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PTOLEMY SPACES WITH STRONG INVERSIONS

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Abstract. It is proved that a compact Ptolemy space with many strong inversions that contains a Ptolemy circle is Möbius equivalent to an extended Euclidean space.

§1. Introduction

This work is motivated by the papers [BS1] and [BS2] of S. Buyalo and V. Schroeder, which give a Möbius characterization of the boundary at infinity of the rank one symmetric spaces of noncompact type. That characterization employs the notion of a *space inversion* with respect to distinct points ω, $\omega' \in X$ and a metric sphere $S \subset X$ between ω, ω'. By definition, such an inversion is a Möbius automorphism $\varphi = \varphi_{\omega,\omega',S} \colon X \to X$ with the following properties:

(1) φ is an involution without fixed points, $\varphi^2 = \mathrm{id}$;
(2) $\varphi(\omega) = \omega'$ (and thus $\varphi(\omega') = \omega$);
(3) φ preserves S, $\varphi(S) = S$;
(4) $\varphi(\sigma) = \sigma$ for any Ptolemy circle $\sigma \subset X$ through ω, ω'.

Recall that, however, a classical inversion of the Euclidean space $\mathbb{R}^n$ with respect to a sphere $S \subset \mathbb{R}^n$ fixes S pointwise. In this paper we impose a stronger condition on an s-inversion, assuming that φ preserves S pointwise, $\varphi(x) = x$ for every $x \in S$; such an inversion will be called a *strong s-inversion*. We study Ptolemy spaces with the following two properties.

(E) Existence: there is at least one Ptolemy circle in X.
(sI) Strong inversions: for any distinct $\omega, \omega' \in X$ and any metric sphere $S \subset X$ between ω, ω' there is a strong space inversion $\varphi_{\omega,\omega',S} \colon X \to X$ with respect to ω, ω' and S.

Our main goal is the proof of the following theorem.

Theorem 1. *Let X be a compact Ptolemy space with properties* (E) *and* (sI). *Then X is Möbius equivalent to the extended Euclidean space $\widehat{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$ for some $n \geq 1$.*

Another Möbius characterization of $\widehat{\mathbb{R}}^n$ was obtained in [FS]: *a compact Ptolemy space X is Möbius equivalent to $\widehat{\mathbb{R}}^n$ if and only if any three points in X lie on a Ptolemy circle.*

Despite the differences in the definition of s-inversions and strong s-inversions, some properties of the spaces under study pertain to both cases. Thus, the definitions of the homotheties and shifts, as well as Lemmas 4, 5, 6 were originally presented in [BS1]. Significant differences between the two classes arise when we consider the symmetry with respect to a horosphere. In general, if we only assume the existence of s-inversions, there is no reason for such a symmetry to exist.

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§2. Basic definitions

2.1. Möbius structures. In this section we follow the definitions in [BS1]. Namely, fix a set X and consider the *extended* metrics on X for which the existence of an *infinitely remote* point $\omega \in X$ is allowed, that is, $d(x, \omega) = \infty$ for all $x \in X$, $x \neq \omega$. We always assume that such a point is unique if it exists, and that $d(\omega, \omega) = 0$.

A quadruple $Q = (x, y, z, u)$ of points in a set X is said to be *admissible* if each entry occurs at most two times in Q. Two metrics d, d' on X are *Möbius equivalent* if for any admissible quadruple $Q = (x, y, z, u) \subset X$ the respective *cross-ratio triples* coincide, $\mathrm{crt}_d(Q) = \mathrm{crt}_{d'}(Q)$, where

$$\mathrm{crt}_d(Q) = (d(x,y)d(z,u) : d(x,z)d(y,u) : d(x,u)d(y,z)) \in \mathbb{R}P^2.$$

If ∞ occurs once in Q, say $u = \infty$, then $\mathrm{crt}_d(x, y, z, \infty) = (d(x, y) : d(x, z) : d(y, z))$. If ∞ occurs twice, say $z = u = \infty$, then $\mathrm{crt}_d(x, y, \infty, \infty) = (0 : 1 : 1)$.

A *Möbius structure* on a set X is a class $\mathcal{M} = \mathcal{M}(X)$ of metrics on X that are pairwise Möbius equivalent.

The topology on (X, d) is that with the base consisting of all open distance balls $B_r(x)$ around points in $x \in X_\omega$ and the complements $X \setminus D$ of all closed distance balls $D = \bar{B}_r(x)$. Möbius equivalent metrics give rise to one and the same topology on X. When a Möbius structure $\mathcal{M}$ on X is fixed, we say that $(X, \mathcal{M})$ or simply X is a *Möbius space.*

A map $f\colon X \to X'$ between two Möbius spaces is said to be *Möbius* if f is injective and for all admissible quadruples $Q \subset X$ we have

$$\mathrm{crt}(f(Q)) = \mathrm{crt}(Q),$$

where the cross-ratio triples are taken with respect to some (and hence an arbitrary) metric of the Möbius structures of X, X'. Möbius maps are continuous. If a Möbius map $f\colon X \to X'$ is bijective, then f^{-1} is Möbius, f is a homeomorphism, and the Möbius spaces X, X' are said to be *Möbius equivalent.*

We note that if two Möbius equivalent metrics have the same infinitely remote point, then they are homothetic, see, e.g., [BS1, FS].

A classical example of a Möbius space is the extended $\widehat{\mathbb{R}}^n = \mathbb{R}^n \cup \infty = S^n$, $n \geq 1$, where the Möbius structure is generated by some extended Euclidean metric on $\widehat{\mathbb{R}}^n$, and $\mathbb{R}^n \cup \infty$ is identified with the unit sphere $S^n \subset \mathbb{R}^{n+1}$ via the stereographic projection.

2.2. Ptolemy spaces. A Möbius space X is called a *Ptolemy space* if it satisfies the Ptolemy property, that is, for all admissible quadruples $Q \subset X$ the entries of the respective cross-ratio triple $\mathrm{crt}(Q) \in \mathbb{R}P^2$ satisfy the triangle inequality.

The Ptolemy property is equivalent to the fact that the Möbius structure $\mathcal{M}$ of X is invariant under the metric inversions, or in other words, $\mathcal{M}$ is Ptolemy if and only if for all $z \in X$ there exists a metric $d_z \in \mathcal{M}$ with infinitely remote point z.

Recall that the metric inversion (or m-inversion for brevity) of a metric $d \in \mathcal{M}(X)$ with respect to $z \in X \setminus \omega$ (ω is infinitely remote for d) of radius $r > 0$ is the function $d_z(x, y) = \frac{r^2 d(x,y)}{d(z,x)d(z,y)}$ for all x, $y \in X$ distinct from z, $d_z(x, z) = \infty$ for all $x \in X \setminus \{z\}$ and $d_z(z, z) = 0$.

The classical example of a Ptolemy space is $\widehat{\mathbb{R}}^n$ with a standard Möbius structure.

An interesting basic fact about Ptolemy spaces is the following Schoenberg theorem.

Theorem 2 ([Sch]). *If a real normed vector space is a Ptolemy space, then it is an inner product space.*

A Ptolemy circle in a Ptolemy space X is a subset $\sigma \subset X$ homeomorphic to S^1 and such that for every quadruple $(x, y, z, u) \in \sigma$ of distinct points we have

$$d(x,z)d(y,u) = d(x,y)d(z,u) + d(x,u)d(y,z)$$

for some (and hence, an arbitrary) metric d of the Möbius structure, where it is assumed that the pair (x, z) separates the pair (y, u), i.e., y and u are in different components of $\sigma \setminus \{x, z\}$.

Given $\omega \in X$, we use the notation $X_\omega = X \setminus \omega$ and always assume that a metric of the Möbius structure on X_ω is fixed. Note that every Ptolemy circle $\sigma \subset X$ that passes through ω is isometric to a geodesic line in X_ω. Such a line $\ell = \sigma_\omega$ is called a *Ptolemy line*.

2.3. Space inversions. Given distinct ω, $\omega' \in X$, we say that a subset $S \subset X$ is a *metric sphere between* ω, ω' if

$$S = \{x \in X \,:\, d(x,\omega) = r\} = S^d_r(\omega)$$

for some metric $d \in \mathcal{M}$ with infinitely remote point ω' and some $r > 0$. Any two such metrics d, $d' \in \mathcal{M}$ are proportional to each other, $d' = \lambda d$ for some $\lambda > 0$. Then $S^d_r(\omega) = S^{d'}_{\lambda r}(\omega)$. Moreover, this notion is symmetric relative to ω, ω', because any metric $d' \in \mathcal{M}$ with infinitely remote point ω is proportional to the m-inversion of d with respect to ω, and we may assume that d' is an m-inversion itself. Then $S = \{x \in X \,:\, d'(x, \omega') = 1/r\}$.

We define a *strong space inversion*, or s-inversion for brevity, with respect to distinct ω, $\omega' \in X$ and a metric sphere $S \subset X$ between ω, ω' as a Möbius automorphism $\varphi = \varphi_{\omega,\omega',S} \colon X \to X$ such that

(1) φ is an involution, $\varphi^2 = \mathrm{id}$;
(2) $\varphi(\omega) = \omega'$ (and thus, $\varphi(\omega') = \omega$);
(3) φ preserves S pointwise, $\varphi(x) = x$ for every $x \in S$;
(4) $\varphi(\sigma) = \sigma$ for any Ptolemy circle $\sigma \subset X$ through ω, ω'.

Let $\omega \in X$. Fix $o \in X_\omega$ and consider a metric sphere $S = S_r(o)$ between o and ω. Let φ be an s-inversion with respect to o, ω and S. Now we prove two technical lemmas.

Lemma 1. *Let $x \in X_\omega$. Then $|ox| \cdot |o\varphi(x)| = r^2$.*

Proof. Let $y \in S$. Then

$$\begin{aligned}\mathrm{crt}(x, y, o, \omega) = (|xy| : |xo| : |yo|) &= \mathrm{crt}(\varphi(x), \varphi(y), \varphi(o), \varphi(\omega)) \\ &= \mathrm{crt}(\varphi(x), y, \omega, o) = (|\varphi(x)y| : |yo| : |\varphi(x)o|).\end{aligned}$$

It follows that $|\varphi(x)o|/|yo| = |yo|/|xo|$ and $|ox| \cdot |o\varphi(x)| = r^2$. □

Lemma 2. *Let $x, y \in X_\omega$. Then $|\varphi(x)\varphi(y)| = r^2 \cdot \frac{|xy|}{|ox|\cdot|oy|}$.*

Proof. Note that

$$\mathrm{crt}(x, y, o, \omega) = (|xy| : |xo| : |yo|) = \mathrm{crt}(\varphi(x), \varphi(y), \omega, o) = (|\varphi(x)\varphi(y)| : |\varphi(y)o| : |\varphi(x)o|).$$

Consequently, $|\varphi(x)\varphi(y)|/|\varphi(x)o| = |xy|/|yo|$. By Lemma 1, $|\varphi(x)o| = r^2/|xo|$. Then $|\varphi(x)\varphi(y)| = |\varphi(x)o| \cdot \frac{|xy|}{|yo|} = r^2 \frac{|xy|}{|ox|\cdot|oy|}$. □

We say that a Möbius space X possesses property (E) if there is a Ptolemy circle in X. And we also say that a Möbius space X possesses property (sI) if for any distinct ω, $\omega' \in X$ and a metric sphere $S \subset X$ between ω, ω' there is an s-inversion $\varphi_{\omega,\omega',S} \colon X \to X$ with respect to ω, ω', and S.

From now on, we assume that X is a compact Ptolemy space with properties (E) and (sI).

§3. Homotheties and shifts

3.1. Homotheties. Fix $\omega \in X$. Let $o \in X_\omega$, $\lambda > 0$. Consider $r_1, r_2 > 0$ such that $\lambda = r_2^2/r_1^2$. Let $S_1 = S_{r_1}(o)$, $S_2 = S_{r_2}(o) \subset X_\omega$ be metric spheres between o, ω. Denote by φ_1, φ_2 s-inversions relative to o, ω, S_1 and o, ω, S_2, respectively.

We define a *homothety with the center o and the coefficient λ* as a Möbius authomorphism $h\colon X \to X$ such that $h = \varphi_2 \circ \varphi_1$.

The next properties follow from the definition of an s-inversion and from Lemma 2.

(1) $h(o) = o, h(\omega) = \omega$.
(2) $h(\sigma) = \sigma$ for any Ptolemy circle $\sigma \subset X$ through o, ω.
(3) $|h(x)h(y)| = \lambda|xy|$ for all $x, y \in X_\omega$.
(4) For each $o \in X_\omega$ and each $\lambda > 0$, there exists a homothety with the center o and the coefficient λ.

We denote a homothety with center o and coefficient λ by $h_{\lambda,o}$.

Proposition 1. *Let ω, $\omega' \in X$, let σ be a Ptolemy circle through ω, ω', and let $\Gamma \subset \sigma$ be a connected component of $\sigma \setminus \{\omega, \omega'\}$. Consider $x, x' \in \Gamma$. Then there exists a homothety h with the center ω' such that $h(x) = x'$.*

Proof. Consider a metric space X_ω. Since $\omega \in \sigma$, Γ is a geodesic ray starting at ω'. Define λ by $|\omega' x'| = \lambda|\omega' x|$. Then $h(x) = x'$ for $h = h_{\lambda,\omega'}$. □

Corollary 1. *Any two distinct Ptolemy circles in a Ptolemy space with properties* (E) *and* (sI) *have at most two points in common.*

Proof. Let $\sigma, \sigma' \subset X$ be intersecting Ptolemy circles with $\omega \in \sigma \cap \sigma'$. Consider the metric space X_ω. Arguing by contradiction, suppose that there exist $x, x' \in (\sigma \cap \sigma') \setminus \{\omega\}$. Let Γ be a connected component of $\sigma \setminus \{x, \omega\}$ such that $x' \in \Gamma$. Also, let Γ' be a connected component of $\sigma' \setminus \{x, \omega\}$ such that $x' \in \Gamma'$. Note that if $x'' \in \Gamma$ and $\lambda = |xx''|/|xx'|$, then for a homothety $h = h_{\lambda,x}$ we have $h(x') = x''$. Then $x'' \in \Gamma'$ and $\Gamma \subset \Gamma'$. Similarly, $\Gamma' \subset \Gamma$, whence $\Gamma \equiv \Gamma'$. In the same way, if Γ_1 is a connected component of $\sigma \setminus \{x', \omega\}$ such that $x \in \Gamma_1$, and Γ_1' is a connected component of $\sigma' \setminus \{x', \omega\}$ such that $x \in \Gamma_1'$, we can check that $\Gamma_1 = \Gamma_1'$. It follows that $\sigma = \sigma'$. □

3.2. Shifts. Note that X is Hausdorff and compact. If we fix a nonprincipal ultrafilter θ on the set of natural numbers $\mathbb{N}$, then for each sequence $x_n \in X$ there exists a unique $x \in X$ such $x = \lim_\theta x_n$. Moreover, $|\lim_\theta(x_n)\lim_\theta(y_n)| = \lim_\theta |x_n y_n|$ for all sequences $x_n, y_n \in X$.

In this section we need the following well-known fact, see, e.g., [BS1, Lemma 6.7].

Lemma 3. *Assume that, for a nondegenerate triple $T = (x, y, z) \subset X$ and a sequence $\varphi_i \in \operatorname{Mob} X$, the sequence $T_i = \varphi_i(T)$ θ-converges to a nondegenerate triple $T' = (x', y', z') \subset X$. Then there exists $\varphi = \lim_\theta \varphi_i \in \operatorname{Mob} X$ with $\varphi(T) = T'$.*

Fix $\omega \in X$ and let $x, x' \in X_\omega$. Let $\lambda_n > 0$, $n \in \mathbb{N}$, be a sequence that goes to zero. Consider the homothety h_n with center x and coefficient λ_n^{-1} and the homothety h_n' with center x' and coefficient λ_n. We denote their composition $h_n' \circ h_n$ by η_n. Note that η_n is an isometry for each $n \in \mathbb{N}$. Then, by Lemma 3, $\eta = \lim_\theta \eta_n$ is a Möbius automorphism with $\eta(x) = x'$ and $\eta(\omega) = \omega$. Moreover $\eta\colon X_\omega \to X_\omega$ is an isometry. We denote it by $\eta_{xx'}$ and call a *shift* from x to x'. For every $x, x' \in X_\omega$, there exists a shift from x to x'.

§4. Foliations by parallel lines

Since each Ptolemy line $\ell \subset X_\omega$ is isometric to $\mathbb{R}$, for every $x_0 \in \ell$ the Busemann functions $b^{\pm}_{\ell,x_0}\colon X_\omega \to \mathbb{R}$ are well defined by the formula

$$b^{\pm}_{\ell,x_0}(x) = \lim_{t\to\pm\infty} |xc(t)| - |x_0c(t)|,$$

where $c(t)\colon \mathbb{R} \to \ell$ is a unit speed parametrization.

Two Ptolemy lines $\ell, \ell' \subset X_\omega$ are said to be *Busemann parallel* if ℓ, ℓ' share Busemann functions, that is, any Busemann function associated with ℓ is also a Busemann function associated with ℓ' and *vice versa.*

The following lemmas were proved in [BS1], and the proofs go without changes in our case.

Lemma 4 ([BS1, Lemma 4.11])**.** *Let $\ell, \ell' \subset X_\omega$ be Ptolemy lines with a common point $o \in l \cap \ell'$ and let $b\colon X_\omega \to \mathbb{R}$ be a Busemann function of ℓ with $b(o) = 0$. Assume that $b \circ c(t) = -t = b \circ c'(t)$ for all $t \geq 0$ and for appropriate unit speed parametrizations $c, c'\colon \mathbb{R} \to X_\omega$ of ℓ, ℓ' (respectively) with $c(0) = o = c'(0)$. Then $l = l'$. In particular, Busemann parallel Ptolemy lines coincide if they have a common point.*

Lemma 5 ([BS1, Lemma 4.12])**.** *Let $c, c'\colon \mathbb{R} \to X_\omega$ be unit speed parametrizations of Ptolemy lines $\ell, \ell' \subset X_\omega$, respectively. If $|c(t_i)c'(t_i)|/|t_i| \to 0$ for some sequences $t_i \to \pm\infty$, then the lines ℓ, ℓ' are Busemann parallel.*

Vice versa, if $\ell, \ell' \subset X_\omega$ are Busemann parallel lines, then

$$\lim_{t\to\infty} |c(t)c'(t)|/t = 0$$

for their unit speed parametrizations $c, c'\colon \mathbb{R} \to X_\omega$ chosen appropriately.

Lemma 6 ([BS1, Lemma 4.13])**.** *A shift $\eta_{xx'}$ moves any Ptolemy line ℓ through x to a Busemann parallel Ptolemy line $\eta_{xx'}(l)$ through x'.*

From Lemma 4 and Lemma 6 we immediately obtain the following claim.

Corollary 2. *Given a Ptolemy line $\ell \subset X_\omega$, through any point $x \in X_\omega$ there is a unique Ptolemy line l_x Busemann parallel to ℓ.*

§5. Symmetries with respect to horospheres

In this section we construct a symmetry with respect to a horosphere.

Fix $\omega \in X$ and a Ptolemy line $\ell \subset X_\omega$, and let $c\colon \mathbb{R} \to X_\omega$ be a unit speed parametrization of ℓ. For $t > 0$, the metric sphere $S_t = \{x \in X_\omega : |xc(t)| = t\}$ passes through $z = c(0)$ and lies between ω and $c(t)$. By (sI), there is an s-inversion $\varphi_t = \varphi_{\omega,c(t),S_t}\colon X \to X$. By the compactness of X, the s-inversions φ_t subconverge as $t \to \infty$ to a map $\varphi_\infty\colon X \to X$. Note that $\varphi_\infty(\omega) = \omega$ because $\varphi_t(c(t)) = \omega$ and $c(t) \to \omega$ as $t \to \infty$.

Lemma 7. *Let $x \in H_z$, where $H_z \subset X_\omega$ is the horosphere through $z \in \ell$ of the Busemann function $b^+(y) = \lim_{t\to\infty}(|yc(t)| - t)$, $y \in X_\omega$. Then $\varphi_\infty(x) = x$.*

Proof. Since $|zc(t)| = t$ for $t \geq 0$, we have $b^+(z) = 0$. Let ℓ_x be a line through x Busemann parallel to ℓ, and let $c'\colon \mathbb{R} \to X_\omega$ be its unit speed parametrization with $c'(0) = x$ such that b^+ is the Busemann function associated with the ray $c'([0,\infty))$. Fix $\varepsilon > 0$ and let $x' = c'(\varepsilon)$. Note that the function $|x'c(t)| - t$ is monotone decreasing and tends to $b^+(x') = -\varepsilon$. On the other hand, $|x'c(0)| > 0$. This means that there exists $t > 0$ such that $|x'c(t)| - t = 0$. Let $x_t = \varphi_t(x)$. Since $\varphi_t(x') = x'$, by Lemma 2 we have

$$|x_tx'| = t^2 \frac{|xx'|}{|c(t)x| \cdot |c(t)x'|} = \frac{t\varepsilon}{|c(t)x|}.$$

Note that the function $|xc(t)| - t$ is monotone decreasing and tends to $b^+(x) = 0$. This means that $|xc(t)| \geq t$ and $|x_t x'| \leq \varepsilon$. It follows that $|xx_t| \leq |xx'| + |x'x_t| \leq 2\varepsilon$. Choosing $\varepsilon \to 0$, we see that $\varphi_t(x) \to x$, and then $\varphi_\infty(x) = x$. □

Now we show that φ_∞ is an isometry of X_ω that, moreover, reflects the Ptolemy line ℓ in z. For each x, $y \in X_\omega$ and every sufficiently large $t > 0$, by Lemma 2 we have

$$|\varphi_t(x)\varphi_t(y)| = \frac{t^2|xy|}{|xc(t)|\,|yc(t)|},$$

and $|xc(t)| = t + b^+(x) + o(1)$, $|yc(t)| = t + b^+(y) + o(1)$. Thus, $|\varphi_\infty(x)\varphi_\infty(y)| = |xy|$ for all x, $y \in X_\omega$, i.e., φ_∞ is an isometry. It preserves the Ptolemy line ℓ because every φ_t preserves the Ptolemy circle $\sigma = l \cup \omega$, and it reflects ℓ in z because $\varphi_\infty(z) = z$ and every φ_t is an s-inversion of σ.

§6. Proof of Theorem 1

6.1. Some metric relations. Recall that a Ptolemy space X is said to be *Busemann flat* if for every Ptolemy circle $\sigma \subset X$ and every point $\omega \in \sigma$, we have $b^+ + b^- \equiv \text{const}$ for opposite Busemann functions $b^\pm \colon X_\omega \to \mathbb{R}$ associated with the Ptolemy line σ_ω, see [BS1, §3.2].

Lemma 8. *X is Busemann flat.*

Proof. Let $\ell \subset X_\omega$ be a Ptolemy line, and let $c\colon \mathbb{R} \to X_\omega$ be a unit speed parametrization of ℓ. Consider the horosphere H_o through $o = c(0)$ of the Busemann function $b^+(x) = \lim_{t\to\infty}(|xc(t)| - t)$, $x \in X_\omega$. Let $b^-(x) = \lim_{t\to\infty}(|xc(-t)| - t)$, $x \in X_\omega$, and let φ be the symmetry with respect to H_o. Note that if $x' = \varphi(x)$, where $x, x' \in X_\omega$, then $b^+(x) = b^-(x')$. Indeed,

$$b^-(x') = \lim_{t\to\infty}(|x'c(-t)| - t) = \lim_{t\to\infty}(|\varphi(x)\varphi(c(t))| - t) = \lim_{t\to\infty}(|xc(t)| - t) = b^+(x).$$

It follows that $b^+(z) = b^-(z)$ for every $z \in H_o$. Therefore, H_o is also a horosphere of the Busemann function b^-, and then $b^+ + b^- \equiv \text{const}$. □

Corollary 3. *For each horosphere H of the Busemann function b^+, the set $\varphi(H)$ is also a horosphere of the Busemann function b^+.*

Lemma 9. *Let ℓ, $\ell' \subset X_\omega$ be Busemann parallel lines, and let $\varphi\colon X_\omega \to X_\omega$ be the symmetry that reflects ℓ at $o \in \ell$. Then φ reflects ℓ' at $o' = H_o \cap \ell'$, where H_o is the horosphere of ℓ through o.*

Proof. H_o is the fixed point set of φ and $\varphi(\ell')$ is Busemann parallel to $\varphi(\ell) = \ell$. Thus, by Lemma 4, $\varphi(\ell') = \ell'$. □

Lemma 10. *Let ℓ, ℓ' be Busemann parallel lines in X_ω, and let $x, y \in \ell$, $x', y' \in \ell'$ be such that $b(x) = b(x')$, $b(y) = b(y')$, where b is a common Busemann function of ℓ and ℓ'. Then $|xy| = |x'y'|$, $|xx'| = |yy'|$, $|xy'| = |yx'|$, and $|x'y| \geq |xx'|$.*

Proof. The first identity is obvious because

$$|xy| = |b(x) - b(y)| = |b(x') - b(y')| = |x'y'|.$$

To prove the other two identities, consider the midpoint $z \in \ell$ between x, y, that is, $|xz| = |zy|$. Let H_x, H_y, H_z be horospheres of b through x, y, z (respectively), and let φ be the symmetry with respect to H_z such that $\varphi(\ell) = \ell$. Note that $\varphi(x) = y$ and $\varphi(\ell') = \ell'$. It follows that $\varphi(H_x) = H_y$. Moreover, $\varphi(x') = y'$ and $\varphi(y') = x'$. Then we have $|xx'| = |yy'|$ and $|xy'| = |yx'|$.

Applying the Ptolemy inequality $|xy|\cdot|x'y'|+|xx'|\cdot|yy'| \geq |xy'|\cdot|yx'|$ to the quadruple (x, x', y, y'), we have

$$(\lozenge) \qquad |xy|^2 + |xx'|^2 \geq |yx'|^2.$$

On the other hand, if y'' is symmetric to y with respect to H_x, then $|xy''| = |xy|$ and $|x'y''| = |x'y|$. Applying the Ptolemy inequality to the quadruple (x, x', y, y''), we have $|x'y| \cdot |xy''| + |x'y''| \cdot |xy| \geq |xx'| \cdot |yy''|$. It follows that $2|xy| \cdot |x'y| \geq 2|xy| \cdot |xx'|$. Thus, $|x'y| \geq |xx'|$. □

Fix $a > 0$ and let $\ell \in X_\omega$ be a Ptolemy line. Consider $x, y \in \ell$ such that $|xy| = a/2$. Let H_x and H_y be horospheres through x and y, and let φ_x and φ_y be the symmetries with respect to H_x and H_y. Consider the isometry $\varphi_y \circ \varphi_x$ and observe that it moves every point along a line Busemann parallel to ℓ at the distance a. We call such an isometry *a-shift along ℓ* and denote it by $\eta_{a,\ell}$. Let ℓ' be a Ptolemy line (not necessarily Busemann parallel to ℓ). Lemma 5 shows that ℓ' and $\eta_{a,\ell}(\ell')$ are Busemann parallel. This means that if H_z is the horosphere with respect to ℓ' through z, then $\eta_{a,\ell}(H_z)$ is the horosphere with respect to ℓ' through $\eta_{a,\ell}(z)$.

6.2. Existence of nonparallel lines. Assume that X is not Möbius equivalent to $\widehat{\mathbb{R}}$.

Lemma 11. *For each $\omega, \omega' \in X$, there exist distinct Ptolemy lines $\ell, \ell' \in X_\omega$ such that $\ell \cap \ell' = \{\omega'\}$.*

Proof. First, we find two Ptolemy circles with exactly two common points. Let $\sigma \subset X$ be a Ptolemy circle, and let $\omega \in \sigma$. Since X is not Möbius equivalent to $\widehat{\mathbb{R}}$, there is $x' \in X \setminus \sigma$. Let $c\colon \mathbb{R} \to X_\omega$ be a unit speed parametrization of the Ptolemy line $\ell = \sigma \setminus \omega$ such that the horosphere H of ℓ through $c(0)$ contains x'. Suppose $z = c(1)$, $z' = c(-1)$, and $|x'z| = |x'z'| = r$. Consider an s-inversion φ with respect to x', ω and the metric sphere $S_r = \{x \in X_\omega : |x'x| = r\}$. By Lemma 2, the image $\varphi(\ell)$ is a Ptolemy circle that intersects ℓ at two points z and z'.

Next let σ_1, σ_2 be the Ptolemy circles described above, $\sigma_1 \cap \sigma_2 = \{z, z'\}$. The lines $\ell_{1,z'} = \sigma_1 \setminus z$, $\ell_{2,z'} = \sigma_2 \setminus z \subset X_z$ through z' are not Busemann parallel. Let $\ell_{1,\omega}, \ell_{2,\omega}$ be the lines in X_z through ω that are Busemann parallel to $\ell_{1,z'}, \ell_{2,z'}$, respectively. Note that $\ell_1' = (\ell_{1,\omega} \setminus \{\omega\}) \cup \{z\}$ and $\ell_2' = (\ell_{2,\omega} \setminus \{\omega\}) \cup \{z\}$ are Ptolemy lines in X_ω. Finally, the Ptolemy lines ℓ_1, ℓ_2 through ω' Busemann parallel to ℓ_1', ℓ_2' (respectively) are distinct. □

6.3. Homotheties preserve a foliation by horospheres. Suppose $c\colon \mathbb{R} \to X_\omega$ is a unit speed parametrization of a Ptolemy line $\ell \subset X_\omega$, $o = c(0)$, $z \in \ell$ and H_z is the horosphere with respect to ℓ through z.

Lemma 12. *Let h be a homothety with the center o. Then $h(H_z)$ is the horosphere with respect to ℓ through $h(z)$.*

Proof. Let $x \in H_z$, and let λ be the coefficient of h. Then $\lim_{t\to\infty}(|xc(t)| - |zc(t)|) = 0$. Multiplying by λ, we have $\lim_{t\to\infty} \lambda(|xc(t)| - |zc(t)|) = 0$. It follows that

$$\lim_{t\to\infty}(|h(x)h(c(t))| - |h(z)h(c(t))|) = \lim_{t\to\infty}(|h(x)c(\lambda t)| - |h(z)c(\lambda t)|) = 0,$$

whence $h(H_z) \subset H_{h(z)}$. On the other hand, for each homothety h we can consider a homothety h' with the same center such that $h' \circ h = \mathrm{id}$. This means that $h(H_z) = H_{h(z)}$. □

6.4. Projection on horospheres. Here we assume that X is not Möbius equivalent to $\widehat{\mathbb{R}}$. Let $o, \omega \in X$, and let $\ell \subset X_\omega$ be a Ptolemy line through o.

Suppose $H_o \subset X_\omega$ is the horosphere with respect to ℓ through o. We define the projection $\pi_o \colon X_\omega \to H_o$ as follows: if $x \in X_\omega$ and ℓ_x is the Ptolemy line through x Busemann parallel to ℓ, then $\pi_o(x) := H_o \cap \ell_x$.

Proposition 2. *Let $\ell' \neq \ell \subset X_\omega$ be Ptolemy lines through o. Then $\pi_o(\ell')$ is a Ptolemy line.*

Proof. We prove that there exists $\alpha > 0$ such that $|\pi_o(c'(t))\pi_o(c'(t'))| = \alpha|t - t'|$ for all $t, t' \in \mathbb{R}$, where $c' \colon \mathbb{R} \to X_\omega$ is a unit speed parametrizations of ℓ' with $c'(0) = o$. Put $z = c'(1)$, $z' = \pi_o(z)$, and $\alpha := |oz'|/|oz|$.

Lemma 13. *Let $x_i = c'(t_i)$, $i = 1, 2, 3$, where $t_1 < t_2 < t_3$. Then*

$$\frac{|\pi_o(x_1)\pi_o(x_2)|}{|x_1x_2|} = \frac{|\pi_o(x_2)\pi_o(x_3)|}{|x_2x_3|} = \frac{|\pi_o(x_1)\pi_o(x_3)|}{|x_1x_3|}.$$

Proof. Let $x_i \in \ell_i$, where ℓ and ℓ_i are Busemann parallel, and let $x_i \in H_i$, where H_i is the horosphere of ℓ_i, $i = 1, 2, 3$.

We observe that the homothety $h_1 \colon X_\omega \to X_\omega$ with the center x_1 and the coefficient $|x_1x_3|/|x_1x_2|$ moves x_2 to x_3, and $h_1(H_1) = H_1$. It follows that $h_1(\ell_2) = \ell_3$. So, if $y_2 = H_1 \cap \ell_2$ and $y_3 = H_1 \cap \ell_3$, then $h_1(y_2) = y_3$. Thus, $|x_1y_3|/|x_1y_2| = |x_1x_3|/|x_1x_2|$. On the other hand, $|x_1y_3| = |\pi_o(x_1)\pi_o(x_3)|$ and $|x_1y_2| = |\pi_o(x_1)\pi_o(x_2)|$. Consequently,

$$\frac{|\pi_o(x_1)\pi_o(x_2)|}{|x_1x_2|} = \frac{|\pi_o(x_1)\pi_o(x_3)|}{|x_1x_3|}.$$

In the same way, considering the homothety h_3 with the center x_3 and the coefficient $|x_1x_3|/|x_2x_3|$, we see that

$$\frac{|\pi_o(x_2)\pi_o(x_3)|}{|x_2x_3|} = \frac{|\pi_o(x_1)\pi_o(x_3)|}{|x_1x_3|}. \qquad \square$$

Now Lemma 13 shows that $|\pi_o(c'(t))\pi_o(c'(t'))| = \alpha|t - t'|$ for all $t, t' \in \mathbb{R}$. $\square$

6.5. Horospheres invariance. Let $H_o \subset X_\omega$ be the horosphere through o with respect to some Ptolemy line $\ell \subset X_\omega$.

Proposition 3. *The subspace $X^1 = H_o \cup \{\omega\}$ is a compact Ptolemy space with properties* (E) *and* (sI).

Proof. Let φ be an s-inversion with respect to $o, \omega \in X$. Note that $\varphi(X^1) = X^1$. Indeed, let $z \in H_o \cup \{\omega\}$, and let $c \colon \mathbb{R} \to X_\omega$ be a unit speed parametrization of ℓ such that $c(0) = o$. Consider the Busemann function $b \colon X_\omega \to \mathbb{R}$ of ℓ such that $b \circ c(t) = -t$. Then $b(z) = 0$. On the other hand, if $z' = \varphi(z)$, then

$$|z'c(t)| = \frac{|zc(1/t)|}{\frac{1}{t} \cdot |xz|} = \frac{t|zc(1/t)|}{|xz|}.$$

Therefore,

$$b(z') = \lim_{t\to\infty}(|z'c(t)| - t) = \lim_{t\to\infty}(t|zc(1/t)|/|xz| - t).$$

Note that by ($\Diamond$), we have

$$|zx|^2 \leq |zc(1/t)|^2 \leq |zx|^2 + 1/t^2.$$

Then

$$0 \leq t|zc(1/t)|/|xz| - t \leq \sqrt{t^2 + 1/|zx|^2} - t.$$

Thus, $b(z') = \lim_{t\to\infty}(t|zc(1/t)|/|xz| - t) = 0$.

It follows that $\varphi_{x,y}(X^1) = X^1$ for any $x, y \in X^1$ and any s-inversion $\varphi_{x,y}$ with respect to x, y.

Let $x, y \in X^1$, and let $S' \subset X^1$ be a metric sphere between x and y in X^1. Note that $S' = S \cap X^1$, where $S \subset X$ is a metric sphere between x and y in X. We define an s-inversion $\varphi'_{x,y,S'} \colon X^1 \to X^1$ with respect to $x, y \in X^1$ and a metric sphere $S' \subset X^1$ between x and y as a restriction of an s-inversion $\varphi_{x,y,S} \colon X \to X$ with respect to $x, y \in X$ and a metric sphere $S \subset X$ to X^1. It follows that X^1 has property (sI).

On the other hand, by Lemma 11, there exists a Ptolemy line $\ell' \neq \ell$ through o. By Proposition 2, $\pi_o(\ell')$ is a Ptolemy line in H_o, and then X^1 has property (E). $\square$

6.6. Coordinates in X_ω. From now on, we fix $o, \omega \in X$ and consider the metric space X_ω. Consider a Ptolemy line ℓ_0 through o with a unit speed parametrization $c_0 \colon \mathbb{R} \to X_\omega$, $c_0(0) = o$. Let H_o be the horosphere with respect to ℓ_0 through o, and let $b_0 \colon X_\omega \to \mathbb{R}$ be the Busemann function of ℓ_0 with $b_0(o) = 0$. For each $z \in H_o$ denote by ℓ_z the line Busemann parallel to ℓ_0 through z and consider the unit speed parametrization $c_z \colon \mathbb{R} \to X_\omega$ of ℓ_z such that $b_0 \circ c_0(t) = -t = b_0 \circ c_z(t)$. From Lemma 4, Corollary 2, and Lemma 8 it follows that the map $i_1 \colon \ell_0 \times H_o \to X_\omega$ such that $i_1(t, z) = c_z(t)$ is a bijection.

Take $x_0 \in \ell_0$ with $|ox_0| = 1$. Recall that $|zx_0| \geq |ox_0| = 1$ for each $z \in H_o$. By Proposition 3, $X^1 = H_o \cup \{\omega\}$ is a compact Ptolemy space with properties (E) and (sI).

Arguing by induction, we obtain a sequence

$$\cdots \subset X^k \subset \cdots \subset X^1 \subset X^0 = X$$

of compact Ptolemy spaces with properties (E) and (sI) and a sequence of points $x_i \in X^i \setminus X^{i+1}$, where $|x_i o| = 1$. Moreover, $|x_i x_k| \geq 1$ for $i \neq k$. Since the ball $B_1(o) = \{x \in X : |xo| \leq 1\}$ is compact, the sequence $\{x_i\}$ is finite, implying the existence of $N \in \mathbb{N}$ such that X^N is Möbius equivalent to $\widehat{\mathbb{R}}$. Then

$$\widehat{\mathbb{R}} = X^N \subset \cdots \subset X^1 \subset X^0 = X.$$

It follows that there is a bijection

$$i \colon \ell_0 \times \ell_1 \times \cdots \times \ell_N \to X_\omega.$$

This bijection induces on X_ω a structure of the vector space $\mathbb{R}^{N+1}$. This means that we can add points of X and multiply them by real numbers. Note that o plays the role of a neutral element.

Let $b_i \colon X_\omega \to \mathbb{R}$, $i = 1, \ldots, N$, be a Busemann function of ℓ_i with $b_i(o) = 0$. Then b_i is the ith coordinate function. Moreover, if $H_i(x)$ is the horosphere with respect to b_i through x, then $x = \bigcap_{i=0}^N H_i(x)$. Denote by $x(i)$ the vector with coordinates $(0, \ldots, b_i(x), \ldots, 0)$, where $b_i(x)$ appears at the ith place. Note that $x = x_0 + \cdots + x_N$.

Let $T_x^i \colon X_\omega \to X_\omega$ be the $b_i(x)$-shift along ℓ_i, and let $T_x \colon X_\omega \to X_\omega$ be defined by $T_x(y) = x + y$ for each $y \in X_\omega$. Observe that $T_{x(i)} = T_x^i$, and then $T_x = T_x^N \circ \cdots \circ T_x^0$. Consequently, T_x is an isometry.

If h_k is the homothety with the center o and coefficient k, then $h_k(x) = kx$, where $k > 0$. Indeed, note that $h_k(x(i)) = kx(i)$. Moreover,

$$h_k(H_i(x)) = h_k(H_i(x(i)) = H_i(kx(i)) = H_i(kx)$$

and

$$h_k(x) = h_k\Big(\bigcap_{i=0}^N H_i(x)\Big) = \bigcap_{i=0}^N H_i(kx) = kx.$$

Therefore, $|o(kx)| = k|ox|$, where $k > 0$.

Let $\nu\colon X_\omega \to \mathbb{R}_+$ be defined by $\nu(x) = |ox|$. We prove that ν is a norm on X_ω. Indeed, if $\nu(x) = 0$, then $|ox| = 0$ and $x = o$. Moreover,

$$\nu(x+y) = |o(x+y)| \leq |ox| + |x(x+y)| = |ox| + |T_x(o)T_x(y)| = |ox| + |oy| = \nu(x) + \nu(y).$$

Finally, observe that $\nu(-x) = |o(-x)| = |T_x(o)T_x(-x)| = |xo| = \nu(x)$. So, if $k \geq 0$, then $\nu(kx) = |o(kx)| = k|ox| = k\nu(x)$. If $k < 0$, then $\nu(kx) = |o(kx)| = \big|o\big(|k|(-x)\big)\big| = |k|\,|o(-x)| = |k|\nu(-x) = |k|\nu(x)$.

Also we note that $\nu(\cdot)$ induces the metric of X_ω. Indeed, $|xy| = |T_x(o)T_x(y-x)| = \nu(y-x)$. Applying the Schoenberg theorem, see Theorem 2, we conclude that X is Möbius equivalent to $\widehat{\mathbb{R}}^N$.

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