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CONGRUENCE PROPERTIES OF INDUCED REPRESENTATIONS AND THEIR APPLICATIONS

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Abstract. Congruence properties of the representations $U_\alpha := U_{\chi_\alpha}^{\mathrm{PSL}(2,\mathbb{Z})}$ are studied for the projective modular group $\mathrm{PSL}(2,\mathbb{Z})$ induced by a family χ_α of characters for the Hecke congruence subgroup $\Gamma_0(4)$, basically introduced by A. Selberg. The interest in the representations U_α stems from their presence in the transfer operator approach to Selberg's zeta function for this Fuchsian group and the character χ_α. Hence, the location of the nontrivial zeros of this function and therefore also the spectral properties of the corresponding automorphic Laplace–Beltrami operator $\Delta_{\Gamma,\chi_\alpha}$ are closely related to their congruence properties. Even if, as expected, these properties of the U_α are easily shown to be equivalent to those well-known for the characters χ_α, surprisingly, both the congruence and the noncongruence groups determined by their kernels are quite different: those determined by χ_α are character groups of type I of the group $\Gamma_0(4)$, whereas those determined by U_α are character groups of the same kind for $\Gamma(4)$. Furthermore, unlike infinitely many of the groups $\ker\chi_\alpha$, whose noncongruence properties follow simply from Zograf's geometric method together with Selberg's lower bound for the lowest nonvanishing eigenvalue of the automorphic Laplacian, such arguments do not apply to the groups $\ker U_\alpha$, for the reason that they can have arbitrary genus $g \geq 0$, unlike the groups $\ker\chi_\alpha$, which all have genus $g = 0$.

§1. Introduction

Whereas for congruence subgroups Γ of the modular group $\mathrm{PSL}(2,\mathbb{Z})$ one expects, in accordance with the general Riemann hypothesis, that the zeros of the Selberg zeta function $Z_\Gamma(s)$ in the half-plane $\mathrm{Re}\, s < \frac{1}{2}$ are located on a few lines parallel to the imaginary axis, for noncongruence subgroups not much is known, and one even expects the zeros to be distributed rather erratically. To understand this distribution better, A. Selberg [25] studied the location of the poles of the Eisenstein series $E_i(z,s;\widetilde{\chi}_\eta)$ for the Hecke congruence subgroup $\Gamma_0(4)$ and a 1-parameter family of characters $\widetilde{\chi}_\eta$, $-\pi/2 \leq \eta \leq \pi/2$, where $\widetilde{\chi}_0$ is the trivial character. He got a remarkable, but often overlooked result concerning the location of the zeros of his zeta function $Z_{\Gamma_0(4)}(s,\widetilde{\chi}_\eta)$, and hence also for the location of the resonances of the automorphic Laplacian $\Delta_{\widetilde{\chi}_\eta}$ in the half-plane $\mathrm{Re}\, s < \frac{1}{2}$. When written as a dynamical zeta function, the Selberg function has the form

$$Z_{\Gamma_0(4)}(s,\widetilde{\chi}_\eta) = \prod_{k=0}^{\infty}\prod_{\{\gamma\}}(1-\widetilde{\chi}_\eta(g_\gamma)e^{-(s+k)l_\gamma}), \quad \mathrm{Re}\, s > 1/2,$$

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$\{\gamma\}$ being the prime periodic orbits of the geodesic flow on the unit tangent bundle of the Hecke surface $\Gamma_0(4)/\mathbb{H}$ and g_γ a hyperbolic element in $\Gamma_0(4)$ determining the periodic orbit γ. Selberg showed that there is no limitation how close to the line $\operatorname{Re} s = \frac{1}{2}$ for $\eta \to 0$, or how far away from it the zeros may lie as $\eta \to \pm\pi/2$. Obviously, this result is closely related to the fact that for each $\eta \neq 0$ the multiplicity of the continuous spectrum of the automorphic Laplacian $\Delta_{\Gamma_0(4),\widetilde{\chi}_\eta}$ is equal to one and for $\eta = 0$ this multiplicity is equal to three. On the other hand, the behavior as $\eta \to \pm\pi/2$ is related to the congruence properties of $\widetilde{\chi}_\eta$ for these parameter values. From the point of view of the spectral theory of automorphic functions, the family of characters $\widetilde{\chi}_\eta$ is singular as η tends to 0.

A similar family $\widetilde{\chi}_\alpha$ of characters for the principal congruence subgroup $\Gamma(2)$, which is conjugate to the Hecke congruence subgroup $\Gamma_0(4)$, but with a slightly different normalization of the parameter $0 \leq \alpha \leq 1$ such that $\widetilde{\chi}_0 = \widetilde{\chi}_1 \equiv 1$, was studied also by R. Phillips and P. Sarnak in [19] in their work on the existence of Maass cusp forms in nonarithmetic situations. Thereby they showed, obviously unaware of an older result by M. Newman [17], that the character $\widetilde{\chi}_\alpha$ is congruent, which means that its kernel is a congruence subgroup with $\ker \widetilde{\chi}_\alpha \geq \Gamma(N)$ for some N, if and only if $\alpha = j/8$, $0 \leq j \leq 8$. The explicit form of these kernels (to be precise, of still another family of conjugate characters χ_α of $\Gamma(2)$ with $\chi_0 = \chi_1 \equiv 1$) was determined by E. Balslev and A. Venkov in [3]. For irrational α, the kernels $\ker \chi_\alpha$ are easily seen to be subgroups of infinite index in $\Gamma(2)$, and hence, the characters in this case are not congruent. For rational $\alpha = n/d$, $d > n$ and $(n,d) = 1$, on the other hand, E. Balslev and A. Venkov constructed in [3] a system of generators for the group $\ker \chi_\alpha$, which for $d = 2, 3, \ldots$ is a cofinite, normal subgroup of $\Gamma(2)$ of index d but not normal in $\mathrm{PSL}(2,\mathbb{Z})$ [17]. Indeed, the group is generated by $d+2$ parabolic elements $S_1, S_2, \ldots, S_{d+2}$ with one relation

$$S_1 S_2 \ldots S_{d+2} = \mathrm{Id}_{2\times 2}\,. \tag{1.1}$$

Since for $\alpha = n/d$, $(n,d) = 1$, $\ker \chi_\alpha$ turns out to be independent of n, it can be simply denoted by Γ_d. Then Γ_d has signature $(h; m_1, m_2, \ldots, m_k; p) = (0; d+2)$ (see [26]), where as usual $2h$ denotes the number of hyperbolic, k the number of elliptic, and p the number of parabolic generators, respectively, m_j being the order of the elliptic generator e_j. By Gauss–Bonnet, its fundamental domain F_d then has hyperbolic area

$$A_d = 2\pi\Big(2h - 2 + \sum_{j=1}^{k}(1 - 1/m_j) + p\Big) = 2\pi d. \tag{1.2}$$

The groups Γ_d, as defined by Balslev and Venkov, coincide with the groups Γ_{6d} of G. Sansone [22]; they were studied in [17] by M. Newman. He solved indeed the congruence problem for these subgroups of $\Gamma(2)$ by showing Γ_{6d} to be congruent only for $d = 1, 2, 4, 8$ with $\Gamma(2d) \leq \Gamma_{6d}$. For the character χ_α of $\Gamma(2)$ this coincides with the above-mentioned result of Phillips and Sarnak, respectively, of Balslev and Venkov.

In a recent paper [5] R. Bruggeman et al. studied the behavior of the zeros of the Selberg zeta function $Z_{\Gamma_0(4)}(s, \chi_\alpha)$ in more detail for the group $\Gamma_0(4)$ and a family of characters conjugate to that of Balslev and Venkov for $\Gamma(2)$. For simplicity, we denote this family again by χ_α. This work was initiated by numerical results in the thesis of M. Fraczek, who was able to trace the zeros of this zeta function as a function of α by using its representation in terms of the Fredholm determinant $\det(1 - \mathcal{L}_{\alpha,s})$ of a family of transfer operators $\mathcal{L}_{\alpha,s}$ [6, 7]. These operators are determined by the geodesic flow on the Hecke surface $\Gamma_0(4)/\mathbb{H}$ and the unitary representations $U_\alpha := U_{\chi_\alpha}^{PSL(2,\mathbb{Z})}$ of the modular group $\mathrm{PSL}(2,\mathbb{Z})$ induced by the family χ_α of characters of the Hecke congruence subgroup $\Gamma_0(4)$. One of the results of [5] is a more detailed description of Selberg's accumulation phenomenon for the resonances on the critical line $\operatorname{Re} s = 1/2$ for noncongruent values α

of the character when it approaches the trivial congruent character χ_0. Fraczek also (see [6]) managed to confirm numerically Selberg's resonances tending to $\operatorname{Re} s = -\infty$ on lines asymptotically equidistant and parallel to the real axis for α approaching the congruent value $\alpha = 1/4$ (see [4]). At the same time, his numerical calculations confirm $\alpha = j/8$, $0 \leq j \leq 8$, as the only congruent values for the induced representation U_{χ_α} as for the character χ_α.

Indeed it is not difficult to show that, as expected, $\ker U_\alpha$ is congruent if and only if $\ker \chi_\alpha$ is congruent. However, these two families of groups have quite different properties: whereas the groups $\ker \chi_\alpha$, which for rational $\alpha = n/d$, $n < d$, are conjugate to the groups Γ_d, and therefore, all have vanishing genus, the groups $\ker U_\alpha$ can have arbitrarily large genus. Hence, contrary to the former ones, their noncongruence nature cannot be deduced simply by applying a recent geometric result of P. Zograf [29, 30, 31] based on previous results of Yang and Yau [28], respectively, Hersh [9], together with A. Selberg's famous lower bound for the eigenvalues of congruence subgroups [24]. Instead, one needs to use a different algebraic approach based on Wohlfahrt's notion of level for general subgroups of the modular group. The fact that there are infinitely many noncongruence groups even in any fixed genus $g \geq 0$ was shown by G. Jones [10].

Many of the known examples of noncongruence subgroups are so-called character groups, i.e., arise as the kernel of some group epimorphism of a congruence subgroup onto some finite Abelian group, or equivalently, as a normal finite index subgroup of a congruence group with Abelian quotient. Most such examples are kernels of unitary characters like the lattice groups of Rankin [20] or the noncongruence groups of Klein and Fricke [8]. Such character groups play an important role in the theory of modular forms for noncongruence subgroups and especially for the Atkin–Swinnerton–Dyer congruence relations (see, for instance, the series of papers [13, 2, 12, 14]). Obviously, the groups Γ_d for the noncongruence values of d belong to this class of noncongruence groups. On the other hand, the group $\ker U_\alpha$ for a noncongruence rational value of α will be shown to be the kernel of an epimorphism $\phi_\alpha \colon \Gamma(4) \to A_\alpha$, where A_α is a finite Abelian group freely generated by three six-dimensional matrices each of order $N(\alpha)$, which for $\alpha = p/q$ and $(p,q) = 1$ is given by $N(\alpha) = \min\{n : q|4n\}$. To our knowledge, not many examples of such noncongruence character groups arising as kernels of epimorphisms onto noncyclic finite Abelian groups have been discussed in the literature.

This paper is organized as follows. In §2 we recall Selberg's character χ_α for the group $\Gamma_0(4)$ and the 6-dim. monomial representation U_α of the modular group $\mathrm{PSL}(2,\mathbb{Z})$ induced by χ_α. We briefly recall Selberg's upper bound for the smallest eigenvalue λ_1 of the automorphic Laplace–Beltrami operator Δ_Γ for congruence subgroups Γ and Zograf's lower bound for this eigenvalue for general finite index subgroups of the modular group. We show how these results imply the noncongrunce property of infinitely many of the groups Γ_d determined by the kernel $\ker \chi_\alpha$ for $\alpha = n/d$. In §3 we introduce the groups $G_\alpha = U_\alpha(\mathrm{PSL}(2,\mathbb{Z}))$, $0 \leq \alpha \leq 1$, and relate them to the group G_0. We determine the kernel $\ker U_0$ of the representation U_0 as the principal congruence subgroup $\Gamma(4)$ and show that $\ker U_\alpha \leq \Gamma(4)$. We introduce the groups $A_\alpha = U_\alpha(\ker U_0)$, which are finitely generated Abelian normal subgroups of G_α and of finite order if and only if α is rational. The factor group G_α/A_α, on the other hand, is isomorphic to G_0, which, by a theorem of M. Millington [16], is itself isomorphic to the modulary group $G(4) = \mathrm{PSL}(2,\mathbb{Z})/\Gamma(4)$. §4 contains the main results of this paper, namely, a complete characterization for rational α of the groups $\ker U_\alpha$ by determining their index in $\Gamma(4)$, their genus, the number of their cusps, their Wohlfahrt level, and the number of their free generators. We show that $\ker U_\alpha$ is congruent if and only if the character χ_α is congruent. Independently from this result, using the above group data leads to another way of finding the α-values for which $\ker U_\alpha$ is congruent. Indeed, for these values the congruence group $\ker U_\alpha$ either coincides with the principle subgroup $\Gamma(4)$, or with $\Gamma(8)$, depending simply on the order of the Abelian group $\Gamma(4)/\ker U_\alpha$.

§2. Selberg's character χ_α for $\Gamma_0(4)$ and the induced representation U_α of $\mathrm{PSL}(2,\mathbb{Z})$

The projective modular group $\mathrm{PSL}(2,\mathbb{Z})$ is defined as

$$\mathrm{PSL}(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} |a,b,c,d \in \mathbb{Z}, ad-bc=1 \right\} / \{\pm \mathrm{Id}\}. \tag{2.1}$$

This group is generated by the elements

$$T = \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathrm{S} = \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{2.2}$$

with the relations $S^2 = (ST)^3 = \pm \mathrm{Id}$. Many of the noncongruence subgroups of the modular group $\mathrm{PSL}(2,\mathbb{Z})$ discussed in the literature are so-called character groups of congruence subgroups G of the modular group . Quite generally, a subgroup Γ of an arithmetic subgroup G, that is $G \le \mathrm{PSL}(2,\mathbb{Z})$ of finite index, is said to be a character group of G if $\Gamma = \ker \phi$ for some group epimorphism $\phi\colon G \to A$ onto a finite Abelian group A. The character group Γ is of type I if $\phi(g) \ne id$ for some parabolic element $g \in G$, otherwise it is of type II. The groups Γ_d are obviously character groups of the congruence subgroup $\Gamma_0(4) \le \mathrm{PSL}(2,\mathbb{Z})$ of type I, because $\Gamma_d = \ker \chi_\alpha$, $\alpha = \frac{n}{d}$, $n < d$, $(n,d) = 1$.

The principal congruence subgroup $\Gamma(n)$ of level n is defined as

$$\Gamma(n) := \left\{ \gamma \in \mathrm{PSL}(2,\mathbb{Z}) \middle| \, \gamma \equiv \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod n \right\}. \tag{2.3}$$

The index $\mu_{\Gamma(n)}$ of $\Gamma(n)$ in $\mathrm{PSL}(2,\mathbb{Z})$ is given by (see, e.g., [21])

$$\mu_{\Gamma(n)} = \left[\mathrm{PSL}(2,\mathbb{Z}) : \Gamma(n) \right] = \begin{cases} \frac{1}{2} n^3 \prod_{p|n} \left(1 - \frac{1}{p^2}\right) & \text{for } n > 2, \\ 6 & \text{for } n = 2, \end{cases} \tag{2.4}$$

where p runs over all primes dividing n. On the other hand, the Hecke congruence subgroup $\Gamma_0(n)$ of $\mathrm{PSL}(2,\mathbb{Z})$ is defined by

$$\Gamma_0(n) = \{\gamma \in \mathrm{PSL}(2,\mathbb{Z}) \,|\, c = 0 \bmod n\}. \tag{2.5}$$

Its index $\mu_{\Gamma_0(n)}$ in $\mathrm{PSL}(2,\mathbb{Z})$ is given by

$$\mu_{\Gamma_0(n)} = n \prod_{p|n} \left(1 + \frac{1}{p}\right). \tag{2.6}$$

In what follows, we are mostly interested in the case where $n = 4$. In this case, $\Gamma_0(4)$ is freely generated by

$$T = \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad ST^4S = \pm \begin{pmatrix} -1 & 0 \\ 4 & -1 \end{pmatrix}, \tag{2.7}$$

and $\mu_{\Gamma_0(4)} = 6$. As representatives of the right cosets $\Gamma_0(4) \setminus \mathrm{PSL}(2,\mathbb{Z})$ of $\Gamma_0(4)$ in $\mathrm{PSL}(2,\mathbb{Z})$ we choose the following set R of elements of $\mathrm{PSL}(2,\mathbb{Z})$:

$$R = \left\{ \mathrm{Id}, S, ST, ST^2, ST^3, ST^2S \right\}. \tag{2.8}$$

Selberg's character

$$\chi_\alpha\colon \Gamma_0(4) \to \mathrm{Aut}\ \mathbb{C}, \quad 0 \le \alpha \le 1, \tag{2.9}$$

for the group $\Gamma_0(4)$ is defined by the following assignments to the above generators:

$$\chi_\alpha(T) = \exp(2\pi i\alpha), \quad \chi_\alpha(ST^4S) = 1. \tag{2.10}$$

This corresponds to the family of conjugate characters considered by Balslev and Venkov for the conjugate group $\Gamma(2)$ in [3]. Since $\chi_0 = \chi_1$ and $\chi_{1-\alpha} = \chi_\alpha^*$, we can restrict ourselves in the sequel to the parameter range $0 \le \alpha \le 1/2$. For our choice of the set R of representatives of $\Gamma_0(4) \setminus \mathrm{PSL}(2,\mathbb{Z})$, the representation $U_\alpha := U_{\chi_\alpha}^{\mathrm{PSL}(2,\mathbb{Z})}$ of $\mathrm{PSL}(2,\mathbb{Z})$ induced by Selberg's character χ_α for $\Gamma_0(4)$ is then given by

$$\big[U_\alpha(g)\big]_{i,j} = \delta_{\Gamma_0(4)}(r_i g r_j^{-1})\chi_\alpha(r_i g r_j^{-1}), \quad r_i \in R,\ 1 \le i,j \le 6, \tag{2.11}$$

where

$$\delta_{\Gamma_0(4)}(\gamma) = \begin{cases} 1 & \text{if } \gamma \in \Gamma_0(4), \\ 0 & \text{if } \gamma \notin \Gamma_0(4). \end{cases} \tag{2.12}$$

Obviously, $U_\alpha(g)$ is a 6-dimensional monomial matrix, i.e., has only one nonvanishing entry in every row and column. For $\alpha = 0$, the matrix $U_0(g)$ becomes a permutation matrix with all nonvanishing entries equal to one, i.e.,

$$\big[U_0(g)\big]_{ij} = \delta_{\Gamma_0(4)}(r_i g r_j^{-1}), \quad r_i \in R,\ 1 \le i,j \le 6. \tag{2.13}$$

For the generators S and T of $\mathrm{PSL}(2,\mathbb{Z})$, we have

$$U_\alpha(S) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \exp(-2\pi i\alpha) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \exp(2\pi i\alpha) & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \tag{2.14}$$

and

$$U_\alpha(T) = \begin{pmatrix} \exp(2\pi i\alpha) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \exp(-2\pi i\alpha) \end{pmatrix}. \tag{2.15}$$

Definition 2.1. The character χ_α (respectively, the representation U_α) is a congruence character (respectively, a congruence representation) or, in short, is congruent if and only if its kernel $\ker\chi_\alpha$ (respectively, $\ker U_\alpha$) is a congruence subgroup, which means that $\Gamma(N) \le \ker\chi_\alpha$ (respectively, $\Gamma(N) \le \ker U_\alpha$) for some N.

It is well known from M. Newman's paper [17] and was later reproved by several other authors that only for $\alpha = j/8$, $0 \le j \le 4$, the character χ_α is congruent. Obviously, for α irrational the group $\ker\chi_\alpha$ has infinite index in the modular group and, hence, cannot be congruent. On the other hand, the group $\ker\chi_\alpha$ for rational $\alpha = n/d$, $1 \le n \le d-1$, does not depend on n and hence can be denoted by Γ_d. In [3], Balslev and Venkov showed that Γ_d has vanishing genus and that the area A_d of its fundamental domain F_d is given by $A_d = 2\pi d$. The fact that Γ_d can be congruent only for finitely many values of d follows indeed already from a remarkable geometric result of P. Zograf [30, 31], based on the previous work of Yang and Yau [28], and, respectively, Hersh [9], together with A. Selberg's famous lower bound for the eigenvalues of the automorphic Laplacian Δ_Γ for congruence subgroups Γ, see [24] (more recent lower bounds can be found in [23]). Let us briefly recall these results.

Theorem 2.2 (Zograf)**.** *Let Γ be a discrete cofinite subgroup of* $\mathrm{PSL}(2,\mathbb{R})$ *of signature* $(h; m_1, m_2, \ldots, m_k; p)$ *and genus* g. *Let* $A(F)$ *be the hyperbolic area of its fundamental domain* F. *Assume that* $A(F) \geq 32\pi(g+1)$. *Then the set of eigenvalues of the automorphic Laplacian* Δ_Γ *in* $(0, 1/4)$ *is not empty, and*

$$\lambda_1 < \frac{8\pi(g+1)}{A(F)}, \tag{2.16}$$

where $\lambda_1 > 0$ *is the smallest nonzero eigenvalue of* Δ_Γ.

On the other hand, Selberg proved the following lower bound for the smallest nonvanishing eigenvalue λ_1 for any congruence subgroup.

Theorem 2.3 (Selberg)**.** *Let* Γ *be a congruence subgroup of* $\mathrm{PSL}(2,\mathbb{Z})$. *Then*

$$(3/16) \leq \lambda_1. \tag{2.17}$$

Selberg's sharper eigenvalue conjecture for congruence subgroups is in fact $\lambda_1 \geq 1/4$ (see [24]). Notice that the interval $[0, 1/4)$ is free from the continuous spectrum of the automorphic Laplacian Δ_Γ, which is real and given by $[1/4, \infty)$. Combining these two theorems, for congruence subgroups we get

$$3/16 < \frac{8\pi(g+1)}{A(F)}. \tag{2.18}$$

If we assume that, for a given d, the group Γ_d, which has vanishing genus g, is a congruence subgroup, then (2.18) shows that $3/16 < 8\pi/2\pi d$ or $d < 64/3$, so that there are only finitely many d with Γ_d a congruence subgroup.

§3. The groups $U_\alpha(\mathrm{PSL}(2,\mathbb{Z})$ and $U_\alpha(\Gamma(4))$

Denote by G_α the group $G_\alpha = U_\alpha(\mathrm{PSL}(2,\mathbb{Z}))$ determined by the induced representation U_α. Since $\mathrm{PSL}(2,\mathbb{Z})$ is generated by S and T, the group G_α is generated by $U_\alpha(S)$ and $U_\alpha(T)$. Next, we denote by $M(6,\mathbb{C})$ the group of all monomial matrices and by $\Delta(6,\mathbb{C})$ the group of all diagonal matrices in $\mathrm{GL}(6,\mathbb{C})$. It is well known that $M(6,\mathbb{C})$ is the normalizer of $\Delta(6,\mathbb{C})$ in $\mathrm{GL}(6,\mathbb{C})$ (see, e.g., [1, p. 48, Exercise 7]). Hence, $\Delta(6,\mathbb{C})$ is obviously normal in $M(6,\mathbb{C})$. Denote furthermore by W the set of all 6-dimensional permutation matrices in $\mathrm{GL}(6,\mathbb{C})$. This is a subgroup of $\mathrm{GL}(6,\mathbb{C})$, also called the Weyl group. The group W is obviously isomorphic to S_6, the symmetric group of degree 6. Then the group $M(6,\mathbb{C})$ of monomial matrices in dimension 6 has the following semidirect product structure (see [1, p. 48, Exercise 7]):

$$M(6,\mathbb{C}) = \Delta(6,\mathbb{C}) \rtimes W, \tag{3.1}$$

therefore, each element $m \in M(6,\mathbb{C})$ can uniquely be expressed as $m = \delta w$, where $\delta \in \Delta(6,\mathbb{C})$ and $w \in W$.

Since, obviously, the generators $U_\alpha(S)$ and $U_\alpha(T)$ of G_α belong to $M(6,\mathbb{C})$, the group G_α is a subgroup of $M(6,\mathbb{C})$

$$G_\alpha \leq M(6,\mathbb{C}). \tag{3.2}$$

Lemma 3.1. *Let* U_0 *be the representation of* $\mathrm{PSL}(2,\mathbb{Z})$ *induced by the trivial character* χ_0 *of* $\Gamma_0(4)$. *Then each element* $U_\alpha(g) \in G_\alpha$ *has a unique representation as*

$$U_\alpha(g) = D_\alpha(g) U_0(g), \tag{3.3}$$

where $D_\alpha(g) \in \Delta(6,\mathbb{C})$.

Proof. For $g \in \mathrm{PSL}(2,\mathbb{Z})$, denote by $D_\alpha(g) \in \Delta(6,\mathbb{C})$ the diagonal matrix

$$\big[D_\alpha(g)\big]_{ik} = \delta_{ik}\chi_\alpha(r_i g r(i)^{-1}), \quad 1 \le i,k \le 6. \tag{3.4}$$

Here, the r_i and $r(i)$ are elements of the set R of representatives of $\Gamma_0(4) \setminus \mathrm{PSL}(2,\mathbb{Z})$, with $r(i)$ uniquely determined by the condition $r_i g r(i)^{-1} \in \Gamma_0(4)$. Then we have

$$\big[D_\alpha(g)U_0(g)\big]_{ij} = \sum_{k=1}^{6} \big[D_\alpha(g)\big]_{ik}\big[U_0(g)\big]_{kj}. \tag{3.5}$$

Inserting (3.4), we get

$$\big[D_\alpha(g)U_0(g)\big]_{ij} = \chi_\alpha(r_i g r(i)^{-1})\big[U_0(g)\big]_{ij}. \tag{3.6}$$

But the definition of U_0 in (2.13) shows that

$$\big[D_\alpha(g)U_0(g)\big]_{ij} = \chi_\alpha(r_i g r(i)^{-1})\delta_{\Gamma_0(4)}(r_i g r_j^{-1}). \tag{3.7}$$

Hence,

$$\big[D_\alpha(g)U_0(g)\big]_{ij} = \begin{cases} \chi_\alpha(r_i g r_j^{-1}) & \text{if } r_i g r_j^{-1} \in \Gamma_0(4), \\ 0 & \text{if } r_i g r_j^{-1} \notin \Gamma_0(4), \end{cases} \tag{3.8}$$

or

$$\big[D_\alpha(g)U_0(g)\big]_{ij} = \delta_{\Gamma_0(4)}(r_i g r_j^{-1})\chi_\alpha(r_i g r_j^{-1}) = U_\alpha(g)_{i,j}. \tag{3.9}$$

Since $U_0(g)$ is a permutation matrix in W and G_α is a subgroup of $M(6,\mathbb{C})$, this decomposition is unique, see (3.1). $\square$

Since $\Delta(6,\mathbb{C})$ is normal in $M(6,\mathbb{C})$ and $G_\alpha \le M(6,\mathbb{C})$, the group $A_\alpha := G_\alpha \cap \Delta(6,\mathbb{C})$ is normal in G_α. By definition, A_α is the group of all diagonal matrices in G_α. Hence, by Lemma 3.1, A_α is the image of the kernel $\ker U_0$ of the representation U_0 under the map U_α, that is

$$A_\alpha = \big\{U_\alpha(\gamma) | \gamma \in \ker U_0\big\}. \tag{3.10}$$

Lemma 3.2. *Let U_0 be the representation of $\mathrm{PSL}(2,\mathbb{Z})$ induced by the trivial character χ_0 of $\Gamma_0(4)$. Then*

$$\ker U_0 = \big\{g \in \mathrm{PSL}(2,\mathbb{Z}) | U_0(g) = \mathrm{Id}_{6\times 6}\big\} = \Gamma(4). \tag{3.11}$$

Proof. Since $\Gamma(4) \lhd \mathrm{PSL}(2,\mathbb{Z})$, for each $\gamma \in \Gamma(4)$ and $r \in R$ there exists $\gamma' \in \Gamma(4)$ such that $r\gamma r^{-1} = \gamma'$. Thus, $\Gamma(4) \le \ker U_0$ by (2.13). To show $\ker U_0 \le \Gamma(4)$, we take $\gamma \in \ker U_0$. Then, by (2.13), for each $r \in R$ we have $r\gamma r^{-1} \in \Gamma_0(4)$. Since $\mathrm{Id} \in R$, necessarily $\gamma \in \Gamma_0(4)$. But for $\gamma \in \Gamma_0(4)$ we have $S\gamma S^{-1} \in \Gamma^0(4)$. On the other hand, $S \in R$, whence $S\gamma S^{-1} \in \Gamma_0(4)$. Hence, γ itself must belong to $\Gamma_0(4) \cap \Gamma^0(4)$. Conjugating $\gamma = \left(\begin{smallmatrix} a & 4b \\ 4c & d \end{smallmatrix}\right) \in \Gamma_0(4) \cap \Gamma^0(4)$ by $ST \in R$ shows that $ST\gamma(ST)^{-1} \in \Gamma_0(4)$ if and only if $a \equiv d \bmod 4$. Since $\det\gamma = 1$, this yields $\gamma \in \Gamma(4)$. Hence, $\ker U_0 \le \Gamma(4)$. This completes the proof. $\square$

Remark 3.3. Obviously, $\ker U_0$ is given by the maximal normal subgroup of the modular group that is contained in $\Gamma_0(4)$. Since, quite generally, the maximal normal subgroup $H(n) \lhd \mathrm{PSL}(2,\mathbb{Z})$ with $H(n) \le \Gamma_0(n)$ is given by

$$H(n) = \left\{ g \in \mathrm{PSL}(2,\mathbb{Z}) \,:\, g = \pm\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \bmod n, \quad \alpha^2 = 1 \bmod n \right\}$$

and since for $n = 4$ there is only the solution $\alpha = \pm 1$, we see that $H(4) = \Gamma(4)$.

Corollary 3.4. *The normal subgroup A_α of G_α in* (3.10) *is given by*

$$A_\alpha = \big\{U_\alpha(\gamma) | \gamma \in \Gamma(4)\big\}. \tag{3.12}$$

In accordance with this corollary, the generators of A_α can be calculated explicitly in terms of generators of $\Gamma(4)$. A set of generators of $\Gamma(4)$ is given for instance by

$$
\begin{aligned}
g_1 = T^4 &= \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}, \\
g_2 = ST^{-4}S &= \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}, \\
g_3 = T^{-1}ST^4ST &= \begin{pmatrix} -5 & -4 \\ 4 & 3 \end{pmatrix}, \\
g_4 = T^{-2}ST^{-4}ST^{-2} &= \begin{pmatrix} 7 & -12 \\ -4 & 7 \end{pmatrix}, \\
g_5 = TST^{-4}ST^{-1} &= \begin{pmatrix} -5 & 4 \\ -4 & 3 \end{pmatrix}
\end{aligned}
\tag{3.13}
$$

(see [11]). The corresponding generators of A_α are obtained by calculating their induced representations $U_\alpha(g_i)$:

$$A_1(\alpha) := U_\alpha(g_1) = \operatorname{diag}(\exp(8\pi i\alpha), 1, 1, 1, 1, \exp(-8\pi i\alpha)), \tag{3.14}$$

$$A_2(\alpha) := U_\alpha(g_2) = \operatorname{diag}(1, \exp(-8\pi i\alpha), 1, \exp(8\pi i\alpha), 1, 1), \tag{3.15}$$

$$A_3(\alpha) := U_\alpha(g_3) = \operatorname{diag}(1, 1, \exp(8\pi i\alpha), 1, \exp(-8\pi i\alpha), 1), \tag{3.16}$$

where $\operatorname{diag}(a_1, \ldots, a_6)$ denotes the 6-dimensional diagonal matrix with entries $\{a_i\}$. It turns out that

$$U_\alpha(g_5) = U_\alpha(g_3), \quad U_\alpha(g_4) = \big[U_\alpha(g_1)U_\alpha(g_2)\big]^{-1}, \tag{3.17}$$

and hence, the group A_α is generated by the three elements $A_k(\alpha)$, $1 \le k \le 3$,

$$A_\alpha = \big\langle A_1(\alpha), A_2(\alpha), A_3(\alpha)\big\rangle. \tag{3.18}$$

Next we consider the factor group G_α/A_α.

Lemma 3.5. *The factor group G_α/A_α is isomorphic to the modulary group $G(4) = \mathrm{PSL}(2,\mathbb{Z})/\Gamma(4)$.*

Proof. Since $G_\alpha = U_\alpha(PSL(2,\mathbb{Z}))$ and $A_\alpha = U_\alpha(\Gamma(4))$, it follows that

$$G_\alpha \cong \mathrm{PSL}(2,\mathbb{Z})/\ker U_\alpha,$$

respectively, $A_\alpha \cong \Gamma(4)/\ker U_\alpha$. Hence,

$$G_\alpha/A_\alpha \cong \mathrm{PSL}(2,\mathbb{Z})/\Gamma(4) = \mathrm{G}(4). \tag{3.19}$$

□

§4. Noncongruence character groups and the induced representation U_α

By the definition of U_α in (2.11), an element $g \in \mathrm{PSL}(2,\mathbb{Z})$ belongs to $\ker U_\alpha$ if and only if for all representatives $r \in R$ in (2.8) we have $\delta_{\Gamma_0(4)}(rgr^{-1})\chi_\alpha(rgr^{-1}) = 1$, and hence, if and only if $rgr^{-1} \in \ker\chi_\alpha$ for all $r \in R$, i.e.,

$$\ker U_\alpha = \big\{g \in \mathrm{PSL}(2,\mathbb{Z}) \,|\, rgr^{-1} \in \ker\chi_\alpha, \ \text{ for all } \ r \in R\big\}. \tag{4.1}$$

Lemma 4.1. *$\ker U_\alpha$ is congruent if and only if $\ker\chi_\alpha$ is congruent.*

Proof. Since $\mathrm{Id} \in R$, relation (4.1) shows that $\ker U_\alpha$ is a subgroup of $\ker\chi_\alpha$,

$$\ker U_\alpha \le \ker\chi_\alpha. \tag{4.2}$$

Thus, if $\ker U_\alpha$ is a congruence subgroup, then so is $\ker \chi_\alpha$. To prove the converse, consider the kernel $\ker U_\alpha$ in (4.1), which is given by the following intersection of sets:

$$\ker U_\alpha = r_1^{-1} \ker \chi_\alpha r_1 \cap r_2^{-1} \ker \chi_\alpha r_2 \cap \cdots \cap r_6^{-1} \ker \chi_\alpha r_6, \quad r_i \in R. \tag{4.3}$$

If $\ker \chi_\alpha$ is congruent, then $\Gamma(n) \leq \ker \chi_\alpha$ for some $n \in \mathbb{N}$. But $\Gamma(n)$ is normal in $\mathrm{PSL}(2,\mathbb{Z})$, so that $\Gamma(n) \leq r^{-1} \ker \chi_\alpha r$ for all $r \in R$. Therefore, by (4.3), $\Gamma(n) \leq \ker U_\alpha$. Hence, $\ker U_\alpha$ is also a congruence subgroup. □

As a consequence of Theorem 4.1, we have the following corollary.

Corollary 4.2. *The representation U_α is congruent if and only if Selberg's character χ_α is congruent.*

Next, we are going to show several properties of $\ker U_\alpha$, which will lead us for rational noncongruence values of α to an infinite family of noncongruence character groups with arbitrarily large genus, rather different from those determined by the character χ_α. At the same time, this will provide us with an independent way of finding the congruence values α of Newman et al. for the character χ_α (respectively, the representation U_α) and the corresponding congruence groups.

For this, denote by $N = N(\alpha)$ the order of the generators of the group A_α defined in (3.18). By Corollary 3.4,

$$\Gamma(4)/\ker U_\alpha \cong A_\alpha, \tag{4.4}$$

whence the index $\mu(\alpha) = \big[\mathrm{PSL}(2,\mathbb{Z}) : \ker U_\alpha\big]$ of $\ker U_\alpha$ in $\mathrm{PSL}(2,\mathbb{Z})$ is equal to the number of elements of A_α times the index of $\Gamma(4)$ in $\mathrm{PSL}(2,\mathbb{Z})$. Thus, we have

$$\mu(\alpha) = 24N^3 = 24N(\alpha)^3. \tag{4.5}$$

For irrational α, the subgroup $\ker U_\alpha$ is therefore of infinite index in $\mathrm{PSL}(2,\mathbb{Z})$ and cannot be a congruence group. In what follows, let α be rational with $N(\alpha) = N$ for some $N \in \mathbb{N}$.

Using the Gauss–Bonnet formula, we can determine the number of generators of $\ker U_\alpha$. Since $\ker U_\alpha \leq \Gamma(4)$, it has no elliptic elements. The Gauss–Bonnet formula for a group Γ without elliptic elements reads (see [26, p. 15]):

$$|F| = 2\pi(2g - 2 + p), \tag{4.6}$$

where $|F|$ is the area of the fundamental domain of Γ, g is its genus and p is the number of its cusps. It is also known that the number of generators of Γ is given by $2g + p$, see [26, p. 14]. But for the group $\ker U_\alpha$ we also have

$$|F| = \mu(\alpha)\frac{\pi}{3}, \tag{4.7}$$

where $\pi/3$ is the area of the fundamental domain of $\mathrm{PSL}(2,\mathbb{Z})$ and $\mu(\alpha)$ is the index of $\ker U_\alpha$ in $\mathrm{PSL}(2,\mathbb{Z})$, determined in (4.5). Hence, the number of generators of $\ker U_\alpha$ is given by

$$2g + p = 4N^3 + 2. \tag{4.8}$$

On the other hand, the number $\mathcal{N}(\alpha)$ of free generators is given by

$$\mathcal{N}(\alpha) = 2g + p - 1 = 4N^3 + 1, \tag{4.9}$$

see [26, p. 14].

Next we recall the concept of the width of a cusp and Wohlfahrt's generalized notion of the level of any subgroup Γ of the modular group [27].

Definition 4.3. For a cusp $x \in \mathbb{Q} \cup \{\infty\}$ of the group $\Gamma \leq \mathrm{PSL}(2,\mathbb{Z})$ and $\sigma \in \mathrm{PSL}(2,\mathbb{Z})$ with $\sigma\infty = x$, let $P \in \Gamma$ be a primitive parabolic element with $Px = x$. If

$$\sigma P \sigma^{-1} = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{Z}), \tag{4.10}$$

then $|m|$ is called the width of the cusp x of Γ.

Definition 4.4. Let $\Gamma \leq \mathrm{PSL}(2,\mathbb{Z})$, and let $W(\Gamma) \leq \mathbb{N}$ be the set of widths of the cusps of Γ. If $W(\Gamma)$ is nonempty and bounded in $\mathbb{N}$, then the Wohlfahrt level $n(\Gamma)$ of Γ is defined to be the least common multiple of the elements of $W(\Gamma)$. Otherwise the level is defined to be zero.

On the other hand, for congruence subgroups Γ, F. Klein defined the level as follows [27].

Definition 4.5. The level of a congruence subgroup is defined to be the smallest integer n such that $\Gamma(n) \subset \Gamma$.

It is known that, for congruence subgroups, Wohlfahrt's and F. Klein's definitions of the level coincide [27], that is, if Γ is a congruence subgroup of Wohlfahrt level n, then $\Gamma(n) \leq \Gamma$. Next, we determine the Wohlfahrt level of the group $\ker U_\alpha$. Since $\ker U_\alpha$ is normal in $\mathrm{PSL}(2,\mathbb{Z})$, all cusps of $\ker U_\alpha$ have the same width, see [26, p. 160]. Thus, it suffices to find the width for one cusp. By (3.14), for α with $N(\alpha) = N$, we have $U_\alpha(g_1)^N = \mathrm{Id}_{6\times 6}$ for the generator

$$g_1 = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} \tag{4.11}$$

of $\Gamma(4)$ in (3.13). Hence,

$$g_1^N = \begin{pmatrix} 1 & 4N \\ 0 & 1 \end{pmatrix} \tag{4.12}$$

belongs to $\ker U_\alpha$ and is obviously primitive. Thus, Wohlfahrt's level $n(\alpha)$ of $\ker U_\alpha$ is given for α with $N(\alpha) = N$ by

$$n(\alpha) = 4N. \tag{4.13}$$

Next we use a formula due to M. Newman [18] to determine the genus of $\ker U_\alpha$. Namely, let Γ be a normal subgroup of $\mathrm{PSL}(2,\mathbb{Z})$ with index μ, genus g, the number of its cusps p, and the Wohlfahrt level n. If $t := \frac{\mu}{n}$, then the following identity is true, see [18] and [26, p. 160]:

$$g = 1 + \frac{\mu}{12} - \frac{t}{2}. \tag{4.14}$$

For the group $\ker U_\alpha$, from (4.5) and (4.13) we see that $t = 6N^2$. Inserting this and (4.5) in (4.14), for the genus $g(\alpha)$ of $\ker U_\alpha$ we obtain

$$g(\alpha) = 1 + 2N^3 - 3N^2, \tag{4.15}$$

and, by (4.8), for the number $p(\alpha)$ of cusps of $\ker U_\alpha$ we get

$$p(\alpha) = 6N^2. \tag{4.16}$$

Summarizing, for the group $\ker U_\alpha$ we arrive at the following theorem.

Theorem 4.6. *If $N(\alpha) = N \in \mathbb{N}$ denotes the order of the generators of the Abelian group $A_\alpha = \Gamma(4)/\ker U_\alpha$ with $|A_\alpha| = N^3$, let $\mu(\alpha)$ be the index of the group $\ker U_\alpha$ in $\mathrm{PSL}(2,\mathbb{Z})$, $g(\alpha)$ its genus, $p(\alpha)$ the number of its cusps, $n(\alpha)$ its Wohlfahrt level, and*

$\mathcal{N}(\alpha)$ the number of its free generators. Then:

- $\mu(\alpha) = 24N^3$,
- $g(\alpha) = 1 + 2N^3 - 3N^2$,
- $p(\alpha) = 6N^2$,
- $n(\alpha) = 4N$,
- $\mathcal{N}(\alpha) = 4N^3 + 1$.

Since the Wohlfahrt level of $\ker U_\alpha$ is given by $n = 4N(\alpha)$, we see that if $\ker U_\alpha$ is a congruence group, then

$$\Gamma(4N) \leq \ker U_\alpha. \tag{4.17}$$

Using this, it is easy to determine the values of α for which $\ker U_\alpha$ is indeed a congruence subgroup. Since the index of $\Gamma(4N)$ in $\mathrm{PSL}(2,\mathbb{Z})$, given by

$$\big[\mathrm{PSL}(2,\mathbb{Z}) \,:\, \Gamma(4N)\big] = \frac{1}{2}(4N)^3 \prod_{p|4N} \Big(1 - \frac{1}{p^2}\Big), \tag{4.18}$$

must then be at least the index $\mu(\alpha)$ of $\ker U_\alpha$ in $\mathrm{PSL}(2,\mathbb{Z})$, we have

$$\frac{1}{2}(4N)^3 \prod_{p|4N} \Big(1 - \frac{1}{p^2}\Big) \geq 24N^3 \tag{4.19}$$

or

$$\frac{4}{3} \prod_{p|4N} \Big(1 - \frac{1}{p^2}\Big) \geq 1. \tag{4.20}$$

Obviously, this inequality is valid if and only if $N = 2^k$, $0 \leq k < \infty$.

Lemma 4.7. *If $N(\alpha) = 2^k$ and $\ker U_\alpha$ is a congruence group, then $\ker U_\alpha = \Gamma(2^{k+2})$ and, hence, $A_\alpha \cong \Gamma(4)/\Gamma(2^{k+2})$.*

Proof. For α with $N(\alpha) = 2^k$, the group A_α has order 2^{3k}. If $\ker U_\alpha$ is a congruence subgroup, then $\Gamma(2^{k+2}) \leq \ker U_\alpha$. On the other hand, for the index $\big[\Gamma(4) \,:\, \Gamma(2^{k+2})\big]$, a simple calculation yields $\big[\Gamma(4) \,:\, \Gamma(2^{k+2})\big] = 2^{3k}$. But $[\Gamma(4) \setminus \Gamma(2^{k+2})] \geq [\Gamma(4) \setminus \ker U_\alpha] \cong |A_\alpha| = 2^3k$. Therefore, $[\ker U_\alpha \,:\, \Gamma(2^{k+2})] = 1$ and hence $\ker U_\alpha = \Gamma(2^{k+2})$. □

Next we show that only for $k = 0, 1, 2$ the principal congruence subgroup $\Gamma(2^{k+2})$ (i.e., only $\Gamma(4), \Gamma(8)$ and $\Gamma(16)$) can coincide with the group $\ker U_\alpha$. This follows immediately from the following lemma.

Lemma 4.8. *The group $\Gamma(4)/\Gamma(2^{k+2})$ is Abelian if and only if $k = 0, 1, 2$.*

Since we did not find this result, which is presumably well known, in the literature, we give a simple proof.

Proof. For $h_i = \left(\begin{smallmatrix} 1+4a_i & 4b_i \\ 4c_i & 1+4d_i \end{smallmatrix}\right) \in \Gamma(4)$, $i = 1, 2$, and $h_{i,j} := h_i h_j$, $i, j = 1, 2$, we have

$$h_{1,2} = h_{2,1} = \begin{pmatrix} 1 + 4(a_1 + a_2) & 4(b_1 + b_2) \\ 4(c_1 + c_2) & 1 + 4(d_1 + d_2) \end{pmatrix} \bmod 16,$$

and

$$h_{1,2}^{-1} h_{2,1} = \begin{pmatrix} 1 + 4(a_1 + a_2 + d_1 + d_2) & 0 \\ 0 & 1 + 4(a_1 + a_2 + d_1 + d_2) \end{pmatrix} \bmod 16.$$

But $4 | (a_i + d_i), i = 1, 2$, whence $h_{1,2} = h_{2,1} \bmod \Gamma(16)$. Consequently, $\Gamma(4)/\Gamma(2^{k+2})$ is Abelian for $k = 0, 1, 2$. To show that this group is not Abelian for $k \geq 3$, we take two elements $h_1 = \left(\begin{smallmatrix} 1 & 4 \\ 0 & 1 \end{smallmatrix}\right)$ and $h_2 = \left(\begin{smallmatrix} -1 & 0 \\ 4 & -1 \end{smallmatrix}\right) \in \Gamma(4)$. Then $h_{1,2}^{-1} h_{2,1} = \left(\begin{smallmatrix} 17 & 64 \\ 64 & 241 \end{smallmatrix}\right)$, which does not belong to $\Gamma(2^{k+2)})$ for $k \geq 3$. □

Next we show that $\Gamma(16)$ cannot be a subgroup of $\ker U_\alpha$. Assume it is a subgroup. By Lemma 4.7, $\ker U_\alpha = \Gamma(16)$. But, by (3.17), $U_\alpha(g_5) = U_\alpha(g_3)$ for the generators g_3 and g_5 of $\Gamma(4)$ in (3.13), whence $g_3^{-1}g_5 \in \ker U_\alpha$. But $g_3^{-1}g_5 = \left(\begin{smallmatrix}1 & 8\\ 8 & 1\end{smallmatrix}\right) \bmod 16$, which does not belong to $\Gamma(16)$. Hence, $\ker U_\alpha > \Gamma(16)$, a contradiction. This proves the next corollary.

Corollary 4.9. *The group* $\ker U_\alpha$ *can be a congruence group only for* $N(\alpha) = 1, 2$.

Then, let α_1 and α_2 denote the α-values for which $N(\alpha_1) = 1$, $N(\alpha_2) = 2$, respectively. We are going to prove that $\ker U_{\alpha_1}$ and $\ker U_{\alpha_2}$ are indeed congruence groups. For this, recall that $\Gamma(4)/\ker U_\alpha \cong A_\alpha$. Since A_{α_1} is the trivial group, $\ker U_{\alpha_1}$ is equal to $\Gamma(4)$ and hence is a congruence group.

It remains to prove the congruence property of $\ker U_{\alpha_2}$. Since $N(\alpha_2) = 2$ and $U_{\alpha_2}(\Gamma(4)) = A_{\alpha_2}$, it follows that $(U_{\alpha_2}(g))^2 = \mathrm{id}$ for all $g \in \Gamma(4)$, so that $g^2 \in \ker U_{\alpha_2}$ for all $g \in \Gamma(4)$. But $g^2 \in \Gamma(8)$ for $g \in \Gamma(4)$. Therefore, the group $\langle g^2, g \in \Gamma(4)\rangle$ generated by $\Gamma(4)^2$ is also included in $\ker U_{\alpha_2}$, whence $\ker U_{\alpha_2} \cap \Gamma(8) \neq \varnothing$. Next we show that the groups $\Gamma(8)$ and $\ker U_{\alpha_2}$ coincide. For this, we note that $A_{\alpha_2} \cong C_2 \times C_2 \times C_2$, where C_2 is the cyclic group of order 2. But $A_{\alpha_2} \cong \Gamma(4)/\ker U_{\alpha_2}$ under the following well known natural group isomorphism $\imath_1 \colon \Gamma(4)/\ker U_{\alpha_2} \to A_{\alpha_2}$:

$$\imath_1(g \ker U_{\alpha_2}) = U_{\alpha_2}(g). \tag{4.21}$$

Thereby, the generators $A_i(\alpha_2), 1 \le i \le 3$, of the group A_{α_2} in (3.14)–(3.16) are mapped to the generators $g_i \ker U_{\alpha_2}, 1 \le i \le 3$, of the group $\Gamma(4)/\ker U_{\alpha_2}$, where the $\{g_i\}$ are as given in (3.13). Indeed, from equation (3.17) it follows that $g_3 = g_5 \bmod \ker U_{\alpha_2}$ and $g_4 = g_2^{-1}g_1^{-1} \bmod \ker U_{\alpha_2}$. On the other hand, it is known [15] that $\Gamma(4)/\Gamma(8)$ is also isomorphic to $C_2 \times C_2 \times C_2$. Indeed, the elements $g_i\Gamma(8), 1 \le i \le 3$, with $\{g_i, 1 \le i \le 5\}$ defined as in (3.13), are generators of the group $\Gamma(4)/\Gamma(8)$: we know that the five elements g_i, $1 \le i \le 5$, generate the group $\Gamma(4)$ and satisfy $g_i^2 = \mathrm{id} \bmod \Gamma(8)$. Furthermore, it is easy to check that $g_3 = g_5 \bmod \Gamma(8)$ and $g_4 = g_2^{-1}g_1^{-1} \bmod \Gamma(8)$. Therefore, the following map of their generators gives rise to an isomorphism $\imath$ of the two groups $\Gamma(4)/\ker U_{\alpha_2}$ and $\Gamma(4)/\Gamma(8)$:

$$\imath \colon \Gamma(4)/\ker U_{\alpha_2} \to \Gamma(4)/\Gamma(8) \tag{4.22}$$

defined by

$$\imath(g_i \ker U_{\alpha_2}) = g_i\Gamma(8). \tag{4.23}$$

Indeed, $\imath(g_i \ker U_{\alpha_2} g_j \ker U_{\alpha_2}) = \imath(g_1 g_2 \ker U_{\alpha_2}) = g_i g_j \Gamma(8) = g_i\Gamma(8) g_j\Gamma(8)$. Since any $g \in \Gamma(4)$ can be expressed both $\bmod \ker U_{\alpha_2}$ and $\bmod \Gamma(8)$ in terms of the generators $g_i, 1 \le i \le 3$, this implies $\imath(g \ker U_{\alpha_2}) = g\Gamma(8)$ for all $g \in \Gamma(4)$. For $g \in \ker U_{\alpha_2}$ this shows that $g \in \Gamma(8)$, implying $\Gamma(8) \ge \ker U_{\alpha_2}$. Then arguments as in Lemma 4.7 show that the two groups coincide. This yields the following corollary.

Corollary 4.10. *The kernel* $\ker U_{\alpha_2}$ *is given by* $\Gamma(8)$ *and, hence,* U_{α_2} *is a congruence representation.*

From the definition of the generators of A_α in (3.14), (3.15), and (3.16) it is clear that $N(\alpha_1) = 1$ if and only if $8\pi i\alpha_1 = 2\pi i k$ and if and only if $\alpha_1 = (1/4)k$ with $k \in \mathbb{Z}$. Moreover, $N(\alpha_2) = 2$ if and only if $8\pi i \alpha_2 = \pi i k$ and if and only if $\alpha_2 = (1/8)k$ with $k \in \mathbb{Z}$ and $(k, 2) = 1$.

Summarizing our discussion of the congruence properties of the kernels $\ker U_\alpha$, we arrive at the following result.

Theorem 4.11. *The representation $U_\alpha, 0 \le \alpha \le 1/2$, defined in* (2.11) *is a congruence representation only for the α-values $0, \frac{1}{8}, \frac{2}{8}, \frac{3}{8}, \frac{4}{8}$. Moreover, we have*

$$\ker U_0 = \ker U_{\frac{2}{8}} = \ker U_{\frac{4}{8}} = \Gamma(4), \tag{4.24}$$

and, respectively,

$$\ker U_{\frac{1}{8}} = \ker U_{\frac{3}{8}} = \Gamma(8). \tag{4.25}$$

Obviously, this implies the well known result of Newman et al. on the congruence properties of the character χ_α. Contrary to the latter case, where the principal congruence groups $\Gamma(2d)$, $d = 1, 2, 4, 8$, appear as subgroups for the congruence character χ_α, for the induced representation U_α only the two groups $\Gamma(4)$ and $\Gamma(8)$ are related to the congruence properties of its representations. On the other hand, the resulting noncongruence groups are of completely different nature in these two cases. It would be of interest to see if something similar happens also for the induced representations of other characters, like for instance the χ_n studied in [13, 2] and [14] for the congruence group $\Gamma^1(5)$.

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