On mappings in Orlicz–Sobolev classes on Riemannian manifolds*, Ukr. Mat. Visn. **8** (2011), no. 3, 319–342; English transl., J. Math. Sci. (N.Y.) **181** (2012), no. 1, 1–17. MR2906695
[13] A. K. Bakhtin, G. P. Bakhtina, and Yu. B. Zelinskii, *Topological-algebraic structures and geometric methods in complex analysis*, Pratsi Inst. Mat. NAN Ukraïni. Mat. Zastosuvannya, vol. 73, NAN Ukraïni, Inst. Mat., Kiev, 2008. (Russian) MR2467439 (2010b:30001)
[14] Z. M. Balogh, *Hausdorff dimension distribution of quasiconformal mappings on the Heisenberg group*, J. Anal. Math. **83** (2001), 289–312. MR1828495 (2002b:30022)
[15] Z. M. Balogh, R. Monti, and J. T. Tyson, *Frequency of Sobolev and qusiconformal dimension distortion*, Research Report 2010-11, 22.07.2010, 1–36.
[16] S. M. Bates, *On the image size of singular maps*, Proc. Amer. Math. Soc. **114** (1992), no. 3, 699–705. MR1074748 (92f:58015)

[17] P. P. Belinskii, *General properties of quasiconformal mappings*, Nauka, Novosibirsk, 1974. (Russian) MR0407275 (53:11054)

[18] P. A. Biluta, *Certain extremal problems for mapping which are quasiconformal in the mean*, Sibirsk. Mat. Zh. **6** (1965), 717–726. (Russian) MR0190328 (32:7741)

[19] Z. Birnbaum and W. Orlicz, *Über die Verallgemeinerungen des Begriffes der zueinander konjugierten Potenzen*, Studia Math. **3** (1931), 1–67.

[20] C. J. Bishop, *Quasiconformal mappings which increase dimension*, Ann. Acad. Sci. Fenn. Math. **24** (1999), no. 2, 397–407. MR1724076 (2000i:30044)

[21] B. Bojarski, V. Ya. Gutlyanskii, O. Martio, and V. I. Ryazanov, *Infinitesimal geometry of quasiconformal and bi-Lipschitz mappings in the plane*, EMS Tracts Math., vol. 19, EMS Publ. House, Zürich, 2013. MR3088451

[22] B. Bojarski, P. Hajlasz, and P. Strzelecki, *Sard's theorem for mappings in Hölder and Sobolev spaces*, Manuscripta Math. **118** (2005), no. 4, 383–397. MR2183045 (2007b:58014)

[23] B. Bojarski and T. Iwaniec, *Analytical foundations of the theory of quasiconformal mappings in* $\mathbb{R}^n$, Ann. Acad. Sci. Fenn. Math. **8** (1983), no. 2, 257–324. MR731786 (85h:30023)

[24] J. Bourgain, M. V. Korobkov, and J. Kristensen, *On the Morse–Sard property and level sets of Sobolev and BV functions*, Rev. Mat. Iberoam. **29** (2013), no. 1, 1–23. MR3010119

[25] S. K. Vodop′yanov, *Closedness of classes of mappings with bounded distortion on Carnot groups*, Mat. Tr. **5** (2002), no. 2, 92–137. (Russian) MR1944067 (2003i:30034)

[26] ———, *Foundations of the theory of mappings with bounded distortion on Carnot groups*, Interaction of Analysis and Geometry, Contemp. Math., vol. 424, Amer. Math. Soc., Providence, RI, 2007, pp. 303–344. MR2316342 (2008j:30031)

[27] S. K. Vodop′yanov and V. M. Gol′dstein, *Sobolev spaces and special classes of mappings*, Novosibirsk. Gos. Univ., Novosibirsk, 1981. (Russian) MR684992 (84c:46037)

[28] S. K. Vodop′yanov, V. M. God′dstein, and Yu. G. Reshetnyak, *The geometric properties of functions with generalized first derivatives*, Uspekhi Mat. Nauk **34** (1979), no. 1, 17–65. (Russian) MR525650 (84b:46036)

[29] S. K. Vodop′yanov and A. D. Ukhlov, *Mappings associated with weighted Sobolev spaces*, Complex Anal. Dynam. Sys. III, Contemp. Math., vol. 455, Amer. Math. Soc., Providence, RI, 2008, pp. 363–382. MR2408182 (2009f:30061)

[30] ———, *Sobolev spaces and* (P, Q)*-quasiconformal mappings of Carnot groups*, Sibirsk. Mat. Zh. **39** (1998), no. 4, 665–682; English transl. Sib. Math. J. **39** (1998), no. 4, 776–795. MR1654140 (99m:46090)

[31] A. P. Calderón, *On the differentiability of absolutely continuous functions*, Riv. Math. Univ. Parma **2** (1951), 203–213. MR0045200 (13:544a)

[32] P. Caraman, *n-Dimensional quasiconformal mappings*, Haessner Publ. Inc., Newfoundland, NJ, 1974. MR0357782 (50:10249)

[33] L. Cesari, *Sulle transformazioni continue*, Ann. Mat. Pura Appl. (4) **21** (1942), 157–188. MR0010605 (6:43c)

[34] A. Cianchi, *A sharp embedding theorem for Orlicz–Sobolev spaces*, Indiana Univ. Math. J. **45** (1996), no. 1, 39–65. MR1406683 (97h:46044)

[35] F. Chiarenza, M. Frasca, and P. Longo, $W^{2,p}$*-solvability of the Dirichlet problem for nondivergence elliptic equations with* VMO *coefficients*, Trans. Amer. Math. Soc. **336** (1993), no. 2, 841–853. MR1088476 (93f:35232)

[36] M. Csörnyei, S. Hencl, and J. Maly, *Homeomorphisms in the Sobolev space* $W^{1,n-1}$, J. Reine Angew. Math. **644** (2010), 221–235. MR2671780 (2011h:46048)

[37] P. T. Church and J. G. Timourian, *Differentiable maps with small critical set or critical set image*, Indiana Univ. Math. J. **27** (1978), no. 6, 953–971. MR511250 (80b:57030)

[38] ———, *Maps having* 0*-dimensional critical set image*, Indiana Univ. Math. J. **27** (1978), no. 5, 813–832. MR503716 (80k:57056)

[39] M. Cristea, *Dilatations of homeomorphisms satisfying some modular inequalities*, Rev. Roumaine Math. Pures Appl. **56** (2011), no. 4, 275–282. MR2962329

[40] ———, *Open discrete mapping having local* ACL^n *inverses*, Complex Var. Elliptic Equ. **55** (2010), no. 1–3, 61–90. MR2599611 (2011b:30054)

[41] ———, *Local homeomorphisms having local* ACL^n *inverses*, Complex Var. Elliptic Equ. **53** (2008), no. 1, 77–99. MR2380822 (2009b:30037)

[42] ———, *Open discrete mappings having local* ACL^n *inverses,* Preprint Inst. Math., no. 6, Romanian Acad., Bucuresti, 2008, pp. 1–29.

[43] ———, *Mappings of finite distortion: Zoric's theorem, and equicontinuity results*, Rev. Roumaine Math. Pures Appl. **52** (2007), no. 5, 539–554. MR2387802 (2009c:30061)

[44] ———, *Mappings of finite distortion: boundary extension*, Rev. Roumaine Math. Pures Appl. **51** (2006), no. 5–6, 607–631. MR2320912 (2008c:30028)

[45] N. Dunford and J. T. Schwartz, *Linear operators.* I*, General theory*, Pure Appl. Math., vol. 7, Intersci. Publ., Inc., New York–London, 1958. MR0117523 (22:8302)

[46] T. Donaldson, *Nonlinear elliptic boundary-value problems in Orlicz–Sobolev spaces*, J. Differential Equations **10** (1971), 507–528. MR0298472 (45:7524)

[47] V. N. Dubinin, *Capacities of condencers and symmetrization in geometric function theory of complex variable*, Dal′nauka, Vladivostok, 2009. (Russian)

[48] A. Ya. Dubovitskii, *On the structure of level sets of differentiable mappings of an n-dimensional cube into a k-dimensional cube*, Izv. Akad. Nauk SSSR Ser. Mat. **21** (1953), no. 3, 371–408. (Russian) MR0094424 (20:942)

[49] A. G. Fadell, *A note on a theorem of Gehring and Lehto*, Proc. Amer. Math. Soc. **49** (1975), 195–198. MR0367132 (51:3374)

[50] H. Federer, *Geometric measure theory*, Grundlehren Math. Wiss., Bd. 153, Springer-Verlag, New York, 1969. MR0257325 (41:1976)

[51] B. Fuglede, *Extremal length and functional completion*, Acta Math. **98** (1957), 171–219. MR0097720 (20:4187)

[52] F. W. Gehring, *Rings and quasiconformal mappings in space*, Trans. Amer. Math. Soc. **103** (1962), 353–393. MR0139735 (25:3166)

[53] F. W. Gehring and O. Lehto, *On the total differentiability of functions of a complex variable*, Ann. Acad. Sci. Fenn. Math. **272** (1959), 3–8. MR0124487 (23:A1799)

[54] F. W. Gehring and O. Martio, *Quasiextremal distance domains and extension of quasiconformal mappings*, J. Anal. Math. **45** (1985), 181–206. MR833411 (87j:30043)

[55] F. W. Gehring and J. Väisälä, *Hausdorff dimension and quasiconformal mappings*, J. London Math. Soc. (2) **6** (1973), no. 2, 504–512. MR0324028 (48:2380)

[56] A. Golberg, *Homeomorphisms with integrally restricted moduli*, Contemp. Math., vol. 553, Amer. Math. Soc., Providence, RI, 2011, 83–98. MR2868590

[57] ———, *Directional dilatations in space*, Complex Var. Elliptic Equ. **55** (2010), no. 1-3, 13–29. MR2599608 (2011a:30062)

[58] ———, *Homeomorphisms with finite mean dilatations*, Contemp. Math., vol. 382, Amer. Math. Soc., Providence, RI, 2005, pp. 177–186. MR2175886 (2007b:30020)

[59] A. Golberg and V. Ya. Gutlyanskii, *On Lipschitz continuity of quasiconformal mappings in space*, J. Anal. Math. **109** (2009), 233–251. MR2585395 (2011f:30045)

[60] V. M. Gol′dshtein and Yu. G. Reshetnyak, *Introduction to the theory of functions with generalized derivatives and quasiconformal mappings*, Nauka, Moscow, 1983; English transl., Quasiconformal mappings and Sobolev speaces, Kluwer, Dordrecht, 1990. MR738784 (85m:46031a)

[61] J.-P. Gossez and V. Mustonen, *Variational inequalities in Orlicz–Sobolev spaces*, Nonlinear Anal. **11** (1987), no. 3, 379–392. MR881725 (88g:35094)

[62] E. L. Grinberg, *On the smoothness hypothesis in Sard's theorem*, Amer. Math. Monthly **92** (1985), no. 10, 733–734. MR820057 (87m:58020)

[63] V. Ya. Gutlyanskii ana V. I. Ryazanov, *Geometric and topological theory of functions and mappings*, Naukova Dumka, Kiev, 2011. (Russian)

[64] V. Ya. Gutlyanskii, V. I. Ryazanov, U. Srebro, and E. Yakubov, *The Beltrami equation: A geometric approach*, Developments Math., vol. 26, Springer, New York, 2012. MR2917642

[65] V. A. Zorich, *Admissible order of growth of the characteristic of quasiconformality in M. A. Lavrent′ev's theorem*, Dokl. Akad. Nauk SSSR **181** (1968), no. 3, 530–533. (Russian) MR0229816 (37:5382)

[66] ——— *Quasiconformal mappings and the asymptotic geometry of manifolds*, Uspekhi Mat. Nauk **57** (2002), no. 3, 3–28; English transl. Russian Math. Surveys **57** (2002), no. 3, 437–264. MR1918854 (2003i:30035)

[67] P. Hajlasz, *Sobolev spaces on an arbitrary metric space*, Potential Anal. **5** (1996), no. 4, 403–415. MR1401074 (97f:46050)

[68] ———, *Whitney's example by way of Assouad's embedding*, Proc. Amer. Math. Soc. **131** (2003), no. 11, 3463–3467. MR1991757 (2004c:26013)

[69] G. H. Hardy, J. E. Littlewood, and G. Polia, *Inequalities*, 2nd ed., Cambridge Univ. Press, Cambridge, 1952. MR0046395 (13:727e)

[70] J. Heinonen, *Lectures on analysis on metric spaces*, Springer, New York etc., 2000. MR1800917 (2002c:30028)

[71] J. Heinonen, T. Kilpelainen, and O. Martio, *Nonlinear potential theory of degenerate elliptic equations*, Oxford Math. Monogr., Clarendon Press, Oxford Univ. Press, New York, 1993. MR1207810 (94e:31003)

[72] J. Heinonen and P. Koskela, *Quasiconformal maps in metric spaces with controlled geometry*, Acta Math. **181** (1998), no. 1, 1–41. MR1654771 (99j:30025)

[73] J. Heinonen, P. Koskela, P. Shanmugalingam, and J. T. Tyson, *Sobolev spaces of Banach space-valued functions and quasiconformal mappings*, J. Anal. Math. **85** (2001), 87–139. MR1869604 (2002k:46090)

[74] D. A. Herron and P. Koskela, *Locally uniform domains and quasiconformal mappings*, Ann. Acad. Sci. Fenn. Math. **20** (1995), no. 1, 187–206. MR1304117 (96h:30037)

[75] J. Hesse, *A p-extremal length and p-capacity equality*, Ark. Mat. **13** (1975), 131–144. MR0379871 (52:776)

[76] M. Hsini, *Existence of solutions to a semilinear elliptic system through generalized Orlicz–Sobolev spaces*, J. Partial Differ. Equ. **23** (2010), no. 2, 168–193. MR2667757 (2011e:35030)

[77] W. Hurewicz and H. Wallman, *Dimension theory*, Princeton Univ. Press, Princeton, N.J., 1941. MR0006493 (3:312b)

[78] A. A. Ignat′ev and V. I. Ryasanov, *Finite mean oscillation in mapping theory*, Ukr. Mat. Visn. **2** (2005), no. 3, 395–417; English transl., Ukr. Math. Bull. **2** (2005), no. 3, 403–424. MR2325895 (2008b:30036)

[79] T. Iwaniec, P. Koskela, and J. Onninen, *Mappings of finite distortion: compactness*, Ann. Acad. Sci. Fenn. Math. **27** (2002), no. 2, 391–417. MR1922197 (2004b:30045)

[80] T. Iwaniec and G. Martin, *Geometrical function theory and non-linear analysis*, Oxford Math. Monogr., Clarendon Press, Oxford Univ. Press, New York, 2001. MR1859913 (2003c:30001)

[81] T. Iwaniec and C. Sbordone, *Riesz transforms and elliptic PDEs with VMO coefficients*, J. Anal. Math. **74** (1998), 183–212. MR1631658 (99e:35034)

[82] T. Iwaniec and V. Sverák, *On mappings with integrable dilatation*, Proc. Amer. Math. Soc. **118** (1993), no. 1, 181–188. MR1160301 (93k:30023)

[83] F. John and L. Nirenberg, *On functions of bounded mean oscillation*, Comm. Pure Appl. Math. **14** (1961), 415–426. MR0131498 (24:A1348)

[84] R. Kaufman, *A singular map of a cube onto a square*, J. Differential Geom. **14** (1979), no. 4, 593–594. MR600614 (82a:26013)

[85] J. Kauhanen, P. Koskela, and J. Maly, *On functions with derivatives in a Lorentz space*, Manuscripta Math. **10** (1999), no. 1, 87–101. MR1714456 (2000j:46064)

[86] E. Ya. Khruslov and L. S. Pankratov, *Homogenization of the Dirichlet variational problems in Orlicz–Sobolev spaces*, Operator Theory Appl.; Fields Inst. Commun., vol. 25, Amer. Math. Soc., Providence, RI, 2000, pp. 345–366. MR1759552 (2001g:35023)

[87] L. V. Kovalev, *Monotonicity of a generalized reduced modulus*, Zap. Nauchn. Sem. S.-Petersburg. Otdel. Mat. Inst. Steklov. (POMI) **276** (2001), 219–236; English transl., J. Math. Sci. (N.Y.) **118** (2003), no. 1, 4861–4870. MR1850369 (2002h:31007)

[88] D. A. Kovtonyuk, I. Petkov, and V. I. Ryazanov, *On homeomorphisms with finite distortion in the plane*, `arXiv:1011.3310v2 [math.CV]`, 18 Nov., 2010, 1–16.

[89] ——— *On the boundary behaviour of solutions to the Beltrami equations*, Complex Var. Elliptic Equ. **58** (2013), no. 5, 647–663. MR3170649

[90] D. A. Kovtonyuk, I. Petkov, V. I. Ryazanov, and R. R. Salimov, *The boundary behaviour and the Dirichlet problem for the Beltrami equations*, Algebra i Analiz **25** (2013), no. 4. 101–124. MR3184618

[91] D. A. Kovtonyuk and V. I. Ryazanov, *On boundaries of space domains*, Proc. Inst. Appl. Math. Mech., vol. 13, NAN Ukraini Inst. Prikl. Mat. Mech., Donetsk 2006, pp. 110–120 (Russian). MR2467526 (2009m:30040)

[92] ———, *On the theory of lower Q-homeomorphisms,* Ukr. Mat. Visn. **5** (2008), no. 2, 159–184; English transl., Ukr. Math. Bull. **5** (2008), no. 2, 157–181. MR2559832 (2010k:30022)

[93] ———, *On the theory of mappings with finite area distortion*, J. Anal. Math. **104** (2008), no. 2, 291–306. MR2403438 (2009e:30042)

[94] ———, *On the boundary behavior of generalized quasi-isometries,* J. Anal. Math. **115** (2011), 103-120. MR2855035

[95] D. A. Kovtonyuk, V. I. Ryazanov, R. R. Salimov, and E. A. Sevost′yanov, *On mappings in the Orlicz–Sobolev classes*, `arXiv:1012.5010v4 [math.CV]`, 12 Jan., 2011, 1–42.

[96] ———, *On mappings in the Orlicz–Sobolev classes*, Ann. Univ. Buchar. Math. Ser. **3(LXI)** (2012), no. 1, 67–78. MR3034963

[97] J. D. Koronel, *Continuity and k-th order differentiability in Orlicz–Sobolev spaces: $W^k L''_A$*, Israel J. Math. **24** (1976), no. 2, 119–138. MR0428030 (55:1060)

[98] M. A. Krasnosel′skii and Ja. B. Rutickii, *Convex functions and Orlicz spaces*, Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1958; English transl., Noordhoff, 1961. MR0126722 (23 #A4016)

[99] V. I. Kruglikov, *Capacities of condensers and quasiconformal in the mean mappings in space*, Mat. Sb. (N.S.) **130** (1986), no. 2, 185–206. (Russian) MR854971 (88d:30028)

[100] S. L. Krushkal′, *On mappings quasiconformal in the mean*, Dokl. Akad. Nauk SSSR **157** (1964), no. 3, 517–519. (Russian) MR0176067 (31:342)

[101] S. L. Krushkal and R. Kühnau, *Quasiconformal mappings—new methods and applications*, Nauka, Novosibirsk, 1984. (Russian) MR753286 (85k:30032a)

[102] V. S. Kud′yavin, *Some estimates of distance disfortion in mappings that are quasoconformal in the mean*, Dinamika Sploshn. Sredy **52** (1981), 168–171. (Russian) MR706457 (84g:30023)

[103] ———, *Local boundary properties of mappings quasiconformal in the mean*, Collection Sci. Works, Inst. Math. Sibirsk. Otdel. Akad. Nauk SSSR, Novosibirsk, 1981, pp. 168–171. (Russian)

[104] ———, *Behavior of certain class of mappings, quasiconformal in the mean, at an isolated singular point*, Dokl. Akad. Nauk SSSR **277** (1984), no. 5, 1056–1058. (Russian) MR759964 (86e:30019)

[105] C. Kuratowski, *Topology*. Vol. 1, Academic Press, New York, 1968. MR0028007 (10:389b)

[106] ———, *Topology*. Vol. 2, Academic Press, New York, 1968. MR0039235 (12:517a)

[107] G. V. Kuz′mina, *Moduli of families of curves and quadratic differentials*, Tr. Mat. Inst. Steklov. **139** (1980), 3–241; English transl., Proc. Steklov. Inst. Math. **1982**, no. 1. MR612632 (84j:30038a)

[108] R. Kühnau, *Uber Extremalprobleme bei im Mittel quasiconformen Abbildungen*, Lecture Notes in Math., vol. 1013, Springer-Verlag, Berlin, 1983, pp. 113–124. MR738087 (87i:30034)

[109] R. Landes and V. Mustonen, *Pseudo-monotone mappings in Sobolev–Orlicz spaces and nonlinear boundary value problems on unbounded domains*, J. Math. Anal. Appl. **88** (1982), 25–36. MR661399 (84f:35057)

[110] V. Lappalainen and A. Lehtonen, *Embedding of Orlicz–Sobolev spaces in Hölder spaces*, Ann. Acad. Sci. Fenn. Math. **14** (1989), no. 1, 41–46. MR997969 (90h:46066)

[111] O. Lehto and K. Virtanen, *Quasiconformal mappings in the plane*, Grundlehren Math. Wiss., Bd. 126, Springer-Verlag, New York, 1973. MR0344463 (49:9202)

[112] J. Lelong-Ferrand, *Representation conforme et transformations à integrale de Dirichlet bornée*, Gauthier-Villars, Paris, 1955. MR0069895 (16:1096b)

[113] T. V. Lomako, *On extension of some generalizations of quasiconformal mappings to a boundary*, Ukr. Mat. Zh. **61** (2009), no. 10, 1329–1337; English transl., Ukr. Math. J. **63** (2009), no. 10, 1568–1577. MR2888544

[114] T. V. Lomako, R. R. Salimov, and E. A. Sevost'yanov, *On equicontinuity of solutions to the Beltrami equations*, Ann. Univ. Buchar. Math. Ser. **1 (LIX)** (2010), no. 2, 263–274. MR2920236

[115] J. Maly, *A simple proof of the Stepanov theorem on differentiability alsmost everywhere*, Exposition Math. **17** (1999), 59–61. MR1687460 (2000a:26012)

[116] J. Maly and O. Martio, *Lusin's condition (N) and mappings of the class $W^{1,n}$*, J. Reine Angew. Math. **485** (1995), 19–36. MR1310951 (95m:26024)

[117] M. Marcus and V. Mizel, *Transformations by functions in Sobolev spaces and lower semicontinuity for parametric variational problems*, Bull. Amer. Math. Soc. **79** (1973), no. 4, 790–795. MR0322651 (48:1013)

[118] O. Martio, *Modern tools in the theory of quasiconformal maps*, Textos de Math. Ser. B, vol. 27, Univ. Coimbra, Dept. Mat., Coimbra, 2000, pp. 1–43. MR1889159 (2002m:30025)

[119] O. Martio, S. Rickman, and J. Väisälä, *Definitions for quasiregular mappings*, Ann. Acad. Sci. Fenn. Math. **448** (1969), 1–40. MR0259114 (41:3756)

[120] O. Martio, V. I. Ryazanov, U. Srebro, and E. Yakubov, *Mappings with finite length distortion*, J. Anal. Math. **93** (2004), 215–236. MR2110329 (2005i:30030)

[121] ———, *Q-homeomorphisms*, Complex Analysis and Dynamical Systems, Contemp. Math., vol. 364, Amer. Math. Soc., Providence, RI, 2004, pp. 193–203. MR2099027 (2005h:30046)

[122] ———, *On Q-homeomorphisms*, Ann. Acad. Sci. Fenn. Math. **30** (2005), no. 1, 49–69. MR2140298 (2006d:30027)

[123] ———, *Moduli in modern mapping theory*, Springer Monogr. Math., Springer, New York, 2009. MR2466579 (2012g:30004)

[124] O. Martio, V. I. Ryazanov, and M. Vuorinen, *BMO and injectivity of space quasiregular mappings*, Math. Nachr. **205** (1999), 149–161. MR1709167 (2001d:30031)

[125] O. Martio and J. Sarvas, *Injectivity theorems in plane and space*, Ann. Acad. Sci. Fenn. Math. **4** (1974), no. 2, 384–401. MR565886 (81i:30039)

[126] O. Martio and M. Vuorinen, *Whitney cubes, p-capacity and Minkowski content*, Exposition. Math. **5** (1987), no. 1, 17–40. MR880256 (88e:28004)
[127] P. Mattila, *Geometry of sets and measures in Euclidean spaces*, Fractals and Rectifiability, Cambridge Stud. Adv. Math., vol. 44, Cambridge Univ. Press, Cambridge, 1995. MR1333890 (96h:28006)
[128] V. G. Maz′ya, *Sobolev spaces*, Leningrad. Univ., Leningrad, 1985; English transl., Springer Ser. Soviet Math., Springer, Berlin, 1985. MR817985 (87g:46056)
[129] D. Menchoff, *Sur les differentielles totales des fonctions univalentes*, Math. Ann. **105** (1931), no. 1, 75–85. MR1512705
[130] R. Näkki, *Boundary behavior of quasiconformal mappings in n-space*, Ann. Acad. Sci. Fenn. Math. **484** (1970), 1–50. MR0285707 (44:2925)
[131] A. Norton, *A critical set with nonnull image has large Hausdorff dimension*, Trans. Amer. Math. Soc. **296** (1986), no. 1, 367–376. MR837817 (87i:26011)
[132] J. Onninen, *Differentiability of monotone Sobolev functions*, Real. Anal. Exchange **26** (2000/2001), no. 2, 761–772. MR1844392 (2002d:26016)
[133] W. Orlicz, *Über eine gewisse Klasse von Räumen vom Typus B.*, Bull. Intern. de l'Acad. Pol. Ser. A, Cracovie, 1932.
[134] ———, *Über Räume* (L^M), Bull. Intern. de l'Acad. Pol. Serie A, Cracovie, 1936.
[135] D. K. Palagachev, *Quasilinear elliptic equations with VMO coefficients*, Trans. Amer. Math. Soc. **347** (1995), no. 7, 2481–2493. MR1308019 (95k:35083)
[136] M. Perovich, *Isolated singularity of the mean quasiconformal mappings*, Lecture Notes in Math., vol. 743, Springer-Verlag, Berlin, 1979, pp. 212–214. MR552884 (81i:30040)
[137] ———, *The global homeomorphism of mappings that are quasiconformal in the mean*, Dokl. Akad. Nauk SSSR **230** (1976), no. 4, 781–784. (Russian) MR0437755 (55:10678)
[138] I. N. Pesin, *Mappings which are quasiconformal in the mean*, Dokl. Akad. Nauk SSSR **187** (1969), no. 4, 740–742. (Russian) MR0249613 (40:2856)
[139] S. P. Ponomarev, *On the N-property of homeomorphisms of the class* W_p^1, Sibirsk. Mat. Zh. **28** (1978), no. 2, 140–148. (Russian) MR890732 (88i:26031)
[140] F. Quinn and A. Sard, *Hausdorff conullity of critical images of Fredholm maps*, Amer. J. Math. **94** (1972), 1101–1110. MR0322899 (48:1260)
[141] T. Rado and P. V. Reichelderfer, *Continuous transformations in analysis*, Grundlehren Math. Wiss., Bd. 75, Springer-Verlag, Berlin, 1955. MR0079620 (18:115C)
[142] M. A. Ragusa, *Elliptic boundary value problem in vanishing mean oscillation hypothesis*, Comment. Math. Univ. Carolin. **40** (1999), no. 4, 651–663. MR1756544 (2001g:35058)
[143] K. Rajala, A. Zapadinskaya, and T. Zürcher, *Generalized Hausdorff dimension distortion in Euclidean spaces under Sobolev mappings*, `arXiv:1007.2091v1 [math.CA]`, 2010, 1–13.
[144] H. M. Reimann, *On the absolute continuity of surface representation*, Comment. Math. Helv. **46** (1971), 44–47. MR0281881 (43:7595)
[145] H. M. Reimann and T. Rychener, *Funktionen beschränkter mittlerer oscillation*, Lecture Notes in Math., vol. 487, Springer-Verlag, Berlin–New York, 1975. MR0511997 (58:23564)
[146] Yu. G. Reshetnyak, *Spatial mappings with bounded distortion*, Nauka, Novosibirsk, 1982; English transl., Trans. Math. Monogr., vol. 73, Amer. Math. Soc., Providence, RI, 1988. MR665590 (84d:30033)
[147] ———, *The N condition for spatial mappings of the class* $W^1_{n,\mathrm{loc}}$, Sibirsk. Mat. Zh. **28** (1987), no. 5, 149–153. (Russian) MR924990 (89b:46049)
[148] ———, *Sobolev classes of functions with values in a metric space*, Sibirsk. Mat. Zh. **38** (1997), no. 3, 657–675; English transl., Sib. Math. J. **38** (1997), no. 3, 567–583. MR1457485 (98h:46031)
[149] ———, *Some geometric properties of functions and mappings with generalized derivatives*, Sibirsk. Mat. Zh. **7** (1966), no. 4, 886–919. (Russian) MR0203013 (34:2872)
[150] S. Rickman, *Quasiregular mappings*, Ergeb. Math. und Grenzgeb. (3), vol. 26, Springer-Verlag, Berlin, 1993. MR1238941 (95g:30026)
[151] W. Rudin, *Function theory in polydisks*, Math. Lect. Notes Ser., W. A. Benjamin Inc., New York, 1969. MR0255841 (41:501)
[152] V. I. Ryazanov, *On mappings that are quasiconformal in the mean*, Sibirsk. Mat. Zh. **37** (1996), no. 2, 378–388; English transl., Sib. Math. J. **37** (1996), no. 2, 325–334. MR1425344 (98a:30022)
[153] V. I. Ryazanov and R. R. Salimov, *Weakly flat spaces and boundaries in the theory of mapping*, Ukr. Mat. Visn. **4** (2007), no. 2, 199–234; English transl., Ukr. Math. Bull. **4** (2007), no. 2, 199–233. MR2503889 (2010c:30034)
[154] V. I. Ryazanov, R. R. Salimov, and E. A. Sevast′yanov, *On convergence analysis of space homeomorphisms*, Siberian Adv. in Math. **23** (2013), no. 4, 1–52.

[155] V. I. Ryzanov and E. A. Sevast′yanov, *Equicontinuous classes of ring Q-homeomorphisms*, Sib. Mat. Zh. **48** (2007), no. 6, 1361–1376; English transl., Sib. Math. J. **48** (2007), no. 6, 1093–1105. MR2397516 (2009k:30030)

[156] ______, *Equicontinuity of mean quasiconformal mappings*, Sibirsk. Mat. Zh. **52** (2011), no. 3, 665–679; English transl., Sib. Math. J. **52** (2011), no. 3, 524–536. MR2858651 (2012h:30121)

[157] ______, *On compactness of Orlicz–Sobolev mappings*, Ann. Univ. Buchar. Ser. Math. **3 (LXI)** (2012), no. 1, 79–87. MR3034964

[158] V. I. Ryazanov, U. Srebro, and E. Yakubov, *On ring solutions of Beltrami equation*, J. Anal. Math. **96** (2005), 117–150. MR2177183 (2006i:30028)

[159] ______, *On convergence theory for Beltrami equations*, Ukr. Mat. Visn. **5** (2008), no. 4, 524–535; English transl., Ukr. Math. Bull. **5** (2008), no. 4, 517–528. MR2573657 (2010k:30023)

[160] ______, *To strong ring solutions of the Beltrami equations*, Uzbek. Math. J. **2009**, no. 1, 127–137. MR2571387

[161] ______, *On strong solutions of the Beltrami equations*, Complex Var. Elliptic Equ. **55** (2010), no. 1-3, 219–236. MR2599622 (2011c:30068)

[162] ______, *Integral conditions in the theory of the Beltrami equations*, Complex Var. Elliptic Equ. **57** (2012), no. 12, 1247–1270. MR2991571

[163] S. Saks, *Theory of the integral*, New-York, 1964. MR0167578 (29:4850)

[164] D. Sarason, *Functions of vanishing mean oscillation*, Trans. Amer. Math. Soc. **207** (1975), 391–405. MR0377518 (51:13690)

[165] A. Sard, *The measure of the critical values of differentiable maps*, Bull. Amer. Math. Soc. **48** (1942), 883–890. MR0007523 (4:153c)

[166] ______, *The equivalence of n-measure and Lebesgue measure in E_n*, Bull. Amer. Math. Soc. **49** (1943), 758–759. MR0008837 (5:62b)

[167] ______, *Images of critical sets*, Ann. of Math. **68** (1958), no. 2, 247–259. MR0100063 (20:6499)

[168] ______, *Hausdorff measure of critical images on Banach manifolds*, Amer. J. Math. **87** (1965), 158–174. MR0173748 (30:3958)

[169] E. S. Smolovaya, *Boundary behavior of ring Q-homeomorphisms in metric spaces*, Ukr. Mat. Zh. **62** (2010), no. 5, 682–689; English transl. Ukr. Math. J. **62** (2010), no. 5, 785–793. MR2888634

[170] S. L. Sobolev, *Applications of functional analysis in mathematical physics*, Izdat. Leningrad. Univ., Leningrad, 1950; English transl., Transl. Math. Monogr., vol. 7, Amer. Math. Soc., Providence, RI, 1963. MR0165337 (29:2624)

[171] A. Yu. Solynin, *Moduli and extremal-metric problems*, Algebra i Analiz **11** (1999), no. 1, 3–86; English transl., St. Petersburg Math. J. **11** (2000), no. 1, 1–65. MR1691080 (2001b:30058)

[172] E. M. Stein, *Editor's note: The differentiability of functions in $\mathbb{R}^n$*, Ann. Math. (2) **113** (1981), no. 2, 383–385. MR607898 (84j:35077)

[173] W. Stepanoff, *Sur la résolution du probléme de Dirichlet á l'aide de l'intégrale de Poisson*, Mat. Sb. **32** (1924), no. 1, 111–114.

[174] G. D. Suvorov, *Generalized principle of the length and area in the mapping theory*, Naukova Dumka, Kiev, 1985. (Russian) MR869064 (88j:30052)

[175] ______, *The metric theory of prime ends and boundary properties of plane mappings with bounded Dirichlet integrals*, Naukova Dumka, Kiev, 1981. (Russian) MR628315 (83a:30027)

[176] ______, *Families of plane topological mappings*, Izdat. Sibirsk. Otdel. Akad. Nauk SSSR, Novosibirsk, 1965. (Russian) MR0199425 (33:7570)

[177] A. V. Sychev, *Moduli and spatial quasiconformal mappings*, Nauka, Novosibirsk, 1983. (Russian) MR0736870 (85i:30047)

[178] H. Tuominen, *Characterization of Orlicz–Sobolev space*, Ark. Mat. **45** (2007), no. 1, 123–139. MR2312957 (2008c:46046)

[179] V. A. Shlyk, *On the equality between p-capacity and p-modulus*, Sibirsk. Mat. Zh. **34** (1993), no. 6, 216–221; English transl., Sib. Math. J. **34** (1993), no. 6, 1196–1200. MR1268174 (95d:31003)

[180] J. Väisälä, *On the null-sets for extremal distances*, Ann. Acad. Sci. Fenn. Math. **322** (1962), 1–12. MR0147633 (26:5148)

[181] ______, *Two new characterizations for quasiconformality*, Ann. Acad. Sci. Fenn. Math. **362** (1965), 1–12. MR0174782 (30:4975)

[182] ______, *Lectures on n-dimensional quasiconformal mappings*, Lecture Notes in Math., vol. 229, Springer-Verlag, Berlin, 1971. MR0454009 (56:12260)

[183] A. Vasil′ev, *Moduli of families of curves for conformal and quasiconformal mappings*, Lecture Notes in Math., vol. 1788, Springer-Verlag, Berlin, 2002. MR1929066 (2003j:30003)

[184] P. A. Vuillermot, *Hölder-regularity for the solutions of strongly nonlinear eigenvalue problems on Orlicz–Sobolev space*, Houston J. Math. **13** (1987), no. 2, 281–287. MR904959 (88m:35020)

[185] M. Vuorinen, *Conformal geometry and quasiregular mappings*, Lecture Notes in Math., vol. 1319, Springer-Verlag, Berlin, 1988. MR950174 (89k:30021)

[186] R. L. Wilder, *Topology of manifolds*, Amer. Math. Soc. Colloq. Publ., vol. 32, Amer. Math. Soc., New York, N.Y., 1949. MR0029491 (10:614c)

[187] H. Whitney, *A function not constant on a connected set of critical points*, Duke Math. J. **1** (1935), no. 4, 514–517. MR1545896

[188] W. P. Ziemer, *Extremal length and conformal capacity*, Trans. Amer. Math. Soc. **126** (1967), no. 3, 460–473. MR0210891 (35:1776)

Institute of Applied Mathematics and Mechanics, National Academy of Sciences of Ukraine, Roze Luxemburg str. 74, Donetsk 83114, Ukraine
E-mail address: denis_kovtonyuk@bk.ru

Institute of Applied Mathematics and Mechanics, National Academy of Sciences of Ukraine, Roze Luxemburg str. 74, Donetsk 83114, Ukraine
E-mail address: vlryazanov1@rambler.ru

Institute of Applied Mathematics and Mechanics, National Academy of Sciences of Ukraine, Roze Luxemburg str. 74, Donetsk 83114, Ukraine
E-mail address: salimov@rambler.ru

Institute of Applied Mathematics and Mechanics, National Academy of Sciences of Ukraine, Roze Luxemburg str. 74, Donetsk 83114, Ukraine
E-mail address: brusin2006@rambler.ru

Received 26/MAY/2013

Translated by V. I. RYAZANOV