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CHARACTERIZATION OF THE INVERSE PROBLEM DATA FOR ONE-DIMENSIONAL TWO-VELOCITY DYNAMICAL SYSTEM

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To the 75th anniversary of A. S. Blagoveshchenskiĭ

Abstract. The evolution of the dynamical system in question is described by the wave equation $\rho u_{tt}-(\gamma u_x)_x+Au_x+Bu=0$, $x>0$, $t>0$, with the zero Cauchy data at $t=0$ and the Dirichlet boundary control at $x=0$. Here ρ, γ, A, B are smooth real 2×2-matrix-valued functions of x; $\rho=\operatorname{diag}\{\rho_1,\rho_2\}$ and $\gamma=\operatorname{diag}\{\gamma_1,\gamma_2\}$ are matrices with positive entries; and $u=u(x,t)$ is a solution (an $\mathbb{R}^2$-valued function). For $x\geq 0$, it is assumed that $\sqrt{\frac{\gamma_2}{\rho_2}}<\sqrt{\frac{\gamma_1}{\rho_1}}$ and $A^{\mathrm{tr}}=-A$, $A_x=B-B^{\mathrm{tr}}$. The "input-output" correspondence is realized by the response operator $R\colon u(0,t)\mapsto \gamma(0)u_x(0,t)$, $t\geq 0$, which plays the role of inverse problem data in applications. In the paper, a constructive characterization is given for the response operators of the systems of this type.

§1. Introduction

1.1. About this paper.

Dynamical system. The system under study is described by the initial-boundary value problem

$$\rho u_{tt}-(\gamma u_x)_x+Au_x+Bu=0, \qquad 0<x<h,\ \ 0<t<T, \tag{1.1}$$

$$u\big|_{t<\tau_1(x)}=0, \tag{1.2}$$

$$u\big|_{x=0}=f, \qquad 0\leq t\leq T. \tag{1.3}$$

Here ρ, γ, A, and B are smooth[1] (2×2-matrix-valued functions of $x\in[0,h]$ satisfying the following conditions:

1) *positivity*: $\rho=\operatorname{diag}\{\rho_1,\rho_2\}$, $\rho_i>0$, and $\gamma=\operatorname{diag}\{\gamma_1,\gamma_2\}$, $\gamma_i>0$,
2) *separation of velocities*: $0<\sqrt{\frac{\gamma_2}{\rho_2}}<\sqrt{\frac{\gamma_1}{\rho_1}}$,
3) *selfadjointness*: $A^{\mathrm{tr}}=-A$, $A_x=B-B^{\mathrm{tr}}$ (tr stands for transposition);

The functions $\tau_i(x):=\int_0^x\sqrt{\frac{\rho_i(s)}{\gamma_i(s)}}\,ds$ are called the eikonals; the numerical parameters h and T are related by the formula $T=\tau_1(h)$; $f=f(t)$ is the boundary control and $u=u^f(x,t)$ is a solution ($\mathbb{R}^2$-valued functions). This problem is hyperbolic, and, in the above setting, it is well posed.

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[1]Throughout, "smooth" means "C^∞-smooth".

Response operator. System (1.1)–(1.3) gives rise to the standard attributes of control theory – spaces and operators. One of them is the *extended response operator* R^{2T}, which realizes the "input-output" correspondence. It is introduced via the problem [2]

$$\begin{aligned} &\rho u_{tt} - (\gamma u_x)_x + Au_x + Bu = 0, && 0 < x < h,\ \ 0 < t < 2T - \tau_1(x),\\ &u\big|_{t<\tau_1(x)} = 0, \\ &u\big|_{x=0} = f, && 0 \le t \le 2T, \end{aligned}$$

by $R^{2T} : f \mapsto \gamma(0) u_x^f\big|_{x=0}$ and can be given by the formula

$$(R^{2T} f)(t) = -\nu f_t(t) + \omega f(t) + \int_0^t r(t-s)\, f(s)\, ds, \quad 0 \le t \le 2T, \tag{1.4}$$

with constant matrices $\nu = \operatorname{diag}\{\nu_1, \nu_2\}$ and ω, and with a smooth matrix-valued function $r(t) = r^{\mathrm{tr}}(t)$, $0 \le t \le 2T$. In the dynamical inverse problem, the operator R^{2T} plays the role of data.

The main result. The operator R^{2T} is determined by the coefficients ρ, γ, A, and B. The following question is natural when we deal with the inverse problem for (1.1)–(1.3): to what extent are these coefficients determined by the response operator? Under conditions **1**–**3**, the parameters ρ, γ, A, and B are described by *eight* independent scalar functions ρ_1, ρ_2, γ_1, γ_2, a_{12}, b_{11}, b_{12}, b_{22}, while the operator R^{2T} is given by the collection ν, ω, r, which involves *three* functions r_{11}, r_{12}, and r_{22}, and *six* numbers forming the matrices ν and ω. Therefore, we can hardly expect that the inverse problem is uniquely solvable, i.e., that the coefficients are determined uniquely, and the *solvability* question arises: what conditions on ν, ω, r ensure the existence of at least one system with this data?

Our main result answers this question. Theorem 1 (see Subsection 3.1) gives necessary and sufficient conditions on the operator (1.4) (on the collection ν, ω, r) that guarantee the existence of a system (1.1)–(1.3) with the prescribed data. Among these conditions, the central one is the positive definiteness of the operator $\mathcal{C}^T$ that acts in $L_2([0,T];\mathbb{R}^2)$ by the rule

$$\big(\mathcal{C}^T f\big)(t) := \nu f(t) + \int_0^T \left[\frac{1}{2}\int_{|t-s|}^{2T-t-s} r(\eta)\, d\eta\right] f(s)\, ds, \quad 0 \le t \le 2T. \tag{1.5}$$

The proof of sufficiency is constructive: we provide a procedure that recovers system (1.1)–(1.3) by ν, ω, r. This procedure involves free parameters, which makes it possible to find *all* systems of this sort with given ν, ω, r.

1.2. Comments. The problems in question describe wave processes in systems for which different wave modes propagate with different velocities and interact with each other. Many-velocity systems occur in various applications: geophysics, acoustics, mechanics, elasticity theory, etc. As an example, we mention a well-known model of elasticity theory, the Timoshenko beam [1, 2]. The corresponding inverse problems consist of recovering the parameters of such systems by some information about the solution, extracted from external observations (measurements), see [3, 4, 5, 6, 7, 8, 9, 11, 19].

The proof of Theorem 1 follows the lines of [6], but now the situation is more complicated, involving a greater number of free parameters that determine the dynamical system in question[3]. The principal difficulty was to choose these parameters consistently. For analogs of the operator (1.5), the positive definiteness condition has been known in

[2]This problem is a natural extension of problem (1.1)–(1.3), which exists and is well posed because the latter problem is hyperbolic.

[3]In particular, in contrast to [6], the mode velocities are not assumed to be constant.

inverse problems since the classical work by M. G. Kreĭn (see [12, 13, 14]), and its presence in the characterization of data can well be expected. In the authors' opinion, the basic achievement in the present paper consists of the procedure of construction of the system starting with ν, ω, r. Given six numbers and three scalar functions, this procedure extracts a fairly complicated object rich with properties: a two-velocity dynamical system. The construction employs a nice physical phenomenon, the existence of slow waves. Such a wave is a mixture of two modes, fast and slow, that propagates with the velocity of the slow one.

The key fragment of our procedure is the so-called *amplitude formula*, which is the main tool for solving inverse problems by the boundary control method.

Since the paper is large, we omit the proofs of some facts and results of auxiliary nature; we refer to these statements as *Propositions* and restrict ourselves to some commentaries on their proofs. Should we have presented these proofs, the paper would have become twice as large: technical complexity is typical of many-velocity systems. The same purpose – to unload the paper – is pursued by overstating the smoothness requirements. However, the following should be mentioned here. As to the data, it suffices to deal with finite smoothness, but, in all statements known to us, the smoothness requirements in the necessary and the sufficient conditions are different. As far as we know, it is an open question how the smoothness of the coefficients ρ, γ, A, B and that of the function r should correspond to each other.

§2. Two-velocity system

2.1. Initial-boundary value problem.

Setting. We consider the problem

$$\rho u_{tt} - (\gamma u_x)_x + Au_x + Bu = 0, \qquad x > 0,\ 0 < t < T, \tag{2.1}$$

$$u\big|_{t=0} = u_t\big|_{t=0} = 0, \qquad x \geq 0, \tag{2.2}$$

$$u\big|_{x=0} = f, \qquad 0 \leq t \leq T, \tag{2.3}$$

in which ρ, γ, A, B are smooth real 2×2-matrix-valued functions of $x \geq 0$; $T < \infty$ is the final moment; $\rho = \operatorname{diag}\{\rho_1(x), \rho_2(x)\}$ and $\gamma = \operatorname{diag}\{\gamma_1(x), \gamma_2(x)\}$ are matrices with positive entries; $f = f(t) = \left(\begin{smallmatrix} f_1(t) \\ f_2(t) \end{smallmatrix}\right)$ is the *boundary control*. A solution $u = u^f(x,t) = \left(\begin{smallmatrix} u_1^f(x,t) \\ u_2^f(x,t) \end{smallmatrix}\right)$ describes a wave initiated by the control f and propagating along the semiaxis $x \geq 0$. The functions $c_i := \sqrt{\frac{\gamma_i(x)}{\rho_i(x}}$ are referred to as *velocities*.

Throughout, we assume that

$$0 < c_2(x) < c_1(x), \quad x \geq 0; \quad \int_0^\infty \frac{dx}{c_1(x)} = \infty, \tag{2.4}$$

and

$$A^{\mathrm{tr}}(x) = -A(x), \quad A_x(x) = B(x) - B^{\mathrm{tr}}(x), \quad x \geq 0, \tag{2.5}$$

where tr stands for transposition. Relations (2.5) are equivalent to

$$A(x) = \begin{pmatrix} 0 & -a(x) \\ a(x) & 0 \end{pmatrix}, \quad b_{21}(x) - b_{12}(x) = a_x(x), \quad x \geq 0.$$

Since the integral in (2.4) diverges, problem (2.1)–(2.3) is well posed for any $T > 0$. By (2.5), the operator $y \mapsto (\gamma y_x)_x - Ay_x - By$ is Lagrange selfadjoint, i.e., it is symmetric in $L_2\big((0,\infty); \mathbb{R}^2\big)$ on the functions with compact support.

The functions $\tau_i(x) := \int_0^x \frac{ds}{c_i(s)}$, called the *eikonals*, are strictly monotone increasing. By (2.4), we have $\tau_1(x) < \tau_2(x)$, $x \geq 0$; $\tau_i(\infty) = \infty$. The functions $x_i(\tau)$ inverse to the eikonals are also strictly monotone increasing, and $x_1(\tau) > x_2(\tau)$ for $\tau \geq 0$.

Generalized solutions. For us, the following agreement will be convenient.

Agreement 1. (a) *All functions depending on time are assumed to be extended by zero for $t < 0$.*

(b) *Let $\Pi \subset \mathbb{R}^2$ be a rectangle or a half-strip with sides parallel to coordinate axes. We say that a function φ is smooth in Π outside of the curves $S_1, \dots, S_p \subset \Pi$ if it is smooth in each connected component of the set $\Pi \setminus \bigcup_{i=1}^p S_i$ and extends up to a smooth function in the vicinity of this component.*

We denote $\theta := \operatorname{diag}\{\theta_1, \theta_2\}$, $\theta_i(x) := \left(\frac{\rho_i(0)\gamma_i(0)}{\rho_i(x)\gamma_i(x)}\right)^{\frac{1}{4}}$ and introduce the linear space

$$\mathcal{M}^T := \left\{ f \in C^\infty\big([0,T];\mathbb{R}^2\big) \;\middle|\; \operatorname{supp} f \subset (0,T] \right\} \tag{2.6}$$

of smooth controls vanishing near $t = 0$. Concerning the solution of problem (2.1)–(2.3), we know the following.

Proposition 1. *For controls $f \in \mathcal{M}^T$, problem (2.1)–(2.3) has a unique classical smooth solution $u^f(x,t)$. This solution can be represented as*

$$u^f(x,t) = \theta(x) \begin{pmatrix} f_1\big(t - \tau_1(x)\big) \\ f_2\big(t - \tau_2(x)\big) \end{pmatrix} + \int_0^{t-\tau_1(x)} \widetilde{w}(x, t-s) f(s)\, ds \tag{2.7}$$

with a matrix kernel $\widetilde{w}(x,t)$, which is smooth in $[0,\infty) \times [0,T]$ outside of the characteristics $t = \tau_i(x)$ of equation (2.1) and satisfies $\widetilde{w}\big|_{t<\tau_1(x)} = 0$, $\widetilde{w}\big|_{x=0} = 0$.

The proof follows the standard lines: problem (2.1)–(2.3) reduces to an equivalent system of integral Volterra equations of the second kind, whose solvability is established in an adequate function class (see [3, 16, 17]). Note that if the velocity separation condition (2.4) is violated, then the representation (2.7) fails.

For $f \in L_2\big([0,T];\mathbb{R}^2\big)$, a (generalized) solution of our problem is defined as the expression on the right in (2.7). This definition implies that

$$u^f\big|_{t<\tau_1(x)} = 0, \tag{2.8}$$

and that the dependence of the solution on the coefficients is local: the values taken by u^f for $0 \leq t \leq T$ are determined by the values of ρ, γ, A, and B for $0 \leq x \leq x_1(T)$ (not depending on the behavior of the coefficients for $x > x_1(T)$). This locality (causality) is a consequence of the hyperbolicity of equation (2.1) and corresponds to the fact that the velocity of the wave propagation is finite.

Since the coefficients in (2.1) are independent of time, we have the well-known relation

$$u^f(\,\cdot\,, s) = u^{\mathcal{T}^T_{T-s} f}(\,\cdot\,, T), \quad 0 \leq s \leq T, \tag{2.9}$$

where $\mathcal{T}^T_{T-s}$ is the delay operator, acting in $L_2\big([0,T];\mathbb{R}^2\big)$ and given by

$$(\mathcal{T}^T_{T-s} f)(t) := f\big(t - (T-s)\big) \tag{2.10}$$

(we have used Agreement 1a).

2.2. System $\mathfrak{s}^T$. Recalling the properties of u^f mentioned above (see (2.8)), we can rewrite problem (2.1)–(2.3) as follows:

$$\rho u_{tt} - (\gamma u_x)_x + Au_x + Bu = 0, \qquad 0 < x < x_1(T),\ \ 0 < t < T, \tag{2.11}$$

$$u\big|_{t<\tau_1(x)} = 0, \tag{2.12}$$

$$u\big|_{x=0} = f, \qquad 0 \le t \le T. \tag{2.13}$$

This is optimal in the sense that it does not involve the coefficients $\{\rho, \gamma, A, B\}\big|_{x>x_1(T)}$, on which the solution does not depend.

We treat problem (2.11)–(2.13) as a dynamical system. It is denoted by $\mathfrak{s}^T$ and is endowed with the standard attributes of control theory.

Spaces and subspaces. The Hilbert space of controls $\mathcal{F}^T := L_2\big([0,T];\mathbb{R}^2\big)$ with the scalar product

$$(f,g)_{\mathcal{F}^T} := \int_0^T f(t)\cdot g(t)\,dt$$

("$\cdot$" denotes the standard product in $\mathbb{R}^2$) is called the *external space* of the system $\mathfrak{s}^T$. It includes the expanding (with the growth of ξ) chain of subspaces

$$\mathcal{F}^{T,\xi} := \big\{f \in \mathcal{F}^T \bigm| \operatorname{supp} f \subset [T-\xi, T]\big\} = \mathcal{T}^T_{T-\xi}\mathcal{F}^T, \quad 0 \le \xi \le T,$$

formed by delayed controls[4]. A delay in control implies that of the wave: we have the relation

$$u^f\big|_{t<\tau_1(x)+T-\xi} = 0, \quad f \in \mathcal{F}^{T,\xi}, \tag{2.14}$$

which follows easily from (2.7) and refines (2.8).

The space $\mathcal{H}^{x_1(T)} := L_{2,\rho}\big([0,x_1(T)];\mathbb{R}^2\big)$ with the scalar product

$$(y,v)_{\mathcal{H}^{x_1(T)}} := \int_0^{x_1(T)} [\rho(x)\,y(x)]\cdot v(x)\,dx$$

is referred to as *internal*. From (2.8) we see that the waves $u^f(\,\cdot\,,t)$ belong to this space for any $t \in [0,T]$. The space $\mathcal{H}^{x_1(T)}$ includes the following two chains of subspaces:

$$\mathcal{H}^{x_i(\xi)} := \big\{y \in \mathcal{H}^{x_1(T)} \bigm| \operatorname{supp} y \subset [0, x_i(\xi)]\big\}, \quad 0 \le \xi \le T,\ \ i = 1,2.$$

Control operator. In terms of control theory, the wave $u^f(\,\cdot\,,t)$ is the state of the system $\mathfrak{s}^T$ at the moment t, and $\{u^t(\,\cdot\,,t) \,|\, 0 \le t \le T\}$ is its trajectory. The correspondence "input $\mapsto$ state" is realized by the *control operator* $W^T : \mathcal{F}^T \to \mathcal{H}^{x_1(T)}$,

$$W^T f := u^f(\,\cdot\,,T).$$

Formula (2.7) shows that

$$(W^T f)(x) = \theta(x)\begin{pmatrix} f_1\big(T-\tau_1(x)\big) \\ f_2\big(T-\tau_2(x)\big)\end{pmatrix} + \int_0^{T-\tau_1(x)} w^T(x,t) f(t)\,dt, \quad x \ge 0, \tag{2.15}$$

with a matrix kernel $w^T(x,t) := \widetilde{w}(x, T-t)$, which is smooth in $[0,x_1(T)] \times [0,T]$ outside of the characteristics $t = T - \tau_i(x)$, and satisfies $w^T\big|_{t>T-\tau_1(x)} = 0$, $w^T\big|_{x=0} = 0$. Obviously, the control operator is bounded. Observe the relations

$$W^T\mathcal{T}^T_{T-\xi} f = u^f(\,\cdot\,,\xi), \quad W^T\mathcal{F}^{T,\xi} \subset \mathcal{H}^{x_1(\xi)}, \quad 0 \le \xi \le T, \tag{2.16}$$

which are another form of (2.9) and (2.8). For the controls that give rise to smooth solutions, we have

$$u^{f_{tt}} = u^f_{tt} \overset{(2.1)}{=} Lu^f,$$

[4] $T - \xi$ is the delay; ξ is the control action time.

where

$$Ly := \rho^{-1}\big((\gamma y_x)_x - Ay_x - By\big).$$

It follows that

$$LW^T = W^T \frac{d^2}{dt^2}, \tag{2.17}$$

which is another form of equation (2.1).

Response operator. The "input-output" correspondence in the system $\mathfrak{s}^T$ is described by the *response operator* $R^T : \mathcal{F}^T \to \mathcal{F}^T$, $\operatorname{Dom} R^T = \mathcal{M}^T$,

$$(R^T f)(t) := \gamma(0)\, u_x^f(0,t), \quad 0 \le t \le T.$$

Since this action involves differentiation, this operator is unbounded.

Proposition 2. *The following representation holds true*:

$$(R^T f)(t) = -\nu f_t(t) + \omega f(t) + \int_0^t r(t-s)\, f(s)\, ds, \quad 0 \le t \le T, \tag{2.18}$$

with the constant matrices

$$\begin{aligned} &\nu = \begin{pmatrix} \nu_1 & 0 \\ 0 & \nu_2 \end{pmatrix}, \quad \nu_i := c_i(0)\rho_i(0), \\ &\omega := \begin{pmatrix} -\frac{c_1(0)}{2}\,(c_1\rho_1)_x\big|_{x=0} & -\frac{c_1(0)}{c_1(0)+c_2(0)}\,a(0) \\ \frac{c_2(0)}{c_1(0)+c_2(0)}\,a(0) & -\frac{c_2(0)}{2}\,(c_2\rho_2)_x\big|_{x=0} \end{pmatrix} \end{aligned} \tag{2.19}$$

$(a(x) = A_{21}(x))$ *and with the smooth matrix-valued function*

$$r(t) := \gamma(0)\widetilde{w}_x(0,t).$$

We have

$$\nu_1, \nu_2 > 0; \quad \omega_{12} = -\alpha\omega_{21}, \quad \alpha > 1; \quad r(t) = r^{\mathrm{tr}}(t) \tag{2.20}$$

and

$$\alpha = \frac{c_1(0)}{c_2(0)}, \quad \omega_{21} - \omega_{12} = a(0).$$

Formula (2.18) is established by differentiating (2.7) in x (see [17]). The symmetry of the matrix-valued *response function* $r(t)$ is deduced from conditions (2.5).

The operator R^{2T}. The system $\mathfrak{s}^T$ gives rise to yet another operator, which is introduced via the initial-boundary problem

$$\rho u_{tt} - (\gamma u_x)_x + Au_x + Bu = 0, \qquad (x,t) \in \Delta^{2T}, \tag{2.21}$$

$$u\big|_{t<\tau_1(x)} = 0, \tag{2.22}$$

$$u\big|_{x=0} = f\,, \qquad 0 \le t \le 2T, \tag{2.23}$$

where $\Delta^{2T} = \big\{(x,t) \,|\, 0 < x < x_1(T),\, 0 < t < 2T - \tau_1(x)\big\}$ (see Figure 1). The fact that this problem is well posed is established with the help of the same techniques as before, by reduction to a system of integral Volterra equations of the 2nd kind. For the controls $f \in \mathcal{M}^{2T}$ (see (2.6)), the solution $u = u^f(x,t)$ is classical and smooth.

Problem (2.21)–(2.23) can be viewed as an extended version of problem (2.11)–(2.13), existing due to the hyperbolicity of the latter. If the control f occurring in (2.23) is such that $f\big|_{[0,T]}$ coincides with f in (2.3), then the solutions of these two problems coincide for $0 \le t \le T$. It should be noted that the solutions of the two problems are well determined by the behavior of the coefficients ρ, γ, A, B for $0 \le x \le x_1(T)$ (they do not depend on the behavior of these coefficients for $x > x_1(T)$).

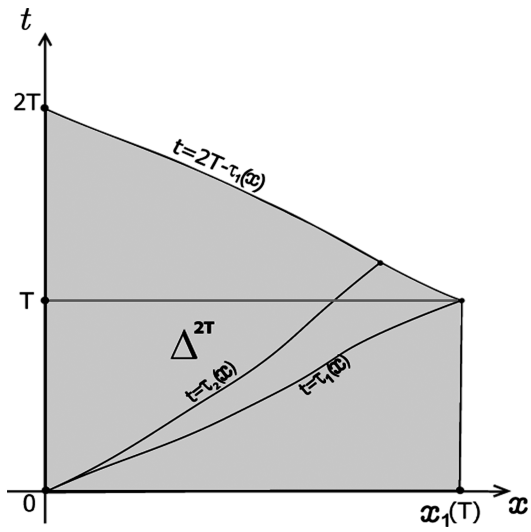

FIGURE 1. The domain Δ^{2T}.

With problem (2.21)–(2.23), we associate the operator $R^{2T} : \mathcal{F}^{2T} \to \mathcal{F}^{2T}$, $\operatorname{Dom} R^{2T} = \mathcal{M}^{2T}$,

$$(R^{2T} f)(t) := \gamma(0)\, u_x^f(0,t), \quad 0 \le t \le 2T,$$

called the *extended* response operator for $\mathfrak{s}^T$.

Proposition 3. *The operator R^{2T} is determined by the coefficients*

$$\{\rho, \gamma, A, B\}\big|_{0 \le x \le x_1(T)}$$

and admits the representation

$$(R^{2T} f)(t) = -\nu f_t(t) + \omega f(t) + \int_0^t r(t-s) f(s)\, ds, \quad 0 \le t \le 2T, \tag{2.24}$$

with matrices ν, ω as in (2.19) *and with a smooth symmetric matrix-valued function $r(t)$, $0 \le t \le 2T$. The function $r\big|_{0 \le t \le T}$ coincides with the function r occurring in* (2.18).

It is easily seen that R^{2T} coincides with the (nonextended) response operator of the system $\mathfrak{s}^{2T}$ with the final moment $t = 2T$[5]. Therefore, formula (2.24) does not require any separate proof; it suffices to reproduce (2.18) with another final moment. The coincidence of these two operators is a consequence of hyperbolicity (finiteness of the impact domains).

The nature of the dependence on coefficients, mentioned in Proposition 3, makes the extended operator an attribute of the two-velocity dynamical system *with the final moment $t = T$*: like the other elements of the system $\mathfrak{s}^T$, this operator is determined by the coefficients $\{\rho, \gamma, A, B\}\big|_{0 \le x \le x_1(T)}$.

Formula (2.24) shows that to define the operator R^{2T} we must have two constant matrices ν and ω and the symmetric matrix $r\big|_{0 \le t \le 2T}$.

Connecting operator. The operator $C^T : \mathcal{F}^T \to \mathcal{F}^T$,

$$C^T := (W^T)^* W^T$$

[5]For that reason, we do not distinguish them notationally.

is said to be *connecting*. The formula

$$(C^T f, g)_{\mathcal{F}^T} = \big(W^T f, W^T g\big)_{\mathcal{H}^{x_1(T)}} = \big(u^f(\,\cdot\,,T), u^g(\,\cdot\,,T)\big)_{\mathcal{H}^{x_1(T)}}, \tag{2.25}$$

which determines C^T, relates the metrics of the external and internal spaces. The operator C^T is bounded (because W^T is), selfadjoint, and nonnegative.

For the boundary control method in inverse problems, a key fact is a simple and explicit relationship between C^T and the extended relation operator, see [18, 6, 15]. To describe this relationship, we introduce the following auxiliary operators:

$$\begin{aligned}
S^T &: \mathcal{F}^T \to \mathcal{F}^{2T}, \quad (S^T f)(t) := \begin{cases} f(t), & 0 \le t \le T, \\ -f(2T-t), & T < t \le 2T, \end{cases} \\
J^{2T} &: \mathcal{F}^{2T} \to \mathcal{F}^{2T}, \quad (J^{2T} f)(t) := \int_0^t f(\eta)\, d\eta, \\
P_-^{2T} &: \mathcal{F}^{2T} \to \mathcal{F}^{2T}, \quad (P_-^{2T} f)(t) := \frac{1}{2}\,[f(t) - f(2T-t)], \\
N^{2T} &: \mathcal{F}^{2T} \to \mathcal{F}^{T}, \quad (N^{2T} f)(t) := f\big|_{[0,T]}.
\end{aligned}$$

It is easy to check that

$$(S^T)^* = 2N^{2T} P_-^{2T}. \tag{2.26}$$

Lemma 1. *We have*

$$\begin{aligned}
C^T &= -\frac{1}{2}\,(S^T)^* R^{2T} J^{2T} S^T; \\
(C^T f)(t) &= \nu f(t) + \int_0^T c^T(t,s) f(s)\, ds, \quad 0 \le t \le T,
\end{aligned} \tag{2.27}$$

with the kernel

$$c^T(t,s) := \frac{1}{2} \int_{|t-s|}^{2T-t-s} r(\eta)\, d\eta,$$

smooth in $[0,T] \times [0,T]$ *outside of the diagonal* $t = s$ *and with* $r\big|_{[0,2T]}$ *as in* (2.24).

Proof. **1.** Choosing controls f, $g \in C_0^\infty((0,T);\mathbb{R}^2)$, we denote

$$f_- := S^T f \in C^\infty((0,2T);\mathbb{R}^2).$$

Let u^g and u^{f_-} be (classical) solutions of problems (2.1)–(2.3) and (2.21)–(2.23) with the controls g and f_-, respectively. Recall that the solution u^{f_-} is defined in the domain Δ^{2T}. Since the supports of f and g are separated away from $t = 0$, for s, $t \le T$ we have

$$\operatorname{supp} u^g(\,\cdot\,,t) \subset [0, x_1(t)) \subset [0, x_1(T)) \ \text{ and } \ \operatorname{supp} u^{f_-}(\,\cdot\,,s) \subset [0, x_1(s)) \subset [0, x_1(T)]$$

by (2.8).

We denote

$$\Theta^T := \big\{(s,t) \,\big|\, 0 < t < T,\ t < s < 2T - t\big\};$$

and fix $(s,t) \in \overline{\Theta^T}$. If $0 \le t \le s \le T$, then the solution $u^{f_-}(\,\cdot\,,s)$ is defined on the segment $0 \le x \le x_1(s)$ containing $\operatorname{supp} u^g(\,\cdot\,,t)$. On the other hand, if $T < s \le 2T - t$, then $u^{f_-}(\,\cdot\,,s)$ is only defined for $0 \le x \le x_1(2T-s)$ (is not defined for $x > x_1(2T-s)$), while $\operatorname{supp} u^g(\,\cdot\,,t) \subset [0, x_1(t)) \subset [0, x_1(2T-s)) \subset [0, x_1(T))$. In both cases, the support of $u^g(\,\cdot\,,t)$ is included in a segment on which the solution $u^{f_-}(\,\cdot\,,s)$ *is defined*, and $u^g(\,\cdot\,,t)$ vanishes near the right end of this segment.

Therefore, the product $u^{f_-}(x,s)u^g(x,t)$ extended by zero from the segment where the first factor is defined to $0 \le x \le x_1(T)$, is well defined and yields a smooth function that

vanishes near $x = x_1(T)$. Together with the above product, the following Blagoveshchenskiĭ function is well defined on $\overline{\Theta^T}$:

$$b(s,t) := \big(u^{f_-}(\,\cdot\,,s), u^g(\,\cdot\,,t)\big)_{\mathcal{H}^{x_1(T)}} = \int_0^{x_1(T)} \big[\rho(x)u^{f_-}(x,s)\big]\cdot u^g(x,t)\,dx.$$

2. Now we differentiate with respect to s, $t \in \Theta^T$ and integrate by parts, using (2.5) and the information about the supports of solutions:

$$\begin{aligned}
&(\partial_t^2 - \partial_s^2)w(s,t)\\
&\quad= \int_0^{x_1(T)} dx\big\{u^{f_-}(x,s)\cdot\big[\rho(x)u^g_{tt}(x,t)\big] - \big[\rho(x)u^{f_-}_{ss}(x,s)\big]\cdot u^g(x,t)\big\}\\
&\quad\overset{(2.1)}{=} \int_0^{x_1(T)} u^{f_-}(x,s)\cdot\big((\gamma u^g_x)_x - Au^g_x - Bu^g\big)(x,t)\,dx\\
&\qquad\qquad - \int_0^{x_1(T)} \big((\gamma u^{f_-}_x)_x - Au^{f_-}_x - Bu^{f_-}\big)(x,s)\cdot u^g(x,t)\,dx\\
&\quad= -\big(u^{f_-}(0,s)\cdot\gamma(0)u^g_x(0,t) + \gamma(0)u^{f_-}_x(0,s)\cdot u^g(0,t)\big)\\
&\qquad - A^{\mathrm{tr}}(0)u^{f_-}(0,s)\cdot u^g(0,t)\\
&\quad\overset{(2.5)}{=} f_-(s)\cdot\big[A(0)g(t) - (R^Tg)(t)\big] + (R^{2T}f_-)(s)\cdot g(t) = F(s,t).
\end{aligned} \tag{2.28}$$

At the same time, (2.3) implies the relations

$$\begin{aligned}
b(s,0) &= \big(u^{f_-}(\,\cdot\,,s), u^g(\,\cdot\,,0)\big)_{\mathcal{H}^{x_1(T)}} = 0,\\
b_t(s,0) &= \big(u^{f_-}(\,\cdot\,,s), u^g_t(\,\cdot\,,0)\big)_{\mathcal{H}^{x_1(T)}} = 0.
\end{aligned}$$

As a result, we get the system

$$\begin{aligned}
&b_{tt} - b_{ss} = F, && (x,t)\in\Theta^T,\\
&b(s,0) = b_t(s,0) = 0, && 0 \le s \le 2T.
\end{aligned}$$

3. D'Alembert's integration yields

$$b(s,t) = \frac{1}{2}\int_0^t d\eta \int_{s-t+\eta}^{s+t-\eta} F(\xi,\eta)\,d\xi.$$

Putting $s = t = T$, we obtain

$$b(T,T) = \frac{1}{2}\int_0^T d\eta \int_\eta^{2T-\eta} F(\xi,\eta)\,d\xi.$$

Since the first term in (2.28), which involves $f_-(s)$, is odd with respect to s relative to $s = T$, integration shows that

$$\begin{aligned}
b(T,T) &= \int_0^T d\eta \int_\eta^{2T-\eta} (R^{2T}f_-)(\xi)\cdot g(\eta)\,d\xi\\
&= -\int_0^T g(\eta)\cdot\big[(J^{2T}R^{2T}f_-)(\eta) - (J^{2T}R^{2T}f_-)(2T-\eta)\big]\,d\eta\\
&= -\frac{1}{2}\,\big(g, N^{2T}2P_-^{2T}J^{2T}R^{2T}S^Tf\big)_{\mathcal{F}^T}\\
&\overset{(2.26)}{=} \Big(-\frac{1}{2}(S^T)^*R^{2T}J^{2T}S^Tf, g\Big)_{\mathcal{F}^T}
\end{aligned} \tag{2.29}$$

(the operators J^{2T} and R^{2T} commute because the latter operator is of convolution nature; see (2.24)).

On the other hand, since, obviously, $u^{f-}(\,\cdot\,,T) = u^f(\,\cdot\,,T)$, we have

$$b(T,T) = \big(u^f(\,\cdot\,,T), u^f(\,\cdot\,,T)\big)_{\mathcal{H}^{x_1(T)}} \overset{(2.25)}{=} \big(C^T f, g\big)_{\mathcal{F}^T}. \tag{2.30}$$

Comparing (2.29) and (2.30), and using the density of the controls $g \in C_0^\infty((0,T);\mathbb{R}^2)$ in $\mathcal{F}^T$, we see that

$$C^T f = -\frac{1}{2}(S^T)^* R^{2T} J^{2T} S^T f, \quad f \in C^\infty\big((0,T);\mathbb{R}^2\big).$$

This identity extends by continuity to all $f \in \mathcal{F}^T$, because the operator $R^{2T}J^{2T}$ is bounded, which is an obvious consequence of (2.24).

This proves the first relation in (2.27). Plugging in it the right-hand side of (2.24) in place of R^{2T}, we get the second relation (in (2.27)) after easy transformations. □

2.3. Controllability.

Reachable sets. The linear sets of the form

$$\mathcal{U}^\xi := \big\{u^f(\,\cdot\,,\xi) \,\big|\, f \in \mathcal{F}^T\big\} \overset{(2.16)}{=} W^T\mathcal{F}^{T,\xi} \overset{(2.16)}{\subset} \mathcal{H}^{x_1(\xi)} \quad (0 \le \xi \le T)$$

are said to be *reachable* (at the moment $t = \xi$). The sets $\mathcal{U}^\xi$ grow with the growth of ξ. Their structure and properties constitute the subject of the boundary control theory. The reachable sets are formed by all states (here, waves) of the system that can be created by using controls of a given class. In the situation where these states exhaust all the space that contains them, we say that the system in question is *controllable*.

Observe that a two-velocity system is not controllable: the set $\mathcal{U}^T$ has a nontrivial orthogonal complement in $\mathcal{H}^{x_1(T)}$ for any $T > 0$. As an example, consider system (2.1)–(2.3) with constant $\rho_1 < \rho_2$, $\gamma_1 = \gamma_2 = 1$, and $A = B = 0$. In this case,

$$u^f(x,T) = \begin{pmatrix} f_1(T - \sqrt{\rho_1}\,x) \\ f_2(T - \sqrt{\rho_2}\,x), \end{pmatrix}$$

$$\mathcal{U}^T = \left\{ y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \,\middle|\, y_1, y_2 \in L_2[0, c_1T],\ \operatorname{supp} y_2 \subset [0, c_2T] \right\}$$

and then, obviously, $\mathcal{H}^{x_1(T)} \ominus \mathcal{U}^T = L_2([0,c_1T];\mathbb{R}^2) \ominus \mathcal{U}^T \ne \{0\}$.

The reachable sets of the system $\mathfrak{s}^T$ admit the following description.

Proposition 4. *There exists a function $K^T(x,s)$, defined and smooth for $x_2(T) \le x \le s \le x_1(T)$, such that the fact that $y = \binom{y_1}{y_2} \in \mathcal{U}^T$ is equivalent to the following relationship between the components y_1 and y_2:*

$$y_2(x) = \int_x^{x_1(T)} K^T(x,s) y_1(s)\, ds, \quad x_2(T) \le x \le x_1(t). \tag{2.31}$$

We explain the origin of this relationship.

Calling off Agreement 1a for some time, consider the following auxiliary initial-boundary value problem:

$$\rho v_{tt} - (\gamma v_x)_x + A v_x + B v = 0, \qquad 0 < x < x_1(T), -\infty < t < T, \tag{2.32}$$

$$v\big|_{x=x_1(T)} = v_x\big|_{x=x_1(T)} = 0, \qquad -\infty < t \le T, \tag{2.33}$$

$$v\big|_{t=T} = y, \qquad 0 \le x \le x_1(T). \tag{2.34}$$

It differs from problem (2.1)–(2.3) only by interchanging[6] the roles of the variables, and the proof that it is well posed can be done as before, by reduction to a system of integral Volterra equations of the 2nd kind. For any $y \in \mathcal{H}^{x_1(T)}$, system (2.32)–(2.34)

[6]In this problem, it is natural to think of x as time and t as the coordinate.

has a unique (adequately defined generalized) solution $v = v^y(x,t)$, for which we have a formula similar to (2.7):

$$v^y(x,t) = \lambda(x,t)\begin{pmatrix} y_1\big(x_1(\tau_1(x)+T-t)\big) \\ y_2\big(x_2(\tau_2(x)+T-t)\big)\end{pmatrix} + \int_{x_2\left(\tau_2(x)+T-t\right)}^{x_1(T)} q^T(x,t,s)y(s)\,ds \tag{2.35}$$

where

$$\lambda = \operatorname{diag}\{\lambda_1,\lambda_2\} \,:\, \lambda_i(x,t) := \left(\frac{\rho_i(x_i(\tau_i(x)+T-t))\gamma_i(x_i(\tau_i(x)+T-t))}{\rho_i(x)\,\gamma_i(x)}\right)^{\frac{1}{4}}.$$

For every $s \in [0,x_1(T)]$, the kernel q^T is smooth in $[0,x_1(T)]\times(-\infty,T]$ outside of the characteristics $t = \tau_i(x)+T-\tau_i(s)$, vanishes for $t < \tau_2(x)+T-\tau_2(s)$, and $q^T(x,T,s) = 0$. Observe that

$$v^y\big|_{x>x_2(t-(T-\xi))} = 0 \quad \text{for} \quad y \in \mathcal{H}^{x_2(\xi)} \quad (0 \le \xi \le T), \tag{2.36}$$

which follows easily from the form of the right-hand side in (2.35) and is an analog of relation (2.14) in problem (2.11)–(2.13).

Comparing problems (2.1)–(2.3) and (2.32)–(2.34), we easily find a relationship between their solutions: if $y = u^f(\,\cdot\,,T)$, then

$$u^f(x,t) = v^y(x,t), \quad 0 \le x \le x_1(T),\ \ 0 \le t \le T, \tag{2.37}$$

and $f = v^y(0,\,\cdot\,)$. It follows that $y \in \mathcal{U}^T$ if and only if $v^y(0,\,\cdot\,) \in \mathcal{F}^T$. The latter relation is equivalent to the condition $v^y(0,t)\big|_{t<0} = 0$. Imposing this condition and putting $x = 0$ in (2.35), we see that the components y_1 and y_2 are not independent. The further analysis results in relation (2.31). In a more general form that takes (2.36) into account, for the sets $\mathcal{U}^\xi$ Proposition 4 looks like this.

Proposition 5. *There exists a function $K^\xi(x,s)$, defined and smooth for $x_2(\xi) \le x \le s \le x_1(\xi)$, such that $y = \left(\begin{smallmatrix} y_1 \\ y_2\end{smallmatrix}\right) \in \mathcal{U}^\xi$ if and only if the components y_1 and y_2 are related to each other in the following way:*

$$y_2(x) = \int_x^{x_1(T)} K^\xi(x,s)y_1(s)\,ds, \quad x_2(\xi) \le x \le x_1(\xi).$$

Now, the operator $V^T\colon \mathcal{H}^{x_1(T)} \to \mathcal{F}^T_{\text{ext}} := L_2\big([T-\tau_2(x_1(T)),T];\mathbb{R}^2\big) \supset \mathcal{F}^T$ arises that solves problem (2.32)–(2.34):

$$V^T y := v^y(0,\,\cdot\,).$$

Formula (2.35) shows that this operator is bounded, and (2.37) implies the relation

$$V^T W^T = \mathbb{I}_{\mathcal{F}^T} \tag{2.38}$$

($\mathbb{I}_{\mathcal{F}^T}$ is the identity operator on $\mathcal{F}^T$). Consequently, the control operator has a bounded left inverse. Since it is Fredholm, we arrive at the following statement[7].

Proposition 6. *The operator W^T acts from $\mathcal{F}^T$ into $\mathcal{H}^{x_1(T)}$ isomorphically onto its range $\operatorname{Ran} W^T = \mathcal{U}^T$, which is a closed subspace. The operator $(W^T)^*\colon \mathcal{H}^{x_1(T)} \to \mathcal{F}^T$ annihilates the defect subspace $\mathcal{H}^{x_1(T)} \ominus \mathcal{U}^T$ and takes $\mathcal{U}^T$ onto $\mathcal{F}^T$ isomorphically. The connecting operator C^T is a positive isomorphism of the space $\mathcal{F}^T$.*

The second claim is a consequence of the known decomposition $\mathcal{H}^{x_1(T)} = \operatorname{Ran} W^T \oplus \operatorname{Ker}(W^T)^*$; the third follows from the first two: C^T is the composition of the isomorphisms W^T and $(W^T)^*\big|_{\operatorname{Ran} W^T}$.

[7]Throughout, by isomorphisms we mean operators that act surjectively and have bounded inverse.

Slow waves. For any $y \in \mathcal{H}^{x_2(T)}$, relation (2.31) is fulfilled trivially, so $\mathcal{H}^{x_2(T)} \subset \mathcal{U}^T$. We describe the inverse image $(W^T)^{-1}\mathcal{H}^{x_2(T)}$ under the isomorphism $W^T\colon \mathcal{F}^T \to \mathcal{U}^T$.

Proposition 7. *There exists a unique smooth function $l = l(t)$, $0 \le t \le T - \tau_1(x_2(T))$, such that $f = \left(\begin{smallmatrix} f_1 \\ f_2 \end{smallmatrix}\right) \in (W^T)^{-1}\mathcal{H}^{x_2(T)}$ if and only if*

$$f_1(t) = \int_0^t l(t-s)\, f_2(s)\, ds, \quad 0 \le t \le T - \tau_1(x_2(T)). \tag{2.39}$$

We briefly describe the arguments leading to the relationship (2.39) between the control components. The condition $f \in (W^T)^{-1}\mathcal{H}^{x_2(T)}$ is equivalent to $u^f(\,\cdot\,, T)\big|_{x>x_2(T)} = 0$. Imposing this condition on the right-hand side of (2.7), we see that the control components f_1 and f_2 are not fully independent of each other. The further analysis of this dependence results in (2.39). A detailed deduction can be found in the paper [17], where the following relations were also established:

$$l(0) = \frac{\sqrt{\gamma_2(0)\rho_2(0)}}{\gamma_1(0)\rho_2(0) - \gamma_2(0)\rho_1(0)}\, a(0) \;=\; \frac{\alpha\omega_{21}}{(\alpha-1)\nu_1} \tag{2.40}$$

(recall that $\alpha := \frac{c_1(0)}{c_2(0)} > 1$). It is noteworthy that the function l *does not depend on* T[8]. Its values in the interval $0 \le t \le T - \tau_1(x_2(T))$, used for the given T, are determined by the values taken by the coefficients ρ, γ, A, B for $0 \le x \le x_2(T)$ only.

Replacing $\mathcal{H}^{x_2(T)}$ with $\mathcal{H}^{x_2(\xi)}$ and using (2.9), we easily get a relation generalizing (2.39). Denote

$$\pi_1^T(\xi) := T - \xi, \quad \pi_2^T(\xi) := T - \tau_1(x_2(\xi)), \quad 0 \le \xi \le T.$$

By (2.4), we have $\pi_1^T(\xi) < \pi_2^T(\xi)$ for $\xi > 0$. We introduce the collection of subspaces

$$\mathcal{F}_l^{T,\xi} := (W^T)^{-1}\mathcal{H}^{x_2(\xi)}, \quad 0 \le \xi \le T, \tag{2.41}$$

and denote $\mathcal{F}_l^T := \mathcal{F}_l^{T,T}$. Observe that $\mathcal{F}_l^{T,\xi} \subset \mathcal{F}^{T,\xi}$, because

$$W^T\mathcal{F}_l^{T,\xi} = \mathcal{H}^{x_2(\xi)} \subset \mathcal{U}^\xi = W^T\mathcal{F}^{T,\xi}.$$

Proposition 8. *Suppose $f = \left(\begin{smallmatrix} f_1 \\ f_2 \end{smallmatrix}\right) \in \mathcal{F}^{T,\xi}$; then $f \in \mathcal{F}_l^{T,\xi}$ if and only if*

$$f_1(t) = \int_{\pi_1^T(\xi)}^t l(t-s) f_2(s)\, ds, \quad \pi_1^T(\xi) \le t \le \pi_2^T(\xi). \tag{2.42}$$

Observe that this relation imposes no restriction on the control components for $\pi_2^T(\xi) < t \le T$.

The solutions u^f initiated by the controls belonging to $\mathcal{F}_l^{T,\xi}$ will be called *slow waves*. This name is motivated by the following. Identity (2.38) implies

$$\mathcal{F}_l^{T,\xi} = V^T\mathcal{H}^{x_2(\xi)}, \quad 0 \le \xi \le T.$$

For $f \in \mathcal{F}^{T,\xi}$, put $y := u^f(\,\cdot\,, T) \in \mathcal{H}^{x_2(\xi)}$. We have

$$u^f\big|_{x>x_2(t-(T-\xi))} \overset{(2.37)}{=} v^y\big|_{x>x_2(t-(T-\xi))} \overset{(2.36)}{=} 0,$$

which means that, for all $t \in [0, T]$, the support of the wave $u^f(\,\cdot\,, t)$ lies in the interval $[0, x_2(t-(T-\xi))]$, which expands at the "slow" rate $c_2(x) = \frac{dx_2}{d\tau}\big|_{\tau=\tau_2(x)}$, and this is not true if condition (2.42) fails.

[8]We may say that $l\big|_{t\ge 0}$ is an attribute of problem (2.1)–(2.3) with $T = \infty$.

The response operator R^T acts as convolution, see (2.18); the relationship (2.42) is also of convolution nature. Since convolution is commutative, we have

$$R^T\left(\operatorname{Dom} R^T \cap \mathcal{F}_l^{T,\xi}\right) \subset \mathcal{F}_l^{T,\xi}, \quad 0 \le \xi \le T. \tag{2.43}$$

The subsystem $\mathfrak{s}_l^T$. In the system $\mathfrak{s}^T$, the slow waves correspond to the subsystem $\mathfrak{s}_l^T$ described by the initial-boundary value problem

$$\rho u_{tt} - (\gamma u_x)_x + A u_x + B u = 0, \qquad 0 < x < x_2(T),\ \ 0 < t < T, \tag{2.44}$$

$$u\big|_{t<\tau_2(x)} = 0, \tag{2.45}$$

$$u\big|_{x=0} = f, \qquad 0 \le t \le T, \tag{2.46}$$

which is well posed for $f \in \mathcal{F}_l^T$. We list the following attributes of $\mathfrak{s}_l^T$:

- the external and internal spaces are $\mathcal{F}_l^T$ and $\mathcal{H}^{x_2(T)}$;
- the control operator $W_l^T \colon \mathcal{F}_l^T \to \mathcal{H}^{x_2(T)}$ acts by the rule $W_l^T := W^T e_l^T$, where $e_l^T \colon \mathcal{F}_l^T \to \mathcal{F}^T$ in the natural embedding operator;
- the response operator $R_l^T \colon \mathcal{F}_l^T \to \mathcal{F}_l^T$ is the *part of the operator* R^T induced in $\mathcal{F}_l^T$, see (2.43);
- the connecting operator $C_l^T := (W_l^T)^* W_l^T : \mathcal{F}_l^T \to \mathcal{F}_l^T$ is the block of the operator C^T in the subspace $\mathcal{F}_l^T$, i.e., $C_l^T = (e_l^T)^* C^T e_l^T$.

By (2.41) we have $W_l^T \mathcal{F}_l^T = \mathcal{H}^{x_2(T)}$, i.e., in contrast to the system itself, the subsystem $\mathfrak{s}_l^T$ *is controllable*[9].

2.4. Amplitude formula. The representation of waves deduced in this subsection is a principal tool for the solution of inverse problems by the boundary control method (see [18, 6, 15]). The deduction of the formula employs the peculiarities of the propagation of discontinuities in the system $\mathfrak{s}^T$, and, in essence, the representation itself is a formula from geometrical optics.

The projections $X_l^{T,\xi}$. Fixing $\xi \in (0, T]$, we decompose the external space of the system $\mathfrak{s}^T$ in the orthogonal sum

$$\mathcal{F}^T = \mathcal{F}_l^{T,\xi} \oplus (\mathcal{F}_l^{T,\xi})^\perp; \tag{2.47}$$

let $X_l^{T,\xi}$ be the projection onto the first summand. To describe its action, we introduce the operator $\Lambda^\xi : L_2[\pi_1^T(\xi), \pi_2^T(\xi)] \to L_2[\pi_1^T(\xi), \pi_2^T(\xi)]$ given by

$$(\Lambda^\xi g)(t) := \int_{\pi_1^T(\xi)}^{t} l(t-s) g(s)\, ds, \quad \pi_1^T(\xi) \le t \le \pi_2^T(\xi). \tag{2.48}$$

Proposition 9. *For $f = \left(\begin{smallmatrix} f_1 \\ f_2 \end{smallmatrix}\right) \in \mathcal{F}^T$, we have*

$$X_l^{T,\xi} f\big|_{0 \le t < \pi_1^T(\xi)} = 0, \tag{2.49}$$

$$\begin{aligned} X_l^{T,\xi} f\big|_{\pi_1^T(\xi) \le t < \pi_2^T(\xi)} &= \begin{pmatrix} \Lambda^\xi \left[\mathbb{I} + (\Lambda^\xi)^* \Lambda^\xi\right]^{-1} \left[(\Lambda^\xi)^* f_1 + f_2\right] \\ \left[\mathbb{I} + (\Lambda^\xi)^* \Lambda^\xi\right]^{-1} \left[(\Lambda^\xi)^* f_1 + f_2\right] \end{pmatrix} \\ &= \begin{pmatrix} (\Lambda^\xi f_2)(t) \\ f_2(t) \end{pmatrix} + \int_{\pi_1^T(\xi)}^{\pi_2^T(\xi)} j^{T,\xi}(t,s) f(s)\, ds, \end{aligned} \tag{2.50}$$

$$X_l^{T,\xi} f\big|_{\pi_2^T(\xi) \le t \le T} = f(t), \tag{2.51}$$

[9] We may say that the passage to slow waves restores controllability, but only on the "slow" segment $[0, x_2(T)]$, captured by such waves by the final moment.

where $j^{T,\xi}$ is a matrix kernel smooth in $[\pi_1^T(\xi),\pi_2^T(\xi)]\times[\pi_1^T(\xi),\pi_2^T(\xi)]$ outside of the diagonal $t=s$ and satisfying $|j^{T,\xi}(t,s)|\le \mathrm{const}$ with a constant independent of ξ, t, s.

These formulas are a result of fairly lengthy calculations, which can be simplified by using the form of the second term in the decomposition $f = X_l^{T,\xi}f + f_\perp^{T,\xi}$ that corresponds to (2.47). This term looks like this:

$$\begin{aligned} f_\perp^{T,\xi}\big|_{0\le t<\pi_1^T(\xi)} &= f(t),\\ f_\perp^{T,\xi}\big|_{\pi_1^T(\xi)\le t<\pi_2^T(\xi)} &= \begin{pmatrix} \big[\mathbb{I}+\Lambda^\xi(\Lambda^\xi)^*\big]^{-1}\big[f_1-\Lambda^\xi f_2\big]\\ -(\Lambda^\xi)^*\big[\mathbb{I}+\Lambda^\xi(\Lambda^\xi)^*\big]^{-1}\big[f_1-\Lambda^\xi f_2\big]\end{pmatrix},\\ f_\perp^{T,\xi}\big|_{\pi_2^T(\xi)\le t\le T} &= 0, \end{aligned}$$

and the proof of (2.49)–(2.51) reduces to the verification of the identities $f = X_l^{T,\xi}f+f_\perp^{T,\xi}$ and $(X_l^{T,\xi}f, f_\perp^{T,\xi})_{\mathcal{F}^T} = 0$. The integral representation in (2.50) is deduced from the preceding operator representation with the use of (2.48).

The projections $\mathcal{P}_l^{T,\xi}$. On the "controllable" part $\mathcal{H}^{x_2(T)}$ of the space $\mathcal{H}^{x_1(T)}$, we define a family of orthogonal projections $P^{x_2(\xi)}$ onto the subspaces $\mathcal{H}^{x_2(\xi)}$. Their action reduces to cutting: for $y\in\mathcal{H}^{x_2(T)}$ we put

$$\big(P^{x_2(\xi)}y\big)(x) = \begin{cases} y(x), & 0\le x<x_2(\xi),\\ 0, & x_2(\xi)\le x\le x_2(T),\end{cases}\qquad 0\le\xi\le T.$$

Recall the relationship (2.41) between the subspaces $\mathcal{H}^{x_2(\xi)}$ and $\mathcal{F}_l^{T,\xi}$.

On the external space $\mathcal{F}^T$ we define the family of operators

$$\mathcal{P}_l^{T,\xi} := (W^T)^{-1}P^{x_2(\xi)}W^T,\quad 0\le\xi\le T. \tag{2.52}$$

Observe the following relations, which follow easily from the definition:

$$(\mathcal{P}_l^{T,\xi})^2 = \mathcal{P}_l^{T,\xi},\quad C^T\mathcal{P}_l^{T,\xi} = (\mathcal{P}_l^{T,\xi})^*C^T,\quad \operatorname{Ran}\mathcal{P}_l^{T,\xi} = \mathcal{F}_l^{T,\xi}. \tag{2.53}$$

As a consequence, we see that $\mathcal{P}_l^{T,\xi}$ is an (oblique) projection from $\mathcal{F}^T$ onto $\mathcal{F}_l^{T,\xi}$ parallel to the subspace $(C^T)^{-1}[\mathcal{F}^T\ominus\mathcal{F}_l^{T,\xi}]$. The properties (2.53) characterize this projection. In order to describe the action of $\mathcal{P}_l^{T,\xi}$, we introduce the embedding operators $e_l^{T,\xi}\colon\mathcal{F}_l^{T,\xi}\to\mathcal{F}^T$[10] and observe the known relations

$$(e_l^{T,\xi})^*e_l^{T,\xi} = \mathbb{I}_{\mathcal{F}_l^{T,\xi}},\quad e_l^{T,\xi}(e_l^{T,\xi})^* = X_l^{T,\xi},\quad 0\le\xi\le T. \tag{2.54}$$

Since C^T is a positive isomorphism, its blocks $(e_l^{T,\xi})^*C^Te_l^{T,\xi}$ in the subspaces $\mathcal{F}_l^{T,\xi}\subset\mathcal{F}_l^T$ are isomorphisms of these subspaces.

Proposition 10. *For $0\le\xi\le T$ we have*

$$\mathcal{P}_l^{T,\xi} = e_l^{T,\xi}\big[\big(e_l^{T,\xi}\big)^*C^Te_l^{T,\xi}\big]^{-1}(e_l^{T,\xi})^*C^T; \tag{2.55}$$

$$(\mathcal{P}_l^{T,\xi}f)(t) = \begin{cases} 0, & 0<t\le\pi_1^T(\xi),\\ \begin{pmatrix}\Lambda^\xi f_2\\ f_2\end{pmatrix}(t) + \int_0^{\pi_2^T(\xi)} p^{T,\xi}(t,s)f(s)\,ds, & \pi_1^T(\xi)<t\le\pi_2^T(\xi),\\ f(t)+\int_0^{\pi_2^T(\xi)} p^{T,\xi}(t,s)f(s)\,ds, & \pi_2^T(\xi)<t\le T.\end{cases} \tag{2.56}$$

The kernel $p^{T,\xi}(t,s)$ is smooth in $[\pi_1^T(\xi),T]\times[0,\pi_2^T(\xi)]$ outside of the diagonal $t=s$ and the lines $t=\pi_2^T(\xi)$ and $s=\pi_1^T(\xi)$, and satisfies $|p^{T,\xi}(t,s)|\le\mathrm{const}$ with a constant independent of ξ, t, s.

[10]Recall that the embedding $e_l^T = e_l^{T,T}\colon\mathcal{F}_l^T\to\mathcal{F}^T$ was used in the description of the subsystem $\mathfrak{s}_l^T$.

The operator representation is established by a direct verification of relations (2.53). Then, using the integral representation in (2.27), we easily deduce the formula

$$\mathcal{P}_l^{T,\xi} = e_l^{T,\xi}(e_l^{T,\xi})^* + K_l^{T,\xi} \overset{(2.54)}{=} X_l^{T,\xi} + K_l^{T,\xi}$$

with an integral operator $K_l^{T,\xi}$. The specific form of the latter and the properties of its kernel $p^{T,\xi}$ are deduced from the formulas of Proposition 9 as a result of a bulky analysis of compositions of the kernels involved. We present a relation that will be used below for characterization of the data. The representations (2.49)–(2.51) and (2.56) show that, for a smooth control f, the discontinuities of the projection $\mathcal{P}_l^{T,\xi} f$ can only occur for $t = \pi_i^T(\xi)$. The size of these discontinuities is found from the same representations, and a simple analysis yields the following result. Denote

$$a\langle s\rangle := a(s+0) - a(s-0), \quad \bar{i} := \begin{cases} 2, & i = 1, \\ 1, & i = 2. \end{cases} \tag{2.57}$$

Proposition 11. *For smooth $f \in \mathcal{F}^T$, the components $\mathcal{P}_l^{T,\xi} f$ satisfy*

$$\lim_{\xi\to 0} (\mathcal{P}_l^{T,\xi} f)_i \langle \pi_{\bar{i}}^T(\xi)\rangle = f_i(T), \quad i = 1, 2. \tag{2.58}$$

In the case where ρ_i and γ_i are constant, formulas (2.58) were established in [6].

Remark 1. The deduction of formulas (2.49)–(2.51), (2.55), (2.56), and (2.58) does not employ the fact that the function l is related to slow waves; only the form (2.48) of the operator Λ^ξ is important. The formulas mentioned above are valid for an arbitrary $l \in C^\infty[0, \pi_2^T(T)]$.

Running ahead, we note that for any function of this sort there exists a system $\mathfrak{s}^T$ in which this function will determine the slow waves (the subsystem $\mathfrak{s}_l^T$).

The space Φ^T. The following interpretation of the projections $\mathcal{P}_l^{T,\xi}$ is useful. In the external space $\mathcal{F}^T$, we introduce the new scalar product (metric)

$$(f,g)_{\Phi^T} := (C^T f, g)_{\mathcal{F}^T} \overset{(2.25)}{=} (W^T f, W^T g)_{\mathcal{H}^{x_1(T)}} = (u^f(\,\cdot\,,T), u^g(\,\cdot\,,T))_{\mathcal{H}^{x_1(T)}}. \tag{2.59}$$

The space emerging in this way will be denoted by Φ^T. Since C^T is an isomorphism (Proposition 6), the new metric is equivalent to the initial one, and $\mathcal{F}^T$ and Φ^T consist of the same elements. This fact will be used notationally: Φ_l^T is $\mathcal{F}_l^T$ viewed as a subspace of Φ^T. The subspaces $\Phi_l^{T,\xi} \equiv \mathcal{F}_l^{T,\xi}$ are understood similarly.

Observe the following relationship between the conjugation operations: for a bounded (in $\mathcal{F}^T$ or, equivalently, in Φ^T) operator A we have

$$C^T A^\star = A^* C^T, \tag{2.60}$$

where $(\,)^\star$ denotes conjugation in Φ^T.

Formulas (2.60) and (2.53) imply the relations

$$(\mathcal{P}_l^{T,\xi})^2 = \mathcal{P}_l^{T,\xi}, \quad (\mathcal{P}_l^{T,\xi})^\star = \mathcal{P}_l^{T,\xi}, \quad \operatorname{Ran} \mathcal{P}_l^{T,\xi} = \mathcal{F}_l^{T,\xi}.$$

It follows that the projection $\mathcal{P}_l^{T,\xi}$ viewed as an operator in Φ^T is an *orthogonal* projection of Φ^T onto the subspace $\Phi_l^{T,\xi}$.

By (2.59), the control operator W_l^T of the subsystem $\mathfrak{s}_l^T$, viewed as an operator from Φ_l^T to $\mathcal{H}^{x_2(T)}$, is unitary. Formula (2.52) shows that this operator plays the role of the transformation that diagonalizes the family of projections $\{\mathcal{P}_l^{T,\xi}\}_{0\le\xi\le T}$ in the sense of the spectral theorem.

Representation of waves. By using (2.15), it is easy to show that the piecewise smooth controls induce piecewise smooth waves. Moreover, the smoothness of the kernel $w^T(x,t)$ in (2.15) ensures that the integral term is continuous (in x) for any $f \in \mathcal{F}^T$. For the same reason, the converse is also true: wave discontinuities may arise only if the corresponding controls are discontinuous. The sizes (*amplitudes*) of the discontinuities are related as follows:

$$(W^T f)_i\langle x_2(\xi)\rangle = -\theta_i(x_2(\xi))\, f_i\langle \pi_{\bar{i}}^T(\xi)\rangle, \quad 0<\xi<T, \;\; i=1,2 \tag{2.61}$$

(see the notation (2.57)); this is an easy consequence of (2.15).

Fixing $\xi \in (0,T)$, we pick a control f that produces a smooth solution u^f. The identities

$$u^{\mathcal{P}_l^{T,\xi} f}(\,\cdot\,,T) = W^T \mathcal{P}_l^{T,\xi} f \overset{(2.52)}{=} P^{x_2(\xi)} W^T f = \begin{cases} u^f(\,\cdot\,,T), & 0 \le x < x_2(\xi), \\ 0, & x \ge x_2(\xi), \end{cases}$$

show that the wave $u^{\mathcal{P}_l^{T,\xi} f}(\,\cdot\,,T)$ is discontinuous at the point $x = x_2(\xi)$, and the corresponding amplitude is

$$\big(u^{\mathcal{P}_l^{T,\xi} f}(\,\cdot\,,T)\big)\langle x_2(\xi)\rangle = -u^f(x_2(\xi),T). \tag{2.62}$$

We apply relations (2.61) to this wave (i.e., replace f by $\mathcal{P}_l^{T,\xi} f$ and $W^T f$ by $W^T \mathcal{P}_l^{T,\xi} f = u^{\mathcal{P}_l^{T,\xi}}(\,\cdot\,,T)$) and employ (2.62) to get

$$\big(u^f(x_2(\xi),T)\big)_i = \theta_i(x_2(\xi))\,\big(\mathcal{P}_l^{T,\xi} f\big)_i\langle \pi_{\bar{i}}^T(\xi)\rangle, \quad 0<\xi<T,$$

or

$$u^f\big(x_2(\xi),T\big) = \theta(x_2(\xi)) \begin{pmatrix} (\mathcal{P}_l^{T,\xi} f)_1\langle \pi_2^T(\xi)\rangle \\ (\mathcal{P}_l^{T,\xi} f)_2\langle \pi_1^T(\xi)\rangle \end{pmatrix}, \quad 0<\xi<T.$$

Now, substituting $x = x_2(\xi) \in (0, x_2(T))$ and using the formulas $\pi_2^T(\xi) = T - \tau_1(x_2(\xi))$ and $\pi_1^T(\xi) = T - \xi = T - \tau_2(x_2(\xi))$, we arrive at the representation

$$u^f(x,T) = \theta(x) \begin{pmatrix} (\mathcal{P}_l^{T,\tau_2(x)} f)_1\langle T-\tau_1(x)\rangle \\ (\mathcal{P}_l^{T,\tau_2(x)} f)_2\langle T-\tau_2(x)\rangle \end{pmatrix}, \quad 0<x<x_2(T), \tag{2.63}$$

called the *amplitude formula* (AF), because the waves in this formula are expressed in terms of the amplitudes of the discontinuities arising when the projections $\mathcal{P}_l^{T,\xi}$ act on controls. Observe the following specific feature of the AF, which is used when solving inverse problems. The column on the right-hand side involves only objects corresponding to the exterior spaces – controls and the projections $\mathcal{P}_l^{T,\xi}$. To construct these projections, it suffices to have the operator $\mathcal{P}_l^{T,\xi}$ and the function l. Recalling (2.27) and (2.24), it is easy to show that the column in the AF is determined by the extended response operator R^{2T} (i.e., by the matrices ν and ω, and by the response function $r\big|_{[0,2T]}$), and the function $l\big|_{[0,\pi_2^T(T)]}$. This fact will play a key role in what follows.

§3. Characterization of data

3.1. The main result. In the inverse problems, the response operator plays the role of the data by which it is required to recover the parameters of the dynamical system in question. A characteristic description of the data provides necessary and sufficient conditions for the inverse problem to be solvable. Applied to the system $\mathfrak{s}^T$[11], these conditions look like this.

[11] Recall that the system $\mathfrak{s}^T$ is determined by the initial-boundary problem (2.11)–(2.13) with smooth coefficients, and that conditions (2.4) and (2.5) are assumed throughout.

Theorem 1. *An operator* $\mathcal{R}^{2T}\colon L_2\big([0,2T];\mathbb{R}^2\big) \to L_2\big([0,2T];\mathbb{R}^2\big)$, $\operatorname{Dom}\mathcal{R}^{2T} = \mathcal{M}^{2T}$, *of the form*

$$\big(\mathcal{R}^{2T} f\big)(t) = -\nu f_t(t) + \omega f(t) + \int_0^t r(t-s) f(s)\,ds, \quad 0 \le t \le 2T, \tag{3.1}$$

with constant matrices $\nu = \operatorname{diag}\{\nu_1, \nu_2\}$ *and* ω *and a smooth matrix-valued function* $r\big|_{0\le t\le 2T}$ *is the extended response operator for some system* $\mathfrak{s}^T$ *if and only if the following conditions are satisfied*:

1) $\nu_1, \nu_2 > 0$, $\omega_{12} = -\alpha\omega_{21}$ *with* $\alpha > 1$;
2) $[r(t)]^{\mathrm{tr}} = r(t)$, $0 \le t \le 2T$;
3) *the operator* $\mathcal{C}^T$ *that acts in* $L_2\big([0,T];\mathbb{R}^2\big)$ *by the rule*

$$\big(\mathcal{C}^T f\big)(t) := \nu f(t) + \int_0^T \left[\frac{1}{2}\int_{|t-s|}^{2T-t-s} r(\eta)\,d\eta\right] f(s)\,ds, \quad 0 \le t \le T, \tag{3.2}$$

is a positive isomorphism.

The "only if" part. If a system $\mathfrak{s}^T$ is such that $R^{2T} = \mathcal{R}^{2T}$, then for its connecting operator we have $C^T = \mathcal{C}^T$, and conditions 1–3 are fulfilled, see Proposition 3, Lemma 1, and Proposition 6.

The "if" part (about the proof). In the remaining part of the paper, we verify that conditions 1–3 suffice. This verification is constructive: given $\mathcal{R}^{2T}$, we build a system $\mathfrak{s}^T$ whose response operator coincides with $\mathcal{R}^{2T}$.

To construct a system of the form (2.11)–(2.13) means to produce the coefficients ρ, γ, A, B that determine this system. Taking condition (2.5) into account, we see that there coefficients are determined by the eight parameters (scalar functions) ρ_1, ρ_2, γ_1, γ_2, a_{12}, b_{11}, b_{12}, b_{22}. At the same time, the operator R^{2T} is determined by the three parameters r_{11}, $r_{12}(= r_{21})$, r_{22} (plus the constant matrices ν and ω). Therefore, the uniqueness of a system $\mathfrak{s}^T$ with a given R^{2T} cannot be expected, and the problem is to describe *all* such systems. Roughly speaking, the construction will be reduced to a "self-consistent" choice of free five ($5 = 8 - 3$) parameters, followed by the verification of the consistency and validity of the choice. As a guideline for our choice, we use the properties and relations of the objects of the system $\mathfrak{s}^T$, established when we studied it in §2. In fact, starting with the data (3.1), we shall construct a "slow" system $\mathfrak{s}_l^{T'}$ of the form (2.44)–(2.46) with $T' > T$, and the system $\mathfrak{s}^T$ will come as a subsystem of this "slow" system.

3.2. The system $\mathfrak{s}_l^{T'}$.

Choice of parameters. So, we start with a collection $\nu, \omega, r\big|_{[0,2T]}$ satisfying conditions 1–3 of Theorem 1. Our considerations are illustrated in Figure 2.

Step 1. The matrix ω allows us to choose $\alpha = \text{const} > 1$ so as to ensure condition 1. Let a segment $[0,h]$ be fixed on the semiaxis $x \ge 0$; all the further considerations will go within this segment.

Step 2. Let two functions c_1, $c_2 \in C^\infty[0,h]$ be such that $0 < c_2(x) < c_1(x)$, $0 \le x \le h$ (cf. (2.4)), and

$$c_1(0) = \alpha c_2(0) \tag{3.3}$$

(cf. (2.20)). We define

$$\tau_i(x) := \int_0^x \frac{ds}{c_i(s)}, \quad T' := \tau_2(h).$$

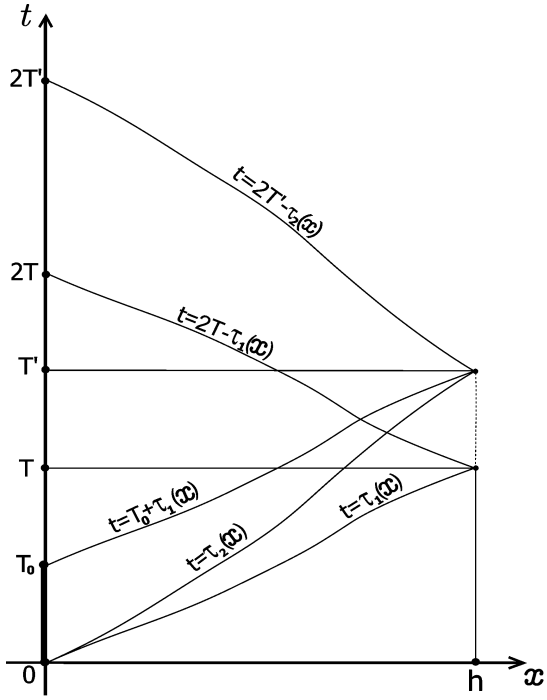

FIGURE 2. Choice of parameters

Let $x_1(\tau)$ and $x_2(\tau)$ be functions inverse to $\tau_1(x)$ and $\tau_2(x)$ and defined on the segments $[0,T]$ and $[0,T']$, respectively. The definitions imply that $T' > T$ and $x_1(T) = x_2(T') = h$.

Step 3. We choose functions ρ_1, $\rho_2 \in C^\infty[0,h]$, $\rho_i > 0$, so that

$$\rho_i(0) = \frac{\nu_i}{c_i(0)}, \quad -\frac{c_i(0)}{2}(c_i\rho_i)_x\Big|_{x=0} = \omega_{ii}. \tag{3.4}$$

Since these relations impose conditions only on $\rho_i(0)$ and $\frac{d\rho_i}{dx}(0)$, such a choice is obviously possible. The above conditions are motivated by (2.19).

Put

$$\gamma_i(x) := \rho_i(x)c_i^2(x), \quad 0 \le x \le h.$$

Step 4. We introduce the functions

$$\pi_1^{T'}(\xi) := T' - \tau_2(x_2(\xi)) = T' - \xi, \quad \pi_2^{T'}(\xi) := T' - \tau_1(x_2(\xi)), \quad 0 \le \xi \le T'.$$

We choose a function $l \in C^\infty[0, \pi_2^{T'}(T')]$ such that

$$l(0) = \frac{\alpha\omega_{21}}{(\alpha-1)\nu_1} \tag{3.5}$$

(cf. (2.40)) and denote $T_0 := \pi_2^{T'}(T')$ (see Figure 2).

At this point, the dynamical system that we construct is supplied with the coefficients (matrices) ρ, γ and the function l, which will give the relationship between the components of the slow waves. Independently of the remaining coefficients A, B (to be constructed in what follows), the response operator will be of the form (2.24) with constant matrices ν and ω coinciding with the similar matrices in (3.1). This coincidence is ensured by imposing conditions (3.3)–(3.5).

Step 5. Here, we extend the matrix-valued function r occurring in (3.1), given for $0 \le t \le 2T$, to the larger segment $0 \le t \le 2T'$. We say that a function $r\big|_{[0,2T']}$ is a Hermite positive extension of the function $r\big|_{[0,2T]}$ if the operator

$$\big(\mathcal{C}^{T'} f\big)(t) := \nu f(t) + \int_0^{T'} \left[\frac{1}{2}\int_{|t-s|}^{2T'-t-s} r(\eta)\,d\eta\right] f(s)\,ds, \quad 0 \le t \le T',$$

is a positive isomorphism in $L_2\big([0,T'];\mathbb{R}^2\big)$ (like $\mathcal{C}^T$ in $L_2\big([0,T];\mathbb{R}^2\big)$). The method of extension presented below was suggested in [6]; it employs an auxiliary *one-velocity* system.

A system of the form (2.1)–(2.3) is said to be *one-velocity* if $\rho_i = \gamma_i = 1$, $A = 0$, and $B^{\mathrm{tr}} = B =: Q$. All its properties are determined by the matrix-valued function (*potential*) $Q\big|_{0\le x\le T}$. For such systems, characterization of the data is known: a smooth symmetric function $\widetilde{r}\big|_{0\le t\le 2T}$ is the response function of a one-velocity system if and only if the operator

$$f \mapsto f(t) + \int_0^{T} \left[\frac{1}{2}\int_{|t-s|}^{2T-t-s} \widetilde{r}(\eta)\,d\eta\right] f(s)\,ds, \quad 0 \le t \le T,$$

is a positive isomorphism in $L_2\big([0,T];\mathbb{R}^2\big)$. The potential $Q\big|_{0\le x\le T}$ is uniquely recovered by the response function with the help of classical tools such as the Gelfand–Levitan–Kreĭn type equation (see [18, 19]).

The required extension of the function r is constructed as follows.

- The operator $\widetilde{C}^T := \nu^{-\frac{1}{2}}\mathcal{C}^T\nu^{-\frac{1}{2}}$ has the form

$$(\widetilde{C}^T f)(t) \overset{(3.2)}{=} f(t) + \int_0^{T} \left[\frac{1}{2}\int_{|t-s|}^{2T-t-s} \widetilde{r}(\eta)\,d\eta\right] f(s)\,ds, \quad 0 \le t \le T,$$

  (here $\widetilde{r} := \nu^{-\frac{1}{2}} r \nu^{-\frac{1}{2}}$) and, obviously, is a positive isomorphism in $L_2\big([0,T];\mathbb{R}^2\big)$. Consequently, $\widetilde{r}\big|_{0\le x\le 2T}$ is the response function for some one-velocity system, and $\widetilde{C}^T$ is its connecting operator.
- We recover the potential $Q\big|_{0\le x\le T}$ by $\widetilde{r}\big|_{0\le t\le 2T}$ and extend it to $[0,T']$ keeping smoothness and symmetry. The extension $Q\big|_{0\le x\le T'}$ gives rise to an extended one-velocity system with a response function $\widetilde{r}\big|_{0\le t\le 2T'}$ that extends $\widetilde{r}\big|_{0\le t\le 2T}$ and with the connecting operator

$$(\widetilde{C}^{T'} f)(t) = f(t) + \int_0^{T'} \left[\frac{1}{2}\int_{|t-s|}^{2T'-t-s} \widetilde{r}(\eta)\,d\eta\right] f(s)\,ds, \quad 0 \le t \le T'.$$

  Like any connecting operator, $\widetilde{C}^{T'}$ is a positive isomorphism of $L_2\big([0,T'];\mathbb{R}^2\big)$. Together with it, such is the operator

$$\begin{aligned}(\mathcal{C}^{T'} f)(t) &:= (\nu^{\frac{1}{2}}\widetilde{C}^{T'}\nu^{\frac{1}{2}} f)(t)\\ &= \nu f(t) + \int_0^{T'} \left[\frac{1}{2}\int_{|t-s|}^{2T'-t-s} r(\eta)\,d\eta\right] f(s)\,ds, \quad 0 \le t \le T',\end{aligned} \tag{3.6}$$

  where $r(t) := \nu^{\frac{1}{2}}\widetilde{r}(t)\nu^{\frac{1}{2}}$, $0 \le t \le 2T'$.
- By construction, the extension $r\big|_{0\le t\le 2T'}$ of the function $r\big|_{0\le t\le 2T}$ is Hermite positive.

It is easily seen that *all* smooth Hermite positive extensions of r can be obtained in this way.

We present an operator formula relating $\mathcal{C}^T$ and $\mathcal{C}^{T'}$. These operators act in the spaces $L_2\big([0,T];\mathbb{R}^2\big)$ and $L_2\big([0,T'];\mathbb{R}^2\big)$, respectively. We agree to view the delay operation introduced in (2.10) as the operator that acts from $L_2\big([0,T];\mathbb{R}^2\big)$ to $L_2\big([0,T'];\mathbb{R}^2\big)$ by the rule

$$(\mathcal{T}^{T'}_{T'-T}f)(t) = \begin{cases} 0, & 0 \le t < T'-T, \\ f(t-(T'-T)), & T'-T \le t \le T'. \end{cases} \tag{3.7}$$

The subspace $\operatorname{Ran}\mathcal{T}^{T'}_{T'-T}$ is formed by the controls that vanish for $0 \le t < T'-T$. The conjugate operator

$$(\mathcal{T}^{T'}_{T'-T})^* \colon L_2\big([0,T'];\mathbb{R}^2\big) \to L_2\big([0,T];\mathbb{R}^2\big)$$

acts by the rule

$$((\mathcal{T}^{T'}_{T'-T})^* f)(t) = f(t+(T'-T)), \quad 0 \le t \le T,$$

and $\operatorname{Ran}(\mathcal{T}^{T'}_{T'-T})^* = L_2\big([0,T];\mathbb{R}^2\big)$.

Lemma 2. *We have*

$$\mathcal{C}^T = (\mathcal{T}^{T'}_{T'-T})^* \mathcal{C}^{T'} \mathcal{T}^{T'}_{T'-T}. \tag{3.8}$$

Proof. Let $\widetilde{W}^T$ and $\widetilde{W}^{T'}$ be the control operators of the one-velocity systems that were used for the extension of r. These operators act in the spaces $L_2\big([0,x_1(T)];\mathbb{R}^2\big)$ and $L_2\big([0,x_1(T')];\mathbb{R}^2\big)$, respectively; we view the former as a subspace of the latter. Under this agreement, we have the identity $\widetilde{W}^T = \widetilde{W}^{T'}\mathcal{T}^{T'}_{T'-T}$, which is another form of relation (2.9) for a one-velocity system with the final moment T' and an intermediate moment $s = T$. The above identity yields

$$\widetilde{C}^T = (\widetilde{W}^T)^*\widetilde{W}^T = (\mathcal{T}^{T'}_{T'-T})^*(\widetilde{W}^{T'})^*\widetilde{W}^{T'}\mathcal{T}^{T'}_{T'-T} = (\mathcal{T}^{T'}_{T'-T})^*\widetilde{C}^{T'}\mathcal{T}^{T'}_{T'-T}.$$

Comparing the beginning and the end, and multiplying by $\nu^{\frac{1}{2}}$ from the right and from the left, we arrive at (3.8). $\square$

At this point, to construct the two-velocity system $\mathfrak{s}^{T'}_l$, we have chosen the matrix-valued functions $\rho, \gamma\big|_{0\le x\le h}$, the function $l\big|_{0\le t\le \pi^{T'}_2(T')}$, and the extension $r\big|_{0\le t\le 2T'}$. This collection is determined by the *eight* parameters (scalar functions) $\rho_1, \rho_2, \gamma_1, \gamma_2, l$ and $\{r_{11}, r_{12}, r_{22}\}\big|_{2T\le t\le 2T'}$. This allows us to expect that the freedom in the choice of parameters is exhausted, and the other elements of the system are determined uniquely.

Spaces, operators, waves.

Spaces. We denote $\mathcal{F}^{T'} := L_2\big([0,T'];\mathbb{R}^2\big)$. In that space, the function l gives rise to the family of subspaces

$$\mathcal{F}^{T',\xi}_l := \left\{ f \in \mathcal{F}^{T'} \;\middle|\; f\big|_{[0,T'-\xi]} = 0,\ f_1(t) = \int_{\pi^{T'}_1(\xi)}^{t} l(t-s) f_2(s)\,ds,\ \pi^{T'}_1(\xi) \le t \le \pi^{T'}_2(\xi) \right\},$$
$$0 < \xi < T'; \quad \mathcal{F}^{T',0}_l := \{0\}, \quad \mathcal{F}^{T',T'}_l =: \mathcal{F}^{T'}_l$$

(cf. (2.41), (2.42)). The largest of this subspaces $\mathcal{F}^{T'}_l$ is said to be *external* and its elements are *controls*.

The space $L_{2,\rho}\big([0,h];\mathbb{R}^2\big) =: \mathcal{H}^h$ is said to be *internal* (recall that $h = x_2(T') = x_1(T)$). It includes the family of subspaces

$$\mathcal{H}^s := \big\{ y \in \mathcal{H}^h \mid \operatorname{supp} y \subset [0,s] \big\}, \quad 0 \le s \le h.$$

Projections. *We define* $\mathcal{P}_l^{T',\xi}$ as the (oblique) projection of $\mathcal{F}^{T'}$ onto $\mathcal{F}_l^{T',\xi}$ parallel to the subspace $(\mathcal{C}^{T'})^{-1}[\mathcal{F}^{T'} \ominus \mathcal{F}_l^{T',\xi}]$. Equivalently,

$$\mathcal{P}_l^{T,\xi} := e_l^{T',\xi}\big[\big(e_l^{T',\xi}\big)^*\mathcal{C}^{T'}e_l^{T',\xi}\big]^{-1}(e_l^{T',\xi})^*\mathcal{C}^{T'}, \quad 0 \le \xi \le T',$$

where $e_l^{T',\xi}\colon \mathcal{F}_l^{T',\xi} \to \mathcal{F}^{T'}$ is the embedding operator (cf. (2.55)). This definition is consistent, because the operator $\mathcal{C}^{T'}$ and all its blocks $\big(e_l^{T',\xi}\big)^*\mathcal{C}^{T'}e_l^{T',\xi}$ are isomorphisms in the corresponding $\mathcal{F}_l^{T',\xi}$. By Proposition 10 (see also Remark 1!) we have the representation

$$(\mathcal{P}_l^{T',\xi}f)(t) = \begin{cases} 0, & 0 < t \le \pi_1^{T'}(\xi), \\ \begin{pmatrix} {}^{\Lambda^\xi f_2} \\ f_2 \end{pmatrix}(t) + \int_0^{\pi_2^{T'}(\xi)} p^{T',\xi}(t,s)f(s)\,ds, & \pi_1^{T'}(\xi) < t \le \pi_2^{T'}(\xi), \\ f(t) + \int_0^{\pi_2^{T'}(\xi)} p^{T',\xi}(t,s)f(s)\,ds, & \pi_2^{T'}(\xi) < t \le T', \end{cases} \tag{3.9}$$

in which the kernel $p^{T',\xi}(t,s)$ is smooth on $[\pi_1^{T'}(\xi), T'] \times [0, \pi_2^{T'}(\xi)]$ outside of the diagonal $t = s$ and the lines $t = \pi_2^{T'}(\xi)$ and $s = \pi_1^{T'}(\xi)$, and satisfies the estimate $|p^{T',\xi}(t,s)| \le$ const with a constant independent of ξ, t, s. Also, we have

$$\lim_{\xi\to 0}(\mathcal{P}_l^{T',\xi}f)_i\langle\pi_{\bar{i}}^{T'}(\xi)\rangle = f_i(T'), \quad i = 1,2, \tag{3.10}$$

(cf. (2.58)). For the proof, see [6, Lemma 8]; now we only note that a specific property of the kernel of the integral part of the operator $\mathcal{C}^{T'}$ should be used: this kernel vanishes for $t = T'$ or $s = T'$ (see (3.2) for $T = T'$).

Waves. We denote

$$\theta := \operatorname{diag}\{\theta_1, \theta_2\}, \quad \theta_i(x) := \left(\frac{\rho_i(0)\gamma_i(0)}{\rho_i(x)\gamma_i(x)}\right)^{\frac{1}{4}},$$

and introduce the following linear space of smooth controls, dense in $\mathcal{F}_l^{T'}$:

$$\mathcal{M}_l^{T'} := \mathcal{F}_l^{T'} \cap \big\{f \in C^\infty\big([0,T'];\mathbb{R}^2\big) \bigm| \operatorname{supp} f \subset (0,T']\big\};$$

these controls vanish in the vicinity of $t = 0$. We *define* an operator $\mathcal{W}_l^{T'}\colon \mathcal{F}_l^{T'} \to \mathcal{H}^h$, $\operatorname{Dom}\mathcal{W}_l^{T'} = \mathcal{M}_l^{T'}$, by the rule

$$\big(\mathcal{W}_l^{T'}f\big)(x) := \theta(x)\begin{pmatrix} (\mathcal{P}^{T',\tau_2(x)}f)_1\langle T' - \tau_1(x)\rangle \\ (\mathcal{P}^{T',\tau_2(x)}f)_2\langle T' - \tau_2(x)\rangle \end{pmatrix}, \quad 0 < x < h \tag{3.11}$$

(cf. (2.63)). The images $u^f(\,\cdot\,,T') := \mathcal{W}_l^{T'}f$ will be called *waves*.

The next claim is a key result for what follows.

Proposition 12. *For* $f \in \mathcal{M}_l^{T'}$, *we have*

$$u^f(x,T') = (\mathcal{W}_l^{T'}f)(x) = \theta(x)\begin{pmatrix} f_1\big(T' - \tau_1(x)\big) \\ f_2\big(T' - \tau_2(x)\big) \end{pmatrix} + \int_0^{T'-\tau_1(x)} w^{T'}(x,t)f(t)\,dt, \quad 0 \le x \le h, \tag{3.12}$$

with a kernel $w^{T'}$ *smooth in* $[0,h] \times [0,T']$ *outside of the curves* $t = T' - \tau_i(x)$, *vanishing for* $t > T' - \tau_1(x)$, *and such that*

$$w^{T'}(0,t) = 0, \quad 0 \le t \le T'. \tag{3.13}$$

The proof of the representation (3.12)[12] can be outlined as follows. Formulas (3.9) are plugged in the right-hand side of (3.11). The discontinuities are calculated by taking

[12]It makes sense to compare (3.12) with (2.15).

into account the possible discontinuities and the position of supports for the kernels $p^{T',\xi}$. Simple but lengthy computations result in (3.12) with a kernel with the entries

$$\begin{aligned}
w_{11}(x,t) &= \theta_1(x)\big(p_{11}^{T',\tau_2(x)}(\,\cdot\,,t)\big)\langle T'-\tau_1(x)\rangle,\\
w_{12}(x,t) &= \theta_1(x)\big[\big(p_{12}^{T',\tau_2(x)}(\,\cdot\,,t)\big)\langle T'-\tau_1(x)\rangle - \widehat{l}\big(T'-\tau_1(x)-t\big)\big],\\
w_{21}(x,t) &= \theta_2(x)\,p_{21}^{T',\tau_2(x)}\big(T'-\tau_2(x),t\big),\\
w_{22}(x,t) &= \theta_2(x)\,p_{22}^{T',\tau_2(x)}\big(T'-\tau_2(x),t\big),
\end{aligned}$$

where the discontinuities of the entries of the kernel $p^{T',\xi}$ in the expressions for w_{11} and w_{12} are taken with respect to the first variable, denoted by the dot, and

$$\widehat{l}(s) := \begin{cases} l(s), & 0<s<\tau_2(x)-\tau_1(x),\\ 0, & s>\tau_2(x)-\tau_1(x).\end{cases}$$

The smoothness nature of $w^{T'}$, as indicated in Proposition 12, can be seen from the above expressions. Property (3.13) is a consequence of identities (3.10), which, combined with (3.11), show that

$$u^f(0,T') = f(T'), \tag{3.14}$$

implying that $w^{T'}(0,t)=0$.

On the operator $\mathcal{W}_l^{T'}$. Formula (3.12) shows that the operator $\mathcal{W}_l^{T'}$ is bounded. Its extension by continuity from $\mathcal{M}_l^{T'}$ to $\mathcal{F}_l^{T'}$ is of the same form, and we keep the notation $\mathcal{W}_l^{T'}$ for the extension. Also, the name *waves* will still be applied to the images under the action of this extension.

Since the functions θ_i, τ_i involved in (3.12) are smooth, the smoothness nature of the kernel $w^{T'}$ shows that the operator $\mathcal{W}_l^{T'}$ preserves smoothness:

$$\mathcal{W}_l^{T'}\mathcal{M}_l^{T'} \subset C^\infty([0,h];\mathbb{R}^2).$$

In what follows, this property will be refined.

Conjugate operator. For a scalar function $g=g(x)$ defined for $0\le x\le h$, we denote

$$\widetilde{g}(x_1(T'-t)) := \begin{cases} 0, & 0\le t<\pi_2^{T'}(T'),\\ g(x_1(T'-t)), & \pi_2^{T'}(T')\le t\le T',\end{cases} \tag{3.15}$$

and define

$$\widehat{x}_1(T'-t) := \begin{cases} h, & 0\le t<T'-\tau_1(h),\\ x_1(T'-t), & T'-\tau_1(h)\le t\le T'.\end{cases}$$

Proposition 13. *The operator $(\mathcal{W}_l^{T'})^*\colon \mathcal{H}^h\to\mathcal{F}_l^{T'}$ acts on the elements $y=\binom{y_1}{y_2}$ by the rule*

$$\big((\mathcal{W}_l^{T'})^*y\big)(t) = \varphi(t)\begin{pmatrix}\widetilde{y}_1\big(x_1(T'-t)\big)\\ y_2\big(x_2(T'-t)\big)\end{pmatrix} + \int_0^{\widehat{x}_1(T'-t)} w_*^{T'}(t,x)\,y(x)\,dx, \quad 0\le t\le T', \tag{3.16}$$

where $\varphi=\operatorname{diag}\{\varphi_1,\varphi_2\}$, $\varphi_i(t) := \big(\rho_i(0)\gamma_i(0)\rho_i\big(x_i(T'-t)\big)\gamma_i\big(x_i(T'-t)\big)\big)^{\frac14}$, and the kernel $w_*^{T'}$ is smooth in $[0,T']\times[0,h]$ outside of the curves $t=T'-\tau_i(x)$ and vanishes for $t>T'-\tau_1(x)$.

For the proof, the right-hand side of (3.12) is substituted in the product $(\mathcal{W}_l^{T'}f,y)_{\mathcal{H}^h}$. Then, by interchanging integrals, we bring this expression to the form $(f,(\mathcal{W}_l^{T'})^*y)_{\mathcal{F}_l^{T'}}$.

Intertwining property. In the exterior space $\mathcal{H}^h$, we consider the projections P^s onto the subspaces $\mathcal{H}^s$. The action of P^s reduces to the cutting of $y \in \mathcal{H}^h$ to the segment $[0, s]$. The next result should be compared with (2.52).

Lemma 3. *We have*

$$\mathcal{W}_l^{T'} \mathcal{P}_l^{T',\xi} = P^{x_2(\xi)} \mathcal{W}_l^{T'}, \quad 0 \le \xi \le T. \tag{3.17}$$

Proof. Take $f \in \mathcal{M}_l^{T'}$; let $\xi, \tau \in (0, T')$. Since the projections $\mathcal{P}_l^{T',\xi}$ expand when ξ grows, we have $\mathcal{P}_l^{T',\xi}\mathcal{P}_l^{T',\tau} = \mathcal{P}_l^{T',\tau}$ for $\tau < \xi$ and $\mathcal{P}_l^{T',\xi}\mathcal{P}_l^{T',\tau} = \mathcal{P}_l^{T',\xi}$ for $\tau > \xi$.

The definition (3.11) shows that, for $\tau < \xi$ and $x = x_2(\tau)$, we have

$$\begin{aligned}(\mathcal{W}_l^{T'} \mathcal{P}_l^{T',\xi} f)(x_2(\tau)) &= \theta(x_2(\tau)) \begin{pmatrix} (\mathcal{P}_l^{T',\tau}\mathcal{P}_l^{T',\xi} f)_1 \langle \pi_2^{T'}(\tau)\rangle \\ (\mathcal{P}_l^{T',\tau}\mathcal{P}_l^{T',\xi} f)_2 \langle \pi_1^{T'}(\tau)\rangle \end{pmatrix} \\ &= \theta(x_2(\tau)) \begin{pmatrix} (\mathcal{P}_l^{T',\tau} f)_1 \langle \pi_2^{T'}(\tau)\rangle \\ (\mathcal{P}_l^{T',\tau} f)_2 \langle \pi_1^{T'}(\tau)\rangle \end{pmatrix} = (W_l^{T'} f)(x_2(\tau)).\end{aligned}$$

If $\tau > \xi$, then

$$\begin{aligned}(\mathcal{W}_l^{T'} \mathcal{P}_l^{T',\xi} f)(x_2(\tau)) &= \theta(x_2(\tau)) \begin{pmatrix} (\mathcal{P}_l^{T',\tau}\mathcal{P}_l^{T',\xi} f)_1 \langle \pi_2^{T'}(\tau)\rangle \\ (\mathcal{P}_l^{T',\tau}\mathcal{P}_2^{T',\xi} f)_2 \langle \pi_1^{T'}(\tau)\rangle \end{pmatrix} \\ &= \theta(x_2(\tau)) \begin{pmatrix} (\mathcal{P}_l^{T',\xi} f)_1 \langle \pi_2^{T'}(\tau)\rangle \\ (\mathcal{P}_l^{T',\xi} f)_2 \langle \pi_1^{T'}(\tau)\rangle \end{pmatrix} = 0,\end{aligned}$$

because the control $\mathcal{P}_l^{T',\xi} f$ has no discontinuities for $t = \pi_i^{T'}(\tau)$ (it can have them for $t = \pi_i^{T'}(\xi)$).

We conclude that

$$(\mathcal{W}_l^{T'} \mathcal{P}_l^{T',\xi} f)(x) = \begin{cases} (\mathcal{W}_l^{T'} f)(x), & x < x_2(\xi) \\ 0, & x > x_2(\xi) \end{cases} = (P^{x_2(\xi)} \mathcal{W}_l^{T'} f)(x)$$

for $0 < x < x_2(T') = h$. Since the controls $f \in \mathcal{M}_l^{T'}$ used above are dense in $\mathcal{F}_l^{T'}$, we arrive at (3.17). □

Relationship with $\mathcal{C}_l^{T'}$. Let $e_l^{T'} : \mathcal{F}_l^{T'} \to \mathcal{F}^{T'}$ denote the corresponding embedding, so that $(e_l^{T'})^* e_l^{T'} = \mathbb{I}_{\mathcal{F}_l^{T'}}$. The operator $\mathcal{C}_l^{T'} : \mathcal{F}_l^{T'} \to \mathcal{F}_l^{T'}$ given by

$$\mathcal{C}_l^{T'} := (e_l^{T'})^* \mathcal{C}^{T'} e_l^{T'} \overset{(3.6)}{=} \nu \mathbb{I}_{\mathcal{F}_l^{T'}} + I_l^{T'}$$

is the block of $\mathcal{C}^{T'}$ in the (sub)space $\mathcal{F}_l^{T'}$. Here $I_l^{T'}$ is a compact integral operator. Since $\mathcal{C}^{T'}$ is an isomorphism, this block is an isomorphism in $\mathcal{F}_l^{T'}$. The last-written identity implies that we can write

$$(\mathcal{C}_l^{T'})^{-1} = \nu^{-1} \mathbb{I}_{\mathcal{F}_l^{T'}} + J_l^{T'} \tag{3.18}$$

with an integral operator

$$(J_l^{T'} f)(t) = \int_0^{T'} j^{T'}(t,s) f(s)\, ds, \quad 0 \le t \le T',$$

which is compact in $\mathcal{F}_l^{T'}$ and has a kernel $j^{T'}$ smooth in $[0, T'] \times [0, T']$ outside of the line $t = s$.

Lemma 4. *We have*

$$\mathcal{C}_l^{T'} = (\mathcal{W}_l^{T'})^* \mathcal{W}_l^{T'}. \tag{3.19}$$

Proof. **1.** Denoting $A := \mathcal{W}_l^{T'}(\mathcal{C}_l^{T'})^{-1}(\mathcal{W}_l^{T'})^*$, we check the relation

$$AP^x = P^x A, \quad 0 < x < h. \tag{3.20}$$

The second identity in (2.53) (with T replaced by T') implies that $(\mathcal{C}_l^{T'})^{-1}(\mathcal{P}_l^{T',\xi})^* = \mathcal{P}_l^{T',\xi}(\mathcal{C}_l^{T'})^{-1}$. Now we write

$$\begin{aligned} AP^x &= \mathcal{W}_l^{T'}(\mathcal{C}_l^{T'})^{-1}(\mathcal{W}_l^{T'})^* P^x \overset{(3.17)}{=} \mathcal{W}_l^{T'}(\mathcal{C}_l^{T'})^{-1}(\mathcal{P}^{T',\tau_2(x)})^*(\mathcal{W}_l^{T'})^* \\ &= \mathcal{W}_l^{T'}\mathcal{P}_l^{T',\xi}(\mathcal{C}_l^{T'})^{-1}(\mathcal{W}_l^{T'})^* \overset{(3.17)}{=} P^x\mathcal{W}_l^{T'}(\mathcal{C}_l^{T'})^{-1}(\mathcal{W}_l^{T'})^* = P^x A. \end{aligned}$$

2. Relations (3.12), (3.16), and (3.18) imply the representation

$$A = \mathbb{I}_{\mathcal{H}^h} + B$$

with a compact integral operator B. By (3.20), we have $BP^x = P^x B$, $0 < x < h$. An operator that commutes with the cutting-projections acts as multiplication by a function. For a compact operator, this is possible only if $B = \mathbb{O}_{\mathcal{H}^h}$, whence $A = \mathbb{I}_{\mathcal{H}^h}$.

3. The map

$$f \mapsto \theta \begin{pmatrix} f_1(T' - \tau_1(\,\cdot\,)) \\ f_2(T' - \tau_2(\,\cdot\,)) \end{pmatrix}$$

related to the right-hand side of (3.12), is an isomorphism from $\mathcal{F}_l^{T'}$ onto $\mathcal{H}^h$, and the integral term in (3.12) corresponds to a compact operator. Therefore, the operator $\mathcal{W}_l^{T'}$ is Fredholm, so that $(\mathcal{W}_l^{T'})^*$ is a Fredholm operator from $\mathcal{H}^h$ to $\mathcal{F}_l^{T'}$.

The identity $\mathcal{W}_l^{T'}(\mathcal{C}_l^{T'})^{-1}(\mathcal{W}_l^{T'})^* = \mathbb{I}_{\mathcal{H}^h}$ shows that the operator $(\mathcal{W}_l^{T'})^*$ is injective. Consequently, it is an isomorphism from $\mathcal{H}^h$ onto the entire space $\mathcal{F}_l^{T'}$. Correspondingly, $\mathcal{W}_l^{T'}$ turns out to be an isomorphism from $\mathcal{F}_l^{T'}$ onto $\mathcal{H}^h$. Now, to get (3.19), it suffices to pass to the inverse operators in the last-written identity. □

Since in the course of the proof we established that $\mathcal{W}_l^{T'}$ is an isomorphism, we can use (3.17) to obtain the relation

$$\mathcal{W}_l^{T'}\mathcal{F}_l^{T',\xi} = \mathcal{H}^{x_2(\xi)}, \quad 0 \le \xi \le T. \tag{3.21}$$

The inverse operator. Now we can get a formula for $(\mathcal{W}_l^{T'})^{-1}$. We shall use the notation (3.15).

Proposition 14. *The operator $(\mathcal{W}_l^{T'})^{-1}\colon \mathcal{H}^h \to \mathcal{F}_l^{T'}$ acts on the elements $y = \left(\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right)$ by the rule*

$$\big((\mathcal{W}_l^{T'})^{-1}y\big)(t) = \eta(t)\begin{pmatrix} \widetilde{y}_1\big(x_1(T'-t)\big) \\ y_2\big(x_2(T'-t)\big) \end{pmatrix} + \int_{x_2(T'-t)}^{h} w_{-1}^{T'}(t,x)\,y(x)\,dx, \;\; 0 \le t \le T', \tag{3.22}$$

where $\eta = \operatorname{diag}\{\eta_1, \eta_2\}$, $\eta_i(t) := \left(\frac{\rho_i(0)\gamma_i(0)}{\rho_i(x_i(T'-t))\gamma_i(x_i(T'-t))}\right)^{-\frac{1}{4}} = \theta_i^{-1}(x_i(T'-t))$, and the kernel $w_{-1}^{T'}$ is smooth in $[0,h]\times[0,T']$ outside of the curves $t = T' - \tau_i(x)$ and vanishes for $x < x_2(T'-t)$.

As in [6], this representation is deduced from the identity

$$(\mathcal{W}_l^{T'})^{-1} \overset{(3.19)}{=} (\mathcal{C}_l^{T'})^{-1}(\mathcal{W}_l^{T'})^*$$

by plugging formulas (3.18) and (3.16) in it. The smoothness nature of the kernel $w_{-1}^{T'}$ results from a thorough and bulky analysis, which we are forced to omit here.

The operator L. Here we introduce the operator that determines the evolution of the dynamical system to be constructed.

Linear spaces. We consider the following family of smooth linear subspaces in the exterior space $\mathcal{F}_l^{T'}$:

$$\mathcal{M}_l^{T',\xi} := \mathcal{M}_l^{T'} \cap \mathcal{F}_l^{T',\xi}, \quad \mathcal{M}_{l,0}^{T',\xi} := \{f \in \mathcal{M}_l^{T',\xi} \mid f(T') = 0\}, \quad 0 \le \xi \le T'.$$

Observe that $\mathcal{M}_l^{T',0} = \{0\}$ and $\mathcal{M}_l^{T',T'} = \mathcal{M}_l^{T'} = \cup_{0<\xi<T'} \mathcal{M}_l^{T',\xi}$. Each of these subspaces is dense in the subspace $\mathcal{F}_l^{T',\xi}$ containing it.

A control belongs to $\mathcal{F}_l^{T',\xi}$ if and only if a relation of convolutive nature is fulfilled for its components; therefore, the linear spaces introduced above are invariant under differentiation and integration, which easily shows that

$$\frac{d^k}{dt^k}\mathcal{M}_l^{T',\xi} = \mathcal{M}_l^{T',\xi}, \quad k = 1, 2, \dots. \tag{3.23}$$

For the same reason, we have

$$\mathcal{T}_s^{T'}\mathcal{F}_l^{T',\xi} \subset \mathcal{F}_l^{T',\xi}, \quad \mathcal{T}_s^{T'}\mathcal{M}_l^{T',\xi} \subset \mathcal{M}_l^{T',\xi}, \quad 0 \le s < \xi \le T'. \tag{3.24}$$

In the internal space $\mathcal{H}^h$ we define the smooth linear subspaces

$$\mathcal{N}^s := \{y \in C^\infty([0,h];\mathbb{R}^2) \mid \operatorname{supp} y \subset [0,s)\},$$
$$\mathcal{N}_0^s := \{y \in \mathcal{N}^s \mid y(0) = 0\}, \quad 0 \le s \le h,$$

observing that $\mathcal{N}^h = \cup_{0<s<h}\mathcal{N}^s$. Each of these subspaces is dense in the corresponding subspace $\mathcal{H}^s$.

Lemma 5. *We have*

$$\mathcal{W}_l^{T'}\mathcal{M}_l^{T',\xi} = \mathcal{N}^{x_2(\xi)}, \quad \mathcal{W}_l^{T'}\mathcal{M}_{l,0}^{T',\xi} = \mathcal{N}_0^{x_2(\xi)}, \quad 0 \le \xi \le T'. \tag{3.25}$$

Proof. **1.** The inclusions $\mathcal{W}_l^{T'}\mathcal{M}_l^{T',\xi} \subset \mathcal{N}^{x_2(\xi)}$ and $\mathcal{W}_l^{T'}\mathcal{M}_{l,0}^{T',\xi} \subset \mathcal{N}_0^{x_2(\xi)}$ follow from the form of the right-hand side of (3.12) and the smoothness nature of the kernel $w^{T'}$ in (3.12).

2. Suppose $y \in \mathcal{N}^{x_2(\xi)}$, $\xi \in (0,T']$. By (3.21) there exists a unique $f \in \mathcal{F}_l^{T',\xi}$ such that $\mathcal{W}_l^{T'}f = \Theta f + If$, where Θf and If are the summands in (3.12), so that I is an integral operator. It is easily seen that $IC^k([0,T'];\mathbb{R}^2) \subset C^{k+1}([0,h];\mathbb{R}^2)$, $k = -1, 0, 1, 2, \dots$, where $C^{-1}([0,T'];\mathbb{R}^2) := L_2([0,T'];\mathbb{R}^2)$, $C^0([0,T'];\mathbb{R}^2) := C([0,T'];\mathbb{R}^2)$.

We write $\Theta f = y - If$. Since y is smooth and $If \in C^0([0,h];\mathbb{R}^2)$, we have $\Theta f \in C^0([0,h];\mathbb{R}^2)$. For the components of f this yields

$$f_1\big|_{[\pi_2^{T'}(\xi),T']} \in C[\pi_2^{T'}(\xi),T'], \ f_2\big|_{[\pi_1^{T'}(\xi),T']} \in C[\pi_1^{T'}(\xi),T'].$$

The relationship between the components for $t \in [\pi_1^{T'}(\xi), \pi_2^{T'}(\xi)]$ shows that

$$f_1\big|_{[\pi_1^{T'}(\xi),\ \pi_2^{T'}(\xi)]} \in C[\pi_1^{T'}(\xi),\ \pi_2^{T'}(\xi)].$$

Therefore, the discontinuities of f_1 and f_2 are possible only if $t = \pi_2^{T'}(\xi)$ or $t = \pi_1^{T'}(\xi)$, respectively. However, from (3.12), we see that the presence of such discontinuities would imply the loss of continuity of the components of y at the point $x = x_2(\xi)$, which is impossible because y is smooth. So, there are no discontinuities. Thus, smoothness improves: from the initial $f \in \mathcal{F}_l^{T',\xi}$ it follows that $f \in \mathcal{F}_l^{T',\xi} \cap C^0([0,T'];\mathbb{R}^2)$.

Arguing much similarly, we can show that f is C^1-smooth piecewise, and then exclude the discontinuities of the derivatives for $t = \pi_i^{T'}(\xi)$, concluding that $f \in \mathcal{F}_l^{T',\xi} \cap C^1([0,T'];\mathbb{R}^2)$.

Continuing in an obvious way, we get $f \in \mathcal{F}_l^{T',\xi} \cap C^k\big([0,T'];\mathbb{R}^2\big)$ with an arbitrary k, which is equivalent to the fact that $f \in \mathcal{M}_l^{T',\xi}$.

3. The second relation in (3.25) is a consequence of the first by (3.14). □

Definition, properties, representation. Having formula (2.17) in mind, in the internal space we *define* an operator $L\colon \mathcal{H}^h \to \mathcal{H}^h$, $\operatorname{Dom} L = \mathcal{N}^h$,

$$L := \mathcal{W}_l^{T'} \frac{d^2}{dt^2} (\mathcal{W}_l^{T'})^{-1}. \tag{3.26}$$

It is easily seen that properties (3.23) and (3.25) ensure that the above definition is consistent. The same properties imply the inclusions

$$L\mathcal{N}^s \subset \mathcal{N}^s, \quad 0 \le s \le h. \tag{3.27}$$

Lemma 6. *The operator L acts locally: for $y \in \operatorname{Dom} L$ we have*

$$\operatorname{supp} Ly \subset \operatorname{supp} y. \tag{3.28}$$

The operator $L_0 := L\big|_{\mathcal{N}_0^h}$ is densely defined and symmetric: for $v, y \in \operatorname{Dom} L_0 = \mathcal{N}_0^h$ we have $(L_0 v, y)_{\mathcal{H}^h} = (v, L_0 y)_{\mathcal{H}^h}$.

Proof. Let $y \in \operatorname{Dom} L$ be such that $\operatorname{supp} y \subset [\alpha,\beta]$, where $0 < \alpha < \beta < h$. We show that $\operatorname{supp} Ly \subset [\alpha,\beta]$.

1. The condition imposed on the support implies that $y \in \mathcal{N}^\beta$, whence $Ly \subset \mathcal{N}^\beta$ by (3.27), i.e., $\operatorname{supp} Ly \subset [0,\beta]$. Thus, L does not extend the support to the right.

2. Let $f \in \mathcal{M}_l^{T',\tau_2(\alpha)}$ be a control vanishing near $t = T'$. Such controls are dense in the subspace $\mathcal{F}_l^{T',\tau_2(\alpha)}$. By (3.25) (with $\xi = \tau_2(\alpha)$), the corresponding waves $\mathcal{W}_l^{T'} f$ lie in the subspace $\mathcal{H}^\alpha$ and are dense in it. Moreover, $y \perp \mathcal{H}^\alpha$ because $\operatorname{supp} y \subset [\alpha,\beta]$.

For y and f as above, we have

$$\begin{aligned}
0 &= \big(\mathcal{W}_l^{T'} f_{tt}, y\big)_{\mathcal{H}^h} \overset{(3.19)}{=} \big(\mathcal{C}_l^{T'} f_{tt}, (\mathcal{W}_l^{T'})^{-1} y\big)_{\mathcal{F}_l^{T'}} \\
&\overset{\flat}{=} \big(\mathcal{C}^{T'} f_{tt}, (\mathcal{W}_l^{T'})^{-1} y\big)_{\mathcal{F}^{T'}} \overset{*}{=} \big(\mathcal{C}^{T'} f, \big((\mathcal{W}_l^{T'})^{-1} y\big)_{tt}\big)_{\mathcal{F}^{T'}} \\
&\overset{\flat\flat}{=} \big(\mathcal{C}^{T'} f, \big((\mathcal{W}_l^{T'})^{-1} y\big)_{tt}\big)_{\mathcal{F}_l^{T'}} \overset{(3.19)}{=} \left(\mathcal{W}_l^{T'} f, \mathcal{W}_l^{T'} \frac{d^2}{dt^2} (\mathcal{W}_l^{T'})^{-1} y\right)_{\mathcal{H}^h} \\
&\overset{(3.26)}{=} (\mathcal{W}_l^{T'} f, Ly)_{\mathcal{H}^h}.
\end{aligned}$$

For the simplicity of notation, in identities ($\flat$) and ($\flat\flat$) we omitted the embedding operator $e_l^{T'}$. Identity ($*$) is a result of integration by parts. When bringing the derivatives through the operator $\mathcal{C}^{T'}$, we have used the vanishing of f near $t = 0$ and $t = T'$ and the relation $(\mathcal{C}^{T'} f)(T') = \nu f(T') = 0$, implied by the form of the kernel of the integral part of $\mathcal{C}^{T'}$ (see (3.2)). Next, comparing the beginning and the end and using the density of the waves $\mathcal{W}_l^{T'} f$ in $\mathcal{H}^\alpha$, we see that $Ly \perp \mathcal{H}^\alpha$, which means that $\operatorname{supp} Ly \subset [\alpha,\beta]$. Consequently, the action of L does not extend the support to the *left*.

Thus, we arrive at (3.28).

3. Let $y, v \in \mathcal{N}_0^h$ and denote $f := (\mathcal{W}_l^{T'})^{-1} y$, $g := (\mathcal{W}_l^{T'})^{-1} v$. Then $f, g \in \mathcal{M}_{l,0}^{T',T'}$ (see (3.25)). We have

$$(L_0 v, y)_{\mathcal{H}^h} \overset{(3.19)}{=} \big(\mathcal{C}_l^{T'} g_{tt}, f\big)_{\mathcal{F}_l^{T'}} \overset{**}{=} \big(\mathcal{C}_l^{T'} g, f_{tt}\big)_{\mathcal{F}_l^{T'}} \overset{(3.19)}{=} (v, L_0 y)_{\mathcal{H}^h}.$$

Identity ($**$) is deduced much as ($*$): we integrate by parts, use the zero conditions for f, g at $t = 0$, $t = T'$, and recall the identities $(\mathcal{C}^{T'} f)(T') = (\mathcal{C}^{T'} g)(T') = 0$. □

Now we find the form of the operator introduced in (3.26).

Lemma 7. *We have*

$$Ly = \rho^{-1}\big[(\gamma y_x)_x - Ay_x - By\big] \tag{3.29}$$

with smooth A and B satisfying (2.5).

Proof. **1.** By rather lengthy calculations, it can be shown that if we plug (3.22) and (3.12) in the right-hand side of (3.26) and then integrate by parts, then we finish with the expression

$$\begin{aligned}(Ly)(x) = \rho^{-1}(x)\big[(\gamma(x)y'(x))' - A(x)y'(x) - B(x)y(x)\big]\\ + C(x)y'(x_2(\tau_1(x))) + D(x)y(x_2(\tau_1(x))) + E(x)\widetilde{y}'(x_1(\tau_2(x)))\\ + F(x)\widetilde{y}(x_1(\tau_2(x))) + \int_{x_2(\tau_1(x))}^{h} G(x,s)\,y(s)\,ds,\end{aligned} \tag{3.30}$$

where we have denoted

$$\widetilde{v}(x_1(\tau_2(x))) := \begin{cases} v(x_1(\tau_2(x))), & 0 \le x \le x_2(\tau_1(h)),\\ 0, & x_2(\tau_1(h)) < x \le h,\end{cases}$$

while A, B, C, D, E, and F stand for smooth matrix-valued functions, and G is a piecewise smooth kernel[13].

2. The operator L can be extended to distributions with the help of the right-hand side of (3.30). Since the distributions admit approximation by smooth functions, the extension of L inherits the locality (3.28). Acting by the right-hand side of (3.30) on the distributions $\delta_s\binom{1}{0}, \delta_s\binom{0}{1}$ ($\delta_s(\,\cdot\,)$ is the Dirac measure supported at $x = s$), it is easy to check that the condition $\operatorname{supp} Ly \subset \operatorname{supp} y = \{s\}$ (which follows from locality) can be fulfilled only if $C(s) = D(s) = E(s) = F(s) = 0$ and $G(\,\cdot\,, s) = 0$. Consequently, (3.29) is true.

3. By Lemma 6, the restriction $L_0 := L\big|_{\mathcal{N}_0^h}$ is a *symmetric* operator in $\mathcal{H}^h = L_{2,\rho}\big([0,h];\mathbb{R}^2\big)$. For a differential operator like (3.29), this is possible only if conditions (2.5) are fulfilled. □

Evolution. With each $f \in \mathcal{F}_l^{T'}$, we associate the function

$$u^f(x,t) := u^{\mathcal{T}_{T'-t}^{T'} f}(\,\cdot\,, T') = (\mathcal{W}_l^{T'}\mathcal{T}_{T'-t}^{T'} f)(x), \quad (x,t) \in [0,h]\times[0,T'], \tag{3.31}$$

(cf. (2.9)). Relations (3.24) and (3.25) show that this definition is consistent.

Lemma 8. *For $f \in \mathcal{M}_l^{T'}$, the function u^f defined as in* (3.31) *is a solution of the following problem*:

$$\rho u_{tt} - (\gamma u_x)_x + Au_x + Bu = 0, \qquad 0 < x < h,\ \ 0 < t < T', \tag{3.32}$$

$$u\big|_{t<\tau_2(x)} = 0, \tag{3.33}$$

$$u\big|_{x=0} = f, \qquad 0 \le t \le T'. \tag{3.34}$$

Proof. We have

$$\begin{aligned} u_{tt}^f &= (\mathcal{W}_l^{T'}\mathcal{T}_{T'-t}^{T'} f)_{tt} = \mathcal{W}_l^{T'}(\mathcal{T}_{T'-t}^{T'} f)_{tt}\\ &= \mathcal{W}_l^{T'}((\mathcal{W}_l^{T'})^{-1}\mathcal{W}_l^{T'}\mathcal{T}_{T'-t}^{T'} f)_{tt} \overset{(3.26)}{=} L\mathcal{W}_l^{T'}\mathcal{T}_{T'-t}^{T'} f = Lu^f.\end{aligned}$$

Multiplying by ρ, we get (3.32).

[13] All these can be expressed in terms of the functions ρ_i, γ_i, the matrix entries of the kernels $w^{T'}$, $w_{-1}^{T'}$, and their derivatives. The expressions are cumbersome, and their specific form plays no role in what follows.

By (3.24), we have $\mathcal{T}^{T'}_{T'-t}f \in \mathcal{M}^{T',t}_l$, whence $u^f(\,\cdot\,,t) \in \mathcal{N}^{x_2(t)} \subset \mathcal{H}^{x_2(t)}$ by (3.25). Therefore, $\operatorname{supp} u^f(\,\cdot\,,t) \subset [0, x_2(t)]$, which is equivalent to (3.33).

The relations

$$u^f(0,t) = u^{\mathcal{T}^{T'}_{T'-t}f}(0,T') \overset{(3.14)}{=} (\mathcal{T}^{T'}_{T'-t}f)(T') = f(t)$$

lead to (3.34). □

We summarize (preliminarily). In the course of constructions presented in Subsections 3.1 and 3.2, from the initial data (the operator (3.1)) we have extracted the matrix-valued functions $\{\rho,\gamma,A,B\}\big|_{0\le x\le h}$ satisfying conditions (2.4) and (2.5). This collection yields a system $\mathfrak{s}^{T'}_l$, via problem (2.44)–(2.46) with final moment $t = T'$. This system has a control operator $W^{T'}_l$ of its own. On the other hand, the same collection gave rise to an operator $\mathcal{W}^{T'}_l$ (see (3.11)), in terms of which the solutions (3.31) of problem (3.32)–(3.34) were defined. Lemma 8 says that these two problems are identical, implying that

$$W^{T'}_l = \mathcal{W}^{T'}_l. \tag{3.35}$$

3.3. System $\mathfrak{s}^T$. Recall that $h = x_1(T) = x_2(T')$ by the choice of parameters in Subsection 3.1.

The collection $\{\rho,\gamma,A,B\}\big|_{0\le x\le h}$ determines a system $\mathfrak{s}^T$ of the form

$$\rho u_{tt} - (\gamma u_x)_x + Au_x + Bu = 0, \qquad 0 < x < h,\ \ 0 < t < T, \tag{3.36}$$

$$u\big|_{t<\tau_1(x)} = 0, \tag{3.37}$$

$$u\big|_{x=0} = f, \qquad 0 \le t \le T, \tag{3.38}$$

(see (2.11)–(2.13)). For the system $\mathfrak{s}^T$ we have:

- the external space $\mathcal{F}^T = L_2\big([0,T];\mathbb{R}^2\big)$;
- the internal space $\mathcal{H}^{x_1(T)} = L_2\big([0,x_1(T)];\mathbb{R}^2\big)$;
- the control operator $W^T\colon \mathcal{F}^T \to \mathcal{H}^{x_1(T)}$;
- the extended response operator
$$(R^{2T}f)(t) = -\widetilde{\nu} f_t(t) + \widetilde{\omega} f(t) + \int_0^t \widetilde{r}(t-s)\, f(s)\, ds, \quad 0 \le t \le 2T,$$
of the form (2.24);
- the connecting operator $C^T\colon \mathcal{F}^T \to \mathcal{F}^T$, $C^T = (W^T)^* W^T$,
$$(C^T f)(t) \overset{(2.27)}{=} \widetilde{\nu} f(t) + \int_0^T \left[\frac{1}{2}\int_{|t-s|}^{2T-t-s} \widetilde{r}(\eta)\, d\eta\right] f(s)\, ds, \quad 0 \le t \le T.$$

The next result establishes the sufficiency of the conditions of Theorem 1, thus completing its proof.

Lemma 9. *We have $R^{2T} = \mathcal{R}^{2T}$.*

Proof. **1.** The identities $\widetilde{\nu} = \nu$ and $\widetilde{\omega} = \omega$ are ensured by the choice of parameters in Subsection 3.1 (see Remark at the end of step 4). It remains to verify that $\widetilde{r} = r$.

2. We establish a relationship between the systems $\mathfrak{s}^T$ and $\mathfrak{s}^{T'}_l$. By comparing the solutions of problems (3.36)–(3.38) and (3.32)–(3.34), it is easy to check that

$$W^T = W^{T'}_l \mathcal{T}^{T'}_{T'-T}, \tag{3.39}$$

where $\mathcal{T}^{T'}_{T'-T}\colon \mathcal{F}^T \to \mathcal{F}^{T'}$ is the operator introduced in (3.7). Note that $\operatorname{Ran} \mathcal{T}^{T'}_{T'-T} \subset \mathcal{F}^{T'}_l$, because the fact that a control belongs to $\mathcal{F}^{T'}_l$ imposes no restriction to the relationship between its components for $T' - T \le t \le T'$. Therefore, the right-hand side of (3.39) is well defined.

Next, we have

$$\begin{aligned} C^T &= (W^T)^* W^T \overset{(3.39)}{=} (\mathcal{T}^{T'}_{T'-T})^* (W^{T'}_l)^* W^{T'}_l \mathcal{T}^{T'}_{T'-T} \\ &\overset{(3.35)}{=} (\mathcal{T}^{T'}_{T'-T})^* (\mathcal{W}^{T'}_l)^* \mathcal{W}^{T'}_l \mathcal{T}^{T'}_{T'-T} \overset{(3.19)}{=} (\mathcal{T}^{T'}_{T'-T})^* \mathcal{C}^{T'}_l \mathcal{T}^{T'}_{T'-T} \\ &\overset{\flat}{=} (\mathcal{T}^{T'}_{T'-T})^* \mathcal{C}^{T'} \mathcal{T}^{T'}_{T'-T} \overset{(3.8)}{=} \mathcal{C}^T. \end{aligned}$$

We explain ($\flat$). Since the components of the controls in $\mathcal{F}^{T'}_l$ are independent for $T'-T \le t \le T'$ (see above), the blocks of $\mathcal{C}^{T'}_l$ and $\mathcal{C}^{T'}$ in the subspace $\mathcal{F}^{T',T} = \operatorname{Ran} \mathcal{T}^{T'}_{T'-T}$ are identical. Identity ($\flat$) expresses this fact.

3. Obviously, the identity $C^T = \mathcal{C}^T$ implies that the kernels of the integral parts of these two operators also coincide:

$$\int_{|t-s|}^{2T-t-s} \widetilde{r}(\eta)\, d\eta = \int_{|t-s|}^{2T-t-s} r(\eta)\, d\eta, \quad 0 \le s, t \le T.$$

Putting $t = s = T - \frac{\sigma}{2}$ and differentiating with respect to σ, we see that $\widetilde{r}(\sigma) = r(\sigma)$ for $0 \le \sigma \le 2T$. □

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