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# HOMOGENIZATION OF THE CAUCHY PROBLEM FOR PARABOLIC SYSTEMS WITH PERIODIC COEFFICIENTS

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Abstract. In $L_2(\mathbb{R}^d;\mathbb{C}^n)$, a class of matrix second order differential operators $\mathcal{B}_\varepsilon$ with rapidly oscillating coefficients (depending on $\mathbf{x}/\varepsilon$) is considered. For a fixed $s>0$ and small $\varepsilon>0$, approximation is found for the operator $\exp(-\mathcal{B}_\varepsilon s)$ in the $(L_2\to L_2)$- and $(L_2\to H^1)$-norm with an error term of order of $\varepsilon$. The results are applied to homogenization of solutions of the parabolic Cauchy problem.

## Introduction

**0.1.** In this paper, we deal with homogenization theory for periodic differential operators (DO's). A broad literature is devoted to homogenization problems (see, for example, [ZhKO, BaPa, BeLP]). We rely on the operator-theoretic (spectral) approach to homogenization problems. This approach was developed in the papers [BSu1, BSu2, BSu3, BSu4] by Birman and Suslina.

**0.2.** We study homogenization in the small period limit $\varepsilon\to 0$ for the following Cauchy problem:

$$\rho(\varepsilon^{-1}\mathbf{x})\partial_s\mathbf{u}_\varepsilon(\mathbf{x},s) = -\widehat{\mathcal{B}}_\varepsilon\mathbf{u}_\varepsilon(\mathbf{x},s)+\mathbf{F}(\mathbf{x},s);\quad \rho(\varepsilon^{-1}\mathbf{x})\mathbf{u}_\varepsilon(\mathbf{x},0)=\boldsymbol{\phi}(\mathbf{x}). \tag{0.1}$$

Here $\boldsymbol{\phi}\in L_2(\mathbb{R}^d;\mathbb{C}^n)$ and $\mathbf{F}\in L_p((0,T);L_2(\mathbb{R}^d;\mathbb{C}^n))$ for some $p$. The solution $\mathbf{u}_\varepsilon(\mathbf{x},s)$ is a $\mathbb{C}^n$-valued function of $\mathbf{x}\in\mathbb{R}^d$ and $s\geq 0$; $\widehat{\mathcal{B}}_\varepsilon$ is a matrix elliptic second order DO acting in $L_2(\mathbb{R}^d;\mathbb{C}^n)$. A measurable $(n\times n)$-matrix-valued function $\rho(\mathbf{x})$ is assumed to be bounded, uniformly positive definite, and periodic relative to some lattice $\Gamma\subset\mathbb{R}^d$. Let $\Omega$ be the cell of the lattice $\Gamma$. We use the notation $\varphi^\varepsilon(\mathbf{x})=\varphi(\varepsilon^{-1}\mathbf{x})$, where $\varphi(\mathbf{x})$ is a measurable $\Gamma$-periodic function in $\mathbb{R}^d$.

The principal part $\widehat{\mathcal{A}}_\varepsilon$ of the operator $\widehat{\mathcal{B}}_\varepsilon$ is given in a factorized form

$$\widehat{\mathcal{A}}_\varepsilon = b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D}), \tag{0.2}$$

where $b(\mathbf{D})$ is a matrix homogeneous first order DO and $g(\mathbf{x})$ is a $\Gamma$-periodic, bounded, and positive definite matrix-valued function in $\mathbb{R}^d$. (The precise assumptions on $b(\mathbf{D})$ and $g(\mathbf{x})$ are given below, see §4.) Homogenization problems for the operator (0.2) were analyzed in detail in [BSu1, BSu2, BSu3, BSu4]. Now we study more general operators $\widehat{\mathcal{B}}_\varepsilon$ that include first and zero order terms:

$$\widehat{\mathcal{B}}_\varepsilon\mathbf{u} = \widehat{\mathcal{A}}_\varepsilon\mathbf{u} + \sum_{j=1}^d\left(a_j^\varepsilon(\mathbf{x})D_j\mathbf{u} + D_j(a_j^\varepsilon(\mathbf{x}))^*\mathbf{u}\right) + \mathcal{Q}^\varepsilon(\mathbf{x})\mathbf{u}+\lambda\mathbf{u}. \tag{0.3}$$

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Here the $a_j(\mathbf{x})$, $j = 1, \dots, d$, are $\Gamma$-periodic $(n \times n)$-matrix-valued functions such that $a_j \in L_\varrho(\Omega)$, $\varrho = 2$ for $d = 1$, $\varrho > d$ for $d \geq 2$. In general, the potential $\mathcal{Q}^\varepsilon(\mathbf{x})$ is a distribution (with values in the class of Hermitian matrices) generated by a rapidly oscillating matrix-valued measure. The constant $\lambda$ is chosen so that the operator $\widehat{\mathcal{B}}_\varepsilon$ is positive definite. The coefficients of the operator (0.3) oscillate rapidly as $\varepsilon \to 0$. Elliptic homogenization problems for the operator (0.3) were studied in [Su3, Su6].

*Our aim in this paper is to find approximation as* $\varepsilon \to 0$ *for the solutions of problem* (0.1). Approximation in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ is given in terms of the solutions of the "homogenized" problem. Approximation in $H^1(\mathbb{R}^d; \mathbb{C}^n)$ requires taking the corrector term into account.

The homogenized problem has the form

$$\overline{\rho}\partial_s \mathbf{u}_0(\mathbf{x}, s) = -\widehat{\mathcal{B}}^0 \mathbf{u}_0(\mathbf{x}, s) + \mathbf{F}(\mathbf{x}, s), \quad \overline{\rho}\mathbf{u}_0(\mathbf{x}, 0) = \boldsymbol{\phi}(\mathbf{x}). \tag{0.4}$$

Here $\overline{\rho}$ is the mean value of the matrix $\rho$ over the cell $\Omega$: $\overline{\rho} = \int_\Omega \rho(\mathbf{x})\, d\mathbf{x}$; $\widehat{\mathcal{B}}^0$ is the *effective operator* with constant coefficients (see (9.4)).

**0.3. Main results.** In the Introduction we only discuss the case where $\rho = \mathbf{1}_n$. In this case the solution of (0.1) is given by $\mathbf{u}_\varepsilon = \exp(-\widehat{\mathcal{B}}_\varepsilon s)\boldsymbol{\phi} + \int_0^s \exp(-\widehat{\mathcal{B}}_\varepsilon(s - \widetilde{s}))\mathbf{F}(\,\cdot\,, \widetilde{s})\, d\widetilde{s}$. So, the problem reduces to the study of the operator exponential $\exp(-\widehat{\mathcal{B}}_\varepsilon s)$ for small $\varepsilon > 0$. (In the general case, we need to study the "bordered" operator exponential $f^\varepsilon e^{-\mathcal{B}_\varepsilon s}(f^\varepsilon)^*$ of the operator $\mathcal{B}_\varepsilon = (f^\varepsilon)^* \widehat{\mathcal{B}}_\varepsilon f^\varepsilon$, where $\rho^{-1} = ff^*$.)

The following estimates are the *main results of the paper*:

$$\left\| e^{-\widehat{\mathcal{B}}_\varepsilon s} - e^{-\widehat{\mathcal{B}}^0 s} \right\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C_1 \varepsilon (\varepsilon^2 + s)^{-1/2} e^{-C_2 s}, \quad s \geq 0; \tag{0.5}$$

$$\left\| e^{-\widehat{\mathcal{B}}_\varepsilon s} - e^{-\widehat{\mathcal{B}}^0 s} - \varepsilon \mathcal{K}(\varepsilon, s) \right\|_{L_2(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \leq C_3 \varepsilon s^{-1} e^{-C_2 s}, \quad \varepsilon \leq s^{1/2}. \tag{0.6}$$

Here $\mathcal{K}(\varepsilon, s)$ is the so-called *corrector*. The corrector has zero order with respect to $\varepsilon$, but involves rapidly oscillating factors. Estimates (0.5) and (0.6) are order-sharp for small $\varepsilon$ and a fixed $s > 0$. The constants in estimates are controlled explicitly in terms of the problem data. Estimate (0.5) makes it possible to prove convergence in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ of the solutions $\mathbf{u}_\varepsilon$ of problem (0.1) to the solution of the effective problem (0.4). Estimate (0.6) makes it possible to find approximation of the solutions $\mathbf{u}_\varepsilon$ in the $H^1(\mathbb{R}^d; \mathbb{C}^n)$-norm. We are interested in the behavior of the solutions $\mathbf{u}_\varepsilon$ for a fixed $s$, and do not strive for accuracy of estimates as $s \to \infty$. So, for our goals it suffices to obtain estimates (0.5), (0.6) with some positive $C_2$.

**0.4.** Homogenization problems for parabolic equations were studied by traditional methods (see [ZhKO, BeLP, BaPa]). We use the spectral approach developed for elliptic problems in [BSu1, BSu2, BSu3, BSu4] and [Su3, Su6]. Parabolic problems were studied by this method in the papers [Su1, Su2, Su4, Su5, V, VSu1, VSu2]. For the operator (0.2), an estimate of the form (0.5) was obtained in [Su2], and an analog of estimate (0.6) was obtained in [Su5] by using that method. By a different method, similar estimates were obtained in [ZhPas] for the acoustics operator $\widehat{\mathcal{A}}_\varepsilon = -\operatorname{div} g^\varepsilon(\mathbf{x})\nabla$. In the present paper, the results of [Su2, Su5] are generalized to the case of the operator family (0.3).

**0.5. The method of investigation.** We explain the method of investigation in the case where $\rho = \mathbf{1}_n$. It is easily seen that estimate (0.6) reduces to the inequality

$$\left\| \widehat{\mathcal{B}}_\varepsilon^{1/2} \left( e^{-\widehat{\mathcal{B}}_\varepsilon s} - e^{-\widehat{\mathcal{B}}^0 s} - \varepsilon \mathcal{K}(\varepsilon, s) \right) \right\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C \varepsilon s^{-1} e^{-C_2 s} \tag{0.7}$$

for $s > 0$, $0 < \varepsilon \leq s^{1/2}$. Using a scaling transformation, we reduce the proof of estimates (0.5), (0.7) to the study of the exponential $\exp(-\widehat{\mathcal{B}}(\varepsilon)\varepsilon^{-2}s)$ of the operator

$$\widehat{\mathcal{B}}(\varepsilon) = b(\mathbf{D})^* g b(\mathbf{D}) + \varepsilon \sum_{j=1}^{d} (a_j D_j + D_j a_j^*) + \varepsilon^2 \mathcal{Q} + \varepsilon^2 \lambda I,$$

which acts in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ and depends on the parameter $\varepsilon$. So, it is necessary to study the behavior of $\exp(-\widehat{\mathcal{B}}(\varepsilon)\widetilde{s})$ for large values of $\widetilde{s} = \varepsilon^{-2}s$.

Applying the Floquet–Bloch theory, we decompose the operator $\widehat{\mathcal{B}}(\varepsilon)$ into the direct integral of operators $\widehat{\mathcal{B}}(\mathbf{k}, \varepsilon)$ acting in $L_2(\Omega; \mathbb{C}^n)$ and depending on the parameter $\mathbf{k} \in \mathbb{R}^d$ (called the quasimomentum). The operator $\widehat{\mathcal{B}}(\mathbf{k}, \varepsilon)$ is given by the expression

$$\widehat{\mathcal{B}}(\mathbf{k}, \varepsilon) = \widehat{\mathcal{A}}(\mathbf{k}) + \varepsilon \sum_{j=1}^{d} \big(a_j (D_j + k_j) + (D_j + k_j) a_j^*\big) + \varepsilon^2 \mathcal{Q} + \varepsilon^2 \lambda I,$$

where $\widehat{\mathcal{A}}(\mathbf{k}) = b(\mathbf{D} + \mathbf{k})^* g b(\mathbf{D} + \mathbf{k})$, with periodic boundary conditions. The spectrum of the operator $\widehat{\mathcal{B}}(\mathbf{k}, \varepsilon)$ is discrete. As in [Su3, Su6], we distinguish the one-dimensional parameter $\tau = (|\mathbf{k}|^2 + \varepsilon^2)^{1/2}$ and study the family $\widehat{\mathcal{B}}(\mathbf{k}, \varepsilon)$ by methods of analytic perturbation theory with respect to $\tau$.

**0.6. The structure of the paper.** The paper consists of three chapters. Chapter 1 (§§1–3) is devoted to the abstract operator-theoretic method. In Chapter 2 (§§4–8) periodic DO's acting in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ are studied. Approximation of the "bordered" operator exponential is obtained in §8. Chapter 3 (§§9–10) is devoted to homogenization of the parabolic Cauchy problem. In §9, by a scaling transformation, the *main results of the paper* are deduced from the results of §8. In §10, the results of §9 are applied to homogenization for parabolic systems.

**0.7. Notation.** Let $\mathfrak{H}$ and $\mathfrak{H}_*$ be separable Hilbert spaces. The symbols $(\cdot\,, \cdot)_{\mathfrak{H}}$ and $\|\cdot\|_{\mathfrak{H}}$ stand for the inner product and the norm in $\mathfrak{H}$, respectively. The symbol $\|\cdot\|_{\mathfrak{H}\to\mathfrak{H}_*}$ denotes the norm of a bounded operator acting from $\mathfrak{H}$ to $\mathfrak{H}_*$. Sometimes we omit indices if this does not lead to confusion. By $I = I_{\mathfrak{H}}$ we denote the identity operator in $\mathfrak{H}$. If $A\colon \mathfrak{H} \to \mathfrak{H}_*$ is a linear operator, then $\operatorname{Dom} A$ and $\operatorname{Ker} A$ denote the domain and the kernel of $A$, respectively. If $\mathfrak{N}$ is a subspace in $\mathfrak{H}$, then $\mathfrak{N}^\perp := \mathfrak{H} \ominus \mathfrak{N}$. If $P$ is the orthogonal projection of $\mathfrak{H}$ onto $\mathfrak{N}$, then $P^\perp$ is the orthogonal projection of $\mathfrak{H}$ onto $\mathfrak{N}^\perp$. The symbols $\langle\cdot\,, \cdot\rangle$ and $|\cdot|$ stand for the usual inner product and the norm in $\mathbb{C}^n$, respectively; $\mathbf{1}_n$ is the identity $(n \times n)$-matrix. If $a$ is an $(n \times n)$-matrix, then $|a|$ is the norm of the matrix $a$ viewed as an operator in $\mathbb{C}^n$, and $a^*$ denotes the adjoint matrix.

Next, $\mathbf{x} = (x^1, \dots, x^d) \in \mathbb{R}^d$, $iD_j = \partial/\partial x^j$, $j = 1, \dots, d$, $\nabla = \operatorname{grad} = (\partial_1, \dots, \partial_d)$, $\mathbf{D} = -i\nabla = (D_1, \dots, D_d)$.

The $L_p$-classes of $\mathbb{C}^n$-valued functions on a domain $\mathcal{O} \subseteq \mathbb{R}^d$ are denoted by $L_p(\mathcal{O}; \mathbb{C}^n)$, $1 \leq p \leq \infty$. By $L_p((0, T); \mathfrak{H})$ we denote the $L_p$-space of $\mathfrak{H}$-valued functions on the interval $(0, T)$. The Sobolev classes of $\mathbb{C}^n$-valued functions (in a domain $\mathcal{O} \subseteq \mathbb{R}^d$) of order $s$ are denoted by $H^s(\mathcal{O}; \mathbb{C}^n)$. If $n = 1$, we write simply $L_p(\mathcal{O})$, $H^s(\mathcal{O})$, but (if this does not lead to confusion) we use this short notation also for the spaces of vector-valued or matrix-valued functions.

By $C$, $c$, $\mathcal{C}$, $\mathfrak{C}$, $\mathfrak{c}$ (possibly, with indices and marks) we denote various constants in estimates.

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# Chapter 1
# Abstract operator-theoretic method

## §1. Quadratic two-parametric operator pencils

We study an operator family $B(t,\varepsilon)$ depending on two real-valued parameters $t \in \mathbb{R}$ and $0 \le \varepsilon \le 1$. The family $B(t,\varepsilon)$ was studied in [Su6, Su7].

**1.1. The operators $X(t)$ and $A(t)$.** Let $\mathfrak{H}$ and $\mathfrak{H}_*$ be complex separable Hilbert spaces. Suppose that $X_0 \colon \mathfrak{H} \to \mathfrak{H}_*$ is a densely defined and closed operator, and $X_1 : \mathfrak{H} \to \mathfrak{H}_*$ is a bounded operator. Then the operator

$$X(t) := X_0 + tX_1 \colon \mathfrak{H} \to \mathfrak{H}_* \tag{1.1}$$

is closed on the domain $\operatorname{Dom} X(t) = \operatorname{Dom} X_0$. In $\mathfrak{H}$, we consider the selfadjoint operator $A(t) = X(t)^* X(t)$ generated by the closed quadratic form $\|X(t)u\|^2_{\mathfrak{H}_*}$, $u \in \operatorname{Dom} X_0$. We put $A_0 := A(0) = X_0^* X_0$ and $\mathfrak{N} := \operatorname{Ker} A_0 = \operatorname{Ker} X_0$. Assume that the following condition is fulfilled.

**Condition 1.1.** *The point $\lambda_0 = 0$ is an isolated point of the spectrum of $A_0$, and $0 < n := \dim \mathfrak{N} < \infty$.*

Let $d^0$ be the *distance from the point $\lambda_0 = 0$ to the rest of the spectrum of $A_0$*. We put $\mathfrak{N}_* = \operatorname{Ker} X_0^*$, $n_* := \dim \mathfrak{N}_*$. *Assume that* $n \le n_* \le \infty$. Let $P$ and $P_*$ be the orthogonal projections of $\mathfrak{H}$ onto $\mathfrak{N}$ and of $\mathfrak{H}_*$ onto $\mathfrak{N}_*$, respectively.

**1.2. The operators $Y(t)$ and $Y_2$.** Let $\widetilde{\mathfrak{H}}$ be yet another separable Hilbert space. Let $Y_0 \colon \mathfrak{H} \to \widetilde{\mathfrak{H}}$ be a densely defined linear operator such that $\operatorname{Dom} X_0 \subset \operatorname{Dom} Y_0$; let $Y_1 \colon \mathfrak{H} \to \widetilde{\mathfrak{H}}$ be a bounded linear operator. We put $Y(t) = Y_0 + tY_1$, $\operatorname{Dom} Y(t) = \operatorname{Dom} Y_0$, and impose the following condition.

**Condition 1.2.** *For some $c_1 > 0$, we have*

$$\|Y(t)u\|_{\widetilde{\mathfrak{H}}} \le c_1 \|X(t)u\|_{\mathfrak{H}_*}, \quad u \in \operatorname{Dom} X_0, \quad t \in \mathbb{R}. \tag{1.2}$$

Estimate (1.2) with $t = 0$ implies that $\operatorname{Ker} X_0 \subset \operatorname{Ker} Y_0$, i.e., $Y_0 P = 0$.

Let $Y_2 \colon \mathfrak{H} \to \widetilde{\mathfrak{H}}$ be a densely defined linear operator such that $\operatorname{Dom} X_0 \subset \operatorname{Dom} Y_2$. We impose the following condition.

**Condition 1.3.** *For any $\nu > 0$ there exists a number $C(\nu) > 0$ such that*

$$\|Y_2 u\|^2_{\widetilde{\mathfrak{H}}} \le \nu \|X(t)u\|^2_{\mathfrak{H}_*} + C(\nu)\|u\|^2_{\mathfrak{H}}, \quad u \in \operatorname{Dom} X_0, \ t \in \mathbb{R}.$$

**1.3. The operator $Q_0$ and the form $\mathfrak{q}$.** Let $Q_0$ be a bounded positive definite linear operator on $\mathfrak{H}$, and let $\mathfrak{q}[u,v]$ be a densely defined Hermitian sesquilinear form in $\mathfrak{H}$ such that $\operatorname{Dom} X_0 \subset \operatorname{Dom} \mathfrak{q}$. The form $\mathfrak{q}$ is subject to the following condition.

**Condition 1.4.** *There exist constants $0 < \kappa \le 1$, $c_0 \in \mathbb{R}$, $c_2 \ge 0$, $c_3 \ge 0$ such that for $u \in \operatorname{Dom} X_0$, $t \in \mathbb{R}$, we have*

$$-(1-\kappa)\|X(t)u\|^2_{\mathfrak{H}_*} - c_0\|u\|^2_{\mathfrak{H}} \le \mathfrak{q}[u,u] \le c_2 \|X(t)u\|^2_{\mathfrak{H}_*} + c_3 \|u\|^2_{\mathfrak{H}}. \tag{1.3}$$

**1.4. The operator $B(t,\varepsilon)$.** In $\mathfrak{H}$, we consider the quadratic form

$$\mathfrak{b}(t,\varepsilon)[u,u] = \|X(t)u\|^2_{\mathfrak{H}_*} + 2\varepsilon \operatorname{Re}(Y(t)u, Y_2 u)_{\widetilde{\mathfrak{H}}} + \varepsilon^2 \mathfrak{q}[u,u] + \lambda \varepsilon^2 (Q_0 u, u)_{\mathfrak{H}}, \quad u \in \operatorname{Dom} X_0. \tag{1.4}$$

The parameter $\lambda \in \mathbb{R}$ is subject to the following restriction:

$$\begin{aligned} &\lambda > \|Q_0^{-1}\|(c_0 + c_4) \ \text{ if } \ \lambda \ge 0, \\ &\lambda > \|Q_0\|^{-1}(c_0 + c_4) \ \text{ if } \ \lambda < 0 \ (\text{and } c_0 + c_4 < 0), \end{aligned} \tag{1.5}$$

where $c_0$ is the constant as in (1.3), and the constant $c_4$ is defined by

$$c_4 := 4\kappa^{-1}c_1^2 C(\nu) \quad \text{for} \quad \nu = \kappa^2(16c_1^2)^{-1}. \tag{1.6}$$

As was noted in [Su7, Subsection 1.4], condition (1.5) implies that

$$\mathfrak{b}(t,\varepsilon)[u,u] \geq \frac{\kappa}{2}\|X(t)u\|^2_{\mathfrak{H}_*} + \beta\varepsilon^2\|u\|^2_{\mathfrak{H}}, \quad u \in \operatorname{Dom} X_0, \tag{1.7}$$

where $\beta > 0$ is defined in terms of $\lambda$ as follows:

$$\begin{aligned} &\beta = \lambda\|Q_0^{-1}\|^{-1} - c_0 - c_4 \quad \text{if} \quad \lambda \geq 0, \\ &\beta = \lambda\|Q_0\| - c_0 - c_4 \quad \text{if} \quad \lambda < 0 \text{ (and } \ c_0 + c_4 < 0). \end{aligned} \tag{1.8}$$

In [Su6, (1.15)], it was shown that

$$\mathfrak{b}(t,\varepsilon)[u,u] \leq (2 + c_1^2 + c_2)\|X(t)u\|^2_{\mathfrak{H}_*} + \big(C(1) + c_3 + |\lambda|\|Q_0\|\big)\varepsilon^2\|u\|^2_{\mathfrak{H}}. \tag{1.9}$$

By (1.7) and (1.9), the form (1.4) is closed and positive definite. The corresponding selfadjoint operator in $\mathfrak{H}$ is denoted by $B(t,\varepsilon)$. Formally, we can write

$$B(t,\varepsilon) = A(t) + \varepsilon(Y_2^* Y(t) + Y(t)^* Y_2) + \varepsilon^2 Q + \lambda\varepsilon^2 Q_0. \tag{1.10}$$

(Here $Q$ is a formal object that corresponds to the form $\mathfrak{q}$.)

**1.5. Passage to the parameter $\tau$.** The family $B(t,\varepsilon)$ is an analytic operator family with respect to the parameters $t$ and $\varepsilon$. If $t = \varepsilon = 0$, the operator (1.10) coincides with $A_0$ and has an isolated eigenvalue $\lambda_0 = 0$ of multiplicity $n$. To apply the methods of analytic perturbation theory, *we introduce the one-dimensional parameter* $\tau = (t^2 + \varepsilon^2)^{1/2}$ and also the additional parameters $\vartheta_1 = t\tau^{-1}$, $\vartheta_2 = \varepsilon\tau^{-1}$, $\vartheta = (\vartheta_1, \vartheta_2)$. Then the operator (1.10) can be rewritten as $B(\tau;\vartheta)$. Formally,

$$\begin{aligned} B(\tau;\vartheta) = (X_0^* + \tau\vartheta_1 X_1^*)(X_0 + \tau\vartheta_1 X_1) + \tau\vartheta_2(Y_2^* Y_0 + Y_0^* Y_2) \\ + \tau^2\vartheta_1\vartheta_2(Y_2^* Y_1 + Y_1^* Y_2) + \tau^2\vartheta_2^2(Q + \lambda Q_0). \end{aligned} \tag{1.11}$$

The corresponding form will be denoted by $\mathfrak{b}(\tau;\vartheta)$. We study the operator $B(\tau;\vartheta)$ as a quadratic operator pencil with respect to the parameter $\tau$ with the help of the tecniques of analytic perturbation theory. Herewith, we should make our constructions and estimates uniform with respect to the parameter $\vartheta$, taking into account that $\vartheta_1^2 + \vartheta_2^2 = 1$. In (1.11) we may assume that $\tau \in \mathbb{R}$.

Let $F(\tau;\vartheta;s)$ be the spectral projection of the operator (1.11) for the closed interval $[0,s]$. We fix a number $\delta \in (0, \kappa d^0/13)$ and put

$$\tau_0 = \delta^{1/2}\left((2 + c_1^2 + c_2)\|X_1\|^2 + C(1) + c_3 + |\lambda|\|Q_0\|\right)^{-1/2}. \tag{1.12}$$

In [Su6, Subsection 1.5], it was proved that

$$F(\tau;\vartheta;\delta) = F(\tau;\vartheta;3\delta), \quad \operatorname{rank} F(\tau;\vartheta;\delta) = n, \tag{1.13}$$

for $|\tau| \leq \tau_0$. Instead of $F(\tau;\vartheta;\delta)$ we shall use the shorter notation $F(\tau;\vartheta)$.

**1.6. The operators $Z$ and $\widetilde{Z}$.** In Subsections 1.6 and 1.7, we introduce some operators that arise in perturbation theory considerations. We denote $\mathcal{D} := \operatorname{Dom} X_0 \cap \mathfrak{N}^\perp$. Since the point $\lambda_0 = 0$ is an isolated point of the spectrum of $A_0$, the form $(X_0\phi, X_0\zeta)$, $\phi, \zeta \in \mathcal{D}$, determines an inner product in $\mathcal{D}$, converting $\mathcal{D}$ into a Hilbert space.

For a given $\omega \in \mathfrak{N}$, we consider the equation $X_0^*(X_0\varphi + X_1\omega) = 0$ for $\varphi \in \mathcal{D}$. This equation is understood in a weak sense. In other words, we look for an element $\varphi \in \mathcal{D}$ satisfying the identity

$$(X_0\varphi, X_0\zeta)_{\mathfrak{H}_*} = -(X_1\omega, X_0\zeta)_{\mathfrak{H}_*} \quad \text{for all} \quad \zeta \in \mathcal{D}. \tag{1.14}$$

Since the right-hand side of (1.14) is an antilinear continuous functional of $\zeta \in \mathcal{D}$, the Riesz theorem shows that there exists a unique solution; denote this solution by $\varphi(\omega)$. *We* *introduce a bounded operator* $Z\colon \mathfrak{H} \to \mathfrak{H}$ *as follows*: $Z\omega = \varphi(\omega)$, $\omega \in \mathfrak{N}$; $Zx = 0$, $x \in \mathfrak{N}^\perp$. Obviously, $PZ = 0$. Note that $\varphi(\omega)$ satisfies the estimate $\|X_0\varphi(\omega)\|_{\mathfrak{H}_*} \le \|X_1\omega\|_{\mathfrak{H}_*}$, whence

$$\|X_0 Z\|_{\mathfrak{H}\to\mathfrak{H}_*} \le \|X_1\|_{\mathfrak{H}\to\mathfrak{H}_*}. \tag{1.15}$$

Similarly, given $\omega \in \mathfrak{N}$, suppose that $\psi \in \mathcal{D}$ satisfies the equation

$$X_0^* X_0 \psi + Y_0^* Y_2 \omega = 0, \tag{1.16}$$

understood in the weak sense. Namely, $\psi \in \mathcal{D}$ satisfies the identity

$$(X_0\psi, X_0\zeta)_{\mathfrak{H}_*} = -(Y_2\omega, Y_0\zeta)_{\widetilde{\mathfrak{H}}} \quad \text{for all} \quad \zeta \in \mathcal{D}. \tag{1.17}$$

By Condition 1.2, the right-hand side of (1.17) is a continuous antilinear functional of $\zeta \in \mathcal{D}$. Therefore, by the Riesz theorem, there exists a unique solution $\psi(\omega)$. *We introduce a bounded operator* $\widetilde{Z}$ *acting in* $\mathfrak{H}$ *by* $\widetilde{Z}\omega = \psi(\omega)$, $\omega \in \mathfrak{N}$; $\widetilde{Z}x = 0$, $x \in \mathfrak{N}^\perp$. Obviously, $P\widetilde{Z} = 0$. We estimate the norm of the operator $X_0\widetilde{Z}$. The solution $\psi(\omega)$ satisfies $\|X_0\psi(\omega)\|_{\mathfrak{H}_*} \le c_1\|Y_2\omega\|_{\widetilde{\mathfrak{H}}}$, whence

$$\|X_0\widetilde{Z}u\|_{\mathfrak{H}_*} = \|X_0\widetilde{Z}Pu\|_{\mathfrak{H}_*} \le c_1\|Y_2Pu\|_{\widetilde{\mathfrak{H}}}, \quad u \in \mathfrak{H}. \tag{1.18}$$

Note that Condition 1.3 with $t = 0$ implies the estimate

$$\|Y_2Pu\|_{\widetilde{\mathfrak{H}}} \le (C(\nu))^{1/2}\|u\|_{\mathfrak{H}}, \quad u \in \mathfrak{H}, \quad \nu > 0. \tag{1.19}$$

Combining (1.18) with (1.19), we obtain

$$\|X_0\widetilde{Z}\|_{\mathfrak{H}\to\mathfrak{H}_*} \le c_1(C(\nu))^{1/2}, \quad \nu > 0. \tag{1.20}$$

**1.7. The operators $R$ and $S$.** We introduce the operator $R := X_0Z|_{\mathfrak{N}} + X_1|_{\mathfrak{N}}\colon \mathfrak{N} \to \mathfrak{N}_*$. As was shown in [BSu1, (1.1.11)], $R = P_*X_1|_{\mathfrak{N}}$. In accordance with [BSu1, Subsection 1.1.3], the operator $S = R^*R\colon \mathfrak{N} \to \mathfrak{N}$ is called the *spectral germ* of the operator family $A(t)$ at $t = 0$. The germ $S$ can be written as $S = PX_1^*P_*X_1|_{\mathfrak{N}}$, so that $\|S\| \le \|X_1\|^2$.

**1.8. The spectral germ of the operator $B(\tau;\vartheta)$.** General facts of the analytic perturbation theory (see [K]) show that for $|\tau| \le \tau_0$ there exist functions $\lambda_l(\tau;\vartheta)$ real-analytic in $\tau$ (the branches of eigenvalues) and real-analytic $\mathfrak{H}$-valued functions $\varphi_l(\tau;\vartheta)$ (the branches of eigenvectors) such that

$$B(\tau;\vartheta)\varphi_l(\tau;\vartheta) = \lambda_l(\tau;\vartheta)\varphi_l(\tau;\vartheta), \quad |\tau| \le \tau_0, \quad l = 1, \dots, n. \tag{1.21}$$

The elements $\varphi_l(\tau;\vartheta)$, $l = 1, \dots, n$, form an orthogonal basis in the eigenspace $F(\tau;\vartheta)\mathfrak{H}$. Relations (1.21) are understood in the weak sense, namely,

$$\mathfrak{b}(\tau;\vartheta)[\varphi_l(\tau;\vartheta), \zeta] = \lambda_l(\tau;\vartheta)(\varphi_l(\tau;\vartheta), \zeta)_{\mathfrak{H}}, \quad \zeta \in \operatorname{Dom} X_0.$$

Moreover, for sufficiently small $\tau_*$ ($\tau_* \le \tau_0$) and $|\tau| \le \tau_*$, we have the following convergent power series expansions:

$$\begin{aligned} &\lambda_l(\tau;\vartheta) = \gamma_l(\vartheta)\tau^2 + \mu_l(\vartheta)\tau^3 + \dots, \quad \gamma_l(\vartheta) \ge 0, \quad l = 1, \dots, n, \\ &\varphi_l(\tau;\vartheta) = \omega_l(\vartheta) + \tau\varphi_l^{(1)}(\vartheta) + \tau^2\varphi_l^{(2)}(\vartheta) + \dots, \quad l = 1, \dots, n. \end{aligned} \tag{1.22}$$

**Definition 1.5** (see [Su6])**.** The operator $S(\vartheta)\colon \mathfrak{N} \to \mathfrak{N}$ defined by

$$\begin{aligned} S(\vartheta) = \vartheta_1^2 S &- \vartheta_1\vartheta_2(X_0Z)^*(X_0\widetilde{Z})|_{\mathfrak{N}} - \vartheta_1\vartheta_2(X_0\widetilde{Z})^*(X_0Z)|_{\mathfrak{N}} \\ &- \vartheta_2^2(X_0\widetilde{Z})^*(X_0\widetilde{Z})|_{\mathfrak{N}} + \vartheta_1\vartheta_2 P(Y_2^*Y_1 + Y_1^*Y_2)|_{\mathfrak{N}} + \vartheta_2^2(Q_{\mathfrak{N}} + \lambda Q_{0\mathfrak{N}}) \end{aligned} \tag{1.23}$$

is called the *spectral germ of the operator pencil* (1.11) *at* $\tau = 0$.

Here $Q_{\mathfrak{N}}$ is the selfadjoint operator in $\mathfrak{N}$ generated by the form $\mathfrak{q}[u,u]$, $u \in \mathfrak{N}$, and $Q_{0\mathfrak{N}} = PQ_0|_{\mathfrak{N}}$. Note that Condition 1.4 with $t = 0$ implies the estimate $\|Q_{\mathfrak{N}}\| \leq \max\{|c_0|; c_3\}$. Hence, by (1.15), (1.19), (1.20) with $\nu = 1$, and by the estimate $\|S\| \leq \|X_1\|^2$, we have

$$\|S(\vartheta)P\| \leq c_5, \tag{1.24}$$

$$c_5 := \left(\|X_1\| + c_1 C(1)^{1/2}\right)^2 + 2C(1)^{1/2}\|Y_1\| + \max\{|c_0|; c_3\} + |\lambda|\|Q_0\|. \tag{1.25}$$

In accordance with [Su6, Proposition 1.6], the numbers $\gamma_l(\vartheta)$ and the elements $\omega_l(\vartheta)$ are eigenvalues and eigenvectors of the selfadjoint operator $S(\vartheta)$:

$$S(\vartheta)\omega_l(\vartheta) = \gamma_l(\vartheta)\omega_l(\vartheta), \quad l = 1, \dots, n. \tag{1.26}$$

**1.9. Threshold approximations.** In [Su6, Theorem 2.2], the following result was obtained.

**Theorem 1.6.** *For $|\tau| \leq \tau_0$ we have*

$$F(\tau;\vartheta) - P = \Phi(\tau;\vartheta), \quad \|\Phi(\tau;\vartheta)\|_{\mathfrak{H}\to\mathfrak{H}} \leq C_1|\tau|, \tag{1.27}$$

$$B(\tau;\vartheta)F(\tau;\vartheta) - \tau^2 S(\vartheta)P = \Psi(\tau;\vartheta), \quad \|\Psi(\tau;\vartheta)\|_{\mathfrak{H}\to\mathfrak{H}} \leq C_2|\tau|^3. \tag{1.28}$$

*The constants $C_1$ and $C_2$ depend on $\delta$, $c_1$, $c_2$, $c_3$, $C(1)$, $\kappa$, $|\lambda|$, $\|X_1\|$, $\|Y_1\|$, and $\|Q_0\|$.*

The constants $C_1$ and $C_2$ can be written explicitly (see [Su6, §2]). We put

$$C_T^{(1)} = \max\left\{2 + c_1^2, (\|X_1\|^2 + C(1))\delta^{-1}\right\}, \tag{1.29}$$

$$C_T^{(2)} = \max\left\{c_2 + 1, (\|X_1\|^2 + \|Y_1\|^2 + C(1) + c_3 + |\lambda|\|Q_0\|)\delta^{-1}\right\}, \tag{1.30}$$

$$C_T = C_T^{(1)} + \tau_0 C_T^{(2)}, \tag{1.31}$$

$$C_T^0 = 32 \cdot 13^2 \kappa^{-1/2}(C_T^{(1)})^2 C_T + 32 \cdot 13\kappa^{-1/2} C_T^{(1)} C_T^{(2)} + 416\kappa^{-1/2} C_T^{(2)} C_T. \tag{1.32}$$

Then

$$C_1 = 32(1 + \pi^{-1})\kappa^{-1/2} C_T, \quad C_2 = 2\delta(1 + \pi^{-1}) C_T^0. \tag{1.33}$$

Besides estimate (1.27), we need a more accurate approximation obtained in [Su6, Subsection 2.5]:

$$F(\tau;\vartheta) - P = \tau F_1(\vartheta) + F_2(\tau;\vartheta), \tag{1.34}$$

where the operator $F_2(\tau;\vartheta)$ is of order of $O(\tau^2)$. In accordance with [Su6, (1.48)], the operator $F_1(\vartheta)$ admits the representation $F_1(\vartheta) = \vartheta_1(Z + Z^*) + \vartheta_2(\widetilde{Z} + \widetilde{Z}^*)$. Hence, the identities $PZ = 0$, $P\widetilde{Z} = 0$ imply that

$$F_1(\vartheta)P = \vartheta_1 Z + \vartheta_2 \widetilde{Z}. \tag{1.35}$$

Comparing (1.24) and (1.28), we obtain

$$\left\|B(\tau;\vartheta)F(\tau;\vartheta)\right\|_{\mathfrak{H}\to\mathfrak{H}} \leq C_3\tau^2, \quad |\tau| \leq \tau_0; \quad C_3 := c_5 + C_2\tau_0. \tag{1.36}$$

Hence, for $|\tau| \leq \tau_0$, the eigenvalues of $B(\tau;\vartheta)$ admit the estimate $\lambda_l(\tau;\vartheta) \leq C_3\tau^2$, $l = 1, \dots, n$. Therefore,

$$\left\|B(\tau;\vartheta)^{1/2}F(\tau;\vartheta)\right\|_{\mathfrak{H}\to\mathfrak{H}} \leq C_3^{1/2}|\tau|, \quad |\tau| \leq \tau_0. \tag{1.37}$$

We also need the following estimate obtained in [Su6, Proposition 2.7]:

$$\begin{aligned} &\left\|B(\tau;\vartheta)^{1/2}F_2(\tau;\vartheta)\right\|_{\mathfrak{H}\to\mathfrak{H}} \leq C_4\delta^{1/2}(1 + \pi^{-1})\tau^2, \quad |\tau| \leq \tau_0, \\ &C_4 := \sqrt{2}(2 + c_1^2 + c_2)^{1/2}(12\kappa^{-1} + 2)^{1/2}(49 C_T^{(1)} C_T + 7 C_T^{(2)}). \end{aligned} \tag{1.38}$$

**1.10. The operator family** $A(t) = M^*\widehat{A}(t)M$**.** Let $\widehat{\mathfrak{H}}$ be yet another Hilbert space, and let $\widehat{X}(t) = \widehat{X}_0 + t\widehat{X}_1 \colon \widehat{\mathfrak{H}} \to \mathfrak{H}_*$ be a family of the form (1.1) satisfying the assumptions of Subsection 1.1. We emphasize that the space $\mathfrak{H}_*$ is the same as before. *All the objects corresponding to* $\widehat{X}(t)$ *are marked by* "$\widehat{\ }$". Suppose that $M \colon \mathfrak{H} \to \widehat{\mathfrak{H}}$ is an isomorphism and that

$$M \operatorname{Dom} X_0 = \operatorname{Dom} \widehat{X}_0, \tag{1.39}$$

$X(t) = \widehat{X}(t)M \colon \mathfrak{H} \to \mathfrak{H}_*$; $X_0 = \widehat{X}_0 M$, $X_1 = \widehat{X}_1 M$. Then $A(t) = M^*\widehat{A}(t)M$, where $\widehat{A}(t) = \widehat{X}(t)^*\widehat{X}(t)$. Observe that $\widehat{\mathfrak{N}} = M\mathfrak{N}$, $\widehat{n} = n$ and $\widehat{\mathfrak{N}}_* = \mathfrak{N}_*$, $\widehat{n}_* = n_*$, $\widehat{P}_* = P_*$. We denote

$$G = (MM^*)^{-1} \colon \widehat{\mathfrak{H}} \to \widehat{\mathfrak{H}}. \tag{1.40}$$

Let $G_{\widehat{\mathfrak{N}}}$ be the block of the operator $G$ in the subspace $\widehat{\mathfrak{N}}$:

$$G_{\widehat{\mathfrak{N}}} = \widehat{P}G\big|_{\widehat{\mathfrak{N}}} \colon \widehat{\mathfrak{N}} \to \widehat{\mathfrak{N}}. \tag{1.41}$$

Obviously, $G_{\widehat{\mathfrak{N}}}$ is an isomorphism in $\widehat{\mathfrak{N}}$. It turns out (see [Su2, Proposition 1.2]) that the orthogonal projections $P$ and $\widehat{P}$ satisfy the relation

$$P = M^{-1}(G_{\widehat{\mathfrak{N}}})^{-1}\widehat{P}(M^*)^{-1}. \tag{1.42}$$

Let $\widehat{S} \colon \widehat{\mathfrak{N}} \to \widehat{\mathfrak{N}}$ be the spectral germ of the operator family $\widehat{A}(t)$ at $t = 0$. In accordance with [BSu1, Subsection 1.1.5], we have

$$S = PM^*\widehat{S}M\big|_{\mathfrak{N}}. \tag{1.43}$$

**1.11. The operator family** $B(t,\varepsilon) = M^*\widehat{B}(t,\varepsilon)M$**.** Let $\widehat{Y}_0 \colon \widehat{\mathfrak{H}} \to \widetilde{\mathfrak{H}}$ satisfy the assumptions of Subsection 1.2. Note that the space $\widetilde{\mathfrak{H}}$ is the same as before. We denote $Y_0 = \widehat{Y}_0 M$, $M \operatorname{Dom} Y_0 = \operatorname{Dom} \widehat{Y}_0$. By (1.39) and the condition $\operatorname{Dom} \widehat{X}_0 \subset \operatorname{Dom} \widehat{Y}_0$, we have $\operatorname{Dom} X_0 \subset \operatorname{Dom} Y_0$. Suppose that $\widehat{Y}_1 \colon \widehat{\mathfrak{H}} \to \widetilde{\mathfrak{H}}$ is a bounded operator and that $Y_1 = \widehat{Y}_1 M \colon \mathfrak{H} \to \widetilde{\mathfrak{H}}$. We put $\widehat{Y}(t) = \widehat{Y}_0 + t\widehat{Y}_1 \colon \widehat{\mathfrak{H}} \to \widetilde{\mathfrak{H}}$, $\operatorname{Dom} \widehat{Y}(t) = \operatorname{Dom} \widehat{Y}_0$, and $Y(t) = \widehat{Y}(t)M = Y_0 + tY_1 \colon \mathfrak{H} \to \widetilde{\mathfrak{H}}$, $\operatorname{Dom} Y(t) = \operatorname{Dom} Y_0$. Suppose that the operators $\widehat{X}(t)$ and $\widehat{Y}(t)$ satisfy Condition 1.2 with some constant $\widehat{c}_1$. Then, automatically, $\|Y(t)u\|_{\widetilde{\mathfrak{H}}} \le c_1\|X(t)u\|_{\mathfrak{H}_*}$, where $c_1 = \widehat{c}_1$.

Let $\widehat{Y}_2 \colon \widehat{\mathfrak{H}} \to \widetilde{\mathfrak{H}}$ be an operator satisfying the assumptions of Subsection 1.2. We put $Y_2 = \widehat{Y}_2 M \colon \mathfrak{H} \to \widetilde{\mathfrak{H}}$, $M \operatorname{Dom} Y_2 = \operatorname{Dom} \widehat{Y}_2$. Since $M$ is an isomorphism and the operator $\widehat{Y}_2$ is densely defined, the operator $Y_2$ is also densely defined. By (1.39), we have $\operatorname{Dom} X_0 \subset \operatorname{Dom} Y_2$. We assume that the operators $\widehat{X}(t)$ and $\widehat{Y}_2$ satisfy Condition 1.3 with some constant $\widehat{C}(\nu) > 0$. Then, automatically, for any $\nu > 0$ there exists a constant $C(\nu) = \widehat{C}(\nu)\|M\|^2 > 0$ such that $\|Y_2 u\|^2_{\widetilde{\mathfrak{H}}} \le \nu\|X(t)u\|^2_{\mathfrak{H}_*} + C(\nu)\|u\|^2_{\mathfrak{H}}$ for $t \in \mathbb{R}$ and $u \in \operatorname{Dom} X_0$.

We put $Q_0 := M^*M$. Then $Q_0$ is a bounded and positive definite operator in $\mathfrak{H}$. (The role of $\widehat{Q}_0$ is played by the identity operator in $\widehat{\mathfrak{H}}$.)

In $\widehat{\mathfrak{H}}$, we consider the quadratic form $\widehat{\mathfrak{q}}$ that satisfies the assumptions of Subsection 1.3. We define the form $\mathfrak{q}$ by the rule $\mathfrak{q}[u,v] = \widehat{\mathfrak{q}}[Mu, Mv]$, $u, v \in \operatorname{Dom} \mathfrak{q}$, $M \operatorname{Dom} \mathfrak{q} = \operatorname{Dom} \widehat{\mathfrak{q}}$. Formally, $Q = M^*\widehat{Q}M$. Assume that the operator $\widehat{X}(t)$ and the form $\widehat{\mathfrak{q}}$ satisfy Condition 1.4 with the constants $\kappa$, $\widehat{c}_0$, $\widehat{c}_2$ and $\widehat{c}_3$. By (1.39), it is easily seen that the operator $X(t) = \widehat{X}(t)M$ and the form $\mathfrak{q}$ also satisfy Condition 1.4 with the constants

$$c_0 = \|M\|^2\widehat{c}_0 \ \text{ if } \ \widehat{c}_0 \ge 0, \quad c_0 = \|M^{-1}\|^{-2}\widehat{c}_0 \ \text{ if } \ \widehat{c}_0 < 0, \tag{1.44}$$

$c_2 = \widehat{c}_2$, $c_3 = \|M\|^2\widehat{c}_3$, and the same constant $\kappa$ as before. By (1.6), the constants $c_4$ and $\widehat{c}_4 = 4\kappa^{-1}\widehat{c}_1^2\widehat{C}(\nu)$ with $\nu = \kappa^2(16\widehat{c}_1^2)^{-1}$ satisfy the relation

$$c_4 = \|M\|^2\widehat{c}_4. \tag{1.45}$$

Under the above assumptions, the operator pencil

$$\widehat{B}(t,\varepsilon) = \widehat{A}(t) + \varepsilon(\widehat{Y}_2^*\widehat{Y}(t) + \widehat{Y}(t)^*\widehat{Y}_2) + \varepsilon^2\widehat{Q} + \lambda\varepsilon^2 I \tag{1.46}$$

and the operator pencil (1.10) satisfy $B(t,\varepsilon) = M^*\widehat{B}(t,\varepsilon)M$. The constant $\lambda$ is chosen in accordance with condition (1.5) for the operator (1.10). Comparing (1.44), (1.45), and the identity $Q_0 = M^*M$, we see that for such $\lambda$ condition (1.5) is also satisfied for the operator (1.46).

Note that for the operator (1.46) relations (1.8) take the form $\widehat{\beta} = \lambda - \widehat{c}_0 - \widehat{c}_4$. Hence, by (1.8), (1.44), (1.45), we have

$$\beta \le \|M^{-1}\|^{-2}\widehat{\beta}. \tag{1.47}$$

**1.12. The relationship between the spectral germs $S(\vartheta)$ and $\widehat{S}(\vartheta)$.** In this subsection, we generalize identity (1.43) to the case of the spectral germs of the operator families (1.46) and (1.10) such that $B(t,\varepsilon) = M^*\widehat{B}(t,\varepsilon)M$. For the family $\widehat{B}(t,\varepsilon)$, we introduce the operators $\widehat{Z}$ and $\widehat{\widetilde{Z}}$ as in Subsection 1.6. We prove the following result.

**Lemma 1.7.** *Under the above assumptions, we have*

$$\widehat{X}_0\widehat{Z}M\big|_{\mathfrak{N}} = X_0Z\big|_{\mathfrak{N}}, \quad \widehat{X}_0\widehat{\widetilde{Z}}M\big|_{\mathfrak{N}} = X_0\widetilde{Z}\big|_{\mathfrak{N}}. \tag{1.48}$$

*Proof.* The operator $R$ is defined by the relation $R := (X_0Z + X_1)|_{\mathfrak{N}}$. On the other hand, $R = P_*X_1\big|_{\mathfrak{N}}$. Therefore, $X_0Z\big|_{\mathfrak{N}} = (P_* - I)X_1\big|_{\mathfrak{N}}$. Similarly, $\widehat{X}_0\widehat{Z}\big|_{\widehat{\mathfrak{N}}} = (P_* - I)\widehat{X}_1\big|_{\widehat{\mathfrak{N}}}$, because $\widehat{P}_* = P_*$. Comparing these relations and recalling that $X_1 = \widehat{X}_1M$ and $\widehat{\mathfrak{N}} = M\mathfrak{N}$, we arrive at the first identity in (1.48).

The second identity in (1.48) is equivalent to

$$\big((X_0\widetilde{Z} - \widehat{X}_0\widehat{\widetilde{Z}}M)\omega, \zeta\big)_{\mathfrak{H}_*} = 0, \quad \omega \in \mathfrak{N}, \quad \zeta \in \mathfrak{H}_*. \tag{1.49}$$

Since $\mathfrak{N}_* = \widehat{\mathfrak{N}}_*$, for $\zeta \in \mathfrak{N}_*$ the identity (1.49) is obvious. Writing $\mathfrak{H}_* = \operatorname{Ran} X_0 \oplus \mathfrak{N}_*$, we see that it suffices to consider $\zeta \in \operatorname{Ran} X_0$. Then $\zeta = X_0\xi$ for some $\xi \in \mathcal{D}$. Since $\zeta = \widehat{X}_0M\xi = \widehat{X}_0\widehat{P}^\perp M\xi$, the required relation can be rewritten as

$$(X_0\widetilde{Z}\omega, X_0\xi)_{\mathfrak{H}_*} = (\widehat{X}_0\widehat{\widetilde{Z}}M\omega, \widehat{X}_0\widehat{P}^\perp M\xi)_{\mathfrak{H}_*}. \tag{1.50}$$

By the definition of the operator $\widetilde{Z}$ (see (1.17)), we have

$$(X_0\widetilde{Z}\omega, X_0\xi)_{\mathfrak{H}_*} = -(Y_2\omega, Y_0\xi)_{\widetilde{\mathfrak{H}}}. \tag{1.51}$$

Similarly, by the definition of the operator $\widehat{\widetilde{Z}}$, we have

$$(\widehat{X}_0\widehat{\widetilde{Z}}M\omega, \widehat{X}_0\widehat{P}^\perp M\xi)_{\mathfrak{H}_*} = -(\widehat{Y}_2M\omega, \widehat{Y}_0\widehat{P}^\perp M\xi)_{\widetilde{\mathfrak{H}}} = -(Y_2\omega, Y_0\xi)_{\widetilde{\mathfrak{H}}}. \tag{1.52}$$

In the last identity we have used the relations $\widehat{Y}_0\widehat{P} = 0$, $Y_0 = \widehat{Y}_0M$, $Y_2 = \widehat{Y}_2M$. Formulas (1.51) and (1.52) imply (1.50). $\square$

Now we return to the operator pencils $B(t,\varepsilon)$ and $\widehat{B}(t,\varepsilon)$ and pass to the parameters $\tau, \vartheta$. Consider the spectral germ (1.23) and a similar spectral germ for the family (1.46):

$$\begin{aligned}\widehat{S}(\vartheta) = \vartheta_1^2\widehat{S} &- \vartheta_1\vartheta_2(\widehat{X}_0\widehat{Z})^*(\widehat{X}_0\widehat{\widetilde{Z}})\big|_{\widehat{\mathfrak{N}}} - \vartheta_1\vartheta_2(\widehat{X}_0\widehat{\widetilde{Z}})^*(\widehat{X}_0\widehat{Z})\big|_{\widehat{\mathfrak{N}}}\\ &- \vartheta_2^2(\widehat{X}_0\widehat{\widetilde{Z}})^*(\widehat{X}_0\widehat{\widetilde{Z}})\big|_{\widehat{\mathfrak{N}}} + \vartheta_1\vartheta_2\widehat{P}(\widehat{Y}_2^*\widehat{Y}_1 + \widehat{Y}_1^*\widehat{Y}_2)\big|_{\widehat{\mathfrak{N}}} + \vartheta_2^2(\widehat{Q}_{\widehat{\mathfrak{N}}} + \lambda I_{\widehat{\mathfrak{N}}}).\end{aligned}$$

The identity $\widehat{\mathfrak{N}} = M\mathfrak{N}$ implies that $PM^* = PM^*\widehat{P}$. Combining this with (1.43), (1.48), and the relations $Y_1 = \widehat{Y}_1 M$, $Y_2 = \widehat{Y}_2 M$, $Q = M^*\widehat{Q}M$, we generalize identity (1.43).

**Proposition 1.8.** *The spectral germs* $S(\vartheta)$ *and* $\widehat{S}(\vartheta)$ *of the operator families* (1.46) *and* (1.10) *satisfy*

$$S(\vartheta) = PM^*\widehat{S}(\vartheta)M\big|_{\mathfrak{N}}. \tag{1.53}$$

**1.13. The operators $\widehat{Z}_G$ and $\widehat{\widetilde{Z}}_G$.** Let $\widehat{Z}_G$ be the operator in $\widehat{\mathfrak{H}}$ that takes an element $\widehat{u} \in \widehat{\mathfrak{H}}$ into a unique solution $\widehat{\phi}_G$ of the problem

$$\widehat{X}_0^*(\widehat{X}_0\widehat{\phi}_G + \widehat{X}_1\widehat{\omega}) = 0, \quad G\widehat{\phi}_G \perp \widehat{\mathfrak{N}}, \tag{1.54}$$

where $\widehat{\omega} = \widehat{P}\widehat{u}$. Problem (1.54) is understood in the weak sense (cf. (1.14)). Then, in accordance with [BSu2, Lemma 6.1],

$$\widehat{Z}_G = MZM^{-1}\widehat{P}. \tag{1.55}$$

Similarly, let $\widehat{\widetilde{Z}}_G$ be the operator in $\widehat{\mathfrak{H}}$ that takes an element $\widehat{u} \in \widehat{\mathfrak{H}}$ to a unique solution $\widehat{\psi}_G$ of the problem

$$\widehat{X}_0^*\widehat{X}_0\widehat{\psi}_G + \widehat{Y}_0^*\widehat{Y}_2\widehat{\omega} = 0, \quad G\widehat{\psi}_G \perp \widehat{\mathfrak{N}}, \tag{1.56}$$

where $\widehat{\omega} = \widehat{P}\widehat{u}$. Problem (1.56) is understood in the weak sense. By recalculation in equation (1.16), we can use the relations $M\mathfrak{N} = \widehat{\mathfrak{N}}$, (1.39), and (1.40) to obtain

$$\widehat{\widetilde{Z}}_G = M\widetilde{Z}M^{-1}\widehat{P}. \tag{1.57}$$

## §2. Approximation of the operator exponential

**2.1.** The principal term of approximation of the operator $\exp(-A(t)s)$ for large values of the parameter $s \geq 0$ was obtained in [Su2, §2.1]. Approximation of the operator $\exp(-A(t)s)$ in the "energy" norm with a corrector term taken into account was obtained in [Su5, §3.2]. Our goal in this section is to approximate the operator $\exp(-B(\tau;\vartheta)s)$ for large values of $s \geq 0$.

In addition to the assumptions of Subsections 1.1–1.4, we impose the condition

$$A(t) \geq c_* t^2 I, \quad c_* > 0, \quad |t| \leq \tau_0. \tag{2.1}$$

Hence, by (1.7), we have

$$B(\tau;\vartheta) \geq \check{c}_* \tau^2 I, \quad |\tau| \leq \tau_0, \quad \check{c}_* = \frac{1}{2}\min\{\kappa c_*, 2\beta\}. \tag{2.2}$$

Therefore, the eigenvalues $\lambda_l(\tau;\vartheta)$ of the operator $B(\tau;\vartheta)$ satisfy the estimates

$$\lambda_l(\tau;\vartheta) \geq \check{c}_* \tau^2, \quad l = 1, \dots, n, \quad |\tau| \leq \tau_0. \tag{2.3}$$

Comparing this with (1.22), we see that $\gamma_l(\vartheta) \geq \check{c}_*$, $l = 1, \dots, n$. Then, by (1.26), $S(\vartheta) \geq \check{c}_* I_{\mathfrak{N}}$. Hence, by (2.2), it follows that

$$\big\|e^{-B(\tau;\vartheta)s}\big\|_{\mathfrak{H}\to\mathfrak{H}} \leq e^{-\check{c}_*\tau^2 s}, \quad \big\|e^{-\tau^2 S(\vartheta)s}P\big\|_{\mathfrak{H}\to\mathfrak{H}} \leq e^{-\check{c}_*\tau^2 s}. \tag{2.4}$$

**2.2. The principal term of approximation.** Let $|\tau| \le \tau_0$. Obviously,

$$e^{-B(\tau;\vartheta)s} = e^{-B(\tau;\vartheta)s}F(\tau;\vartheta) + e^{-B(\tau;\vartheta)s}F(\tau;\vartheta)^{\perp}, \tag{2.5}$$

where $F(\tau;\vartheta)^{\perp}$ is the spectral projection of the operator $B(\tau;\vartheta)$ for the interval $(\delta;\infty)$. Then, by using the inequality $\exp(-\delta s/2) \le (\delta s)^{-1/2}$, we get

$$\left\| e^{-B(\tau;\vartheta)s}F(\tau;\vartheta)^{\perp}\right\|_{\mathfrak{H}\to\mathfrak{H}} \le e^{-\delta s} \le (\delta s)^{-1/2}e^{-\delta s/2}, \quad s \ge 0. \tag{2.6}$$

Next,

$$e^{-B(\tau;\vartheta)s}F(\tau;\vartheta) = Pe^{-B(\tau;\vartheta)s}F(\tau;\vartheta) + P^{\perp}e^{-B(\tau;\vartheta)s}F(\tau;\vartheta). \tag{2.7}$$

By (1.27), $P^{\perp}F(\tau;\vartheta) = (F(\tau;\vartheta) - P)F(\tau;\vartheta) = \Phi(\tau;\vartheta)F(\tau;\vartheta)$. Combining this with (1.27) and (2.2), we obtain

$$\left\| P^{\perp}e^{-B(\tau;\vartheta)s}F(\tau;\vartheta)\right\|_{\mathfrak{H}\to\mathfrak{H}} = \left\| \Phi(\tau;\vartheta)e^{-B(\tau;\vartheta)s}F(\tau;\vartheta)\right\|_{\mathfrak{H}\to\mathfrak{H}} \le C_1|\tau|e^{-\check{c}_*\tau^2 s}. \tag{2.8}$$

We put

$$\Sigma(s) := Pe^{-B(\tau;\vartheta)s}F(\tau;\vartheta) - Pe^{-\tau^2 S(\vartheta)Ps}, \tag{2.9}$$

$$\mathcal{E}(s) := e^{\tau^2 S(\vartheta)Ps}\Sigma(s) = e^{\tau^2 S(\vartheta)Ps}Pe^{-B(\tau;\vartheta)s}F(\tau;\vartheta) - P. \tag{2.10}$$

Differentiating (2.10) with respect to $s$ and using (1.28), we obtain

$$\begin{aligned}\mathcal{E}'(s) &= e^{\tau^2 S(\vartheta)Ps}P\left(\tau^2 S(\vartheta)P - B(\tau;\vartheta)F(\tau;\vartheta)\right)e^{-B(\tau;\vartheta)s}F(\tau;\vartheta)\\ &= -e^{\tau^2 S(\vartheta)Ps}P\Psi(\tau;\vartheta)e^{-B(\tau;\vartheta)s}F(\tau;\vartheta).\end{aligned}$$

From the identity $\mathcal{E}(s) = \mathcal{E}(0) + \int_0^s \mathcal{E}'(\widetilde{s})\,d\widetilde{s}$, it follows that

$$\mathcal{E}(s) = PF(\tau;\vartheta) - P - \int_0^s e^{\tau^2 S(\vartheta)P\widetilde{s}}P\Psi(\tau;\vartheta)e^{-B(\tau;\vartheta)\widetilde{s}}F(\tau;\vartheta)\,d\widetilde{s}.$$

Hence, by (1.27), the operator $\Sigma(s) = e^{-\tau^2 S(\vartheta)Ps}\mathcal{E}(s)$ satisfies the identity

$$\Sigma(s) = e^{-\tau^2 S(\vartheta)Ps}P\Phi(\tau;\vartheta) - \int_0^s e^{-\tau^2 S(\vartheta)P(s-\widetilde{s})}P\Psi(\tau;\vartheta)e^{-B(\tau;\vartheta)\widetilde{s}}F(\tau;\vartheta)\,d\widetilde{s}.$$

Combining this with (2.4) and (1.27), (1.28), we arrive at the estimate

$$\|\Sigma(s)\|_{\mathfrak{H}\to\mathfrak{H}} \le C_1|\tau|e^{-\check{c}_*\tau^2 s} + C_2|\tau|^3 s e^{-\check{c}_*\tau^2 s}. \tag{2.11}$$

Relations (2.7), (2.8), (2.9), and (2.11) imply that

$$\left\| e^{-B(\tau;\vartheta)s}F(\tau;\vartheta) - Pe^{-\tau^2 S(\vartheta)Ps}\right\|_{\mathfrak{H}\to\mathfrak{H}} \le (2C_1|\tau| + C_2|\tau|^3 s)e^{-\check{c}_*\tau^2 s}. \tag{2.12}$$

We put $|\tau|\sqrt{s} =: \alpha$ and write $(2C_1|\tau| + C_2|\tau|^3 s)e^{-\check{c}_*\tau^2 s/2} = s^{-1/2}\varphi(\alpha)$, where $\varphi(\alpha) := (2C_1\alpha + C_2\alpha^3)e^{-\check{c}_*\alpha^2/2}$. Denote

$$C_5 := \max_{\alpha\ge 0}\varphi(\alpha) = \max_{\alpha\ge 0}(2C_1\alpha + C_2\alpha^3)e^{-\check{c}_*\alpha^2/2}. \tag{2.13}$$

Then

$$\left\| e^{-B(\tau;\vartheta)s}F(\tau;\vartheta) - Pe^{-\tau^2 S(\vartheta)Ps}\right\|_{\mathfrak{H}\to\mathfrak{H}} \le C_5 s^{-1/2}e^{-\check{c}_*\tau^2 s/2}, \quad s > 0. \tag{2.14}$$

By (2.5), (2.6), and (2.14), we obtain

$$\left\| e^{-B(\tau;\vartheta)s} - Pe^{-\tau^2 S(\vartheta)Ps}\right\|_{\mathfrak{H}\to\mathfrak{H}} \le C_5 s^{-1/2}e^{-\check{c}_*\tau^2 s/2} + \delta^{-1/2}s^{-1/2}e^{-\delta s/2}. \tag{2.15}$$

Note that for $|\tau| \le \tau_0$ we have

$$e^{-\delta s/2} \le e^{-\tau^2 C_* s}, \quad e^{-\check{c}_*\tau^2 s/2} \le e^{-\tau^2 C_* s}, \quad |\tau| \le \tau_0, \tag{2.16}$$

with the constant

$$C_* := \frac{1}{2}\min\{\breve{c}_*; \delta\tau_0^{-2}\}. \tag{2.17}$$

From (2.16) and (2.15) it follows that

$$\bigl\|e^{-B(\tau;\vartheta)s} - Pe^{-\tau^2 S(\vartheta)Ps}\bigr\|_{\mathfrak{H}\to\mathfrak{H}} \le (C_5 + \delta^{-1/2})s^{-1/2}e^{-\tau^2 C_* s}. \tag{2.18}$$

Moreover, by (2.4) and (2.17), for all $s \ge 0$ the left-hand side of (2.18) satisfies the estimate

$$\bigl\|e^{-B(\tau;\vartheta)s} - Pe^{-\tau^2 S(\vartheta)Ps}\bigr\|_{\mathfrak{H}\to\mathfrak{H}} \le 2e^{-C_*\tau^2 s}.$$

For $s > 0$ we have $\min\{2; (C_5 + \delta^{-1/2})s^{-1/2}\} \le C_6(1+s)^{-1/2}$, where

$$C_6 := \sqrt{2}\max\{2; C_5 + \delta^{-1/2}\}. \tag{2.19}$$

Thus, we have proved that

$$\bigl\|e^{-B(\tau;\vartheta)s} - Pe^{-\tau^2 S(\vartheta)Ps}\bigr\|_{\mathfrak{H}\to\mathfrak{H}} \le C_6(1+s)^{-1/2}e^{-\tau^2 C_* s},\ s \ge 0,\ |\tau| \le \tau_0. \tag{2.20}$$

In accordance with [Su6, (3.26)], we denote $L(t,\varepsilon) := \tau^2 S(\vartheta)$:

$$\begin{aligned} L(t,\varepsilon) = t^2 S + t\varepsilon\bigl( -(X_0 Z)^*(X_0\widetilde{Z})|_{\mathfrak{N}} - (X_0\widetilde{Z})^*(X_0 Z)\big|_{\mathfrak{N}}\bigr) \\ + t\varepsilon P(Y_2^* Y_1 + Y_1^* Y_2)\big|_{\mathfrak{N}} + \varepsilon^2\bigl( -(X_0\widetilde{Z})^*(X_0\widetilde{Z})\big|_{\mathfrak{N}} + Q_{\mathfrak{N}} + \lambda Q_{0\mathfrak{N}}\bigr). \end{aligned} \tag{2.21}$$

Cf. (1.23). Note that the estimate $S(\vartheta) \ge \breve{c}_* I_{\mathfrak{N}}$ implies that

$$L(t,\varepsilon) \ge \breve{c}_*(t^2 + \varepsilon^2)I_{\mathfrak{N}}. \tag{2.22}$$

Now we formulate (2.20) in terms of the operator $L(t,\varepsilon)$.

**Theorem 2.1.** *Let $B(t,\varepsilon)$ be the operator defined in Subsection* 1.4*. Suppose that condition* (2.1) *is satisfied. Let $L(t,\varepsilon)$ be the operator* (2.21)*. Then*

$$\bigl\|e^{-B(t,\varepsilon)s} - e^{-L(t,\varepsilon)s}P\bigr\|_{\mathfrak{H}\to\mathfrak{H}} \le C_6(1+s)^{-1/2}e^{-\tau^2 C_* s}, \quad s \ge 0, \quad |\tau| \le \tau_0.$$

*The constant $C_6$ depends only on $\delta$, $\kappa$, $c_*$, $c_0$, $c_1$, $c_2$, $c_3$, $c_4$, $C(1)$, $\lambda$, $\|X_1\|$, $\|Y_1\|$, $\|Q_0\|$, and $\|Q_0^{-1}\|$. The constant $C_*$ is defined by* (2.17)*.*

**2.3. Approximation with the corrector term taken into account.** Approximation of the operator $\exp(-B(\tau;\vartheta)s)$ with the corrector term taken into account is given by the following theorem.

**Theorem 2.2.** *Under the assumptions of Theorem* 2.1*, let $Z$ and $\widetilde{Z}$ be the operators defined in Subsection* 1.6*. Then*

$$\begin{aligned} \bigl\|B(t,\varepsilon)^{1/2}\bigl(e^{-B(t,\varepsilon)s} - \bigl(I + tZ + \varepsilon\widetilde{Z}\bigr)e^{-L(t,\varepsilon)s}P\bigr)\bigr\|_{\mathfrak{H}\to\mathfrak{H}} \le C_8 s^{-1}e^{-\tau^2 C_* s}, \\ |\tau| \le \tau_0, \quad s > 0. \end{aligned} \tag{2.23}$$

*The constant $C_8$ is defined below in* (2.32)*.*

*Proof.* We put $\mathfrak{U}(\tau;\vartheta;s) := B(\tau;\vartheta)^{1/2}e^{-B(\tau;\vartheta)s}$. Obviously,

$$\begin{aligned} \mathfrak{U}(\tau;\vartheta;s) = \mathfrak{U}(\tau;\vartheta;s)F(\tau;\vartheta)^{\perp} + \mathfrak{U}(\tau;\vartheta;s)F(\tau;\vartheta)(F(\tau;\vartheta) - P) \\ + F(\tau;\vartheta)\mathfrak{U}(\tau;\vartheta;s)P. \end{aligned} \tag{2.24}$$

Relations (1.13), (2.16), and the inequality $e^{-\alpha} \le \alpha^{-1}$, $\alpha > 0$, imply

$$\begin{aligned} \bigl\|\mathfrak{U}(\tau;\vartheta;s)F(\tau;\vartheta)^{\perp}\bigr\|_{\mathfrak{H}\to\mathfrak{H}} &\le \sup_{\mu \ge 3\delta}\mu^{1/2}e^{-\mu s} \le 2(3\delta)^{-1/2}s^{-1}e^{-3\delta s/2} \\ &\le 2(3\delta)^{-1/2}s^{-1}e^{-\tau^2 C_* s}, \quad |\tau| \le \tau_0. \end{aligned} \tag{2.25}$$

Next, using (2.3) and (2.17), for $s > 0$ and $|\tau| \le \tau_0$ we get

$$\|\mathfrak{U}(\tau;\vartheta;s)F(\tau;\vartheta)\|_{\mathfrak{H}\to\mathfrak{H}} \le \sup_{1\le l\le n} (\lambda_l(\tau;\vartheta))^{1/2} e^{-\lambda_l(\tau;\vartheta)s} \le 2s^{-1} \sup_{1\le l\le n} (\lambda_l(\tau;\vartheta))^{-1/2} e^{-\lambda_l(\tau;\vartheta)s/2} \le 2\check{c}_*^{-1/2}|\tau|^{-1}s^{-1}e^{-\tau^2 C_* s}.$$

Combining this with (1.27), we see that

$$\|\mathfrak{U}(\tau;\vartheta;s)F(\tau;\vartheta)(F(\tau;\vartheta)-P)\|_{\mathfrak{H}\to\mathfrak{H}} \le 2C_1\check{c}_*^{-1/2}s^{-1}e^{-\tau^2 C_* s} \tag{2.26}$$

for $s > 0$ and $|\tau| \le \tau_0$. The last term on the right-hand side in (2.24) is represented as

$$F(\tau;\vartheta)\mathfrak{U}(\tau;\vartheta;s)P = B(\tau;\vartheta)^{1/2}F(\tau;\vartheta)e^{-\tau^2 S(\vartheta)s}P + B(\tau;\vartheta)^{1/2}F(\tau;\vartheta)\big(e^{-B(\tau;\vartheta)s}F(\tau;\vartheta) - e^{-\tau^2 S(\vartheta)s}P\big)P. \tag{2.27}$$

By (1.37), (2.12), and (2.17), we have

$$\big\|B(\tau;\vartheta)^{1/2}F(\tau;\vartheta)\big(e^{-B(\tau;\vartheta)s}F(\tau;\vartheta) - e^{-\tau^2 S(\vartheta)s}P\big)P\big\|_{\mathfrak{H}\to\mathfrak{H}} \le C_3^{1/2}|\tau|(2C_1|\tau| + C_2|\tau|^3 s)e^{-\check{c}_*\tau^2 s} \le C_7 s^{-1}e^{-\tau^2 C_* s}, \tag{2.28}$$

where

$$C_7 := C_3^{1/2}\sup_{\alpha>0}(2C_1\alpha + C_2\alpha^2)e^{-\check{c}_*\alpha/2}. \tag{2.29}$$

Relations (2.24)–(2.28) yield

$$\begin{aligned} &\big\|B(\tau;\vartheta)^{1/2}\big(e^{-B(\tau;\vartheta)s} - \big(I+\tau(\vartheta_1 Z + \vartheta_2\widetilde{Z})\big)e^{-\tau^2 S(\vartheta)s}P\big)\big\|_{\mathfrak{H}\to\mathfrak{H}} \\ &\quad\le (2(3\delta)^{-1/2} + 2C_1\check{c}_*^{-1/2} + C_7)s^{-1}e^{-\tau^2 C_* s} \\ &\qquad + \big\|B(\tau;\vartheta)^{1/2}\big(F(\tau;\vartheta)P - P - \tau(\vartheta_1 Z + \vartheta_2\widetilde{Z})\big)e^{-\tau^2 S(\vartheta)s}P\big\|_{\mathfrak{H}\to\mathfrak{H}}. \end{aligned} \tag{2.30}$$

From (1.34) and (1.35) it follows that $F(\tau;\vartheta)P - P - \tau(\vartheta_1 Z + \vartheta_2\widetilde{Z}) = F_2(\tau;\vartheta)P$. By using (1.38), (2.4), and (2.17), we estimate the last term in (2.30):

$$\begin{aligned} \big\|B(\tau;\vartheta)^{1/2}F_2(\tau;\vartheta)e^{-\tau^2 S(\vartheta)s}P\big\|_{\mathfrak{H}\to\mathfrak{H}} &\le C_4\delta^{1/2}(1+\pi^{-1})\tau^2 e^{-\tau^2\check{c}_* s} \\ &\le C_4\delta^{1/2}(1+\pi^{-1})2\check{c}_*^{-1}s^{-1}e^{-\tau^2 C_* s}, \quad s>0, \quad |\tau|\le\tau_0. \end{aligned} \tag{2.31}$$

Combining (2.30) and (2.31), we arrive at estimate (2.23) with the constant

$$C_8 := 2(3\delta)^{-1/2} + 2C_1\check{c}_*^{-1/2} + C_7 + 2C_4\delta^{1/2}(1+\pi^{-1})\check{c}_*^{-1}. \tag{2.32}$$

□

## §3. Approximation of the "bordered" operator exponential

**3.1. The principal term of approximation.** Suppose that the assumptions of Subsections 1.10 and 1.11 are satisfied, i.e., $B(\tau;\vartheta) = M^*\widehat{B}(\tau;\vartheta)M$. Our goal in this section is to find an approximation for the operator $Me^{-B(\tau;\vartheta)s}M^*$ acting in $\widehat{\mathfrak{H}}$. The principal term of approximation for $Me^{-A(t)s}M^*$ was found in [Su2, Subsection 2.2], approximation with the corrector term taken into account was obtained in [Su5, Theorem 4.1]. We generalize these considerations to the case of the family $B(\tau;\vartheta)$.

We use the notation (1.40), (1.41) and put

$$M_0 := (G_{\widehat{\mathfrak{N}}})^{-1/2} \colon \widehat{\mathfrak{N}} \to \widehat{\mathfrak{N}}. \tag{3.1}$$

From (2.20) it follows that for $s \ge 0$ and $|\tau| \le \tau_0$ we have

$$\big\|Me^{-B(\tau,\vartheta)s}M^* - Me^{-\tau^2 S(\vartheta)s}PM^*\big\|_{\widehat{\mathfrak{H}}\to\widehat{\mathfrak{H}}} \le C_6\|M\|^2(1+s)^{-1/2}e^{-\tau^2 C_* s}. \tag{3.2}$$

**Proposition 3.1.** *The operator* $\Lambda(\tau;\vartheta;s) := Me^{-\tau^2 S(\vartheta)s}PM^*$ *acting in the Hilbert space* $\widehat{\mathfrak{H}}$ *admits the representation*

$$\Lambda(\tau;\vartheta;s) = M_0 e^{-\tau^2 M_0 \widehat{S}(\vartheta) M_0 s} M_0 \widehat{P}. \tag{3.3}$$

*Proof.* Let $\widehat{\eta} \in \widehat{\mathfrak{H}}$, and let $\widehat{\xi}(s) = \Lambda(\tau;\vartheta;s)\widehat{\eta}$. Then $M^{-1}\widehat{\xi}(s) \in \mathfrak{N}$, $\widehat{\xi}(s) \in \widehat{\mathfrak{N}}$, and $M^{-1}\widehat{\xi}(s)$ is the solution of the Cauchy problem

$$\frac{d}{ds} M^{-1}\widehat{\xi}(s) = -\tau^2 S(\vartheta) M^{-1}\widehat{\xi}(s), \quad M^{-1}\widehat{\xi}(0) = PM^*\widehat{\eta}. \tag{3.4}$$

By (1.53), $S(\vartheta)M^{-1}\widehat{\xi}(s) = PM^*\widehat{S}(\vartheta)\widehat{\xi}(s)$. Next, from (1.42) we deduce that $PM^* = M^{-1}(G_{\widehat{\mathfrak{N}}})^{-1}\widehat{P}$. Then (3.4) and (3.1) show that

$$\frac{d}{ds}\widehat{\xi}(s) = -\tau^2 M_0^2 \widehat{S}(\vartheta)\widehat{\xi}(s), \quad \widehat{\xi}(0) = M_0^2 \widehat{P}\widehat{\eta},$$

or equivalently, $\frac{d}{ds} M_0^{-1}\widehat{\xi}(s) = -\tau^2 M_0 \widehat{S}(\vartheta)\widehat{\xi}(s)$, $M_0^{-1}\widehat{\xi}(0) = M_0\widehat{P}\widehat{\eta}$. Hence, $M_0^{-1}\widehat{\xi}(s) = e^{-\tau^2 M_0 \widehat{S}(\vartheta) M_0 s} M_0 \widehat{P}\widehat{\eta}$, which implies (3.3). □

We introduce the operator $\widehat{L}(t,\varepsilon) := \tau^2 \widehat{S}(\vartheta)$. The following result is a consequence of (3.2) and (3.3).

**Theorem 3.2.** *Under the above assumptions, we have*

$$\big\| Me^{-B(t,\varepsilon)s}M^* - M_0 e^{-M_0 \widehat{L}(t,\varepsilon) M_0 s} M_0 \widehat{P} \big\|_{\widehat{\mathfrak{H}}\to\widehat{\mathfrak{H}}} \le C_6 \|M\|^2 (1+s)^{-1/2} e^{-\tau^2 C_* s}, \quad s \ge 0, \quad |\tau| \le \tau_0. \tag{3.5}$$

### 3.2. Approximation with the corrector term taken into account.

**Theorem 3.3.** *Under the assumptions of Subsections* 1.10 *and* 1.11, *let* $\widehat{Z}_G$ *and* $\widehat{\widetilde{Z}}_G$ *be the operators* (1.55) *and* (1.57), *respectively. Then*

$$\big\| \widehat{B}(t,\varepsilon)^{1/2} \big( Me^{-B(t,\varepsilon)s}M^* - (I + t\widehat{Z}_G + \varepsilon \widehat{\widetilde{Z}}_G) M_0 e^{-M_0 \widehat{L}(t,\varepsilon) M_0 s} M_0 \widehat{P} \big) \big\|_{\widehat{\mathfrak{H}}\to\widehat{\mathfrak{H}}} \le C_8 \|M\| s^{-1} e^{-\tau^2 C_* s}, \quad s > 0, \quad 0 < \varepsilon \le 1, \quad |\tau| \le \tau_0.$$

*Proof.* The required estimate follows from (2.23) by recalculation. Combining (1.55), (1.57), and Proposition 3.1, we obtain

$$\begin{aligned}
&\big\| \widehat{B}(\tau;\vartheta)^{1/2} \big( Me^{-B(\tau;\vartheta)s}M^* - (I + \tau(\vartheta_1 \widehat{Z}_G + \vartheta_2 \widehat{\widetilde{Z}}_G))\Lambda(\tau;\vartheta;s) \big) \big\|_{\widehat{\mathfrak{H}}\to\widehat{\mathfrak{H}}} \\
&= \big\| \widehat{B}(\tau;\vartheta)^{1/2} M \big( e^{-B(\tau;\vartheta)s} - (I + \tau(\vartheta_1 Z + \vartheta_2 \widetilde{Z})) e^{-\tau^2 S(\vartheta)s} P \big) M^* \big\|_{\widehat{\mathfrak{H}}\to\widehat{\mathfrak{H}}} \\
&= \big\| B(\tau;\vartheta)^{1/2} \big( e^{-B(\tau;\vartheta)s} - (I + \tau(\vartheta_1 Z + \vartheta_2 \widetilde{Z})) e^{-\tau^2 S(\vartheta)s} P \big) M^* \big\|_{\widehat{\mathfrak{H}}\to\mathfrak{H}} \\
&\le \|M\| \big\| B(\tau;\vartheta)^{1/2} \big( e^{-B(\tau;\vartheta)s} - (I + \tau(\vartheta_1 Z + \vartheta_2 \widetilde{Z})) e^{-\tau^2 S(\vartheta)s} P \big) \big\|_{\mathfrak{H}\to\mathfrak{H}}.
\end{aligned}$$

Together with (2.23), this implies the claim. □

# Chapter 2
# Periodic differential operators in $L_2(\mathbb{R}^d;\mathbb{C}^n)$

## §4. Basic definitions

**4.1. The lattices $\Gamma$ and $\widetilde{\Gamma}$.** Let $\Gamma$ be a lattice in $\mathbb{R}^d$ generated by a basis $\mathbf{a}_1,\dots,\mathbf{a}_d$: $\Gamma = \big\{\mathbf{a}\in\mathbb{R}^d : \mathbf{a} = \sum_{j=1}^d n^j \mathbf{a}_j,\ n^j \in \mathbb{Z}\big\}$. Let $\Omega$ denote the elementary cell of the lattice $\Gamma$: $\Omega = \big\{\mathbf{x}\in\mathbb{R}^d : \mathbf{x} = \sum_{j=1}^d \xi^j \mathbf{a}_j,\ 0<\xi^j<1\big\}$. The basis $\mathbf{b}^1,\dots,\mathbf{b}^d$ dual to $\mathbf{a}_1,\dots,\mathbf{a}_d$ is defined by the relations $\langle \mathbf{b}^l, \mathbf{a}_j\rangle = 2\pi\delta_j^l$. This basis generates the lattice $\widetilde{\Gamma}$ dual to the lattice $\Gamma$. Let $\widetilde{\Omega}$ denote the *Brillouin zone* of the lattice $\widetilde{\Gamma}$: $\widetilde{\Omega} = \{\mathbf{k} \in \mathbb{R}^d : |\mathbf{k}| < |\mathbf{k}-\mathbf{b}|,\ 0 \neq \mathbf{b} \in \widetilde{\Gamma}\}$. The domain $\widetilde{\Omega}$ is a fundamental domain for $\widetilde{\Gamma}$. We use the notation $|\Omega| = \operatorname{meas}\Omega$, $|\widetilde{\Omega}| = \operatorname{meas}\widetilde{\Omega}$. Let $r_0$ be the radius of the ball inscribed in $\operatorname{clos}\widetilde{\Omega}$, and let $2r_1 = \operatorname{diam}\widetilde{\Omega}$.

**4.2. Factorized second order operators.** (See [BSu1].) Let $b(\mathbf{D}) = \sum_{l=1}^d b_l D_l$: $L_2(\mathbb{R}^d;\mathbb{C}^n) \to L_2(\mathbb{R}^d;\mathbb{C}^m)$ be a first order DO. Here the $b_l$ are constant $(m\times n)$-matrices. *We assume that* $m \geq n$. The symbol $b(\boldsymbol{\xi}) = \sum_{l=1}^d b_l\xi_l$ *is assumed to be* such that $\operatorname{rank} b(\boldsymbol{\xi}) = n$, $0 \neq \boldsymbol{\xi} \in \mathbb{R}^d$. Then for some $\alpha_0$, $\alpha_1 > 0$ we have

$$\alpha_0 \mathbf{1}_n \leq b(\boldsymbol{\theta})^* b(\boldsymbol{\theta}) \leq \alpha_1 \mathbf{1}_n, \quad \boldsymbol{\theta} \in \mathbb{S}^{d-1}, \quad 0 < \alpha_0 \leq \alpha_1 < \infty. \tag{4.1}$$

Let an $(n\times n)$-matrix-valued function $f(\mathbf{x})$ and an $(m\times m)$-matrix-valued function $h(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^d$, be bounded together with the inverses:

$$f,\ f^{-1} \in L_\infty(\mathbb{R}^d); \quad h,\ h^{-1} \in L_\infty(\mathbb{R}^d). \tag{4.2}$$

The functions $f$ and $h$ are assumed to be $\Gamma$-periodic. Consider the DO

$$\mathcal{X} := hb(\mathbf{D})f\colon L_2(\mathbb{R}^d;\mathbb{C}^n) \to L_2(\mathbb{R}^d;\mathbb{C}^m), \tag{4.3}$$

$$\operatorname{Dom}\mathcal{X} := \{\mathbf{u} \in L_2(\mathbb{R}^d;\mathbb{C}^n) : f\mathbf{u} \in H^1(\mathbb{R}^d;\mathbb{C}^n)\}. \tag{4.4}$$

The operator (4.3) is closed on the domain (4.4). Consider the selfadjoint operator $\mathcal{A} := \mathcal{X}^*\mathcal{X}$ in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ corresponding to the quadratic form $\mathfrak{a}[\mathbf{u},\mathbf{u}] = \|\mathcal{X}\mathbf{u}\|^2_{L_2}$, $\mathbf{u} \in \operatorname{Dom}\mathcal{X}$. Formally, we can write $\mathcal{A} = f^* b(\mathbf{D})^* g b(\mathbf{D}) f$, where $g = h^*h$. Using the Fourier transformation and (4.1), (4.2), it is easy to show that for $\mathbf{u} \in \operatorname{Dom}\mathcal{X}$ we have

$$\alpha_0 \|g^{-1}\|^{-1}_{L_\infty} \|\mathbf{D}(f\mathbf{u})\|^2_{L_2(\mathbb{R}^d)} \leq \mathfrak{a}[\mathbf{u},\mathbf{u}] \leq \alpha_1 \|g\|_{L_\infty} \|\mathbf{D}(f\mathbf{u})\|^2_{L_2(\mathbb{R}^d)}. \tag{4.5}$$

**4.3. The operators $\mathcal{Y}$ and $\mathcal{Y}_2$.** Now we proceed to the description of lower order terms. We introduce the operator $\mathcal{Y}\colon L_2(\mathbb{R}^d;\mathbb{C}^n) \to L_2(\mathbb{R}^d;\mathbb{C}^{dn})$ defined by

$$\mathcal{Y}\mathbf{u} = \mathbf{D}(f\mathbf{u}) = \operatorname{col}\{D_1(f\mathbf{u}),\dots,D_d(f\mathbf{u})\}, \quad \operatorname{Dom}\mathcal{Y} = \operatorname{Dom}\mathcal{X}.$$

The lower estimate (4.5) means that

$$\|\mathcal{Y}\mathbf{u}\|_{L_2(\mathbb{R}^d)} \leq c_1 \|\mathcal{X}\mathbf{u}\|_{L_2(\mathbb{R}^d)}, \quad \mathbf{u} \in \operatorname{Dom}\mathcal{X}, \tag{4.6}$$

$$c_1 = \alpha_0^{-1/2} \|g^{-1}\|^{1/2}_{L_\infty}. \tag{4.7}$$

Let $a_j(\mathbf{x})$, $j = 1,\dots,d$, be bounded $\Gamma$-periodic $(n\times n)$-matrix-valued functions in $\mathbb{R}^d$ such that

$$a_j \in L_\varrho(\Omega), \quad \varrho = 2 \ \text{ for } \ d = 1, \quad \varrho > d \ \text{ for } \ d \geq 2; \quad j = 1,\dots,d. \tag{4.8}$$

Consider the operator $\mathcal{Y}_2\colon L_2(\mathbb{R}^d;\mathbb{C}^n) \to L_2(\mathbb{R}^d;\mathbb{C}^{dn})$ acting on the domain $\operatorname{Dom}\mathcal{Y}_2 = \operatorname{Dom}\mathcal{X}$ and defined by $\mathcal{Y}_2\mathbf{u} = \operatorname{col}\{a_1^* f\mathbf{u},\dots,a_d^* f\mathbf{u}\}$. Formally, we have $(\mathcal{Y}_2^*\mathcal{Y} + \mathcal{Y}^*\mathcal{Y}_2)\mathbf{u} = \sum_{j=1}^d \big(f^* a_j D_j(f\mathbf{u}) + f^* D_j(a_j^* f\mathbf{u})\big)$.

By using the Hölder inequality, conditions (4.2), (4.8), and the compactness of the embedding $H^1(\Omega) \subset L_p(\Omega)$ for $p = 2\varrho(\varrho-2)^{-1}$, one can check (cf. [Su6, Subsection 5.2]) that for any $\nu > 0$ there exists a constant $C(\nu) > 0$ such that

$$\|\mathcal{Y}_2\mathbf{u}\|^2_{L_2(\mathbb{R}^d)} \leq \nu \|\mathcal{X}\mathbf{u}\|^2_{L_2(\mathbb{R}^d)} + C(\nu)\|\mathbf{u}\|^2_{L_2(\mathbb{R}^d)}, \quad \mathbf{u} \in \operatorname{Dom}\mathcal{X}. \tag{4.9}$$

For a fixed $\nu$, the constant $C(\nu)$ depends on the norms $\|a_j\|_{L_\varrho(\Omega)}$, $j = 1,\dots,d$, $\|f\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, on $\alpha_0$, $d$, $\varrho$, and on the parameters of the lattice $\Gamma$.

Using (4.6), (4.9), it is easy to check that

$$2\varepsilon|\operatorname{Re}(\mathcal{Y}\mathbf{u},\mathcal{Y}_2\mathbf{u})_{L_2}| \le \frac{\kappa}{2}\|\mathcal{X}\mathbf{u}\|^2_{L_2} + c_4\varepsilon^2\|\mathbf{u}\|^2_{L_2}, \quad \mathbf{u}\in\operatorname{Dom}\mathcal{X}, \tag{4.10}$$

$$c_4 := 4\kappa^{-1}c_1^2 C(\nu) \quad \text{for} \quad \nu = \kappa^2(16c_1^2)^{-1}. \tag{4.11}$$

**4.4. The operator $\mathcal{Q}_0$ and the form $q[\mathbf{u},\mathbf{u}]$.** Let $\mathcal{Q}_0$ be the operator in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ that acts as multiplication by the $\Gamma$-periodic positive definite and bounded matrix-valued function $\mathcal{Q}_0(\mathbf{x}) := f(\mathbf{x})^* f(\mathbf{x})$.

Suppose that $d\mu(\mathbf{x})$ is a $\Gamma$-periodic $\sigma$-finite Borel measure in $\mathbb{R}^d$ with values in the class of Hermitian $(n\times n)$-matrices. Then $d\mu(\mathbf{x}) = \{d\mu_{jl}(\mathbf{x})\}$, $j,l = 1,\dots,n$. In other words, $d\mu_{jl}(\mathbf{x})$ is a complex-valued $\Gamma$-periodic measure in $\mathbb{R}^d$, and $d\mu_{jl} = d\mu_{lj}^*$. Suppose that the measure $d\mu$ is such that the function $|v(\mathbf{x})|^2$ is integrable with respect to each measure $d\mu_{jl}$ for any $v \in H^1(\mathbb{R}^d)$.

In $L_2(\mathbb{R}^d;\mathbb{C}^n)$, we consider the form $q[\mathbf{u},\mathbf{u}] = \int_{\mathbb{R}^d}\langle d\mu(\mathbf{x}) f\mathbf{u}, f\mathbf{u}\rangle$, $\mathbf{u}\in\operatorname{Dom}\mathcal{X}$. The measure $d\mu$ is subject to the following condition.

**Condition 4.1.** *For any $\mathbf{v}\in H^1(\Omega;\mathbb{C}^n)$, we have*

$$-\widetilde{c}\|\mathbf{D}\mathbf{v}\|^2_{L_2(\Omega)} - \widehat{c}_0\|\mathbf{v}\|^2_{L_2(\Omega)} \le \int_\Omega \langle d\mu(\mathbf{x})\mathbf{v},\mathbf{v}\rangle \le \widetilde{c}_2\|\mathbf{D}\mathbf{v}\|^2_{L_2(\Omega)} + \widehat{c}_3\|\mathbf{v}\|^2_{L_2(\Omega)},$$

*where $\widehat{c}_0\in\mathbb{R}$, $\widetilde{c}_2\ge 0$, $\widehat{c}_3\ge 0$, and $0\le\widetilde{c}<\alpha_0\|g^{-1}\|^{-1}_{L_\infty}$.*

Note that Condition 4.1 implies the estimate

$$\begin{aligned} -\widetilde{c}\|\mathbf{D}(f\mathbf{u})\|^2_{L_2(\Omega)} - c_0\|\mathbf{u}\|^2_{L_2(\Omega)} &\le \int_\Omega\langle d\mu(\mathbf{x}) f\mathbf{u}, f\mathbf{u}\rangle \\ &\le \widetilde{c}_2\|\mathbf{D}(f\mathbf{u})\|^2_{L_2(\Omega)} + c_3\|\mathbf{u}\|^2_{L_2(\Omega)} \end{aligned} \tag{4.12}$$

with the constants

$$c_0 = \widehat{c}_0\|f\|^2_{L_\infty} \ \text{ if } \ \widehat{c}_0\ge 0, \quad c_0 = \widehat{c}_0\|f^{-1}\|^{-2}_{L_\infty} \ \text{ if } \ \widehat{c}_0 < 0; \tag{4.13}$$

$$c_3 = \|f\|^2_{L_\infty}\widehat{c}_3. \tag{4.14}$$

For $\mathbf{u}\in\operatorname{Dom}\mathcal{X}$, writing inequality (4.12) for the shifted cells $\Omega+\mathbf{a}$, $\mathbf{a}\in\Gamma$, and summing up, we obtain

$$-\widetilde{c}\|\mathbf{D}(f\mathbf{u})\|^2_{L_2(\mathbb{R}^d)} - c_0\|\mathbf{u}\|^2_{L_2(\mathbb{R}^d)} \le q[\mathbf{u},\mathbf{u}] \le \widetilde{c}_2\|\mathbf{D}(f\mathbf{u})\|^2_{L_2(\mathbb{R}^d)} + c_3\|\mathbf{u}\|^2_{L_2(\mathbb{R}^d)}.$$

Hence, by (4.5),

$$\begin{aligned} -(1-\kappa)\|\mathcal{X}\mathbf{u}\|^2_{L_2(\mathbb{R}^d)} - c_0\|\mathbf{u}\|^2_{L_2(\mathbb{R}^d)} \le q[\mathbf{u},\mathbf{u}] \le c_2\|\mathcal{X}\mathbf{u}\|^2_{L_2(\mathbb{R}^d)} + c_3\|\mathbf{u}\|^2_{L_2(\mathbb{R}^d)}, \\ \mathbf{u}\in\operatorname{Dom}\mathcal{X}, \end{aligned} \tag{4.15}$$

where

$$c_2 = \widetilde{c}_2\alpha_0^{-1}\|g^{-1}\|_{L_\infty}, \quad \kappa = 1 - \widetilde{c}\alpha_0^{-1}\|g^{-1}\|_{L_\infty}, \quad 0<\kappa\le 1. \tag{4.16}$$

**4.5. The operator $\mathcal{B}(\varepsilon)$.** In $L_2(\mathbb{R}^d;\mathbb{C}^n)$, we consider the quadratic form

$$\begin{aligned} \mathfrak{b}(\varepsilon)[\mathbf{u},\mathbf{u}] = \mathfrak{a}[\mathbf{u},\mathbf{u}] + 2\varepsilon\operatorname{Re}(\mathcal{Y}\mathbf{u},\mathcal{Y}_2\mathbf{u})_{L_2(\mathbb{R}^d)} + \varepsilon^2 q[\mathbf{u},\mathbf{u}] + \lambda\varepsilon^2(\mathcal{Q}_0\mathbf{u},\mathbf{u})_{L_2(\mathbb{R}^d)}, \\ \mathbf{u}\in\operatorname{Dom}\mathcal{X}, \end{aligned} \tag{4.17}$$

where $0<\varepsilon\le 1$ and the *parameter* $\lambda\in\mathbb{R}$ *satisfies the following restriction*:

$$\begin{aligned} &\lambda > \|\mathcal{Q}_0^{-1}\|_{L_\infty}(c_0+c_4) \ \text{ if } \ \lambda\ge 0, \\ &\lambda > \|\mathcal{Q}_0\|^{-1}_{L_\infty}(c_0+c_4) \ \text{ if } \ \lambda<0 \ (\text{and } \ c_0+c_4<0). \end{aligned} \tag{4.18}$$

Now we estimate the form (4.17) from below. Let $\beta > 0$ be defined by

$$\begin{aligned} &\beta = \lambda \|\mathcal{Q}_0^{-1}\|_{L_\infty}^{-1} - c_0 - c_4 \quad \text{if} \quad \lambda \geq 0, \\ &\beta = \lambda \|\mathcal{Q}_0\|_{L_\infty} - c_0 - c_4 \quad \text{if} \quad \lambda < 0 \ (\text{and} \quad c_0 + c_4 < 0). \end{aligned} \tag{4.19}$$

Combining (4.10), the lower estimate in (4.15), (4.18), and (4.19), we arrive at

$$\mathfrak{b}(\varepsilon)[\mathbf{u},\mathbf{u}] \geq \frac{\kappa}{2}\mathfrak{a}[\mathbf{u},\mathbf{u}] + \beta\varepsilon^2 \|\mathbf{u}\|_{L_2}^2, \quad \mathbf{u} \in \operatorname{Dom}\mathcal{X}, \quad 0 < \varepsilon \leq 1. \tag{4.20}$$

Thus, the form $\mathfrak{b}(\varepsilon)$ is positive definite. From (4.6), (4.9) for $\nu = 1$, and the upper estimate in (4.15) it follows that

$$\mathfrak{b}(\varepsilon)[\mathbf{u},\mathbf{u}] \leq (2 + c_1^2 + c_2)\mathfrak{a}[\mathbf{u},\mathbf{u}] + (C(1) + c_3 + |\lambda| \|\mathcal{Q}_0\|_{L_\infty})\varepsilon^2 \|\mathbf{u}\|_{L_2}^2, \mathbf{u} \in \operatorname{Dom}\mathcal{X}. \tag{4.21}$$

By (4.20) and (4.21), the form $\mathfrak{b}(\varepsilon)$ is closed. *The corresponding positive definite operator in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ is denoted by $\mathcal{B}(\varepsilon)$.* Formally, we can write

$$\begin{aligned} \mathcal{B}(\varepsilon) &= \mathcal{A} + \varepsilon(\mathcal{Y}_2^*\mathcal{Y} + \mathcal{Y}^*\mathcal{Y}_2) + \varepsilon^2 f^*\mathcal{Q}f + \varepsilon^2\lambda\mathcal{Q}_0 \\ &= f^*b(\mathbf{D})^*gb(\mathbf{D})f + \varepsilon\sum_{j=1}^d f^*(a_jD_j + D_ja_j^*)f + \varepsilon^2 f^*\mathcal{Q}f + \varepsilon^2\lambda\mathcal{Q}_0, \end{aligned} \tag{4.22}$$

where $\mathcal{Q}$ can be interpreted as the generalized matrix-valued potential generated by the measure $d\mu$.

*For further references, by the "initial data" we mean the following set of parameters*:

$$\begin{aligned} &d,\ m,\ n,\ \varrho;\ \alpha_0,\ \alpha_1,\ \|g\|_{L_\infty},\ \|g^{-1}\|_{L_\infty},\ \|f\|_{L_\infty},\ \|f^{-1}\|_{L_\infty},\ \|a_j\|_{L_\varrho(\Omega)}, \\ &\qquad j = 1,\dots,d;\ \widetilde{c},\ \widehat{c}_0,\ \widetilde{c}_2,\ \widehat{c}_3 \ \text{ from Condition 4.1};\ \lambda. \end{aligned} \tag{4.23}$$

We shall trace the dependence of constants in estimates on the initial data and the parameters of the lattice. The constants $c_1$, $C(1)$, $\kappa$, $c_2$, $c_3$, $c_4$, $c_0$, $\beta$ are determined by the initial data and the lattice.

## §5. Direct integral decomposition for the operator $\mathcal{B}(\varepsilon)$

**5.1. The Gelfand transformation.** Initially, the Gelfand transformation $\mathcal{U}$ is defined on the functions of the Schwartz class $\mathbf{v} \in \mathcal{S}(\mathbb{R}^d;\mathbb{C}^n)$ by the formula

$$\widetilde{\mathbf{v}}(\mathbf{k},\mathbf{x}) = (\mathcal{U}\mathbf{v})(\mathbf{k},\mathbf{x}) = |\widetilde{\Omega}|^{-1/2}\sum_{\mathbf{a}\in\Gamma}\exp(-i\langle\mathbf{k},\mathbf{x}+\mathbf{a}\rangle)\mathbf{v}(\mathbf{x}+\mathbf{a}), \quad \mathbf{x}\in\Omega,\ \mathbf{k}\in\widetilde{\Omega}.$$

Herewith, $\int_{\widetilde{\Omega}}\int_\Omega |\widetilde{\mathbf{v}}(\mathbf{k},\mathbf{x})|^2\,d\mathbf{x}\,d\mathbf{k} = \int_{\mathbb{R}^d}|\mathbf{v}(\mathbf{x})|^2\,d\mathbf{x}$, and $\mathcal{U}$ extends by continuity to a unitary operator

$$\mathcal{U} \,:\, L_2(\mathbb{R}^d;\mathbb{C}^n) \to \int_{\widetilde{\Omega}} \oplus L_2(\Omega;\mathbb{C}^n)\,d\mathbf{k} =: \mathcal{H}. \tag{5.1}$$

*Let $\widetilde{H}^1(\Omega;\mathbb{C}^n)$ denote the subspace of all functions in $H^1(\Omega;\mathbb{C}^n)$ whose $\Gamma$-periodic extension to $\mathbb{R}^d$ belongs to the class $H^1_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{C}^n)$.* The relation $\mathbf{v} \in H^1(\mathbb{R}^d;\mathbb{C}^n)$ is equivalent to the fact that $\widetilde{\mathbf{v}}(\mathbf{k},\,\cdot\,) \in \widetilde{H}^1(\Omega;\mathbb{C}^n)$ for a. e. $\mathbf{k} \in \widetilde{\Omega}$, and

$$\int_{\widetilde{\Omega}}\int_\Omega \left(|(\mathbf{D}+\mathbf{k})\widetilde{\mathbf{v}}(\mathbf{k},\mathbf{x})|^2 + |\widetilde{\mathbf{v}}(\mathbf{k},\mathbf{x})|^2\right)\,d\mathbf{x}\,d\mathbf{k} < \infty.$$

Under the Gelfand transformation $\mathcal{U}$, the operator of multiplication by a bounded periodic matrix-valued function in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ turns into multiplication by the same function on the fibers of the direct integral $\mathcal{H}$. On these fibers, the operator $b(\mathbf{D})$ applied to $\mathbf{v} \in H^1(\mathbb{R}^d;\mathbb{C}^n)$ turns into the operator $b(\mathbf{D}+\mathbf{k})$ applied to $\widetilde{\mathbf{v}}(\mathbf{k},\,\cdot\,) \in \widetilde{H}^1(\Omega;\mathbb{C}^n)$.

**5.2. The operators $\mathcal{A}(\mathbf{k})$.** (See [BSu1, Subsection 2.2.1].) We put

$$\mathfrak{H} = L_2(\Omega; \mathbb{C}^n), \quad \mathfrak{H}_* = L_2(\Omega; \mathbb{C}^m), \quad \widetilde{\mathfrak{H}} = L_2(\Omega; \mathbb{C}^{dn}) \tag{5.2}$$

and consider the closed operator $\mathcal{X}(\mathbf{k}) : \mathfrak{H} \to \mathfrak{H}_*$, $\mathbf{k} \in \mathbb{R}^d$, defined by the relations

$$\mathcal{X}(\mathbf{k}) = hb(\mathbf{D} + \mathbf{k})f, \quad \mathbf{k} \in \mathbb{R}^d, \tag{5.3}$$

$$\mathfrak{d} := \operatorname{Dom} \mathcal{X}(\mathbf{k}) = \{\mathbf{u} \in \mathfrak{H} \, : \, f\mathbf{u} \in \widetilde{H}^1(\Omega; \mathbb{C}^n)\}. \tag{5.4}$$

The selfadjoint operator $\mathcal{A}(\mathbf{k}) := \mathcal{X}(\mathbf{k})^* \mathcal{X}(\mathbf{k}) : \mathfrak{H} \to \mathfrak{H}$, $\mathbf{k} \in \mathbb{R}^d$, is generated by the quadratic form $\mathfrak{a}(\mathbf{k})[\mathbf{u}, \mathbf{u}] := \|\mathcal{X}(\mathbf{k})\mathbf{u}\|^2_{\mathfrak{H}_*}$, $\mathbf{u} \in \mathfrak{d}$, $\mathbf{k} \in \mathbb{R}^d$. From (4.1) and (4.2) it follows that

$$\begin{gathered}\alpha_0 \|g^{-1}\|^{-1}_{L_\infty} \|(\mathbf{D} + \mathbf{k})\mathbf{v}\|^2_{L_2(\Omega)} \leq \mathfrak{a}(\mathbf{k})[\mathbf{u}, \mathbf{u}] \leq \alpha_1 \|g\|_{L_\infty} \|(\mathbf{D} + \mathbf{k})\mathbf{v}\|^2_{L_2(\Omega)}, \\ \mathbf{v} = f\mathbf{u} \in \widetilde{H}^1(\Omega; \mathbb{C}^n).\end{gathered} \tag{5.5}$$

By (5.5) and the compactness of the embedding of $\widetilde{H}^1(\Omega; \mathbb{C}^n)$ into $\mathfrak{H}$, the spectrum of $\mathcal{A}(\mathbf{k})$ is discrete. We put $\mathfrak{N} := \operatorname{Ker} \mathcal{A}(0) = \operatorname{Ker} \mathcal{X}(0)$. Inequality (5.5) for $\mathbf{k} = 0$ implies that

$$\mathfrak{N} = \operatorname{Ker} \mathcal{A}(0) = \{\mathbf{u} \in L_2(\Omega; \mathbb{C}^n) \, : \, f\mathbf{u} = \mathbf{c} \in \mathbb{C}^n\}, \quad \dim \mathfrak{N} = n. \tag{5.6}$$

As was shown in [BSu1, (2.2.11), (2.2.12)],

$$\mathcal{A}(\mathbf{k}) \geq c_* |\mathbf{k}|^2 I, \quad \mathbf{k} \in \operatorname{clos} \widetilde{\Omega}, \quad c_* = \alpha_0 \|f^{-1}\|^{-2}_{L_\infty} \|g^{-1}\|^{-1}_{L_\infty}. \tag{5.7}$$

In accordance with [BSu1, (2.2.14)], the distance $d^0$ from the point $\lambda_0 = 0$ to the rest of the spectrum of $\mathcal{A}(0)$ satisfies the estimate

$$d^0 \geq 4 c_* r_0^2. \tag{5.8}$$

**5.3. The operators $\mathcal{Y}(\mathbf{k})$ and $Y_2$.** Consider the operator $\mathcal{Y}(\mathbf{k})\colon \mathfrak{H} \to \widetilde{\mathfrak{H}}$, that acts on the domain $\operatorname{Dom} \mathcal{Y}(\mathbf{k}) = \mathfrak{d}$ and is defined by

$$\mathcal{Y}(\mathbf{k})\mathbf{u} = (\mathbf{D} + \mathbf{k}) f\mathbf{u} = \operatorname{col}\{(D_1 + k_1) f\mathbf{u}, \dots, (D_d + k_d) f\mathbf{u}\}, \quad \mathbf{u} \in \mathfrak{d}. \tag{5.9}$$

The lower estimate (5.5) implies that

$$\|\mathcal{Y}(\mathbf{k})\mathbf{u}\|_{\widetilde{\mathfrak{H}}} \leq c_1 \|\mathcal{X}(\mathbf{k})\mathbf{u}\|_{\mathfrak{H}_*}, \quad \mathbf{u} \in \mathfrak{d}, \tag{5.10}$$

where the constant $c_1$ is as in (4.7).

Consider the operator $Y_2\colon \mathfrak{H} \to \widetilde{\mathfrak{H}}$ defined by the relation

$$Y_2 \mathbf{u} = \operatorname{col}\{a_1^* f\mathbf{u}, \dots, a_d^* f\mathbf{u}\}, \quad \operatorname{Dom} Y_2 = \mathfrak{d}. \tag{5.11}$$

As was shown in [Su6, Subsection 5.7], for any $\nu > 0$ there exist constants $C_j(\nu) > 0$, $j = 1, \dots, d$, such that for $\mathbf{k} \in \mathbb{R}^d$ we have

$$\|a_j^* \mathbf{v}\|^2_{L_2(\Omega)} \leq \nu \|(\mathbf{D} + \mathbf{k})\mathbf{v}\|^2_{L_2(\Omega)} + C_j(\nu) \|\mathbf{v}\|^2_{L_2(\Omega)}, \ \mathbf{v} \in \widetilde{H}^1(\Omega; \mathbb{C}^n), \ j = 1, \dots, d.$$

Let $\mathbf{v} = f\mathbf{u}$, $\mathbf{u} \in \mathfrak{d}$. Then, summing these inequalities over $j$ and using (4.2), (5.5), we see that for any $\nu > 0$ there exists a constant $C(\nu) > 0$ (the same as in (4.9)) such that

$$\|Y_2 \mathbf{u}\|^2_{\widetilde{\mathfrak{H}}} \leq \nu \|\mathcal{X}(\mathbf{k})\mathbf{u}\|^2_{\mathfrak{H}_*} + C(\nu) \|\mathbf{u}\|^2_{\mathfrak{H}}, \quad \mathbf{u} \in \mathfrak{d}, \, \mathbf{k} \in \mathbb{R}^d. \tag{5.12}$$

**5.4. The operator $\mathcal{Q}_0$ and the form $q_\Omega[\mathbf{u},\mathbf{u}]$.** Let $\mathcal{Q}_0$ be the bounded operator in $\mathfrak{H}$ acting as multiplication by the matrix-valued function $\mathcal{Q}_0(\mathbf{x}) = f(\mathbf{x})^* f(\mathbf{x})$.

In $L_2(\Omega;\mathbb{C}^n)$, we consider the form $q_\Omega[\mathbf{u},\mathbf{u}] = \int_\Omega \langle d\mu(\mathbf{x}) f\mathbf{u}, f\mathbf{u}\rangle$, $\mathbf{u}\in\mathfrak{d}$. Replacing $f(\mathbf{x})\mathbf{u}(\mathbf{x})$ by $f(\mathbf{x})\mathbf{u}(\mathbf{x})\exp(i\langle\mathbf{k},\mathbf{x}\rangle)$ in (4.12) (these functions belong to $H^1(\Omega;\mathbb{C}^n)$ simultaneously) and using (5.5), we get

$$-(1-\kappa)\|\mathcal{X}(\mathbf{k})\mathbf{u}\|^2_{\mathfrak{H}_*} - c_0\|\mathbf{u}\|^2_{\mathfrak{H}} \le q_\Omega[\mathbf{u},\mathbf{u}] \le c_2\|\mathcal{X}(\mathbf{k})\mathbf{u}\|^2_{\mathfrak{H}_*} + c_3\|\mathbf{u}\|^2_{\mathfrak{H}},\ \mathbf{u}\in\mathfrak{d},\ \mathbf{k}\in\mathbb{R}^d. \tag{5.13}$$

Here the constants $\kappa$, $c_0$, $c_2$, $c_3$ are the same as in (4.15).

**5.5. The operator pencil $\mathcal{B}(\mathbf{k},\varepsilon)$.** In the space $\mathfrak{H}$, we consider the quadratic form

$$\mathfrak{b}(\mathbf{k},\varepsilon)[\mathbf{u},\mathbf{u}] = \mathfrak{a}(\mathbf{k})[\mathbf{u},\mathbf{u}] + 2\varepsilon\,\mathrm{Re}(\mathcal{Y}(\mathbf{k})\mathbf{u}, Y_2\mathbf{u})_{\widetilde{\mathfrak{H}}} + \varepsilon^2 q_\Omega[\mathbf{u},\mathbf{u}] + \lambda\varepsilon^2(\mathcal{Q}_0\mathbf{u},\mathbf{u})_{\mathfrak{H}}, \quad \mathbf{u}\in\mathfrak{d}.$$

From (4.18), (4.19), (5.10), (5.12), and (5.13) it follows that

$$\mathfrak{b}(\mathbf{k},\varepsilon)[\mathbf{u},\mathbf{u}] \ge \frac{\kappa}{2}\mathfrak{a}(\mathbf{k})[\mathbf{u},\mathbf{u}] + \beta\varepsilon^2\|\mathbf{u}\|^2_{\mathfrak{H}}, \quad \mathbf{u}\in\mathfrak{d}. \tag{5.14}$$

Next, using (5.10), (5.12) for $\nu=1$, and the upper estimate in (5.13), we obtain

$$\mathfrak{b}(\mathbf{k},\varepsilon)[\mathbf{u},\mathbf{u}] \le (2+c_1^2+c_2)\mathfrak{a}(\mathbf{k})[\mathbf{u},\mathbf{u}] + (C(1)+c_3+|\lambda|\|\mathcal{Q}_0\|_{L_\infty})\varepsilon^2\|\mathbf{u}\|^2_{\mathfrak{H}}, \quad \mathbf{u}\in\mathfrak{d}. \tag{5.15}$$

The inequalities (5.14) and (5.15) show that the form $\mathfrak{b}(\mathbf{k},\varepsilon)$ is closed on the domain (5.4) and positive definite. The selfadjoint operator in $\mathfrak{H}$ generated by this form is denoted by $\mathcal{B}(\mathbf{k},\varepsilon)$. Formally, we can write

$$\begin{aligned}\mathcal{B}(\mathbf{k},\varepsilon) &= \mathcal{A}(\mathbf{k}) + \varepsilon(Y_2^*\mathcal{Y}(\mathbf{k}) + \mathcal{Y}(\mathbf{k})^*Y_2) + \varepsilon^2 f^*\mathcal{Q}f + \lambda\varepsilon^2\mathcal{Q}_0\\ &= f^*b(\mathbf{D}+\mathbf{k})^*gb(\mathbf{D}+\mathbf{k})f + \varepsilon\sum_{j=1}^d f^*(a_j(D_j+k_j) + (D_j+k_j)a_j^*)f\\ &\quad + \varepsilon^2 f^*\mathcal{Q}f + \lambda\varepsilon^2 f^*f.\end{aligned} \tag{5.16}$$

**5.6. Direct integral expansion for the operator $\mathcal{B}(\varepsilon)$.** Under the Gelfand transformation $\mathcal{U}$, the operator (4.22) acting in the space $L_2(\mathbb{R}^d;\mathbb{C}^n)$ expands into the direct integral of the operators (5.16) acting in $L_2(\Omega;\mathbb{C}^n)$:

$$\mathcal{U}\mathcal{B}(\varepsilon)\mathcal{U}^{-1} = \int_{\widetilde{\Omega}} \oplus\mathcal{B}(\mathbf{k},\varepsilon)\,d\mathbf{k}.$$

This means the following. Let $\widetilde{\mathbf{u}} = \mathcal{U}\mathbf{u}$, where $\mathbf{u}\in\mathrm{Dom}\,\mathfrak{b}(\varepsilon)$. Then

$$\widetilde{\mathbf{u}}(\mathbf{k},\,\cdot\,)\in\mathfrak{d} \text{ for a. e. } \mathbf{k}\in\widetilde{\Omega}, \tag{5.17}$$

$$\mathfrak{b}(\varepsilon)[\mathbf{u},\mathbf{u}] = \int_{\widetilde{\Omega}} \mathfrak{b}(\mathbf{k},\varepsilon)[\widetilde{\mathbf{u}}(\mathbf{k},\,\cdot\,), \widetilde{\mathbf{u}}(\mathbf{k},\,\cdot\,)]\,d\mathbf{k}. \tag{5.18}$$

Conversely, if $\widetilde{\mathbf{u}}\in\mathcal{H}$ satisfies (5.17) and the integral in (5.18) is finite, then $\mathbf{u}\in\mathrm{Dom}\,\mathfrak{b}(\varepsilon)$ and we have (5.18).

## §6. Incorporation of the operators $\mathcal{B}(\mathbf{k},\varepsilon)$ into the abstract method

**6.1.** For $d>1$, the operators $\mathcal{B}(\mathbf{k},\varepsilon)$ depend on the multidimensional parameter $\mathbf{k}$. As in [BSu1, Chapter 2], we distinguish a one-dimensional parameter $t$ by putting $\mathbf{k}=t\boldsymbol{\theta}$, $t=|\mathbf{k}|$, $\boldsymbol{\theta}\in\mathbb{S}^{d-1}$. We apply the method of Chapter 1. Now, all the objects depend on the additional parameter $\boldsymbol{\theta}$. We must make our considerations and estimates uniform in $\boldsymbol{\theta}$. The spaces $\mathfrak{H}$, $\mathfrak{H}_*$, and $\widetilde{\mathfrak{H}}$ are defined by (5.2). We put $X(t)=X(t;\boldsymbol{\theta}) := \mathcal{X}(t\boldsymbol{\theta})$. By (5.3), $X(t;\boldsymbol{\theta}) = X_0 + tX_1(\boldsymbol{\theta})$, where $X_0=\mathcal{X}(0)=h(\mathbf{x})b(\mathbf{D})f(\mathbf{x})$, $\mathrm{Dom}\,X_0=\mathfrak{d}$, and $X_1(\boldsymbol{\theta})$ is the bounded operator acting as multiplication by the matrix $h(\mathbf{x})b(\boldsymbol{\theta})f(\mathbf{x})$. Next, we put $A(t)=A(t;\boldsymbol{\theta}) := \mathcal{A}(t\boldsymbol{\theta})$. By (5.6), the kernel $\mathfrak{N}=\mathrm{Ker}\,X_0=\mathrm{Ker}\,\mathcal{A}(0)$ is

$n$-dimensional. Condition 1.1 is satisfied, and $d^0$ obeys (5.8). As was shown in [BSu1, Chapter 2, §3], the condition $n \le n_* = \dim \operatorname{Ker} X_0^*$ is also satisfied.

Next, the role of $Y(t)$ is played by the operator $Y(t;\boldsymbol{\theta}) := \mathcal{Y}(t\boldsymbol{\theta})$. By (5.9), we have $Y(t;\boldsymbol{\theta}) = Y_0 + tY_1(\boldsymbol{\theta})$, where

$$(6.1)\qquad \begin{aligned} &Y_0\mathbf{u} = \mathbf{D}(f\mathbf{u}) = \operatorname{col}\{D_1 f\mathbf{u}, \dots, D_d f\mathbf{u}\}, \quad \operatorname{Dom} Y_0 = \mathfrak{d};\\ &Y_1(\boldsymbol{\theta})\mathbf{u} = \operatorname{col}\{\theta_1 f\mathbf{u}, \dots, \theta_d f\mathbf{u}\}. \end{aligned}$$

Condition 1.2 is ensured by (5.10). The operator $Y_2$ is defined by (5.11). By (5.12), Condition 1.3 is fulfilled. The role of the form $\mathfrak{q}$ from Subsection 1.3 is played by the form $q_\Omega$. By (5.13), Condition 1.4 is fulfilled. The role of the operator $Q_0$ from Subsection 1.3 is played by the operator of multiplication by the matrix-valued function $\mathcal{Q}_0(\mathbf{x})$. By (4.18), the parameter $\lambda$ satisfies (1.5). Estimates (5.14) and (5.15) correspond to (1.7) and (1.9).

Finally, the role of the operator pencil $B(t,\varepsilon)$ (see (1.10)) is played by the operator family (5.16): $B(t,\varepsilon;\boldsymbol{\theta}) := \mathcal{B}(t\boldsymbol{\theta},\varepsilon)$.

Thus, all the assumptions of Chapter 1 are satisfied.

**6.2.** In accordance with Subsection 1.5, we should fix a positive number $\delta$ such that $\delta < \kappa d^0/13$. Taking (5.7) and (5.8) into account, we put

$$(6.2)\qquad \delta = \frac{1}{4}\kappa c_* r_0^2 = \frac{1}{4}\kappa\alpha_0 \|f^{-1}\|_{L_\infty}^{-2}\|g^{-1}\|_{L_\infty}^{-1} r_0^2.$$

Relations (4.1), (4.2), and (6.1) show that

$$(6.3)\qquad \|X_1(\boldsymbol{\theta})\| \le \alpha_1^{1/2}\|g\|_{L_\infty}^{1/2}\|f\|_{L_\infty}, \quad \|Y_1(\boldsymbol{\theta})\| = \|f\|_{L_\infty}, \quad \boldsymbol{\theta}\in\mathbb{S}^{d-1}.$$

Instead of the sharp value of the constant (1.12), which depends on $\boldsymbol{\theta}$ and is equal to $\delta^{1/2}((2+c_1^2+c_2)\|X_1(\boldsymbol{\theta})\|^2 + C(1) + c_3 + |\lambda|\|f\|_{L_\infty}^2)^{-1/2}$, we take the following value, which is suitable for all $\boldsymbol{\theta}\in\mathbb{S}^{d-1}$:

$$(6.4)\qquad \tau_0 = \delta^{1/2}\big((2+c_1^2+c_2)\alpha_1\|g\|_{L_\infty}\|f\|_{L_\infty}^2 + C(1) + c_3 + |\lambda|\|f\|_{L_\infty}^2\big)^{-1/2}.$$

Condition (2.1) is satisfied due to (5.7). Then, by (5.14), the operator $B(t,\varepsilon;\boldsymbol{\theta})$ satisfies a condition of the form (2.2):

$$(6.5)\qquad B(t,\varepsilon;\boldsymbol{\theta}) \ge \check{c}_*(t^2+\varepsilon^2)I, \quad \mathbf{k} = t\boldsymbol{\theta}\in\widetilde{\Omega}, \quad 0<\varepsilon\le 1,$$

$$(6.6)\qquad \check{c}_* = \frac{1}{2}\min\{\kappa c_*, 2\beta\}.$$

**6.3. The effective characteristics.** In the case where $f = \mathbf{1}_n$, the effective characteristics were constructed in [Su6, Subsections 6.3, 6.4, 7.1]. In this subsection, we formulate the necessary results.

*Below, all the objects corresponding to $f = \mathbf{1}_n$ are marked by the upper hat "$\widehat{\ }$".* We have $\widehat{\mathfrak{H}} = \mathfrak{H} = L_2(\Omega;\mathbb{C}^n)$. By Subsection 6.1, $\widehat{X}(t;\boldsymbol{\theta}) = \widehat{X}_0 + t\widehat{X}_1(\boldsymbol{\theta})$, $\widehat{X}_0 = h(\mathbf{x})b(\mathbf{D})$, $\operatorname{Dom}\widehat{X}_0 = \widetilde{H}^1(\Omega;\mathbb{C}^n)$, and $\widehat{X}_1(\boldsymbol{\theta})$ is the bounded operator of multiplication by the matrix $h(\mathbf{x})b(\boldsymbol{\theta})$. Formally, $\widehat{A}(t;\boldsymbol{\theta}) = \widehat{X}(t;\boldsymbol{\theta})^*\widehat{X}(t;\boldsymbol{\theta})$. If $f = \mathbf{1}_n$, the kernel (5.6) coincides with the subspace of constants $\widehat{\mathfrak{N}} = \{\mathbf{u}\in\mathfrak{H} : \mathbf{u} = \mathbf{c}\in\mathbb{C}^n\}$. The orthogonal projection $\widehat{P}$ of $\mathfrak{H} = L_2(\Omega;\mathbb{C}^n)$ onto the subspace $\widehat{\mathfrak{N}} = \mathbb{C}^n$ is the operator of averaging over the cell $\Omega$: $\widehat{P}\mathbf{u} = |\Omega|^{-1}\int_\Omega \mathbf{u}(\mathbf{x})\,d\mathbf{x}$.

Next, $\widehat{Y}(t;\boldsymbol{\theta}) = \widehat{Y}_0 + t\widehat{Y}_1(\boldsymbol{\theta})\colon \mathfrak{H}\to\widetilde{\mathfrak{H}}$, where $\widehat{Y}_0\mathbf{u} = \mathbf{D}\mathbf{u} = \operatorname{col}\{D_1\mathbf{u},\dots,D_d\mathbf{u}\}$, $\operatorname{Dom}\widehat{Y}_0 = \widetilde{H}^1(\Omega;\mathbb{C}^n)$, and $\widehat{Y}_1(\boldsymbol{\theta})\mathbf{u} = \operatorname{col}\{\theta_1\mathbf{u},\dots,\theta_d\mathbf{u}\}$. The operator $\widehat{Y}_2\colon\mathfrak{H}\to\widetilde{\mathfrak{H}}$ acts on the domain $\operatorname{Dom}\widehat{Y}_2 = \widetilde{H}^1(\Omega;\mathbb{C}^n)$ and is defined by $\widehat{Y}_2\mathbf{u} = \operatorname{col}\{a_1^*\mathbf{u},\dots,a_d^*\mathbf{u}\}$. The role of the form $\widehat{\mathfrak{q}}[\mathbf{u},\mathbf{u}]$ is played by the form $\int_\Omega\langle d\mu(\mathbf{x})\mathbf{u},\mathbf{u}\rangle$; the role of the operator $\widehat{Q}_0$ is played by the identity operator $I$.

The operator pencil $\widehat{B}(t,\varepsilon;\boldsymbol{\theta})$ is formally given by the expression

$$\widehat{B}(t,\varepsilon;\boldsymbol{\theta}) = \widehat{A}(t;\boldsymbol{\theta}) + \varepsilon(\widehat{Y}_2^*\widehat{Y}(t;\boldsymbol{\theta}) + \widehat{Y}(t;\boldsymbol{\theta})^*\widehat{Y}_2) + \varepsilon^2 Q + \lambda\varepsilon^2 I.$$

In accordance with Subsection 1.6, we introduce the operators $\widehat{Z}$ and $\widehat{\widetilde{Z}}$. Now the operator $\widehat{Z}$ depends on $\boldsymbol{\theta}$. As was shown in [BSu3, (4.2)], $\widehat{Z}(\boldsymbol{\theta}) = \Lambda b(\boldsymbol{\theta})\widehat{P}$, where $\Lambda(\mathbf{x})$ is a $\Gamma$-periodic $(n\times m)$-matrix-valued function satisfying

$$b(\mathbf{D})^* g(\mathbf{x})(b(\mathbf{D})\Lambda(\mathbf{x}) + \mathbf{1}_m) = 0, \quad \int_\Omega \Lambda(\mathbf{x})\,d\mathbf{x} = 0. \tag{6.7}$$

In accordance with [Su6, Subsection 6.3], $\widehat{\widetilde{Z}} = \widetilde{\Lambda}\widehat{P}$, where $\widetilde{\Lambda}(\mathbf{x})$ is a $\Gamma$-periodic $(n\times n)$-matrix-valued function satisfying

$$b(\mathbf{D})^* g(\mathbf{x}) b(\mathbf{D})\widetilde{\Lambda}(\mathbf{x}) + \sum_{j=1}^d D_j a_j(\mathbf{x})^* = 0, \quad \int_\Omega \widetilde{\Lambda}(\mathbf{x})\,d\mathbf{x} = 0. \tag{6.8}$$

Now the spectral germ $\widehat{S}$ defined in Subsection 1.7 depends on $\boldsymbol{\theta}$. By [BSu1, Chapter 3, §1], the operator $\widehat{S}(\boldsymbol{\theta})\colon \widehat{\mathfrak{N}} \to \widehat{\mathfrak{N}}$ acts as the operator of multiplication by the matrix $b(\boldsymbol{\theta})^* g^0 b(\boldsymbol{\theta})$, $\boldsymbol{\theta}\in\mathbb{S}^{d-1}$. Here $g^0$ is a constant positive $(m\times m)$-matrix called the *effective matrix* and defined by

$$g^0 = |\Omega|^{-1}\int_\Omega g(\mathbf{x})(b(\mathbf{D})\Lambda(\mathbf{x}) + \mathbf{1}_m)\,d\mathbf{x}. \tag{6.9}$$

As in [Su6, (7.2), (7.3)], we define the constant matrices

$$V := |\Omega|^{-1}\int_\Omega (b(\mathbf{D})\Lambda(\mathbf{x}))^* g(\mathbf{x}) b(\mathbf{D})\widetilde{\Lambda}(\mathbf{x})\,d\mathbf{x}, \tag{6.10}$$

$$W := |\Omega|^{-1}\int_\Omega (b(\mathbf{D})\widetilde{\Lambda}(\mathbf{x}))^* g(\mathbf{x}) b(\mathbf{D})\widetilde{\Lambda}(\mathbf{x})\,d\mathbf{x}. \tag{6.11}$$

Now the operator $\widehat{L}(t,\varepsilon)$ defined by (2.21) depends on $\boldsymbol{\theta}$. We return to the parameter $\mathbf{k} = t\boldsymbol{\theta}$: $\widehat{L}(t,\varepsilon;\boldsymbol{\theta}) = \widehat{L}(\mathbf{k},\varepsilon)$. It turns out (see [Su6, (7.8)]) that

$$\widehat{L}(\mathbf{k},\varepsilon) = b(\mathbf{k})^* g^0 b(\mathbf{k}) + \varepsilon(-b(\mathbf{k})^* V - V^* b(\mathbf{k})) + \varepsilon\sum_{j=1}^d (\overline{a_j + a_j^*}) k_j + \varepsilon^2(-W + \overline{\mathcal{Q}} + \lambda I), \tag{6.12}$$

where $(\overline{a_j + a_j^*}) := |\Omega|^{-1}\int_\Omega (a_j(\mathbf{x}) + a_j(\mathbf{x})^*)\,d\mathbf{x}$ and

$$\overline{\mathcal{Q}} := |\Omega|^{-1}\int_\Omega d\mu(\mathbf{x}). \tag{6.13}$$

We put

$$\widehat{\mathcal{A}}^0(\mathbf{k}) = b(\mathbf{D}+\mathbf{k})^* g^0 b(\mathbf{D}+\mathbf{k}), \quad \widehat{\mathcal{Y}}^0(\mathbf{k}) = -b(\mathbf{D}+\mathbf{k})^* V + \sum_{j=1}^d \overline{a_j}(D_j + k_j),$$
$$\widehat{\mathcal{B}}^0(\mathbf{k},\varepsilon) = \widehat{\mathcal{A}}^0(\mathbf{k}) + \varepsilon(\widehat{\mathcal{Y}}^0(\mathbf{k}) + \widehat{\mathcal{Y}}^0(\mathbf{k})^*) + \varepsilon^2(\overline{\mathcal{Q}} - W + \lambda I).$$

Then

$$\widehat{L}(\mathbf{k},\varepsilon)\widehat{P} = \widehat{\mathcal{B}}^0(\mathbf{k},\varepsilon)\widehat{P}. \tag{6.14}$$

**6.4. The case where $f \neq \mathbf{1}_n$.** Now we consider the operators $\mathcal{B}(\varepsilon)$ of the general form (4.22) and the corresponding families $B(t,\varepsilon;\boldsymbol{\theta})$ described in Subsection 6.1. To mark the objects corresponding to the case of $f = \mathbf{1}_n$ with the same $b$, $g$, $a_j$, $j = 1,\dots,d$, $\lambda$, $\mathcal{Q}$, we use the upper hat "$\widehat{\ }$".

We apply the approach of Subsections 1.10–1.12. Now $\widehat{\mathfrak{H}} = \mathfrak{H} = L_2(\Omega;\mathbb{C}^n)$, and the isomorphism $M$ is the operator of multiplication by the matrix-valued function $f$. The role of the operator $G$ of Subsection 1.10 (see (1.40)) is played by the operator $\rho$ acting as multiplication by the matrix-valued function $\rho(\mathbf{x}) := (f(\mathbf{x})f(\mathbf{x})^*)^{-1}$. The block of $\rho$ in the kernel $\widehat{\mathfrak{N}} = \mathbb{C}^n$ is the operator of multiplication by the constant matrix $\overline{\rho} = |\Omega|^{-1}\int_\Omega (f(\mathbf{x})f(\mathbf{x})^*)^{-1}\,d\mathbf{x}$. The role of the operator $M_0$ (see (3.1)) is played by the operator of multiplication by the constant matrix $f_0 := (\overline{\rho})^{-1/2}$. Note that

$$|f_0| \leq \|f\|_{L_\infty}, \quad |f_0^{-1}| \leq \|f^{-1}\|_{L_\infty}. \tag{6.15}$$

By (5.7), $\widehat{\mathcal{A}}(\mathbf{k}) \geq \widehat{c}_*|\mathbf{k}|^2 I$, $\mathbf{k} \in \widetilde{\Omega}$, where $\widehat{c}_* = \alpha_0\|g^{-1}\|^{-1}_{L_\infty}$. The constants $c_*$ and $\widehat{c}_*$ satisfy $c_* = \|f^{-1}\|^{-2}_{L_\infty}\widehat{c}_*$. As in (1.47), $\beta \leq \|f^{-1}\|^{-2}_{L_\infty}\widehat{\beta}$, and by (6.6), $\check{c}_* = \frac{1}{2}\min\{\kappa c_*, 2\beta\}$, $\widehat{\check{c}}_* = \frac{1}{2}\min\{\kappa\widehat{c}_*, 2\widehat{\beta}\}$. Thus, $\check{c}_* \leq \|f^{-1}\|^{-2}_{L_\infty}\widehat{\check{c}}_*$. In accordance with (2.22), $\widehat{L}(\mathbf{k},\varepsilon) \geq \widehat{\check{c}}_*(|\mathbf{k}|^2+\varepsilon^2)\mathbf{1}_n$. Hence, by (6.15), we have

$$f_0\widehat{L}(\mathbf{k},\varepsilon)f_0 \geq \check{c}_*(|\mathbf{k}|^2+\varepsilon^2)\mathbf{1}_n, \quad \mathbf{k} \in \mathbb{R}^d. \tag{6.16}$$

## §7. Approximation of the operator $f\exp(-\mathcal{B}(\mathbf{k},\varepsilon)s)f^*$

**7.1. The principal term of approximation.** The principal term of approximation for the operator $f\exp(-\mathcal{A}(\mathbf{k})s)f^*$ was obtained in [Su2, Subsection 6.2], approximation with the corrector term taken into account was found in [Su5, §8]. Now we consider the exponential of the operator

$$\mathcal{B}(\mathbf{k},\varepsilon) = f^*\widehat{\mathcal{B}}(\mathbf{k},\varepsilon)f. \tag{7.1}$$

To apply Theorem 3.2 to the operator (7.1), we need to specify the constants in estimates. The constants $c_1$, $C(\nu)$, $\kappa$, $c_0$, $c_2$, $c_3$, $c_4$ were defined in §4 (see (4.7), (4.9), (4.11), (4.13), (4.14), (4.16)). The constant $\lambda$ satisfies condition (4.18), $\beta$ was defined in (4.19), $c_*$ and $\check{c}_*$ were defined in (5.7) and (6.6). The constants $\delta$ and $\tau_0$ are given by (6.2) and (6.4).

In accordance with (1.29) and (1.30), we introduce the constants $C_T^{(1)}$ and $C_T^{(2)}$, which now depend on the additional parameter $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$. Using (4.7) and (6.3), we take the following overstated constants suitable for all $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$:

$$C_T^{(1)} = \max\bigl\{2+\alpha_0^{-1}\|g^{-1}\|_{L_\infty}, \bigl(\alpha_1\|g\|_{L_\infty}\|f\|^2_{L_\infty} + C(1)\bigr)\delta^{-1}\bigr\},$$
$$C_T^{(2)} = \max\bigl\{c_2+1, \bigl(\alpha_1\|g\|_{L_\infty}\|f\|^2_{L_\infty} + \|f\|^2_{L_\infty} + C(1) + c_3 + |\lambda|\|f\|^2_{L_\infty}\bigr)\delta^{-1}\bigr\}.$$

Using these $C_T^{(1)}$ and $C_T^{(2)}$, we define the constants $C_T$, $C_T^0$, $C_1$, $C_2$, $C_5$, $C_6$ by (1.31), (1.32), (1.33), (2.13), and (2.19); then these constants do not depend on $\boldsymbol{\theta}$. As in (2.17), we put

$$C_* = \frac{1}{2}\min\{\check{c}_*; \delta\tau_0^{-2}\}. \tag{7.2}$$

*We denote* $\mathcal{E}^0(\mathbf{k},\varepsilon,s) := f_0 e^{-f_0\widehat{\mathcal{B}}^0(\mathbf{k},\varepsilon)f_0 s}f_0$ and apply Theorem 3.2. By (6.14), from (3.5) it follows that

$$\|fe^{-\mathcal{B}(\mathbf{k},\varepsilon)s}f^* - \mathcal{E}^0(\mathbf{k},\varepsilon,s)\widehat{P}\|_{\mathfrak{H}\to\mathfrak{H}} \leq C_6\|f\|^2_{L_\infty}(1+s)^{-1/2}e^{-(|\mathbf{k}|^2+\varepsilon^2)C_*s}, \quad s \geq 0, \quad |\mathbf{k}|^2+\varepsilon^2 \leq \tau_0^2. \tag{7.3}$$

Now we obtain estimates in the case where $|\mathbf{k}|^2+\varepsilon^2>\tau_0^2$. By (6.5),

$$\|fe^{-\mathcal{B}(\mathbf{k},\varepsilon)s}f^*\|_{\mathfrak{H}\to\mathfrak{H}}\le\|f\|^2_{L_\infty}e^{-\check{c}_*(|\mathbf{k}|^2+\varepsilon^2)s}. \tag{7.4}$$

Using (6.14), (6.15), and (6.16), we get

$$\|\mathcal{E}^0(\mathbf{k},\varepsilon,s)\widehat{P}\|_{\mathfrak{H}\to\mathfrak{H}}\le|f_0|^2e^{-\check{c}_*(|\mathbf{k}|^2+\varepsilon^2)s}\le\|f\|^2_{L_\infty}e^{-\check{c}_*(|\mathbf{k}|^2+\varepsilon^2)s}. \tag{7.5}$$

Combining (7.4), (7.5), and (7.2) and using the inequality $e^{-\alpha}\le(1+\alpha)^{-1/2}$, $\alpha\ge0$, we see that for $s\ge0$ and $|\mathbf{k}|^2+\varepsilon^2>\tau_0^2$ the following is true:

$$\begin{aligned}&\|fe^{-\mathcal{B}(\mathbf{k},\varepsilon)s}f^*-\mathcal{E}^0(\mathbf{k},\varepsilon,s)\widehat{P}\|_{\mathfrak{H}\to\mathfrak{H}}\\&\qquad\le2\|f\|^2_{L_\infty}\max\{1;\sqrt{2}\check{c}_*^{-1/2}\tau_0^{-1}\}(1+s)^{-1/2}e^{-(|\mathbf{k}|^2+\varepsilon^2)C_*s}.\end{aligned} \tag{7.6}$$

Estimates (7.3) and (7.6) imply

$$\begin{aligned}&\|fe^{-\mathcal{B}(\mathbf{k},\varepsilon)s}f^*-\mathcal{E}^0(\mathbf{k},\varepsilon,s)\widehat{P}\|_{\mathfrak{H}\to\mathfrak{H}}\\&\qquad\le\|f\|^2_{L_\infty}\max\{C_6;2\sqrt{2}\check{c}_*^{-1/2}\tau_0^{-1}\}(1+s)^{-1/2}e^{-(|\mathbf{k}|^2+\varepsilon^2)C_*s},\quad\mathbf{k}\in\widetilde{\Omega}.\end{aligned} \tag{7.7}$$

Now we show that the operator $\widehat{P}$ can be replaced by $I$ in (7.7). Since $\mathcal{E}^0(\mathbf{k},\varepsilon,s)$ is the operator with the symbol $f_0\exp(-f_0\widehat{L}(\mathbf{b}+\mathbf{k},\varepsilon)f_0s)f_0$, relations (6.15), (6.16), and (7.2) yield

$$\begin{aligned}\|\mathcal{E}^0(\mathbf{k},\varepsilon,s)(I-\widehat{P})\|_{\mathfrak{H}\to\mathfrak{H}}&\le\|f\|^2_{L_\infty}\sup_{0\ne\mathbf{b}\in\widetilde{\Gamma}}e^{-\check{c}_*(|\mathbf{k}+\mathbf{b}|^2+\varepsilon^2)s}\\&\le\|f\|^2_{L_\infty}\max\{1;\sqrt{2}\check{c}_*^{-1/2}r_0^{-1}\}(1+s)^{-1/2}e^{-(|\mathbf{k}|^2+\varepsilon^2)C_*s},\quad\mathbf{k}\in\widetilde{\Omega}.\end{aligned} \tag{7.8}$$

Combining (7.7) and (7.8), we arrive at the following result.

**Theorem 7.1.** *For $s\ge0$, $\mathbf{k}\in\operatorname{clos}\widetilde{\Omega}$, and $0<\varepsilon\le1$, we have*

$$\|fe^{-\mathcal{B}(\mathbf{k},\varepsilon)s}f^*-\mathcal{E}^0(\mathbf{k},\varepsilon,s)\|_{\mathfrak{H}\to\mathfrak{H}}\le\mathcal{C}_1(1+s)^{-1/2}e^{-(|\mathbf{k}|^2+\varepsilon^2)C_*s}.$$

*Here* $\mathcal{C}_1:=\|f\|^2_{L_\infty}\max\{C_6;2\sqrt{2}\check{c}_*^{-1/2}\tau_0^{-1}\}+\|f\|^2_{L_\infty}\max\{1;\sqrt{2}\check{c}_*^{-1/2}r_0^{-1}\}$.

**7.2. Approximation with the corrector term taken into account.** To apply Theorem 3.3 to the operator family $\mathcal{B}(\mathbf{k},\varepsilon)$, we need to specify the values of the constants. The constants $C_T$, $C_1$, $C_2$ were defined in Subsection 7.1. In accordance with (1.25), recalling (6.3), we can take the following overstated value of the constant $c_5$:

$$c_5:=\big(\alpha_1^{1/2}\|g\|^{1/2}_{L_\infty}\|f\|_{L_\infty}+c_1C(1)^{1/2}\big)^2+2C(1)^{1/2}\|f\|_{L_\infty}+\max\{|c_0|;c_3\}+|\lambda|\|f\|^2_{L_\infty}.$$

For this $c_5$, we define the constants $C_3$, $C_4$, $C_7$, $C_8$ in accordance with (1.36), (1.38), (2.29), and (2.32); then these constants do not depend on $\boldsymbol{\theta}$.

Now, by using the method of Subsection 1.13, we introduce the operators $\widehat{Z}_\rho(\boldsymbol{\theta})$ and $\widehat{\widehat{Z}}_\rho$ acting in $\mathfrak{H}$. Let a $\Gamma$-periodic $(n\times m)$-matrix-valued function $\Lambda_\rho(\mathbf{x})$ be the solution of the problem

$$b(\mathbf{D})^*g(\mathbf{x})\,(b(\mathbf{D})\Lambda_\rho(\mathbf{x})+\mathbf{1}_m)=0,\qquad\int_\Omega\rho(\mathbf{x})\Lambda_\rho(\mathbf{x})\,d\mathbf{x}=0.$$

Here the equation is understood in the weak sense. Cf. [BSu3, §5]. Obviously, $\Lambda_\rho(\mathbf{x})$ differs from the solution $\Lambda(\mathbf{x})$ of problem (6.7) by a constant summand:

$$\Lambda_\rho(\mathbf{x})=\Lambda(\mathbf{x})+\Lambda_\rho^0,\quad\Lambda_\rho^0=-(\overline{\rho})^{-1}(\overline{\rho\Lambda}). \tag{7.9}$$

In [BSu3, Subsection 7.3], it was checked that

$$|\Lambda_\rho^0|\le C_\rho:=m^{1/2}(2r_0)^{-1}\alpha_0^{-1/2}\|g\|^{1/2}_{L_\infty}\|g^{-1}\|^{1/2}_{L_\infty}\|f\|^2_{L_\infty}\|f^{-1}\|^2_{L_\infty}. \tag{7.10}$$

As in [BSu3, §5], the role of the operator $\widehat{Z}_G$ of Subsection 1.13 is played by the operator $\widehat{Z}_\rho(\boldsymbol{\theta}) = \Lambda_\rho b(\boldsymbol{\theta})\widehat{P}$. Since $b(\mathbf{D})\widehat{P} = 0$, we have $t\widehat{Z}_\rho(\boldsymbol{\theta}) = \Lambda_\rho b(\mathbf{D}+\mathbf{k})\widehat{P}$, $\mathbf{k} \in \mathbb{R}^d$.

In accordance with (1.56), we introduce the operator $\widehat{\widetilde{Z}}_\rho$ in $\mathfrak{H}$ that takes an element $\widehat{\mathbf{u}} \in \mathfrak{H}$ to the solution $\mathbf{w}^{(\rho)} \in \widetilde{H}^1(\Omega;\mathbb{C}^n)$ of the problem

$$b(\mathbf{D})^* g(\mathbf{x}) b(\mathbf{D})\mathbf{w}^{(\rho)} + \sum_{j=1}^d D_j a_j(\mathbf{x})^* \mathbf{c} = 0, \quad \int_\Omega \rho(\mathbf{x})\mathbf{w}^{(\rho)}(\mathbf{x})\,d\mathbf{x} = 0, \quad \mathbf{c} = \widehat{P}\widehat{\mathbf{u}}.$$

Let a $\Gamma$-periodic $(n\times n)$-matrix-valued function $\widetilde{\Lambda}_\rho(\mathbf{x})$ be the solution of the problem

$$b(\mathbf{D})^* g(\mathbf{x}) b(\mathbf{D})\widetilde{\Lambda}_\rho(\mathbf{x}) + \sum_{j=1}^d D_j a_j(\mathbf{x})^* = 0, \quad \int_\Omega \rho(\mathbf{x})\widetilde{\Lambda}_\rho(\mathbf{x})\,d\mathbf{x} = 0.$$

This equation is understood in the weak sense. Note that

$$\widetilde{\Lambda}_\rho(\mathbf{x}) = \widetilde{\Lambda}(\mathbf{x}) + \widetilde{\Lambda}_\rho^0, \quad \widetilde{\Lambda}_\rho^0 = -(\overline{\rho})^{-1}(\overline{\rho\widetilde{\Lambda}}), \tag{7.11}$$

where $\widetilde{\Lambda}$ is the $\Gamma$-periodic solution of problem (6.8). As was shown in [Su6, (7.52)],

$$\|\widetilde{\Lambda}\|_{L_2(\Omega)} \le (2r_0)^{-1} C_a n^{1/2}\alpha_0^{-1}\|g^{-1}\|_{L_\infty},$$

where the constant $C_a$ is defined below in (7.24). Hence,

$$|\overline{\rho\widetilde{\Lambda}}| \le \|f^{-1}\|^2_{L_\infty}|\Omega|^{-1/2}\|\widetilde{\Lambda}\|_{L_2(\Omega)} \le (2r_0)^{-1} C_a n^{1/2}\alpha_0^{-1}\|g^{-1}\|_{L_\infty}\|f^{-1}\|^2_{L_\infty}|\Omega|^{-1/2}.$$

Thus, $\widetilde{\Lambda}_\rho^0$ satisfies the estimate

$$|\widetilde{\Lambda}_\rho^0| \le \widetilde{C}_\rho := (2r_0)^{-1} C_a n^{1/2}\alpha_0^{-1}\|g^{-1}\|_{L_\infty}\|f\|^2_{L_\infty}\|f^{-1}\|^2_{L_\infty}|\Omega|^{-1/2}. \tag{7.12}$$

By the definitions of $\widehat{\widetilde{Z}}_\rho$ and $\widetilde{\Lambda}_\rho$, we have $\widehat{\widetilde{Z}}_\rho = \widetilde{\Lambda}_\rho\widehat{P}$.

Since $t\widehat{Z}_\rho(\boldsymbol{\theta}) = \Lambda_\rho b(\mathbf{D}+\mathbf{k})\widehat{P}$ and $\widehat{\widetilde{Z}}_\rho = \widetilde{\Lambda}_\rho\widehat{P}$, Theorem 3.3 implies the estimate

$$\begin{aligned} &\big\|\widehat{\mathcal{B}}(\mathbf{k},\varepsilon)^{1/2}\big(f e^{-\mathcal{B}(\mathbf{k},\varepsilon)s} f^* - \big(I + \Lambda_\rho b(\mathbf{D}+\mathbf{k}) + \varepsilon\widetilde{\Lambda}_\rho\big)\mathcal{E}^0(\mathbf{k},\varepsilon,s)\widehat{P}\big)\big\|_{\mathfrak{H}\to\mathfrak{H}} \\ &\qquad \le C_8\|f\|_{L_\infty} s^{-1} e^{-(|\mathbf{k}|^2+\varepsilon^2)C_* s}, \quad s>0, \quad 0<\varepsilon\le 1, \quad |\mathbf{k}|^2+\varepsilon^2 \le \tau_0^2. \end{aligned} \tag{7.13}$$

Now, using (7.9) and (7.11), we show that, in (7.13), $\Lambda_\rho$ and $\widetilde{\Lambda}_\rho$ can be replaced by $\Lambda$ and $\widetilde{\Lambda}$, respectively. By referring to (5.15) with $f = \mathbf{1}_n$, it is easy to check (see [Su6, (7.32)]) that

$$\big\|\widehat{\mathcal{B}}(\mathbf{k},\varepsilon)^{1/2}\widehat{P}\big\|_{\mathfrak{H}\to\mathfrak{H}} \le C_P(|\mathbf{k}|^2+\varepsilon^2)^{1/2}, \quad \mathbf{k} \in \widetilde{\Omega}, \tag{7.14}$$

where $C_P = \max\{(2+c_1^2+c_2)^{1/2}\alpha_1^{1/2}\|g\|^{1/2}_{L_\infty}; (\widehat{C}(1)+\widehat{c}_3+|\lambda|)^{1/2}\}$. Combining (4.1), (7.5), (7.10), (7.12), (7.14), the identity $b(\mathbf{D})\widehat{P} = 0$, and (7.2), we obtain

$$\begin{aligned} &\big\|\widehat{\mathcal{B}}(\mathbf{k},\varepsilon)^{1/2}\big(\Lambda_\rho^0 b(\mathbf{D}+\mathbf{k}) + \varepsilon\widetilde{\Lambda}_\rho^0\big)\mathcal{E}^0(\mathbf{k},\varepsilon,s)\widehat{P}\|_{\mathfrak{H}\to\mathfrak{H}} \\ &\qquad \le \big\|\widehat{\mathcal{B}}(\mathbf{k},\varepsilon)^{1/2}\widehat{P}\big\|(\alpha_1^{1/2}|\Lambda_\rho^0||\mathbf{k}| + |\widetilde{\Lambda}_\rho^0|\varepsilon)\|f\|^2_{L_\infty} e^{-\check{c}_*(|\mathbf{k}|^2+\varepsilon^2)s} \\ &\qquad \le 2C_P\check{c}_*^{-1}\|f\|^2_{L_\infty}(\alpha_1^{1/2}C_\rho + \widetilde{C}_\rho)s^{-1}e^{-(|\mathbf{k}|^2+\varepsilon^2)C_* s}, \quad s>0, \quad \mathbf{k}\in\widetilde{\Omega}. \end{aligned} \tag{7.15}$$

From (7.13), (7.15), and (7.9) it follows that

$$\begin{aligned} &\big\|\widehat{\mathcal{B}}(\mathbf{k},\varepsilon)^{1/2}\big(f e^{-\mathcal{B}(\mathbf{k},\varepsilon)s} f^* - \big(I + \Lambda b(\mathbf{D}+\mathbf{k}) + \varepsilon\widetilde{\Lambda}\big)\mathcal{E}^0(\mathbf{k},\varepsilon,s)\widehat{P}\big)\big\|_{\mathfrak{H}\to\mathfrak{H}} \\ &\qquad \le C_9 s^{-1} e^{-(|\mathbf{k}|^2+\varepsilon^2)C_* s}, \quad s>0, \quad 0<\varepsilon\le 1, \quad |\mathbf{k}|^2+\varepsilon^2 \le \tau_0^2, \end{aligned} \tag{7.16}$$

where $C_9 = C_8\|f\|_{L_\infty} + 2C_P\check{c}_*^{-1}\|f\|^2_{L_\infty}(\alpha_1^{1/2}C_\rho + \widetilde{C}_\rho)$.

**7.3. Estimates for $|\mathbf{k}|^2+\varepsilon^2>\tau_0^2$.** Now we estimate each term under the norm sign in (7.16). From (7.1) it follows that

$$\begin{aligned}\big\|\widehat{\mathcal{B}}(\mathbf{k},\varepsilon)^{1/2}fe^{-\mathcal{B}(\mathbf{k},\varepsilon)s}f^*\mathbf{u}\big\|^2_{\mathfrak{H}}&=\big(\widehat{\mathcal{B}}(\mathbf{k},\varepsilon)fe^{-\mathcal{B}(\mathbf{k},\varepsilon)s}f^*\mathbf{u},fe^{-\mathcal{B}(\mathbf{k},\varepsilon)s}f^*\mathbf{u}\big)_{\mathfrak{H}}\\&=\|\mathcal{B}(\mathbf{k},\varepsilon)^{1/2}e^{-\mathcal{B}(\mathbf{k},\varepsilon)s}f^*\mathbf{u}\|^2_{\mathfrak{H}}\le\big\|\mathcal{B}(\mathbf{k},\varepsilon)^{1/2}e^{-\mathcal{B}(\mathbf{k},\varepsilon)s}\big\|^2_{\mathfrak{H}\to\mathfrak{H}}\|f\|^2_{L_\infty}\|\mathbf{u}\|^2_{\mathfrak{H}}.\end{aligned}\tag{7.17}$$

By (6.5) and (7.2),

$$\begin{aligned}\big\|\mathcal{B}(\mathbf{k},\varepsilon)^{1/2}e^{-\mathcal{B}(\mathbf{k},\varepsilon)s}\big\|_{\mathfrak{H}\to\mathfrak{H}}&\le\sup_{\alpha\ge\check{c}_*(|\mathbf{k}|^2+\varepsilon^2)}2\alpha^{-1/2}s^{-1}e^{-\alpha s/2}\\&\le2\check{c}_*^{-1/2}\tau_0^{-1}s^{-1}e^{-(|\mathbf{k}|^2+\varepsilon^2)C_*s},\quad s>0,\ \ |\mathbf{k}|^2+\varepsilon^2>\tau_0^2.\end{aligned}$$

Hence, by (7.17), for $s>0$ and $|\mathbf{k}|^2+\varepsilon^2>\tau_0^2$ we have

$$\big\|\widehat{\mathcal{B}}(\mathbf{k},\varepsilon)^{1/2}fe^{-\mathcal{B}(\mathbf{k},\varepsilon)s}f^*\big\|_{\mathfrak{H}\to\mathfrak{H}}\le2\|f\|_{L_\infty}\check{c}_*^{-1/2}\tau_0^{-1}s^{-1}e^{-(|\mathbf{k}|^2+\varepsilon^2)C_*s}.\tag{7.18}$$

By (7.2), (7.5), and (7.14), for $s>0$ and $|\mathbf{k}|^2+\varepsilon^2>\tau_0^2$ we obtain

$$\big\|\widehat{\mathcal{B}}(\mathbf{k},\varepsilon)^{1/2}\mathcal{E}^0(\mathbf{k},\varepsilon,s)\widehat{P}\big\|_{\mathfrak{H}\to\mathfrak{H}}\le2\|f\|^2_{L_\infty}C_P\check{c}_*^{-1}\tau_0^{-1}s^{-1}e^{-(|\mathbf{k}|^2+\varepsilon^2)C_*s}.\tag{7.19}$$

Now we estimate the norm of the corrector term:

$$\begin{aligned}&\big\|\widehat{\mathcal{B}}(\mathbf{k},\varepsilon)^{1/2}\big(\Lambda b(\mathbf{D}+\mathbf{k})+\varepsilon\widetilde{\Lambda}\big)\mathcal{E}^0(\mathbf{k},\varepsilon,s)\widehat{P}\big\|_{\mathfrak{H}\to\mathfrak{H}}\\&\le\big\|\widehat{\mathcal{B}}(\mathbf{k},\varepsilon)^{1/2}\Lambda\widehat{P}_m\big\|_{\mathfrak{H}\to\mathfrak{H}}\big\|b(\mathbf{D}+\mathbf{k})\mathcal{E}^0(\mathbf{k},\varepsilon,s)\widehat{P}\big\|_{\mathfrak{H}\to\mathfrak{H}}\\&\quad+\varepsilon\big\|\widehat{\mathcal{B}}(\mathbf{k},\varepsilon)^{1/2}\widetilde{\Lambda}\widehat{P}\big\|_{\mathfrak{H}\to\mathfrak{H}}\big\|\mathcal{E}^0(\mathbf{k},\varepsilon,s)\widehat{P}\big\|_{\mathfrak{H}\to\mathfrak{H}}.\end{aligned}\tag{7.20}$$

To estimate the norm of the operator $b(\mathbf{D}+\mathbf{k})\mathcal{E}^0(\mathbf{k},\varepsilon,s)\widehat{P}$, we use (4.1), (7.2), (7.5), and the identity $b(\mathbf{D})\widehat{P}=0$:

$$\begin{aligned}\big\|b(\mathbf{D}+\mathbf{k})\mathcal{E}^0(\mathbf{k},\varepsilon,s)\widehat{P}\big\|_{\mathfrak{H}\to\mathfrak{H}}&\le\alpha_1^{1/2}|\mathbf{k}|\|f\|^2_{L_\infty}e^{-\check{c}_*(|\mathbf{k}|^2+\varepsilon^2)s}\\&\le2\alpha_1^{1/2}\check{c}_*^{-1}\|f\|^2_{L_\infty}|\mathbf{k}|(|\mathbf{k}|^2+\varepsilon^2)^{-1}s^{-1}e^{-(|\mathbf{k}|^2+\varepsilon^2)C_*s},\quad\mathbf{k}\in\widetilde{\Omega}.\end{aligned}\tag{7.21}$$

The operators $\widehat{\mathcal{B}}(\mathbf{k},\varepsilon)^{1/2}\Lambda\widehat{P}_m$ and $\widehat{\mathcal{B}}(\mathbf{k},\varepsilon)^{1/2}\widetilde{\Lambda}\widehat{P}$ were estimated in [Su6, Lemmas 7.2 and 7.3]. Now we formulate the results.

**Lemma 7.2.** *For $\mathbf{k}\in\widetilde{\Omega}$, $0<\varepsilon\le1$ we have*

$$\big\|\widehat{\mathcal{B}}(\mathbf{k},\varepsilon)^{1/2}\Lambda\widehat{P}_m\big\|_{\mathfrak{H}\to\mathfrak{H}}\le C_\Lambda(\mathbf{k},\varepsilon),\tag{7.22}$$

$$\big\|\widehat{\mathcal{B}}(\mathbf{k},\varepsilon)^{1/2}\widetilde{\Lambda}\widehat{P}\big\|_{\mathfrak{H}\to\mathfrak{H}}\le C_{\widetilde{\Lambda}}(\mathbf{k},\varepsilon),\tag{7.23}$$

*where $C_\Lambda(\mathbf{k},\varepsilon)$ and $C_{\widetilde{\Lambda}}(\mathbf{k},\varepsilon)$ are defined by*

$$\begin{aligned}C_\Lambda(\mathbf{k},\varepsilon)^2&=(2+c_1^2+c_2)m\big(\|g\|^{1/2}_{L_\infty}+c^{(1)}|\mathbf{k}|\big)^2+c^{(2)}\varepsilon^2,\\C_{\widetilde{\Lambda}}(\mathbf{k},\varepsilon)^2&=(2+c_1^2+c_2)n|\Omega|^{-1}\big(c^{(3)}+c^{(4)}|\mathbf{k}|\big)^2+c^{(5)}\varepsilon^2.\end{aligned}$$

*Here*

$$C_a^2=\sum_{j=1}^d\int_\Omega|a_j(\mathbf{x})|^2\,d\mathbf{x},\tag{7.24}$$

$$\begin{aligned}c^{(1)}&=(2r_0)^{-1}\alpha_1^{1/2}\alpha_0^{-1/2}\|g^{-1}\|^{1/2}_{L_\infty}\|g\|_{L_\infty},\\c^{(2)}&=\big(\widehat{C}(1)+\widehat{c}_3+|\lambda|\big)m(2r_0)^{-2}\alpha_0^{-1}\|g^{-1}\|_{L_\infty}\|g\|_{L_\infty},\\c^{(3)}&=C_a\alpha_0^{-1/2}\|g^{-1}\|^{1/2}_{L_\infty},\quad c^{(4)}=(2r_0)^{-1}C_a\alpha_0^{-1}\alpha_1^{1/2}\|g\|^{1/2}_{L_\infty}\|g^{-1}\|_{L_\infty},\\c^{(5)}&=\big(\widehat{C}(1)+\widehat{c}_3+|\lambda|\big)(2r_0)^{-2}C_a^2n\alpha_0^{-2}\|g^{-1}\|^2_{L_\infty}|\Omega|^{-1}.\end{aligned}$$

**Corollary 7.3.** *For* $\mathbf{k} \in \widetilde{\Omega}$, $0 < \varepsilon \le 1$ *we have*

$$\big\|\widehat{\mathcal{B}}(\mathbf{k},\varepsilon)^{1/2}\Lambda\widehat{P}_m\big\|_{\mathfrak{H}\to\mathfrak{H}} \le C_\Lambda(r_1,1), \quad \big\|\widehat{\mathcal{B}}(\mathbf{k},\varepsilon)^{1/2}\widetilde{\Lambda}\widehat{P}\big\|_{\mathfrak{H}\to\mathfrak{H}} \le C_{\widetilde{\Lambda}}(r_1,1). \tag{7.25}$$

Note that relations (7.21) and (7.22) imply the estimate

$$\begin{aligned} &\big\|\widehat{\mathcal{B}}(\mathbf{k},\varepsilon)^{1/2}\Lambda\widehat{P}_m\big\|_{\mathfrak{H}\to\mathfrak{H}}\big\|b(\mathbf{D}+\mathbf{k})\mathcal{E}^0(\mathbf{k},\varepsilon,s)\widehat{P}\big\|_{\mathfrak{H}\to\mathfrak{H}} \\ &\quad\le C_\Lambda(\mathbf{k},\varepsilon)\alpha_1^{1/2}2\check{c}_*^{-1}\|f\|^2_{L_\infty}|\mathbf{k}|(|\mathbf{k}|^2+\varepsilon^2)^{-1}s^{-1}e^{-(|\mathbf{k}|^2+\varepsilon^2)C_*s} \\ &\quad\le \check{c}_*^{-1}C_\Lambda\|f\|^2_{L_\infty}s^{-1}e^{-(|\mathbf{k}|^2+\varepsilon^2)C_*s}, \quad |\mathbf{k}|^2+\varepsilon^2 > \tau_0^2, \end{aligned} \tag{7.26}$$

where

$$C_\Lambda^2 = 4\alpha_1(2+c_1^2+c_2)m\big(\|g\|^{1/2}_{L_\infty}\tau_0^{-1}+c^{(1)}\big)^2+\alpha_1 c^{(2)}.$$

Similarly, by (7.2), (7.5), and (7.23), for $|\mathbf{k}|^2+\varepsilon^2 > \tau_0^2$ we have

$$\varepsilon\big\|\widehat{\mathcal{B}}(\mathbf{k},\varepsilon)^{1/2}\widetilde{\Lambda}\widehat{P}\big\|_{\mathfrak{H}\to\mathfrak{H}}\big\|\mathcal{E}^0(\mathbf{k},\varepsilon,s)\widehat{P}\|_{\mathfrak{H}\to\mathfrak{H}} \le \check{c}_*^{-1}C_{\widetilde{\Lambda}}\|f\|^2_{L_\infty}s^{-1}e^{-(|\mathbf{k}|^2+\varepsilon^2)C_*s}, \tag{7.27}$$

where

$$C_{\widetilde{\Lambda}}^2 = (2+c_1^2+c_2)n|\Omega|^{-1}(2c^{(3)}\tau_0^{-1}+c^{(4)})^2+4c^{(5)}.$$

Now we summarize the results. From (7.20), (7.26), and (7.27) it follows that

$$\begin{aligned} &\big\|\widehat{\mathcal{B}}(\mathbf{k},\varepsilon)^{1/2}\big(\Lambda b(\mathbf{D}+\mathbf{k})+\varepsilon\widetilde{\Lambda}\big)\mathcal{E}^0(\mathbf{k},\varepsilon,s)\widehat{P}\big\|_{\mathfrak{H}\to\mathfrak{H}} \\ &\quad\le \check{c}_*^{-1}\|f\|^2_{L_\infty}(C_\Lambda+C_{\widetilde{\Lambda}})s^{-1}e^{-(|\mathbf{k}|^2+\varepsilon^2)C_*s}, \quad s>0, \quad |\mathbf{k}|^2+\varepsilon^2 > \tau_0^2. \end{aligned} \tag{7.28}$$

Relations (7.18), (7.19), and (7.28) yield

$$\begin{aligned} &\big\|\widehat{\mathcal{B}}(\mathbf{k},\varepsilon)^{1/2}\big(fe^{-\mathcal{B}(\mathbf{k},\varepsilon)s}f^* - \big(I+\Lambda b(\mathbf{D}+\mathbf{k})+\varepsilon\widetilde{\Lambda}\big)\mathcal{E}^0(\mathbf{k},\varepsilon,s)\widehat{P}\big)\big\|_{\mathfrak{H}\to\mathfrak{H}} \\ &\quad\le C_{10}s^{-1}e^{-(|\mathbf{k}|^2+\varepsilon^2)C_*s}, \quad s>0, \quad |\mathbf{k}|^2+\varepsilon^2 > \tau_0^2, \end{aligned} \tag{7.29}$$

where $C_{10} = 2\|f\|_{L_\infty}\check{c}_*^{-1/2}\tau_0^{-1} + 2\|f\|^2_{L_\infty}C_P\check{c}_*^{-1}\tau_0^{-1} + \check{c}_*^{-1}\|f\|^2_{L_\infty}(C_\Lambda+C_{\widetilde{\Lambda}})$ .

**7.4.** Combining (7.16) and (7.29), we arrive at the estimate

$$\begin{aligned} &\big\|\widehat{\mathcal{B}}(\mathbf{k},\varepsilon)^{1/2}\big(fe^{-\mathcal{B}(\mathbf{k},\varepsilon)s}f^* - \big(I+\Lambda b(\mathbf{D}+\mathbf{k})+\varepsilon\widetilde{\Lambda}\big)\mathcal{E}^0(\mathbf{k},\varepsilon,s)\widehat{P}\big)\big\|_{\mathfrak{H}\to\mathfrak{H}} \\ &\qquad\le \max\{C_9;C_{10}\}s^{-1}e^{-(|\mathbf{k}|^2+\varepsilon^2)C_*s}, \quad s>0, \quad \mathbf{k}\in\widetilde{\Omega}. \end{aligned} \tag{7.30}$$

Now we show that the operator $\widehat{P}$ can be replaced by $I$ in the principal term of approximation. For that, we estimate the norm of the operator $\widehat{\mathcal{B}}(\mathbf{k},\varepsilon)^{1/2}\mathcal{E}^0(\mathbf{k},\varepsilon,s)\widehat{P}^\perp$. By (5.15) with $f=\mathbf{1}_n$, we have

$$\begin{aligned} \big\|\widehat{\mathcal{B}}(\mathbf{k},\varepsilon)^{1/2}\mathcal{E}^0(\mathbf{k},\varepsilon,s)\widehat{P}^\perp\mathbf{u}\big\|^2_{\mathfrak{H}} &\le (2+c_1^2+c_2)\big\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2}\mathcal{E}^0(\mathbf{k},\varepsilon,s)\widehat{P}^\perp\mathbf{u}\big\|^2_{\mathfrak{H}} \\ &\quad+(\widehat{C}(1)+\widehat{c}_3+|\lambda|)\varepsilon^2\|\mathcal{E}^0(\mathbf{k},\varepsilon,s)\widehat{P}^\perp\mathbf{u}\|^2_{\mathfrak{H}}, \quad \mathbf{u}\in\mathfrak{H}. \end{aligned} \tag{7.31}$$

Since $\mathcal{E}^0(\mathbf{k},\varepsilon,s)$ is the operator with the symbol $f_0\exp(-f_0\widehat{L}(\mathbf{b}+\mathbf{k},\varepsilon)f_0 s)f_0$, we can use (4.1), (6.15), (6.16), (7.2), and the estimate $|\mathbf{b}+\mathbf{k}|\ge r_0$, for $\mathbf{k}\in\widetilde{\Omega}$, $0\ne\mathbf{b}\in\widetilde{\Gamma}$, to obtain

$$\begin{aligned} \big\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2}\mathcal{E}^0(\mathbf{k},\varepsilon,s)\widehat{P}^\perp\big\|_{\mathfrak{H}\to\mathfrak{H}} &\le \|g\|^{1/2}_{L_\infty}\|f\|^2_{L_\infty}\alpha_1^{1/2}\sup_{0\ne\mathbf{b}\in\widetilde{\Gamma}}|\mathbf{b}+\mathbf{k}|e^{-\check{c}_*(|\mathbf{b}+\mathbf{k}|^2+\varepsilon^2)s} \\ &\le 2\|g\|^{1/2}_{L_\infty}\|f\|^2_{L_\infty}\alpha_1^{1/2}\check{c}_*^{-1}r_0^{-1}s^{-1}e^{-(|\mathbf{k}|^2+\varepsilon^2)C_*s}, \ s>0. \end{aligned} \tag{7.32}$$

Similarly, by (6.16) and (7.2),

$$\varepsilon\big\|\mathcal{E}^0(\mathbf{k},\varepsilon,s)\widehat{P}^\perp\big\|_{\mathfrak{H}\to\mathfrak{H}} \le \|f\|^2_{L_\infty}\check{c}_*^{-1}r_0^{-1}s^{-1}e^{-(|\mathbf{k}|^2+\varepsilon^2)C_*s}, \quad s>0. \tag{7.33}$$

Substituting (7.32) and (7.33) in (7.31) yields

$$\big\|\widehat{\mathcal{B}}(\mathbf{k},\varepsilon)^{1/2}\mathcal{E}^0(\mathbf{k},\varepsilon,s)\widehat{P}^\perp\big\|_{\mathfrak{H}\to\mathfrak{H}} \le C_{11}s^{-1}e^{-(|\mathbf{k}|^2+\varepsilon^2)C_*s}, \quad s>0, \tag{7.34}$$

where $C_{11} = r_0^{-1}\check{c}_*^{-1}\|f\|^2_{L_\infty}(4\|g\|_{L_\infty}\alpha_1(2+c_1^2+c_2)+\widehat{C}(1)+\widehat{c}_3+|\lambda|)^{1/2}$.

Combining (7.30) and (7.34), we obtain the estimate

$$\begin{aligned}\bigl\|\widehat{\mathcal{B}}(\mathbf{k},\varepsilon)^{1/2}\bigl(fe^{-\mathcal{B}(\mathbf{k},\varepsilon)s}f^* - \bigl(I+\Lambda b(\mathbf{D}+\mathbf{k})\widehat{P}+\varepsilon\widetilde{\Lambda}\widehat{P}\bigr)\mathcal{E}^0(\mathbf{k},\varepsilon,s)\bigr)\bigr\|_{\mathfrak{H}\to\mathfrak{H}} \\ \le \mathcal{C}_2 s^{-1}e^{-(|\mathbf{k}|^2+\varepsilon^2)C_* s}, \quad s>0, \quad \mathbf{k}\in\widetilde{\Omega},\end{aligned} \tag{7.35}$$

with the constant $\mathcal{C}_2 = \max\{C_9; C_{10}\} + C_{11}$.

**7.5. Estimates for** $0<s<1$**.** Now we show that for $s>0$ the left-hand side of (7.35) can be also estimated by $\mathcal{C}_3 s^{-1/2}\exp(-(|\mathbf{k}|^2+\varepsilon^2)C_* s)$, where $\mathcal{C}_3$ is some constant. For $0<s<1$ this estimate is more preferable compared to (7.35), but estimate (7.35) is preferable for $s\ge 1$. Now we estimate each term under the norm sign in (7.35) separately.

Using (6.5), (7.2), (7.17), and the inequality $e^{-\alpha/2}\le\alpha^{-1/2}$, $\alpha>0$, for $s>0$, $\mathbf{k}\in\widetilde{\Omega}$ we get

$$\bigl\|\widehat{\mathcal{B}}(\mathbf{k},\varepsilon)^{1/2}fe^{-\mathcal{B}(\mathbf{k},\varepsilon)s}f^*\bigr\|_{\mathfrak{H}\to\mathfrak{H}} \le \|f\|_{L_\infty}s^{-1/2}e^{-(|\mathbf{k}|^2+\varepsilon^2)C_* s}. \tag{7.36}$$

By (5.15) with $f=\mathbf{1}_n$, we obtain

$$\begin{aligned}\bigl\|\widehat{\mathcal{B}}(\mathbf{k},\varepsilon)^{1/2}\mathcal{E}^0(\mathbf{k},\varepsilon,s)\bigr\|^2_{\mathfrak{H}\to\mathfrak{H}} \le (2+c_1^2+c_2)\bigl\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2}\mathcal{E}^0(\mathbf{k},\varepsilon,s)\bigr\|^2_{\mathfrak{H}\to\mathfrak{H}} \\ + (\widehat{C}(1)+\widehat{c}_3+|\lambda|)\varepsilon^2\bigl\|\mathcal{E}^0(\mathbf{k},\varepsilon,s)\bigr\|^2_{\mathfrak{H}\to\mathfrak{H}}.\end{aligned} \tag{7.37}$$

Since $\mathcal{E}^0(\mathbf{k},\varepsilon,s)$ is the operator with the symbol $f_0\exp(-f_0\widehat{L}(\mathbf{b}+\mathbf{k},\varepsilon)f_0 s)f_0$, we can use (4.1), (6.15), (6.16), (7.2), and the inequality $e^{-\alpha/2}\le\alpha^{-1/2}$ to show that

$$\begin{aligned}\bigl\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2}\mathcal{E}^0(\mathbf{k},\varepsilon,s)\bigr\|_{\mathfrak{H}\to\mathfrak{H}} \le \|g\|^{1/2}_{L_\infty}\|f\|^2_{L_\infty}\alpha_1^{1/2}\sup_{\mathbf{b}\in\widetilde{\Gamma}}|\mathbf{b}+\mathbf{k}|e^{-\check{c}_*(|\mathbf{b}+\mathbf{k}|^2+\varepsilon^2)s} \\ \le \|g\|^{1/2}_{L_\infty}\|f\|^2_{L_\infty}\alpha_1^{1/2}\check{c}_*^{-1/2}s^{-1/2}e^{-(|\mathbf{k}|^2+\varepsilon^2)C_* s}, \quad s>0, \quad \mathbf{k}\in\widetilde{\Omega}.\end{aligned} \tag{7.38}$$

Similarly,

$$\varepsilon\|\mathcal{E}^0(\mathbf{k},\varepsilon,s)\|_{\mathfrak{H}\to\mathfrak{H}} \le \|f\|^2_{L_\infty}\check{c}_*^{-1/2}s^{-1/2}e^{-(|\mathbf{k}|^2+\varepsilon^2)C_* s}, \quad s>0, \quad \mathbf{k}\in\widetilde{\Omega}. \tag{7.39}$$

From (7.37), (7.38), and (7.39) it follows that

$$\bigl\|\widehat{\mathcal{B}}(\mathbf{k},\varepsilon)^{1/2}\mathcal{E}^0(\mathbf{k},\varepsilon,s)\bigr\|_{\mathfrak{H}\to\mathfrak{H}} \le C_{12}s^{-1/2}e^{-(|\mathbf{k}|^2+\varepsilon^2)C_* s}, \quad s>0, \quad \mathbf{k}\in\widetilde{\Omega}, \tag{7.40}$$

where $C_{12} = \check{c}_*^{-1/2}\|f\|^2_{L_\infty}(\|g\|_{L_\infty}\alpha_1(2+c_1^2+c_2)+\widehat{C}(1)+\widehat{c}_3+|\lambda|)^{1/2}$.

Now we estimate the norm of the corrector term. Substituting (7.25) in (7.20) and using (4.1), (7.2), (7.5), for $s>0$ and $\mathbf{k}\in\widetilde{\Omega}$ we get

$$\begin{aligned}\bigl\|\widehat{\mathcal{B}}(\mathbf{k},\varepsilon)^{1/2}\bigl(\Lambda b(\mathbf{D}+\mathbf{k})+\varepsilon\widetilde{\Lambda}\bigr)\mathcal{E}^0(\mathbf{k},\varepsilon,s)\widehat{P}\bigr\|_{\mathfrak{H}\to\mathfrak{H}} \\ \le \|f\|^2_{L_\infty}\check{c}_*^{-1/2}(\alpha_1^{1/2}C_\Lambda(r_1,1)+C_{\widetilde{\Lambda}}(r_1,1))s^{-1/2}e^{-(|\mathbf{k}|^2+\varepsilon^2)C_* s}.\end{aligned} \tag{7.41}$$

Combining (7.36), (7.40), and (7.41) yields

$$\begin{aligned}\bigl\|\widehat{\mathcal{B}}(\mathbf{k},\varepsilon)^{1/2}\bigl(fe^{-\mathcal{B}(\mathbf{k},\varepsilon)s}f^* - \bigl(I+\Lambda b(\mathbf{D}+\mathbf{k})\widehat{P}+\varepsilon\widetilde{\Lambda}\widehat{P}\bigr)\mathcal{E}^0(\mathbf{k},\varepsilon,s)\bigr)\bigr\|_{\mathfrak{H}\to\mathfrak{H}} \\ \le \mathcal{C}_3 s^{-1/2}e^{-(|\mathbf{k}|^2+\varepsilon^2)C_* s}, \quad s>0, \quad \mathbf{k}\in\widetilde{\Omega},\end{aligned} \tag{7.42}$$

where $\mathcal{C}_3 = \|f\|_{L_\infty} + C_{12} + \|f\|^2_{L_\infty}\check{c}_*^{-1/2}(\alpha_1^{1/2}C_\Lambda(r_1,1)+C_{\widetilde{\Lambda}}(r_1,1))$.

Using (7.42) for $0<s<1$ and (7.35) for $s\ge 1$, we obtain the following result.

**Theorem 7.4.** *Under the above assumptions,*

$$\begin{aligned}\|\widehat{\mathcal{B}}(\mathbf{k},\varepsilon)^{1/2}\bigl(fe^{-\mathcal{B}(\mathbf{k},\varepsilon)s}f^* - \bigl(I+\Lambda b(\mathbf{D}+\mathbf{k})\widehat{P}+\varepsilon\widetilde{\Lambda}\widehat{P}\bigr)\mathcal{E}^0(\mathbf{k},\varepsilon,s)\bigr)\|_{\mathfrak{H}\to\mathfrak{H}} \le \Phi_1(\mathbf{k},s,\varepsilon), \\ s>0, \quad \mathbf{k}\in\operatorname{clos}\widetilde{\Omega}, \quad 0<\varepsilon\le 1,\end{aligned} \tag{7.43}$$

*where*

$$\Phi_1(\mathbf{k},s,\varepsilon)=\begin{cases}\mathcal{C}_2 s^{-1}e^{-(|\mathbf{k}|^2+\varepsilon^2)C_* s} & \textit{if } \ s\ge 1,\\ \mathcal{C}_3 s^{-1/2}e^{-(|\mathbf{k}|^2+\varepsilon^2)C_* s} & \textit{if } \ 0<s<1.\end{cases}$$

## §8. Approximation of the operator $f\exp(-\mathcal{B}(\varepsilon)s)f^*$

**8.1. The principal term of approximation.** Now we return to the study of the operator $\mathcal{B}(\varepsilon)$ acting in $L_2(\mathbb{R}^d;\mathbb{C}^n)$. We also consider the operator $\widehat{\mathcal{B}}(\varepsilon)$ corresponding to the case where $f=\mathbf{1}_n$. The operator $\widehat{\mathcal{B}}(\varepsilon)$ is generated by the quadratic form $\widehat{\mathfrak{b}}(\varepsilon)$ given by (4.17) with $f=\mathbf{1}_n$.

In accordance with [BSu1, Chapter 3, §1], the operator

$$\widehat{\mathcal{A}}^0=b(\mathbf{D})^*g^0b(\mathbf{D}) \tag{8.1}$$

is called the *effective operator* for $\widehat{\mathcal{A}}=b(\mathbf{D})^*g(\mathbf{x})b(\mathbf{D})$. The effective matrix $g^0$ is given by (6.9). Next, we put

$$\widehat{\mathcal{Y}}^0=-b(\mathbf{D})^*V+\sum_{j=1}^d\overline{a_j}D_j, \tag{8.2}$$

where $V$ is the matrix defined by (6.10). Consider the operator

$$\widehat{\mathcal{B}}^0(\varepsilon)=\widehat{\mathcal{A}}^0+\varepsilon(\widehat{\mathcal{Y}}^0+(\widehat{\mathcal{Y}}^0)^*)+\varepsilon^2(\overline{\mathcal{Q}}-W+\lambda I).$$

Here $W$ is the matrix (6.11). The operator $\widehat{\mathcal{B}}^0(\varepsilon)$ is a second order DO with constant coefficients. The symbol of the operator $\widehat{\mathcal{B}}^0(\varepsilon)$ is the matrix (6.12).

We denote $\mathcal{E}^0(\varepsilon,s):=f_0e^{-f_0\widehat{\mathcal{B}}^0(\varepsilon)f_0s}f_0$. By the direct integral expansions of the operators $\mathcal{B}(\varepsilon)$ and $\widehat{\mathcal{B}}^0(\varepsilon)$ (see §5), Theorem 7.1 implies the following result.

**Theorem 8.1.** *Suppose that the operator $\mathcal{B}(\varepsilon)$ satisfies the assumptions of Subsection* 4.5. *Then for $s\ge 0$ and $0<\varepsilon\le 1$, we have*

$$\big\|fe^{-\mathcal{B}(\varepsilon)s}f^*-\mathcal{E}^0(\varepsilon,s)\big\|_{L_2(\mathbb{R}^d;\mathbb{C}^n)\to L_2(\mathbb{R}^d;\mathbb{C}^n)}\le\mathcal{C}_1(1+s)^{-1/2}e^{-\varepsilon^2C_*s}.$$

**8.2. Approximation with the corrector term taken into account.** In this subsection we use Theorem 7.4 to obtain a more accurate approximation for the operator $fe^{-\mathcal{B}(\varepsilon)s}f^*$. Note that the operator $b(\mathbf{D})$ expands in the direct integral of the operators $b(\mathbf{D}+\mathbf{k})$. Under the Gelfand transformation, the operators of multiplication by the $\Gamma$-periodic matrices $\Lambda$ and $\widetilde{\Lambda}$ turn into operators of multiplication by the same matrices $\Lambda$ and $\widetilde{\Lambda}$. Next, we put $\Pi=\mathcal{U}^{-1}[\widehat{P}]\mathcal{U}$, where $[\widehat{P}]$ is an operator in $\mathcal{H}$ (see (5.1)) that acts layerwise as the operator $\widehat{P}$ of averaging over the cell. In [BSu3, Subsection 6.1], it was shown that $\Pi$ is a pseudodifferential operator in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ and its symbol is $\chi_{\widetilde{\Omega}}(\boldsymbol{\xi})$. Here $\chi_{\widetilde{\Omega}}(\boldsymbol{\xi})$ is the characteristic function of the set $\widetilde{\Omega}$. In other words,

$$(\Pi\mathbf{u})(\mathbf{x})=(2\pi)^{-d/2}\int_{\widetilde{\Omega}}e^{i\langle\mathbf{x},\boldsymbol{\xi}\rangle}(\mathcal{F}\mathbf{u})(\boldsymbol{\xi})\,d\boldsymbol{\xi},$$

where $\mathcal{F}$ stands for the Fourier transformation.

Thus, under the Gelfand transformation, the operator

$$\widehat{\mathcal{B}}(\varepsilon)^{1/2}\big(fe^{-\mathcal{B}(\varepsilon)s}f^*-\big(I+\Lambda b(\mathbf{D})\Pi+\varepsilon\widetilde{\Lambda}\Pi\big)\mathcal{E}^0(\varepsilon,s)\big)$$

expands in the direct integral of the operators under the norm sign in (7.43). Hence, by (7.43), we obtain the following result.

**Theorem 8.2.** *We have*

$$\big\|\widehat{\mathcal{B}}(\varepsilon)^{1/2}\big(fe^{-\mathcal{B}(\varepsilon)s}f^* - \big(I+\Lambda b(\mathbf{D})\Pi+\varepsilon\widetilde{\Lambda}\Pi\big)\mathcal{E}^0(\varepsilon,s)\big)\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le \Phi_2(s,\varepsilon), \quad s>0,\quad 0<\varepsilon\le 1, \tag{8.3}$$

*where*

$$\Phi_2(s,\varepsilon) = \begin{cases} \mathcal{C}_2 s^{-1} e^{-\varepsilon^2 C_* s} & \text{if } s\ge 1,\\ \mathcal{C}_3 s^{-1/2} e^{-\varepsilon^2 C_* s} & \text{if } 0<s<1.\end{cases} \tag{8.4}$$

**8.3. Elimination of the operator $\Pi$ from the corrector term for $s \ge 1$.** Now we analyze the possibility of replacing the operator $\Pi$ by the identity operator $I$ in the corrector term. For this, we estimate the norm of the operator

$$\widehat{\mathcal{B}}(\varepsilon)^{1/2}\big(\Lambda b(\mathbf{D}) + \varepsilon\widetilde{\Lambda}\big)\mathcal{E}^0(\varepsilon,s)(I-\Pi).$$

**Proposition 8.3.** *Denote $\Xi(\varepsilon,s) = \mathcal{E}^0(\varepsilon,s)(I-\Pi)$. Then, for any $l>0$, the operators $b(\mathbf{D})\Xi(\varepsilon,s)$ and $\varepsilon\Xi(\varepsilon,s)$ are continuous mappings of $L_2(\mathbb{R}^d;\mathbb{C}^n)$ to $H^l(\mathbb{R}^d;\mathbb{C}^n)$, and*

$$\|b(\mathbf{D})\Xi(\varepsilon,s)\|_{L_2(\mathbb{R}^d)\to H^l(\mathbb{R}^d)} \le \alpha_1^{1/2}\mathcal{C}_l s^{-(l+1)/2} e^{-\varepsilon^2 C_* s}, \quad s>0, \tag{8.5}$$

$$\varepsilon\|\Xi(\varepsilon,s)\|_{L_2(\mathbb{R}^d)\to H^l(\mathbb{R}^d)} \le \mathcal{C}_l s^{-(l+1)/2} e^{-\varepsilon^2 C_* s}, \quad s>0. \tag{8.6}$$

*Proof.* Since $\Xi(\varepsilon,s)$ is the pseudodifferential operator with the symbol

$$f_0 e^{-f_0\widehat{L}(\boldsymbol{\xi},\varepsilon)f_0 s} f_0(1-\chi_{\widetilde{\Omega}}(\boldsymbol{\xi})),$$

by (4.1), (6.15), and (6.16), we have

$$\begin{aligned} \|b(\mathbf{D})\Xi(\varepsilon,s)\|_{L_2\to H^l} &\le \alpha_1^{1/2}\|f\|^2_{L_\infty} \sup_{|\boldsymbol{\xi}|>r_0} |\boldsymbol{\xi}|(1+|\boldsymbol{\xi}|^2)^{l/2} e^{-\check{c}_*(|\boldsymbol{\xi}|^2+\varepsilon^2)s},\\ \varepsilon\|\Xi(\varepsilon,s)\|_{L_2\to H^l} &\le \|f\|^2_{L_\infty} \sup_{|\boldsymbol{\xi}|>r_0} \varepsilon(1+|\boldsymbol{\xi}|^2)^{l/2} e^{-\check{c}_*(|\boldsymbol{\xi}|^2+\varepsilon^2)s}. \end{aligned} \tag{8.7}$$

Here we have used the relation $1-\chi_{\widetilde{\Omega}}(\boldsymbol{\xi}) = 0$ for $|\boldsymbol{\xi}|\le r_0$. Applying (7.2) and (8.7), we obtain estimates (8.5) and (8.6) with $C_l = \|f\|^2_{L_\infty}\check{c}_*^{-(l+1)/2}(r_0^{-2}+1)^{l/2}\gamma_l$ and $\gamma_l = \sup_{\alpha>0}\alpha^{(l+1)/2}e^{-\alpha/2} = (l+1)^{(l+1)/2}e^{-(l+1)/2}$. □

**Proposition 8.4.** *Suppose $l=1$ for $d=1$, $l>1$ for $d=2$, and $l=d/2$ for $d\ge 3$. Let $[\Lambda]$ and $[\widetilde{\Lambda}]$ be the operators of multiplication by the matrix-valued functions $\Lambda(\mathbf{x})$ and $\widetilde{\Lambda}(\mathbf{x})$, respectively. Then the operators $g^{1/2}b(\mathbf{D})[\Lambda]\colon H^l(\mathbb{R}^d;\mathbb{C}^m)\to L_2(\mathbb{R}^d;\mathbb{C}^n)$ and $g^{1/2}b(\mathbf{D})[\widetilde{\Lambda}]\colon H^l(\mathbb{R}^d;\mathbb{C}^n)\to L_2(\mathbb{R}^d;\mathbb{C}^n)$ are continuous mappings, and*

$$\big\|g^{1/2}b(\mathbf{D})[\Lambda]\big\|_{H^l(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le \mathfrak{C}_d, \tag{8.8}$$

$$\big\|g^{1/2}b(\mathbf{D})[\widetilde{\Lambda}]\big\|_{H^l(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le \widetilde{\mathfrak{C}}_d. \tag{8.9}$$

*The constants $\mathfrak{C}_d$ and $\widetilde{\mathfrak{C}}_d$ depend only on $l$, the initial data (4.23), and the parameters of the lattice $\Gamma$.*

*Proof.* Estimate (8.8) was obtained in [Su5, Proposition 9.3]. The constant $\mathfrak{C}_d$ can be written explicitly (see [Su5, Subsection 9.2]).

Now we prove (8.9). Let $\mathbf{v}_i(\mathbf{x})$, $i=1,\dots,n$, be the columns of the matrix $\widetilde{\Lambda}(\mathbf{x})$. Then $\mathbf{v}_i \in \widetilde{H}^1(\Omega;\mathbb{C}^n)$ is a weak $\Gamma$-periodic solution of the problem

$$b(\mathbf{D})^* g(\mathbf{x}) b(\mathbf{D})\mathbf{v}_i + \sum_{j=1}^d D_j a_j(\mathbf{x})^*\mathbf{e}_i = 0, \quad \int_\Omega \mathbf{v}_i(\mathbf{x})\,d\mathbf{x} = 0. \tag{8.10}$$

Here $\{\mathbf{e}_i\}_{i=1,\dots,n}$ is the standard orthogonal basis in $\mathbb{C}^n$. Since $\mathbf{v}_i$ is a $\Gamma$-periodic function with zero mean value, we have $\|\mathbf{v}_i\|_{L_2(\Omega)} \le (2r_0)^{-1}\|\mathbf{D}\mathbf{v}_i\|_{L_2(\Omega)}$. Hence, by using the "energy" inequality, it is easy to check that (see [Su6, (7.51) and (7.52)])

$$\|\mathbf{v}_i\|_{H^1(\Omega)} \le (1+(2r_0)^{-2})^{1/2} C_a \alpha_0^{-1}\|g^{-1}\|_{L_\infty}, \tag{8.11}$$

where $C_a$is the constant (7.24).

Recall that $b(\mathbf{D}) = \sum_{k=1}^d b_k D_k$ and, by (4.1), $|b_k| \le \alpha_1^{1/2}$. Let $u \in H^l(\mathbb{R}^d)$. We have

$$g^{1/2} b(\mathbf{D})(\mathbf{v}_i u) = g^{1/2}(b(\mathbf{D})\mathbf{v}_i)u + \sum_{k=1}^d g^{1/2} b_k (D_k u)\mathbf{v}_i. \tag{8.12}$$

We estimate the right-hand side in (8.12):

$$\Big\| \sum_{k=1}^d g^{1/2} b_k (D_k u)\mathbf{v}_i \Big\|_{L_2(\mathbb{R}^d)} \le \|g\|_{L_\infty}^{1/2} \alpha_1^{1/2} d^{1/2} \Big( \int_{\mathbb{R}^d} |\mathbf{D}u|^2 |\mathbf{v}_i|^2 \, d\mathbf{x} \Big)^{1/2}. \tag{8.13}$$

Next,

$$\int_{\mathbb{R}^d} |\mathbf{D}u|^2 |\mathbf{v}_i|^2 \, d\mathbf{x} = \sum_{\mathbf{a}\in\Gamma} \int_{\Omega+\mathbf{a}} |\mathbf{D}u|^2 |\mathbf{v}_i|^2 \, d\mathbf{x}. \tag{8.14}$$

Now we use the embedding $H^1(\Omega;\mathbb{C}^n) \subset L_q(\Omega;\mathbb{C}^n)$, where $q=\infty$ for $d=1$, $q<\infty$ for $d=2$, and $q = 2d/(d-2)$ for $d \ge 3$. For $d=2$ we choose $q = 2/(l-1)$. Let $C(d,n)$ be the norm of the corresponding embedding operator. Then

$$\|\mathbf{v}_i\|_{L_q(\Omega)} \le C(d,n)\|\mathbf{v}_i\|_{H^1(\Omega)}. \tag{8.15}$$

By the Hölder inequality,

$$\int_\Omega |\mathbf{v}_i|^2 |\mathbf{D}u|^2 \, d\mathbf{x} \le \|\mathbf{v}_i\|^2_{L_q(\Omega)} \|\mathbf{D}u\|^2_{L_p(\Omega)}, \tag{8.16}$$

where $p=2$ for $d=1$, $p = 2q/(q-2) = 2/(2-l)$ for $d=2$, and $p=d$ for $d\ge 3$.

Also, we use the embedding $H^{l-1}(\Omega;\mathbb{C}^d) \subset L_p(\Omega;\mathbb{C}^d)$, where $l=1$ and $p=2$ for $d=1$, $1<l<2$ and $p=2/(2-l)$ for $d=2$, $l=d/2$ and $p=d$ for $d\ge 3$. Let $\widetilde{c}_d$ be the norm of the corresponding embedding operator. Then

$$\|\mathbf{D}u\|_{L_p(\Omega)} \le \widetilde{c}_d \|u\|_{H^l(\Omega)}. \tag{8.17}$$

Substituting (8.15) and (8.17) in (8.16), we obtain

$$\int_\Omega |\mathbf{v}_i|^2 |\mathbf{D}u|^2 \, d\mathbf{x} \le C(d,n)^2 \widetilde{c}_d^{\,2} \|\mathbf{v}_i\|^2_{H^1(\Omega)} \|u\|^2_{H^l(\Omega)}.$$

Hence, by (8.14) and the periodicity of $\mathbf{v}_i$, we have

$$\int_{\mathbb{R}^d} |\mathbf{D}u|^2 |\mathbf{v}_i|^2 \, d\mathbf{x} \le C(d,n)^2 \widetilde{c}_d^{\,2} \|\mathbf{v}_i\|^2_{H^1(\Omega)} \|u\|^2_{H^l(\mathbb{R}^d)}. \tag{8.18}$$

By (8.13), inequality (8.18) implies the estimate

$$\Big\| \sum_{k=1}^d g^{1/2} b_k (D_k u)\mathbf{v}_i \Big\|_{L_2(\mathbb{R}^d)} \le \|g\|_{L_\infty}^{1/2} \alpha_1^{1/2} d^{1/2} C(d,n)\widetilde{c}_d \|\mathbf{v}_i\|_{H^1(\Omega)} \|u\|_{H^l(\mathbb{R}^d)}. \tag{8.19}$$

Next, (8.10) yields the identity

$$\int_{\mathbb{R}^d} \langle g(\mathbf{x}) b(\mathbf{D})\mathbf{v}_i, b(\mathbf{D})\mathbf{w}\rangle \, d\mathbf{x} + \int_{\mathbb{R}^d} \sum_{j=1}^d \langle a_j(\mathbf{x})^* \mathbf{e}_i, D_j \mathbf{w}\rangle \, d\mathbf{x} = 0 \tag{8.20}$$

for all $\mathbf{w} \in H^1(\mathbb{R}^d;\mathbb{C}^n)$ such that $\mathbf{w}(\mathbf{x}) = 0$ for $|\mathbf{x}| > R$ (with some $R>0$).

Let $u \in C_0^\infty(\mathbb{R}^d)$. We put $\mathbf{w}(\mathbf{x}) = |u(\mathbf{x})|^2 \mathbf{v}_i$. Substituting this in (8.20), we obtain (cf. [Su6, (8.36)])

$$\mathcal{J}_0 := \int_{\mathbb{R}^d} |g^{1/2} b(\mathbf{D})\mathbf{v}_i|^2 |u|^2 \, d\mathbf{x} = \mathcal{J}_1 + \mathcal{J}_2,$$

$$\mathcal{J}_1 = -\int_{\mathbb{R}^d} \left\langle g^{1/2} b(\mathbf{D})\mathbf{v}_i, \sum_{k=1}^d g^{1/2} b_k((D_k u)\bar{u} + u(D_k \bar{u}))\mathbf{v}_i \right\rangle d\mathbf{x}, \tag{8.21}$$

$$\mathcal{J}_2 = -\int_{\mathbb{R}^d} \sum_{j=1}^d \langle a_j^* \mathbf{e}_i, D_j(|u|^2 \mathbf{v}_i)\rangle \, d\mathbf{x} = -\int_{\mathbb{R}^d} \sum_{j=1}^d \langle a_j^* \mathbf{e}_i, (D_j(u\mathbf{v}_i))\bar{u} + \mathbf{v}_i u(D_j \bar{u})\rangle \, d\mathbf{x}.$$

We follow [Su6] to estimate the term $\mathcal{J}_1$:

$$|\mathcal{J}_1| \le \frac{1}{2} \int_{\mathbb{R}^d} |g^{1/2} b(\mathbf{D})\mathbf{v}_i|^2 |u|^2 \, d\mathbf{x} + 2\|g\|_{L_\infty} \alpha_1 \, d \int_{\mathbb{R}^d} |\mathbf{D}u|^2 |\mathbf{v}_i|^2 \, d\mathbf{x}.$$

Combining this with (8.18), we see that

$$|\mathcal{J}_1| \le \frac{1}{2}\mathcal{J}_0 + 2\|g\|_{L_\infty} \alpha_1 d C(d,n)^2 \widetilde{c}_d^{\,2} \|\mathbf{v}_i\|^2_{H^1(\Omega)} \|u\|^2_{H^l(\mathbb{R}^d)}. \tag{8.22}$$

Now we proceed to estimating the term $\mathcal{J}_2$. By condition (4.8) on the coefficients $a_j$ and the condition on $l$,

$$\int_{\mathbb{R}^d} |a_j(\mathbf{x})|^2 |u|^2 \, d\mathbf{x} \le C^2_{\Omega,l,\varrho} \|a_j\|^2_{L_\varrho(\Omega)} \|u\|^2_{H^l(\mathbb{R}^d)}. \tag{8.23}$$

Here $C_{\Omega,l,\varrho}$ is the norm of the embedding operator $H^l(\Omega) \subset L_{2\varrho/(\varrho-2)}(\Omega)$. We have (cf. [Su6])

$$\begin{aligned}|\mathcal{J}_2| &\le \sum_{j=1}^d \int_{\mathbb{R}^d} (|D_j(\mathbf{v}_i u)||a_j||u| + |\mathbf{v}_i||D_j u||a_j||u|) \, d\mathbf{x} \\ &\le \mu \int_{\mathbb{R}^d} |\mathbf{D}(\mathbf{v}_i u)|^2 \, d\mathbf{x} + \int_{\mathbb{R}^d} |\mathbf{v}_i|^2 |\mathbf{D}u|^2 \, d\mathbf{x} + \Big(\frac{1}{4\mu} + \frac{1}{4}\Big) \sum_{j=1}^d \int_{\mathbb{R}^d} |a_j|^2 |u|^2 \, d\mathbf{x}\end{aligned}$$

for any $\mu > 0$. Combining this with (8.18) and (8.23), we arrive at the estimate

$$\begin{aligned}|\mathcal{J}_2| \le \mu \int_{\mathbb{R}^d} |\mathbf{D}(\mathbf{v}_i u)|^2 \, d\mathbf{x} &+ C(d,n)^2 \widetilde{c}_d^{\,2} \|\mathbf{v}_i\|^2_{H^1(\Omega)} \|u\|^2_{H^l(\mathbb{R}^d)} \\ &+ \Big(\frac{1}{4} + \frac{1}{4\mu}\Big) C^2_{\Omega,l,\varrho} \sum_{j=1}^d \|a_j\|^2_{L_\varrho(\Omega)} \|u\|^2_{H^l(\mathbb{R}^d)}.\end{aligned} \tag{8.24}$$

From (8.21), (8.22), and (8.24) it follows that

$$\begin{aligned}\frac{1}{2}\mathcal{J}_0 \le \mu \int_{\mathbb{R}^d} |\mathbf{D}(\mathbf{v}_i u)|^2 \, d\mathbf{x} &+ (2\|g\|_{L_\infty} \alpha_1 d + 1) C(d,n)^2 \widetilde{c}_d^{\,2} \|\mathbf{v}_i\|^2_{H^1(\Omega)} \|u\|^2_{H^l(\mathbb{R}^d)} \\ &+ \Big(\frac{1}{4} + \frac{1}{4\mu}\Big) C^2_{\Omega,l,\varrho} \sum_{j=1}^d \|a_j\|^2_{L_\varrho(\Omega)} \|u\|^2_{H^l(\mathbb{R}^d)}.\end{aligned} \tag{8.25}$$

Comparing (8.12), (8.19), and (8.25), we obtain the inequality

$$
\begin{aligned}
\|g^{1/2}b(\mathbf{D})(\mathbf{v}_i u)\|^2_{L_2(\mathbb{R}^d)} &\leq 2\mathcal{J}_0 + 2\Big\| \sum_{k=1}^d g^{1/2} b_k (D_k u)\mathbf{v}_i \Big\|^2_{L_2(\mathbb{R}^d)} \\
&\leq (10\|g\|_{L_\infty}\alpha_1 d + 4) C(d,n)^2 \widetilde{c}_d^{\,2} \|\mathbf{v}_i\|^2_{H^1(\Omega)} \|u\|^2_{H^l(\mathbb{R}^d)} \\
&\quad + (1+\mu^{-1}) C^2_{\Omega,l,\varrho} \sum_{j=1}^d \|a_j\|^2_{L_\varrho(\Omega)} \|u\|^2_{H^l(\mathbb{R}^d)} + 4\mu \int_{\mathbb{R}^d} |\mathbf{D}(\mathbf{v}_i u)|^2\, d\mathbf{x}.
\end{aligned}
\tag{8.26}
$$

The lower estimate (4.5) with $f = \mathbf{1}_n$ implies

$$
4\mu \int_{\mathbb{R}^d} |\mathbf{D}(\mathbf{v}_i u)|^2\, d\mathbf{x} \leq \frac{1}{2} \|g^{1/2} b(\mathbf{D})(\mathbf{v}_i u)\|^2_{L_2(\mathbb{R}^d)}, \quad \mu = \frac{1}{8}\alpha_0 \|g^{-1}\|^{-1}_{L_\infty}.
$$

Together with (8.26) and (8.11) this yields $\|g^{1/2}b(\mathbf{D})(\mathbf{v}_i u)\|_{L_2(\mathbb{R}^d)} \leq \mathfrak{C}_v \|u\|_{H^l(\mathbb{R}^d)}$, where

$$
\begin{aligned}
\mathfrak{C}_v^2 = (20\|g\|_{L_\infty}\alpha_1 d + 8) C(d,n)^2 \widetilde{c}_d^{\,2} (1 + (2r_0)^{-2}) C_a^2 \alpha_0^{-2} \|g^{-1}\|^2_{L_\infty} \\
+ (2 + 16\alpha_0^{-1}\|g^{-1}\|_{L_\infty}) C^2_{\Omega,l,\varrho} \sum_{j=1}^d \|a_j\|^2_{L_\varrho(\Omega)}.
\end{aligned}
$$

Thus, $\|g^{1/2}b(\mathbf{D})[\mathbf{v}_i]\|_{H^l(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leq \mathfrak{C}_v$, $i = 1, \dots, n$, whence we see that (8.9) is fulfilled with the constant $\widetilde{\mathfrak{C}}_d = n^{1/2}\mathfrak{C}_v$. □

**Proposition 8.5.** *Suppose* $\mathfrak{r} = 0$ *for* $d = 1$, $\mathfrak{r} > 0$ *for* $d = 2$, *and* $\mathfrak{r} = d/2 - 1$ *for* $d \geq 3$. *Then* $[\Lambda]\colon H^{\mathfrak{r}}(\mathbb{R}^d;\mathbb{C}^m) \to L_2(\mathbb{R}^d;\mathbb{C}^n)$ *and* $[\widetilde{\Lambda}]\colon H^{\mathfrak{r}}(\mathbb{R}^d;\mathbb{C}^n) \to L_2(\mathbb{R}^d;\mathbb{C}^n)$ *are continuous mappings, and*

$$
\|[\Lambda]\|_{H^{\mathfrak{r}}(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leq \mathfrak{C}_\Lambda, \tag{8.27}
$$

$$
\|[\widetilde{\Lambda}]\|_{H^{\mathfrak{r}}(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leq \mathfrak{C}_{\widetilde{\Lambda}}. \tag{8.28}
$$

*The constants* $\mathfrak{C}_\Lambda$ *and* $\mathfrak{C}_{\widetilde{\Lambda}}$ *depend only on the initial data* (4.23) *and the parameters of the lattice* $\Gamma$; *in the case where* $d = 2$ *they depend also on* $\mathfrak{r}$.

*Proof.* Estimate (8.27) was obtained in [Su5, Proposition 11.3]. The constant $\mathfrak{C}_\Lambda$ can be written explicitly (see [Su5, Subsection 11.2]).

Now we prove (8.28). Assume that $0 < \mathfrak{r} < 1$ in the case where $d = 2$. As in (8.14)–(8.18) with $l - 1$ replaced by $\mathfrak{r}$, we obtain

$$
\int_{\mathbb{R}^d} |\mathbf{v}_i(\mathbf{x})|^2 |u|^2\, d\mathbf{x} \leq C(d,n)^2 \check{c}_d^{\,2} \|\mathbf{v}_i\|^2_{H^1(\Omega)} \|u\|^2_{H^{\mathfrak{r}}(\mathbb{R}^d)}.
$$

Here $\check{c}_d$ is the norm of the embedding $H^{\mathfrak{r}}(\Omega) \subset L_p(\Omega)$, where $\mathfrak{r} = 0$ and $p = 2$ for $d = 1$; $0 < \mathfrak{r} < 1$ and $p = 2/(1-\mathfrak{r})$ for $d = 2$; and $\mathfrak{r} = d/2 - 1$ and $p = d$ for $d \geq 3$. Together with (8.11), this implies (8.28) with the constant

$$
\mathfrak{C}_{\widetilde{\Lambda}} = n^{1/2} C(d,n) \check{c}_d (1 + (2r_0)^{-2})^{1/2} C_a \alpha_0^{-1} \|g^{-1}\|_{L_\infty}. \qquad □
$$

Now, using Propositions 8.4 and 8.5, we arrive at the following result.

**Proposition 8.6.** *Suppose* $l = 1$ *for* $d = 1$, $l > 1$ *for* $d = 2$, *and* $l = d/2$ *for* $d \geq 3$. *Let* $0 < \varepsilon \leq 1$. *Then* $\widehat{\mathcal{B}}(\varepsilon)^{1/2}[\Lambda]\colon H^l(\mathbb{R}^d;\mathbb{C}^m) \to L_2(\mathbb{R}^d;\mathbb{C}^n)$ *and* $\widehat{\mathcal{B}}(\varepsilon)^{1/2}[\widetilde{\Lambda}]\colon H^l(\mathbb{R}^d;\mathbb{C}^n) \to L_2(\mathbb{R}^d;\mathbb{C}^n)$ *are continuous mappings, and*

$$
\|\widehat{\mathcal{B}}(\varepsilon)^{1/2}[\Lambda]\|_{H^l(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leq \mathfrak{C}_{\mathcal{B}}, \quad \|\widehat{\mathcal{B}}(\varepsilon)^{1/2}[\widetilde{\Lambda}]\|_{H^l(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leq \widetilde{\mathfrak{C}}_{\mathcal{B}}. \tag{8.29}
$$

*The constants* $\mathfrak{C}_{\mathcal{B}}$ *and* $\widetilde{\mathfrak{C}}_{\mathcal{B}}$ *depend only on the initial data* (4.23) *and the parameters of the lattice* $\Gamma$; *in the case where* $d = 2$, *they depend also on* $l$.

*Proof.* Let $\mathbf{u} \in H^l(\mathbb{R}^d;\mathbb{C}^m)$. By (4.21) with $f = \mathbf{1}_n$, we have

$$\|\widehat{\mathcal{B}}(\varepsilon)^{1/2}\Lambda\mathbf{u}\|^2_{L_2} \leq (2 + c_1^2 + c_2)\|\widehat{\mathcal{A}}^{1/2}\Lambda\mathbf{u}\|^2_{L_2} + \big(\widehat{C}(1) + \widehat{c}_3 + |\lambda|\big)\varepsilon^2\|\Lambda\mathbf{u}\|^2_{L_2}. \tag{8.30}$$

Obviously, $\mathfrak{r} := l-1$ satisfies the assumptions of Proposition 8.5, so that (8.27) implies

$$\|\Lambda\mathbf{u}\|^2_{L_2(\mathbb{R}^d)} \leq (\mathfrak{C}_\Lambda)^2\, \|\mathbf{u}\|^2_{H^{l-1}(\mathbb{R}^d)} \leq (\mathfrak{C}_\Lambda)^2\, \|\mathbf{u}\|^2_{H^l(\mathbb{R}^d)}. \tag{8.31}$$

By (8.8), we have $\|\widehat{\mathcal{A}}^{1/2}\Lambda\mathbf{u}\|^2_{L_2(\mathbb{R}^d)} \leq \mathfrak{C}_d^2\|\mathbf{u}\|^2_{H^l(\mathbb{R}^d)}$. Together with (8.30) and (8.31), this yields the first estimate in (8.29) with the constant $\mathfrak{C}^2_{\mathcal{B}} = (2 + c_1^2 + c_2)\mathfrak{C}_d^2 + \big(\widehat{C}(1) + \widehat{c}_3 + |\lambda|\big)\mathfrak{C}^2_\Lambda$.

Similarly, by using (8.9) and (8.28), one can prove the second estimate (8.29) with the constant $\widetilde{\mathfrak{C}}^2_{\mathcal{B}} = (2 + c_1^2 + c_2)\widetilde{\mathfrak{C}}_d^2 + \big(\widehat{C}(1) + \widehat{c}_3 + |\lambda|\big)\mathfrak{C}^2_{\widetilde{\Lambda}}$. □

Combining (8.5), (8.6), and (8.29), we obtain

$$\begin{aligned}\big\|\widehat{\mathcal{B}}(\varepsilon)^{1/2}\big(\Lambda b(\mathbf{D}) + \varepsilon\widetilde{\Lambda}\big)\mathcal{E}^0(\varepsilon,s)(I-\Pi)\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}&\\ \leq \big(\mathfrak{C}_{\mathcal{B}}\alpha_1^{1/2}\mathcal{C}_l + \widetilde{\mathfrak{C}}_{\mathcal{B}}\mathcal{C}_l\big)s^{-(l+1)/2}e^{-\varepsilon^2 C_* s},& \quad s>0, \quad 0<\varepsilon\leq 1,\end{aligned}$$

where $l = 1$ for $d = 1$, $l > 1$ for $d = 2$, and $l = d/2$ for $d \geq 3$. If $s \geq 1$, then $s^{-(l+1)/2} \leq s^{-1}$. In the case where $d = 2$, we fix $l$ (for instance, $l = 3/2$). Combined with Theorem 8.2, this implies the following result.

**Theorem 8.7.** *We have*

$$\begin{aligned}\big\|\widehat{\mathcal{B}}(\varepsilon)^{1/2}\big(fe^{-\mathcal{B}(\varepsilon)s}f^* - \big(I + \Lambda b(\mathbf{D}) + \varepsilon\widetilde{\Lambda}\big)\mathcal{E}^0(\varepsilon,s)\big)\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leq \mathcal{C}_2' s^{-1}e^{-\varepsilon^2 C_* s},&\\ s\geq 1, \quad 0<\varepsilon\leq 1,&\end{aligned}$$

*where* $\mathcal{C}_2' = \mathcal{C}_2 + \mathfrak{C}_{\mathcal{B}}\alpha_1^{1/2}\mathcal{C}_l + \widetilde{\mathfrak{C}}_{\mathcal{B}}\mathcal{C}_l$.

# Chapter 3
# Homogenization of periodic differential operators

## §9. Approximation of the operator $f^\varepsilon \exp(-\mathcal{B}_\varepsilon s)(f^\varepsilon)^*$

**9.1. The operators $\widehat{\mathcal{B}}_\varepsilon$ and $\mathcal{B}_\varepsilon$.** For any $\Gamma$-periodic function $\phi(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^d$, we denote $\phi^\varepsilon(\mathbf{x}) := \phi(\varepsilon^{-1}\mathbf{x})$. Consider the operator $\widehat{\mathcal{A}}_\varepsilon = b(\mathbf{D})^* g^\varepsilon b(\mathbf{D})$ in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ generated by the closed quadratic form $\widehat{\mathfrak{a}}_\varepsilon[\mathbf{u},\mathbf{u}] = (g^\varepsilon b(\mathbf{D})\mathbf{u}, b(\mathbf{D})\mathbf{u})_{L_2(\mathbb{R}^d)}$, $\mathbf{u} \in H^1(\mathbb{R}^d;\mathbb{C}^n)$. The form $\widehat{\mathfrak{a}}_\varepsilon$ satisfies the following estimates similar to (4.5):

$$\alpha_0\|g^{-1}\|^{-1}_{L_\infty}\|\mathbf{D}\mathbf{u}\|^2_{L_2} \leq \widehat{\mathfrak{a}}_\varepsilon[\mathbf{u},\mathbf{u}] \leq \alpha_1\|g\|_{L_\infty}\|\mathbf{D}\mathbf{u}\|^2_{L_2}, \quad \mathbf{u} \in H^1(\mathbb{R}^d;\mathbb{C}^n). \tag{9.1}$$

Next, let $\widehat{\mathcal{Y}}\colon L_2(\mathbb{R}^d;\mathbb{C}^n) \to L_2(\mathbb{R}^d;\mathbb{C}^{dn})$ be defined by $\widehat{\mathcal{Y}}\mathbf{u} = \operatorname{col}\{D_1\mathbf{u},\dots,D_d\mathbf{u}\}$, where $\mathbf{u} \in H^1(\mathbb{R}^d;\mathbb{C}^n)$. Let $\widehat{\mathcal{Y}}_{2,\varepsilon}\colon L_2(\mathbb{R}^d;\mathbb{C}^n) \to L_2(\mathbb{R}^d;\mathbb{C}^{dn})$ be the operator acting as follows: $\widehat{\mathcal{Y}}_{2,\varepsilon}\mathbf{u} = \operatorname{col}\{(a_1^\varepsilon(\mathbf{x}))^*\mathbf{u},\dots,(a_d^\varepsilon(\mathbf{x}))^*\mathbf{u}\}$, $\mathbf{u} \in H^1(\mathbb{R}^d;\mathbb{C}^n)$.

Let $d\mu(\mathbf{x})$ be the matrix-valued measure in $\mathbb{R}^d$ defined in Subsection 4.4. We define a measure $d\mu^\varepsilon(\mathbf{x})$ as follows. For any Borel set $\Delta \subset \mathbb{R}^d$, we consider the set $\varepsilon^{-1}\Delta := \{\mathbf{y} = \varepsilon^{-1}\mathbf{x}\colon \mathbf{x} \in \Delta\}$ and put $\mu^\varepsilon(\Delta) := \varepsilon^d\mu(\varepsilon^{-1}\Delta)$. Consider the quadratic form $\widehat{q}_\varepsilon$ defined by $\widehat{q}_\varepsilon[\mathbf{u},\mathbf{u}] = \int_{\mathbb{R}^d}\langle d\mu^\varepsilon(\mathbf{x})\mathbf{u},\mathbf{u}\rangle$, $\mathbf{u} \in H^1(\mathbb{R}^d;\mathbb{C}^n)$.

Suppose that all the assumptions of Subsections 4.1–4.5 are satisfied. In $L_2(\mathbb{R}^d;\mathbb{C}^n)$, we consider the quadratic form

$$\widehat{\mathfrak{b}}_\varepsilon[\mathbf{u},\mathbf{u}] = \widehat{\mathfrak{a}}_\varepsilon[\mathbf{u},\mathbf{u}] + 2\operatorname{Re}(\widehat{\mathcal{Y}}\mathbf{u},\widehat{\mathcal{Y}}_{2,\varepsilon}\mathbf{u})_{L_2} + \widehat{q}_\varepsilon[\mathbf{u},\mathbf{u}] + \lambda\|\mathbf{u}\|^2_{L_2}, \quad \mathbf{u} \in H^1(\mathbb{R}^d).$$

Let $T_\varepsilon$ be the unitary *scaling transformation* in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ defined by $(T_\varepsilon \mathbf{u})(\mathbf{y}) = \varepsilon^{d/2}\mathbf{u}(\varepsilon\mathbf{y})$. For any $\mathbf{u}\in H^1(\mathbb{R}^d;\mathbb{C}^n)$, we have

$$\widehat{\mathfrak{a}}_\varepsilon[\mathbf{u},\mathbf{u}] = \varepsilon^{-2}\widehat{\mathfrak{a}}[T_\varepsilon\mathbf{u},T_\varepsilon\mathbf{u}],\quad \widehat{\mathfrak{b}}_\varepsilon[\mathbf{u},\mathbf{u}] = \varepsilon^{-2}\widehat{\mathfrak{b}}(\varepsilon)[T_\varepsilon\mathbf{u},T_\varepsilon\mathbf{u}], \tag{9.2}$$

where $\widehat{\mathfrak{a}}$ is the form defined in Subsection 4.2 with $f=\mathbf{1}_n$ and $\widehat{\mathfrak{b}}(\varepsilon)$ is the form (4.17) with $f=\mathbf{1}_n$. From (9.2) and estimates (4.20) and (4.21), it follows that

$$\begin{aligned} &\widehat{\mathfrak{b}}_\varepsilon[\mathbf{u},\mathbf{u}] \geq \frac{\kappa}{2}\widehat{\mathfrak{a}}_\varepsilon[\mathbf{u},\mathbf{u}] + \widehat{\beta}\|\mathbf{u}\|^2_{L_2},\quad \mathbf{u}\in H^1(\mathbb{R}^d;\mathbb{C}^n),\\ &\widehat{\mathfrak{b}}_\varepsilon[\mathbf{u},\mathbf{u}] \leq (2+c_1^2+c_2)\widehat{\mathfrak{a}}_\varepsilon[\mathbf{u},\mathbf{u}] + (\widehat{C}(1)+\widehat{c}_3+|\lambda|)\|\mathbf{u}\|^2_{L_2},\quad \mathbf{u}\in H^1(\mathbb{R}^d;\mathbb{C}^n). \end{aligned} \tag{9.3}$$

Thus, the form $\widehat{\mathfrak{b}}_\varepsilon$ is closed and positive definite. The selfadjoint operator in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ generated by the form $\widehat{\mathfrak{b}}_\varepsilon$ is denoted by $\widehat{\mathcal{B}}_\varepsilon$. Formally, we can write

$$\widehat{\mathcal{B}}_\varepsilon = b(\mathbf{D})^* g^\varepsilon b(\mathbf{D}) + \sum_{j=1}^d (a_j^\varepsilon D_j + D_j(a_j^\varepsilon)^*) + \mathcal{Q}^\varepsilon + \lambda I,$$

where $\mathcal{Q}^\varepsilon$ should be viewed as the generalized matrix-valued potential generated by the measure $d\mu^\varepsilon$.

Next, in the space $L_2(\mathbb{R}^d;\mathbb{C}^n)$, we consider the selfadjoint positive definite operator $\mathcal{B}_\varepsilon = (f^\varepsilon)^*\widehat{\mathcal{B}}_\varepsilon f^\varepsilon$ generated by the quadratic form

$$\mathfrak{b}_\varepsilon[\mathbf{u},\mathbf{u}] := \widehat{\mathfrak{b}}_\varepsilon[f^\varepsilon\mathbf{u}, f^\varepsilon\mathbf{u}],\quad \operatorname{Dom}\mathfrak{b}_\varepsilon = \{\mathbf{u}\in L_2(\mathbb{R}^d;\mathbb{C}^n)\,:\, f^\varepsilon\mathbf{u}\in H^1(\mathbb{R}^d;\mathbb{C}^n)\}.$$

**9.2. The effective operator for $\widehat{\mathcal{B}}_\varepsilon$.** Suppose that the operator $\widehat{\mathcal{A}}^0$ is defined by (8.1), and that $\widehat{\mathcal{Y}}^0$, $\bar{\mathcal{Q}}$, and $W$ are defined by (8.2), (6.13), and (6.11), respectively. The operator

$$\widehat{\mathcal{B}}^0 = \widehat{\mathcal{A}}^0 + \widehat{\mathcal{Y}}^0 + (\widehat{\mathcal{Y}}^0)^* + \bar{\mathcal{Q}} - W + \lambda I \tag{9.4}$$

is called the *effective operator* for $\widehat{\mathcal{B}}_\varepsilon$. In other words,

$$\widehat{\mathcal{B}}^0 = b(\mathbf{D})^* g^0 b(\mathbf{D}) - b(\mathbf{D})^* V - V^* b(\mathbf{D}) + \sum_{j=1}^d (\overline{a_j + a_j^*})D_j + \bar{\mathcal{Q}} - W + \lambda I.$$

**9.3. The principal term of approximation.** Denote

$$\mathcal{E}^0(s) := f_0 e^{-f_0\widehat{\mathcal{B}}^0 f_0 s} f_0. \tag{9.5}$$

Observe that

$$f^\varepsilon e^{-\mathcal{B}_\varepsilon s}(f^\varepsilon)^* = T_\varepsilon^* f e^{-\mathcal{B}(\varepsilon)\widetilde{s}} f^* T_\varepsilon,\quad \mathcal{E}^0(s) = T_\varepsilon^*\mathcal{E}^0(\varepsilon,\widetilde{s})T_\varepsilon, \tag{9.6}$$

where $\mathcal{B}(\varepsilon)$ is the operator (4.22) and $\widetilde{s} = \varepsilon^{-2}s$. So, by the scaling transformation, Theorem 8.1 implies the following result.

**Theorem 9.1.** *Under the assumptions of Subsections* 4.1–4.5*, let $\mathcal{B}_\varepsilon$ be the operator defined in Subsection* 9.1*, and let $\mathcal{E}^0(s)$ be the operator* (9.5)*. Then*

$$\left\| f^\varepsilon e^{-\mathcal{B}_\varepsilon s}(f^\varepsilon)^* - \mathcal{E}^0(s)\right\|_{L_2(\mathbb{R}^d;\mathbb{C}^n)\to L_2(\mathbb{R}^d;\mathbb{C}^n)} \leq \mathcal{C}_1\varepsilon(\varepsilon^2+s)^{-1/2}e^{-C_* s},\ 0<\varepsilon\leq 1,\ s\geq 0. \tag{9.7}$$

*The constants $C_*$ and $\mathcal{C}_1$ depend only on the initial data* (4.23) *and the parameters of the lattice $\Gamma$.*

**9.4. Approximation in the $(L_2 \to H^1)$-norm.** First, by Theorem 8.2, we obtain approximation with the corrector term taken into account.

Let $\Pi_\varepsilon$ denote the pseudodifferential operator in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ with the symbol $\chi_{\widetilde{\Omega}/\varepsilon}(\boldsymbol{\xi})$:

$$(\Pi_\varepsilon \mathbf{f})(\mathbf{x}) = (2\pi)^{-d/2} \int_{\widetilde{\Omega}/\varepsilon} e^{i\langle \mathbf{x},\boldsymbol{\xi}\rangle} (\mathcal{F}\mathbf{f})(\boldsymbol{\xi})\, d\boldsymbol{\xi}. \tag{9.8}$$

Using (9.6) and the identities $[\widetilde{\Lambda}^\varepsilon] = T_\varepsilon^*[\widetilde{\Lambda}]T_\varepsilon$, $\Lambda^\varepsilon b(\mathbf{D}) = \varepsilon^{-1}T_\varepsilon^*\Lambda b(\mathbf{D})T_\varepsilon$, $\Pi_\varepsilon = T_\varepsilon^*\Pi T_\varepsilon$, we get

$$\begin{aligned}&\widehat{\mathcal{B}}_\varepsilon^{1/2}\big(f^\varepsilon e^{-\mathcal{B}_\varepsilon s}(f^\varepsilon)^* - (I + \varepsilon\Lambda^\varepsilon b(\mathbf{D})\Pi_\varepsilon + \varepsilon\widetilde{\Lambda}^\varepsilon\Pi_\varepsilon)\mathcal{E}^0(s)\big)\\ &\quad = \varepsilon^{-1}T_\varepsilon^*\widehat{\mathcal{B}}(\varepsilon)^{1/2}\big(f e^{-\mathcal{B}(\varepsilon)\widetilde{s}} f^* - (I + \Lambda b(\mathbf{D})\Pi + \varepsilon\widetilde{\Lambda}\Pi)\mathcal{E}^0(\varepsilon,\widetilde{s})\big)T_\varepsilon,\end{aligned}$$

where $\widetilde{s} = \varepsilon^{-2}s$. Hence, replacing $s$ by $\widetilde{s}$ in (8.3) and recalling that $T_\varepsilon$ is a unitary operator, we obtain the following estimate:

$$\begin{aligned}\big\|\widehat{\mathcal{B}}_\varepsilon^{1/2}\big(f^\varepsilon e^{-\mathcal{B}_\varepsilon s}(f^\varepsilon)^* - (I+\varepsilon(\Lambda^\varepsilon b(\mathbf{D})+\widetilde{\Lambda}^\varepsilon)\Pi_\varepsilon)\mathcal{E}^0(s)\big)\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} &\le \varepsilon^{-1}\Phi_2(\widetilde{s},\varepsilon),\\ 0<\varepsilon\le 1,\quad s>0.&\end{aligned} \tag{9.9}$$

Now, by (9.9), we obtain approximation for the operator $f^\varepsilon e^{-\mathcal{B}_\varepsilon s}(f^\varepsilon)^*$ in the norm of the space of operators acting from $L_2(\mathbb{R}^d;\mathbb{C}^n)$ to $H^1(\mathbb{R}^d;\mathbb{C}^n)$.

**Theorem 9.2.** *Under the assumptions of Theorem* 9.1, *suppose that the matrix-valued function* $\Lambda(\mathbf{x})$ *is the periodic solution of problem* (6.7), *and the matrix-valued function* $\widetilde{\Lambda}(\mathbf{x})$ *is the periodic solution of problem* (6.8). *We put* $\Lambda^\varepsilon(\mathbf{x}) = \Lambda(\varepsilon^{-1}\mathbf{x})$ *and* $\widetilde{\Lambda}^\varepsilon(\mathbf{x}) = \widetilde{\Lambda}(\varepsilon^{-1}\mathbf{x})$. *Let* $\Pi_\varepsilon$ *be the operator* (9.8). *Then*

$$\begin{aligned}\big\|f^\varepsilon e^{-\mathcal{B}_\varepsilon s}(f^\varepsilon)^* - (I + \varepsilon(\Lambda^\varepsilon b(\mathbf{D}) + \widetilde{\Lambda}^\varepsilon)\Pi_\varepsilon)\mathcal{E}^0(s)\big\|_{L_2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} &\le \Psi(s,\varepsilon),\\ 0<\varepsilon\le 1,\ s>0.&\end{aligned} \tag{9.10}$$

*Here* $\Psi(s,\varepsilon)$ *is defined by*

$$\Psi(s,\varepsilon) = \begin{cases}\mathcal{C}_4\varepsilon s^{-1}e^{-C_* s} & \text{if } s>0,\ 0<\varepsilon\le s^{1/2},\\ \mathcal{C}_5 s^{-1/2}e^{-C_* s} & \text{if } s>0,\ \varepsilon > s^{1/2},\end{cases} \tag{9.11}$$

*where* $\mathcal{C}_4 = \mathcal{C}_2\mathfrak{c}$, $\mathcal{C}_5 = \mathcal{C}_3\mathfrak{c}$, *and* $\mathfrak{c} = \max\{\sqrt{2}\kappa^{-1/2}\alpha_0^{-1/2}\|g^{-1}\|_{L_\infty}^{1/2};\widehat{\beta}^{\,-1/2}\}$. *The constants* $\mathcal{C}_4$, $\mathcal{C}_5$, *and* $C_*$ *depend only on the problem data* (4.23) *and the parameters of the lattice* $\Gamma$.

*Proof.* Denote

$$\Upsilon(\varepsilon,s) := f^\varepsilon e^{-\mathcal{B}_\varepsilon s}(f^\varepsilon)^* - (I + \varepsilon(\Lambda^\varepsilon b(\mathbf{D}) + \widetilde{\Lambda}^\varepsilon)\Pi_\varepsilon)\mathcal{E}^0(s).$$

By (9.3) and (9.9), we have

$$\begin{aligned}\frac{\kappa}{2}\|(g^\varepsilon)^{1/2}b(\mathbf{D})\Upsilon(\varepsilon,s)\boldsymbol{\eta}\|^2_{L_2(\mathbb{R}^d)} + \widehat{\beta}\|\Upsilon(\varepsilon,s)\boldsymbol{\eta}\|^2_{L_2(\mathbb{R}^d)} &\le \|\widehat{\mathcal{B}}_\varepsilon^{1/2}\Upsilon(\varepsilon,s)\boldsymbol{\eta}\|^2_{L_2(\mathbb{R}^d)}\\ &\le \varepsilon^{-2}\Phi_2(\widetilde{s},\varepsilon)^2\|\boldsymbol{\eta}\|^2_{L_2(\mathbb{R}^d)},\quad \boldsymbol{\eta}\in L_2(\mathbb{R}^d;\mathbb{C}^n),\quad s>0.\end{aligned}$$

Combining this with the lower estimate (9.1), we obtain

$$\begin{aligned}\frac{\kappa}{2}\alpha_0\|g^{-1}\|^{-1}_{L_\infty}\|\mathbf{D}\Upsilon(\varepsilon,s)\boldsymbol{\eta}\|^2_{L_2(\mathbb{R}^d)} + \widehat{\beta}\|\Upsilon(\varepsilon,s)\boldsymbol{\eta}\|^2_{L_2(\mathbb{R}^d)} &\le \varepsilon^{-2}\Phi_2(\widetilde{s},\varepsilon)^2\|\boldsymbol{\eta}\|^2_{L_2(\mathbb{R}^d)},\\ \boldsymbol{\eta}\in L_2(\mathbb{R}^d;\mathbb{C}^n),\quad s>0.&\end{aligned} \tag{9.12}$$

Obviously,

$$\begin{aligned}\|\Upsilon(\varepsilon,s)\boldsymbol{\eta}\|^2_{H^1(\mathbb{R}^d)} &\le \max\{2\kappa^{-1}\alpha_0^{-1}\|g^{-1}\|_{L_\infty};\widehat{\beta}^{\,-1}\}\\ &\quad\times\Big(\frac{\kappa}{2}\alpha_0\|g^{-1}\|^{-1}_{L_\infty}\|\mathbf{D}\Upsilon(\varepsilon,s)\boldsymbol{\eta}\|^2_{L_2(\mathbb{R}^d)} + \widehat{\beta}\|\Upsilon(\varepsilon,s)\boldsymbol{\eta}\|^2_{L_2(\mathbb{R}^d)}\Big).\end{aligned} \tag{9.13}$$

Estimate (9.10) is a consequence of (9.12), (9.13), and (8.4). □

**9.5. Approximation for $\varepsilon \le s^{1/2}$.** Similarly, by using Theorem 8.7, one can prove the following statement.

**Theorem 9.3.** *Under the assumptions of Theorem* 9.2, *we have*

$$\left\| f^\varepsilon e^{-\mathcal{B}_\varepsilon s}(f^\varepsilon)^* - (I + \varepsilon(\Lambda^\varepsilon b(\mathbf{D}) + \widetilde{\Lambda}^\varepsilon))\mathcal{E}^0(s) \right\|_{L_2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \le \varepsilon \mathcal{C}_4' s^{-1} e^{-C_* s}, \quad 0 < \varepsilon \le s^{1/2}, \quad 0 < \varepsilon \le 1. \tag{9.14}$$

*The constants $\mathcal{C}_4' := \mathcal{C}_2' \mathfrak{c}$ and $C_*$ depend only on the problem data* (4.23) *and the parameters of the lattice* $\Gamma$.

## §10. Application to homogenization of the parabolic Cauchy problem

**10.1. The Cauchy problem.** Let $\rho(\mathbf{x})$ be a measurable $\Gamma$-periodic $(n \times n)$-matrix-valued function in $\mathbb{R}^d$; we assume that it is bounded and uniformly positive definite. Let $0 < T \le \infty$. Consider the following Cauchy problem:

$$\rho(\varepsilon^{-1}\mathbf{x}) \frac{\partial \mathbf{u}_\varepsilon(\mathbf{x}, s)}{\partial s} = -\widehat{\mathcal{B}}_\varepsilon \mathbf{u}_\varepsilon(\mathbf{x}, s) + \mathbf{F}(\mathbf{x}, s), \quad \rho(\varepsilon^{-1}\mathbf{x})\mathbf{u}_\varepsilon(\mathbf{x}, 0) = \boldsymbol{\phi}(\mathbf{x}), \tag{10.1}$$

$\mathbf{x} \in \mathbb{R}^d$, $s \in (0, T)$, where $\boldsymbol{\phi} \in L_2(\mathbb{R}^d; \mathbb{C}^n)$ and $\mathbf{F} \in \mathcal{H}_p(T) := L_p((0,T); L_2(\mathbb{R}^d; \mathbb{C}^n))$ for some $1 < p \le \infty$. We factorize the matrix $\rho(\mathbf{x})$ as $\rho(\mathbf{x})^{-1} = f(\mathbf{x}) f(\mathbf{x})^*$. Then $\mathbf{v}_\varepsilon := (f^\varepsilon)^{-1}\mathbf{u}_\varepsilon$ is the solution of the problem

$$\begin{aligned} \frac{\partial \mathbf{v}_\varepsilon(\mathbf{x}, s)}{\partial s} &= -(f^\varepsilon(\mathbf{x}))^* \widehat{\mathcal{B}}_\varepsilon f^\varepsilon(\mathbf{x}) \mathbf{v}_\varepsilon(\mathbf{x}, s) + (f^\varepsilon(\mathbf{x}))^* \mathbf{F}(\mathbf{x}, s), \\ \mathbf{v}_\varepsilon(\mathbf{x}, 0) &= (f^\varepsilon(\mathbf{x}))^* \boldsymbol{\phi}(\mathbf{x}). \end{aligned}$$

Since $\mathcal{B}_\varepsilon = (f^\varepsilon(\mathbf{x}))^* \widehat{\mathcal{B}}_\varepsilon f^\varepsilon(\mathbf{x})$, we have

$$\mathbf{v}_\varepsilon = \exp(-\mathcal{B}_\varepsilon s)(f^\varepsilon)^* \boldsymbol{\phi} + \int_0^s \exp(-\mathcal{B}_\varepsilon (s - \widetilde{s}))(f^\varepsilon)^* \mathbf{F}(\,\cdot\,, \widetilde{s})\, d\widetilde{s},$$

$$\mathbf{u}_\varepsilon = f^\varepsilon \exp(-\mathcal{B}_\varepsilon s)(f^\varepsilon)^* \boldsymbol{\phi} + \int_0^s f^\varepsilon \exp(-\mathcal{B}_\varepsilon (s - \widetilde{s}))(f^\varepsilon)^* \mathbf{F}(\,\cdot\,, \widetilde{s})\, d\widetilde{s}. \tag{10.2}$$

Let $\mathbf{u}_0(\mathbf{x}, s)$ be the solution of the "homogenized" problem

$$\overline{\rho} \frac{\partial \mathbf{u}_0(\mathbf{x}, s)}{\partial s} = -\widehat{\mathcal{B}}^0 \mathbf{u}_0(\mathbf{x}, s) + \mathbf{F}(\mathbf{x}, s), \quad \overline{\rho}\mathbf{u}_0(\mathbf{x}, 0) = \boldsymbol{\phi}(\mathbf{x}), \tag{10.3}$$

where $\overline{\rho} = |\Omega|^{-1} \int_\Omega \rho(\mathbf{x})\, d\mathbf{x}$. Note that $\overline{\rho} = f_0^{-2}$. As in (10.2), we obtain

$$\mathbf{u}_0 = f_0 \exp(-f_0 \widehat{\mathcal{B}}^0 f_0 s) f_0 \boldsymbol{\phi} + \int_0^s f_0 \exp(-f_0 \widehat{\mathcal{B}}^0 f_0 (s - \widetilde{s})) f_0 \mathbf{F}(\,\cdot\,, \widetilde{s})\, d\widetilde{s}. \tag{10.4}$$

**10.2. Convergence of the solutions in $L_2(\mathbb{R}^d; \mathbb{C}^n)$.** By (9.7),

$$\begin{aligned} \|\mathbf{u}_\varepsilon(\,\cdot\,, s) - \mathbf{u}_0(\,\cdot\,, s)\|_{L_2(\mathbb{R}^d)} &\le \mathcal{C}_1 \varepsilon (\varepsilon^2 + s)^{-1/2} e^{-C_* s} \|\boldsymbol{\phi}\|_{L_2(\mathbb{R}^d)} \\ &\quad + \mathcal{C}_1 \varepsilon \int_0^s (\varepsilon^2 + s - \widetilde{s})^{-1/2} e^{-C_*(s - \widetilde{s})} \|\mathbf{F}(\,\cdot\,, \widetilde{s})\|_{L_2(\mathbb{R}^d)}\, d\widetilde{s}. \end{aligned} \tag{10.5}$$

For $1 < p \le \infty$, we estimate the integral on the right-hand side of (10.5) by using the Hölder inequality ($p^{-1} + (p')^{-1} = 1$):

$$\begin{aligned} &\int_0^s (\varepsilon^2 + s - \widetilde{s})^{-1/2} e^{-C_*(s - \widetilde{s})} \|\mathbf{F}(\,\cdot\,, \widetilde{s})\|_{L_2(\mathbb{R}^d)}\, d\widetilde{s} \\ &\qquad \le \|\mathbf{F}\|_{\mathcal{H}_p(s)} \left( \int_0^s (\varepsilon^2 + s - \widetilde{s})^{-p'/2} e^{-C_* p'(s - \widetilde{s})}\, d\widetilde{s} \right)^{1/p'}. \end{aligned} \tag{10.6}$$

In the case where $2 < p \le \infty$ $(1 \le p' < 2)$, the right-hand side of (10.6) can be estimated by $\|\mathbf{F}\|_{\mathcal{H}_p(s)}(C_* p')^{1/2-1/p'}\big(\Gamma(1-p'/2)\big)^{1/p'}$. For $1 < p < 2$, we estimate the integral with the help of the inequality $e^{-C_* p'(s-\widetilde{s})} \le 1$:

$$\int_0^s (\varepsilon^2 + s - \widetilde{s})^{-p'/2} e^{-C_* p'(s-\widetilde{s})}\, d\widetilde{s} \le \varepsilon^{2-p'}(p'/2-1)^{-1}. \tag{10.7}$$

For $p = 2$, we substitute $\zeta = s - \widetilde{s}$ and split the interval of integration:

$$\begin{aligned}\int_0^s (\varepsilon^2 + s - \widetilde{s})^{-1} e^{-2C_*(s-\widetilde{s})}\, d\widetilde{s} &\le \int_0^1 (\varepsilon^2+\zeta)^{-1}\, d\zeta + \int_1^s e^{-2C_*\zeta}\, d\zeta \\ &\le \ln 2 + 2|\ln\varepsilon| + (2C_*)^{-1}, \quad 0 < \varepsilon \le 1.\end{aligned} \tag{10.8}$$

Combining estimates (10.5)–(10.8), we arrive at the following result.

**Theorem 10.1.** *Suppose $\mathbf{F} \in \mathcal{H}_p(T)$ for some $1 < p \le \infty$. Then for any $s \in (0,T)$ the solutions $\mathbf{u}_\varepsilon(\,\cdot\,, s)$ tend to $\mathbf{u}_0(\,\cdot\,, s)$ in the $L_2(\mathbb{R}^d;\mathbb{C}^n)$-norm. For $0 < \varepsilon \le 1$, we have*

$$\|\mathbf{u}_\varepsilon(\,\cdot\,,s) - \mathbf{u}_0(\,\cdot\,,s)\|_{L_2(\mathbb{R}^d)} \le \mathcal{C}_1 \varepsilon(\varepsilon^2+s)^{-1/2} e^{-C_* s}\|\boldsymbol{\phi}\|_{L_2(\mathbb{R}^d)} + \theta_1(\varepsilon,p)\|\mathbf{F}\|_{\mathcal{H}_p(s)}.$$

*Here $\theta_1(\varepsilon,p)$ is given by*

$$\theta_1(\varepsilon,p) = \begin{cases} \varepsilon^{2-2/p}\mathcal{C}_1(p'/2-1)^{-1/p'} & \text{if } 1 < p < 2, \\ \varepsilon\mathcal{C}_1\left(\ln 2 + 2|\ln\varepsilon| + (2C_*)^{-1}\right)^{1/2} & \text{if } p = 2, \\ \varepsilon\mathcal{C}_1 (C_* p')^{-1/2+1/p}\left(\Gamma(1-p'/2)\right)^{1/p'} & \text{if } 2 < p \le \infty, \end{cases}$$

*where $p^{-1} + (p')^{-1} = 1$.*

**10.3. Approximation in $H^1(\mathbb{R}^d;\mathbb{C}^n)$ for solutions of the homogeneous Cauchy problem.** Now we consider the homogeneous Cauchy problem

$$\rho(\varepsilon^{-1}\mathbf{x})\frac{\partial \mathbf{u}_\varepsilon(\mathbf{x},s)}{\partial s} = -\widehat{\mathcal{B}}_\varepsilon \mathbf{u}_\varepsilon(\mathbf{x},s), \quad \rho(\varepsilon^{-1}\mathbf{x})\mathbf{u}_\varepsilon(\mathbf{x},0) = \boldsymbol{\phi}(\mathbf{x}), \tag{10.9}$$

where $\boldsymbol{\phi} \in L_2(\mathbb{R}^d;\mathbb{C}^n)$. The corresponding "homogenized" problem has the form

$$\overline{\rho}\frac{\partial \mathbf{u}_0(\mathbf{x},s)}{\partial s} = -\widehat{\mathcal{B}}^0 \mathbf{u}_0(\mathbf{x},s), \quad \overline{\rho}\mathbf{u}_0(\mathbf{x},0) = \boldsymbol{\phi}(\mathbf{x}). \tag{10.10}$$

The following result is a direct consequence of (9.14).

**Theorem 10.2.** *Under the assumptions of Subsections* 4.1–4.4*, let $\mathbf{u}_\varepsilon$ be the solution of problem* (10.9)*, and let $\mathbf{u}_0$ be the solution of problem* (10.10)*. Then*

$$\begin{aligned}\big\|\mathbf{u}_\varepsilon(\,\cdot\,,s) - \mathbf{u}_0(\,\cdot\,,s) - \varepsilon(\Lambda^\varepsilon b(\mathbf{D}) + \widetilde{\Lambda}^\varepsilon)\mathbf{u}_0(\,\cdot\,,s)\big\|_{H^1(\mathbb{R}^d)} &\le \mathcal{C}_4' \varepsilon s^{-1} e^{-C_* s}\|\boldsymbol{\phi}\|_{L_2(\mathbb{R}^d)}, \\ 0 < \varepsilon \le 1, \quad 0 < \varepsilon &\le s^{1/2}.\end{aligned}$$

*The constants $\mathcal{C}_4'$ and $C_*$ depend only on the problem data* (4.23) *and the parameters of the lattice $\Gamma$.*

**10.4. Approximation in $H^1(\mathbb{R}^d;\mathbb{C}^n)$ for solutions of the nonhomogeneous Cauchy problem.** We return to problem (10.1).

**Theorem 10.3.** *Let $\mathbf{u}_\varepsilon$ be the solution of problem* (10.1)*, where $\boldsymbol{\phi} \in L_2(\mathbb{R}^d;\mathbb{C}^n)$ and $\mathbf{F} \in \mathcal{H}_p(T)$, $2 < p \le \infty$, and let $\mathbf{u}_0$ be the solution of problem* (10.3)*. Let $\Pi_\varepsilon$ be the operator* (9.8)*. Let $0 < \varepsilon \le 1$. Then for $0 < s \le T$ and $0 < \varepsilon \le s^{1/2}$ we have*

$$\begin{aligned}&\big\|\mathbf{u}_\varepsilon(\,\cdot\,,s) - \mathbf{u}_0(\,\cdot\,,s) - \varepsilon(\Lambda^\varepsilon b(\mathbf{D}) + \widetilde{\Lambda}^\varepsilon)\Pi_\varepsilon\mathbf{u}_0(\,\cdot\,,s)\big\|_{H^1(\mathbb{R}^d)} \\ &\qquad\le \mathcal{C}_4 \varepsilon s^{-1} e^{-C_* s}\|\boldsymbol{\phi}\|_{L_2(\mathbb{R}^d)} + \theta_2(\varepsilon,p)\|\mathbf{F}\|_{\mathcal{H}_p(s)},\end{aligned} \tag{10.11}$$

*where*

$$\theta_2(\varepsilon,p)=\begin{cases}\varepsilon^{1-2/p}\big(\mathcal{C}_4(p'-1)^{-1/p'}+\mathcal{C}_5(1-p'/2)^{-1/p'}\big) & \textit{if}\ \ 2<p<\infty,\\ 2\mathcal{C}_4\varepsilon|\ln\varepsilon|+\mathcal{C}_4\varepsilon C_*^{-1}e^{-C_*}+2\mathcal{C}_5\varepsilon & \textit{if}\ \ p=\infty.\end{cases}$$

*Here* $p^{-1}+(p')^{-1}=1$.

*Proof.* Let $0<s\le T$, and let $0<\varepsilon\le\min\{s^{1/2},1\}$. From (9.10) and (10.2), (10.4) it follows that

$$\begin{aligned}&\big\|\mathbf{u}_\varepsilon(\,\cdot\,,s)-\mathbf{u}_0(\,\cdot\,,s)-\varepsilon\big(\Lambda^\varepsilon b(\mathbf{D})+\widetilde{\Lambda}^\varepsilon\big)\Pi_\varepsilon\mathbf{u}_0(\,\cdot\,,s)\|_{H^1(\mathbb{R}^d)}\\ &\le\mathcal{C}_4\varepsilon s^{-1}e^{-C_*s}\|\boldsymbol{\phi}\|_{L_2(\mathbb{R}^d)}+\int_0^s\Psi(s-\widetilde{s},\varepsilon)\|\mathbf{F}(\,\cdot\,,\widetilde{s})\|_{L_2(\mathbb{R}^d)}\,d\widetilde{s},\end{aligned}\tag{10.12}$$

where $\Psi(s,\varepsilon)$ is defined by (9.11). Denote

$$\mathcal{I}:=\int_0^s\Psi(s-\widetilde{s},\varepsilon)\|\mathbf{F}(\,\cdot\,,\widetilde{s})\|_{L_2(\mathbb{R}^d)}\,d\widetilde{s}.\tag{10.13}$$

The integral $\mathcal{I}$ can be rewritten as

$$\begin{aligned}\mathcal{I}=\mathcal{C}_4\varepsilon&\int_0^{s-\varepsilon^2}(s-\widetilde{s})^{-1}e^{-C_*(s-\widetilde{s})}\|\mathbf{F}(\,\cdot\,,\widetilde{s})\|_{L_2(\mathbb{R}^d)}\,d\widetilde{s}\\ &+\mathcal{C}_5\int_{s-\varepsilon^2}^s(s-\widetilde{s})^{-1/2}e^{-C_*(s-\widetilde{s})}\|\mathbf{F}(\,\cdot\,,\widetilde{s})\|_{L_2(\mathbb{R}^d)}\,d\widetilde{s}.\end{aligned}\tag{10.14}$$

For $2<p<\infty$, the estimate $e^{-C_*(s-\widetilde{s})}\le 1$ and the Hölder inequality ($p^{-1}+(p')^{-1}=1$) show that

$$\mathcal{I}\le\|\mathbf{F}\|_{\mathcal{H}_p(s)}\varepsilon^{1-2/p}\big(\mathcal{C}_4(p'-1)^{-1/p'}+\mathcal{C}_5(1-p'/2)^{-1/p'}\big).\tag{10.15}$$

For $p=\infty$, identity (10.14) yields the estimate

$$\mathcal{I}\le\|\mathbf{F}\|_{\mathcal{H}_\infty(s)}\bigg(\mathcal{C}_4\varepsilon\int_0^{s-\varepsilon^2}(s-\widetilde{s})^{-1}e^{-C_*(s-\widetilde{s})}\,d\widetilde{s}+\mathcal{C}_5\int_{s-\varepsilon^2}^s(s-\widetilde{s})^{-1/2}\,d\widetilde{s}\bigg).\tag{10.16}$$

Note that

$$\int_0^{s-\varepsilon^2}(s-\widetilde{s})^{-1}e^{-C_*(s-\widetilde{s})}\,d\widetilde{s}\le 2|\ln\varepsilon|+C_*^{-1}e^{-C_*}.\tag{10.17}$$

Using (10.16) and (10.17), we obtain

$$\mathcal{I}\le\varepsilon\|\mathbf{F}\|_{\mathcal{H}_\infty(s)}\left(2\mathcal{C}_4|\ln\varepsilon|+\mathcal{C}_4C_*^{-1}e^{-C_*}+2\mathcal{C}_5\right).\tag{10.18}$$

Combining (10.12), (10.13), (10.15), and (10.18), we arrive at (10.11). □

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