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HOMOGENIZATION OF ELLIPTIC SYSTEMS WITH PERIODIC COEFFICIENTS: OPERATOR ERROR ESTIMATES IN $L_2(\mathbb{R}^d)$ WITH CORRECTOR TAKEN INTO ACCOUNT

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Abstract. A matrix elliptic selfadjoint second order differential operator (DO) $\mathcal{B}_\varepsilon$ with rapidly oscillating coefficients is considered in $L_2(\mathbb{R}^d;\mathbb{C}^n)$. The principal part $b(\mathbf{D})^* g(\varepsilon^{-1}\mathbf{x}) b(\mathbf{D})$ of this operator is given in a factorized form, where g is a periodic, bounded, and positive definite matrix-valued function and $b(\mathbf{D})$ is a matrix first order DO whose symbol is a matrix of maximal rank. The operator $\mathcal{B}_\varepsilon$ also includes first and zero order terms with unbounded coefficients. The problem of homogenization in the small period limit is studied. For the generalized resolvent of $\mathcal{B}_\varepsilon$, approximation in the $L_2(\mathbb{R}^d;\mathbb{C}^n)$-operator norm with an error $O(\varepsilon^2)$ is obtained. The principal term of this approximation is given by the generalized resolvent of the effective operator $\mathcal{B}^0$ with constant coefficients. The first order corrector is taken into account. The error estimate obtained is order sharp; the constants in estimates are controlled in terms of the problem data. General results are applied to homogenization problems for the Schrödinger operator and the two-dimensional Pauli operator with singular rapidly oscillating potentials.

Introduction

The paper concerns homogenization theory of periodic differential operators (DOs). A broad literature is devoted to this field; at the first place, we mention the books [BeLP, BaPa, ZhKO].

0.1. Statement of the problem. Main results. We study matrix elliptic DOs in $\mathbb{R}^d$. Let Γ be a lattice in $\mathbb{R}^d$, and let Ω be the elementary cell of the lattice Γ. For Γ-periodic functions on $\mathbb{R}^d$, we denote $\varphi^\varepsilon(\mathbf{x}) := \varphi(\varepsilon^{-1}\mathbf{x})$, $\varepsilon > 0$; $\overline{\varphi} := |\Omega|^{-1}\int_\Omega \varphi(\mathbf{x})\,d\mathbf{x}$.

Consider a selfadjoint DO $\mathcal{B}_\varepsilon$ acting in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ and formally defined by

$$\mathcal{B}_\varepsilon = b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D}) + \sum_{j=1}^d \left(a_j^\varepsilon(\mathbf{x}) D_j + D_j (a_j^\varepsilon(\mathbf{x}))^* \right) + Q^\varepsilon(\mathbf{x}). \tag{0.1}$$

(The precise definition is given in terms of quadratic forms.) Here the principal term $\mathcal{A}_\varepsilon = b(\mathbf{D})^* g^\varepsilon b(\mathbf{D})$ is presented in a factorized form, where $g(\mathbf{x})$ is a Γ-periodic, bounded, and positive definite matrix-valued function, and $b(\mathbf{D})$ is a matrix homogeneous first order DO with constant coefficients. (The precise assumptions on g and $b(\mathbf{D})$ are listed below in §3.) Homogenization problems for the operator $\mathcal{A}_\varepsilon$ have been studied in detail in [BSu1, BSu2, BSu3, BSu4]. The coefficients $a_j(\mathbf{x})$ in (0.1) are Γ-periodic matrix-valued functions; in general, they are unbounded. In general, $Q(\mathbf{x})$ is a distribution generated by some periodic measure (with values in the class of Hermitian matrices). The precise

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assumptions on the coefficients are formulated in §3. The coefficients of the operator (0.1) oscillate rapidly as $\varepsilon \to 0$.

A typical homogenization problem for the operator (0.1) *is to approximate the resolvent* $(\mathcal{B}_\varepsilon + \lambda I)^{-1}$ *or the generalized resolvent* $(\mathcal{B}_\varepsilon + \lambda Q_0^\varepsilon)^{-1}$ *for small* ε. Here $Q_0(\mathbf{x})$ is a bounded and positive definite Γ-periodic matrix-valued function. The parameter λ is subject to some restriction ensuring the positive definiteness of the operator $\mathcal{B}_{\lambda,\varepsilon} := \mathcal{B}_\varepsilon + \lambda Q_0^\varepsilon$. Before, the homogenization problem for $\mathcal{B}_\varepsilon$ was studied by D. I. Borisov [Bo] (in the case of sufficiently smooth coefficients) and by the author [Su1, Su2]. It was shown that, as $\varepsilon \to 0$, the operator $\mathcal{B}_{\lambda,\varepsilon}^{-1}$ converges in the $L_2(\mathbb{R}^d;\mathbb{C}^n)$-operator norm to the operator $(\mathcal{B}_\lambda^0)^{-1}$, where $\mathcal{B}_\lambda^0 = \mathcal{B}^0 + \lambda \overline{Q_0}$ is the effective operator with constant coefficients. The following sharp order estimate was obtained:

$$\left\| \mathcal{B}_{\lambda,\varepsilon}^{-1} - (\mathcal{B}_\lambda^0)^{-1} \right\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C\varepsilon, \quad 0 < \varepsilon \leq 1. \tag{0.2}$$

The effective operator is given by

$$\mathcal{B}^0 = b(\mathbf{D})^* g^0 b(\mathbf{D}) - b(\mathbf{D})^* V - V^* b(\mathbf{D}) + \sum_{j=1}^{d} (\overline{a_j + a_j^*}) D_j + \overline{Q} - W. \tag{0.3}$$

Here g^0 is the so-called effective matrix (see the definition in §4). Note that the operator (0.3) involves nontrivial lower order terms (the constant matrices V and W are defined below in §5).

Besides estimate (0.2), in [Bo] and [Su2] approximation of the operator $\mathcal{B}_{\lambda,\varepsilon}^{-1}$ in the norm of operators acting from $L_2(\mathbb{R}^d;\mathbb{C}^n)$ to the Sobolev space $H^1(\mathbb{R}^d;\mathbb{C}^n)$ was found, and the following sharp order error estimate was obtained:

$$\left\| \mathcal{B}_{\lambda,\varepsilon}^{-1} - (\mathcal{B}_\lambda^0)^{-1} - \varepsilon K_{1,\varepsilon} \right\|_{L_2(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \leq C\varepsilon, \quad 0 < \varepsilon \leq 1. \tag{0.4}$$

Here the corrector $K_{1,\varepsilon}$ defined by

$$K_{1,\varepsilon} = (\Lambda^\varepsilon b(\mathbf{D}) + \widetilde{\Lambda}^\varepsilon) \Pi_\varepsilon (\mathcal{B}_\lambda^0)^{-1} \tag{0.5}$$

is taken into account. The periodic matrix-valued functions Λ and $\widetilde{\Lambda}$ are described below in §4 as solutions of some boundary value problems on the cell Ω; Π_ε is an auxiliary smoothing operator. The corrector (0.5) differs from the standard corrector of the homogenization theory only by the operator Π_ε.

In the present paper, we obtain approximation of the operator $\mathcal{B}_{\lambda,\varepsilon}^{-1}$ in the $L_2(\mathbb{R}^d;\mathbb{C}^n)$-operator norm with error $O(\varepsilon^2)$. *Our main result* is the following sharp order estimate:

$$\left\| \mathcal{B}_{\lambda,\varepsilon}^{-1} - (\mathcal{B}_\lambda^0)^{-1} - \varepsilon K_\varepsilon \right\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C\varepsilon^2, \quad 0 < \varepsilon \leq 1. \tag{0.6}$$

The constant C is controlled in terms of the problem data. The structure of the corrector K_ε is more complicated than the structure of the corrector (0.5), which is the first term of the new corrector. Namely, we have

$$K_\varepsilon = K_{1,\varepsilon} + K_{1,\varepsilon}^* + K_3. \tag{0.7}$$

The third term in (0.7) does not depend on ε and has the form $K_3 = -(\mathcal{B}_\lambda^0)^{-1} \mathcal{N} (\mathcal{B}_\lambda^0)^{-1}$, where $\mathcal{N}$ is a selfadjoint third order operator with constant coefficients; see §7. The expression for $\mathcal{N}$ that we have found is rather bulky, before it was known only in the absence of lower order terms (see [BSu3]).

Comparison of estimates (0.4) and (0.6) does not lead to a contradiction. As was shown in [BSu4] for the operator $\mathcal{A}_\varepsilon$, estimate (0.4) remains true if $K_{1,\varepsilon}$ is replaced by K_ε; only the constant in the estimate will change under such a replacement. This means that the second and third terms of the corrector (0.7) are of order of $O(1)$ in the $(L_2 \to H^1)$-operator norm, and so they “move to the error term”. At the same time, these terms should be taken into account when we approximate the operator $\mathcal{B}_{\lambda,\varepsilon}^{-1}$ in the L_2-operator norm with error $O(\varepsilon^2)$.

In general, the first two terms of the corrector (0.7) involve the smoothing operator Π_ε, which arises necessarily in the corrector because the functions Λ and $\widetilde{\Lambda}$ are not sufficiently regular. We trace the cases where it is possible to get rid of Π_ε.

Note that the operator

$$\mathcal{B}_\varepsilon = \mathcal{A}_\varepsilon + \varepsilon^{-1} v^\varepsilon + Q^\varepsilon \tag{0.8}$$

with a singular potential $\varepsilon^{-1} v^\varepsilon$ can be represented in the form (0.1) (see §8). Here $v(\mathbf{x})$ is a Hermitian periodic matrix-valued function with zero mean value. In general, the corresponding effective operator has nontrivial first order terms.

We consider some applications of the general results. In §9, we study a homogenization problem for the Schrödinger operator with rapidly oscillating metric g^ε, magnetic potential $\mathbf{A}^\varepsilon$, and electric potential $\varepsilon^{-2}\breve{v}^\varepsilon + \varepsilon^{-1} v^\varepsilon + \mathcal{V}^\varepsilon$ involving singular summands. In §10, the general results are applied to the two-dimensional Pauli operator with the magnetic potential $\varepsilon^{-1}\breve{\mathbf{A}}^\varepsilon + \mathbf{A}^\varepsilon$ (with a singular summand) perturbed by a matrix electric potential $\varepsilon^{-1} v^\varepsilon + \mathcal{V}^\varepsilon$.

0.2. Estimates of the form (0.2), (0.4), and (0.6) are called operator error estimates in homogenization theory. In the case where $\mathcal{B}_\varepsilon = \mathcal{A}_\varepsilon$ (i.e., in the absence of lower order terms), an estimate of the form (0.2) was obtained for the first time in [BSu1] by a spectral method. Estimates (0.4) and (0.6) for the operator $\mathcal{A}_\varepsilon$ were obtained in [BSu4] and [BSu3], respectively.

By a different method suggested by V. V. Zhikov, estimates of the form (0.2) and (0.4) were obtained in [Zh, ZhPas] for the acoustics and elasticity operators (in the absence of lower order terms and for $Q_0 = I$).

In the presence of the first and zero order terms, the homogenization problem for the operator $\mathcal{B}_\varepsilon$ was studied in [Bo], where estimates (0.2) and (0.4) were obtained. Moreover, it was assumed that the coefficients depend not only on the rapid variable but also on the slow variable. However, in [Bo] the coefficients of the operator were assumed to be sufficiently smooth, and the constants in estimates depended on "smooth" norms of the coefficients.

In [Su1], estimate (0.2) was proved in the case where the coefficients a_j and Q were bounded. In [Su2] this restriction was relaxed, and estimates (0.2) and (0.4) were proved under mild assumptions on the coefficients (the same as in the present paper).

In the absence of the lower order terms, estimate (0.6) was obtained in [BSu3], where it was found out that the corrector in estimate (0.6) differs from the standard one and has a three-terms structure like in (0.7).

0.3. Our method is a further development of the operator-theoretic approach in the homogenization theory. This method was suggested and developed in the papers [BSu1, BSu2, BSu3, BSu4], where the operator $\mathcal{A}_\varepsilon$ was studied, and was modified in [Su1, Su2] as applied to $\mathcal{B}_\varepsilon$.

By a scaling transformation, the problem is reduced to the study of the pencil of selfadjoint operators

$$\mathcal{B}_\lambda(\varepsilon) = b(\mathbf{D})^* g(\mathbf{x}) b(\mathbf{D}) + \varepsilon \sum_{j=1}^{d} (a_j(\mathbf{x}) D_j + D_j a_j(\mathbf{x})^*) + \varepsilon^2 Q(\mathbf{x}) + \lambda \varepsilon^2 Q_0(\mathbf{x}), \tag{0.9}$$

acting in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ and depending on the parameter ε. Here the coefficients depend on $\mathbf{x}$ (but not on $\varepsilon^{-1}\mathbf{x}$). We need to approximate the operator $\mathcal{B}_\lambda(\varepsilon)^{-1}$ with error $O(1)$. Next, by the Floquet-Bloch theory the operator (0.9) is decomposed in the direct integral of the operators $\mathcal{B}_\lambda(\mathbf{k},\varepsilon)$ acting in $L_2(\Omega;\mathbb{C}^n)$ and depending on the parameter $\mathbf{k}\in\mathbb{R}^d$ (the quasimomentum). The operator $\mathcal{B}_\lambda(\mathbf{k},\varepsilon)$ is defined by the expression

$$(0.10)\quad \mathcal{B}_\lambda(\mathbf{k},\varepsilon)=b(\mathbf{D}+\mathbf{k})^*gb(\mathbf{D}+\mathbf{k})+\varepsilon\sum_{j=1}^d\bigl(a_j(D_j+k_j)+(D_j+k_j)a_j^*\bigr)+\varepsilon^2Q+\lambda\varepsilon^2Q_0$$

with periodic boundary conditions. The problem is reduced to approximation of the operator $\mathcal{B}_\lambda(\mathbf{k},\varepsilon)^{-1}$ in the $L_2(\Omega;\mathbb{C}^n)$-operator norm with error $O(1)$ uniformly with respect to the quasimomentum. The operator $\mathcal{B}_\lambda(\mathbf{k},\varepsilon)$ is an elliptic operator in a bounded domain; its spectrum is discrete. The family (0.10) is a quadratic function of the multidimensional parameter $(\mathbf{k},\varepsilon)$. For $\mathbf{k}=0$ and $\varepsilon=0$, the point $\lambda_0=0$ is an eigenvalue of this operator of multiplicity n (and is the bottom of its spectrum). For approximation of the operator $\mathcal{B}_\lambda(\mathbf{k},\varepsilon)^{-1}$, only a small spectral interval $[0,\delta]$ is important. In order to find the required approximation, we apply methods of the analytic perturbation theory and study (0.10) as an analytic operator family with respect to the *one-dimensional parameter* $\tau=(|\mathbf{k}|^2+\varepsilon^2)^{1/2}$. Here, a good deal of constructions can be done in the framework of an abstract operator-theoretic approach. The corresponding abstract considerations were made in advance in [Su3]; the required approximation of the operator $\mathcal{B}_\lambda(\mathbf{k},\varepsilon)^{-1}$ is deduced from the results of [Su3].

0.4. The plan of the paper. The paper consists of 10 sections. In §1, the results of the abstract operator-theoretic method, as obtained in [Su3], are presented. Chapter 1 (§§2–6) is devoted to the study of the operator family (0.9). In §2, some preliminaries about lattices and the Gelfand transformation are given. §3 contains the definition of the operator $\mathcal{B}_\lambda(\varepsilon)$ and the direct integral expansion for this operator. In §4, the operator pencil $\mathcal{B}_\lambda(\mathbf{k},\varepsilon)$ is studied, and the objects of the abstract approach are realized as applied to this pencil. In §5, approximation of the operator $\mathcal{B}_\lambda(\mathbf{k},\varepsilon)^{-1}$ in the $L_2(\Omega;\mathbb{C}^n)$-operator norm with error $O(1)$ is obtained on the basis of the abstract results. In §6, we use the results of §5 and the direct integral expansion to deduce approximation of the operator $\mathcal{B}_\lambda(\varepsilon)^{-1}$ in the $L_2(\mathbb{R}^d;\mathbb{C}^n)$-operator norm with error $O(1)$. A separate attention is paid to the conditions that allow one to get rid of the smoothing operator in the corrector. Chapter 2 (§§7–10) is devoted to the homogenization problem for the operator $\mathcal{B}_{\lambda,\varepsilon}$. §7 contains the statement of the problem and the main result: by a scaling transformation, estimate (0.6) is deduced from the results of §6. In §8, we consider the homogenization problem for the operator (0.8) with a singular potential, and also obtain approximation of the generalized resolvent of the operator $\widetilde{\mathcal{B}}_\varepsilon=(f^\varepsilon)^*\mathcal{B}_\varepsilon f^\varepsilon$ (which is interesting for applications). In §9 and §10, the general results are applied to the specific operators of mathematical physics, namely, to the Schrödinger operator and the two-dimensional Pauli operator with singular rapidly oscillating potentials.

0.5. Notation. Let $\mathfrak{H}$ and $\mathfrak{H}_*$ be separable complex Hilbert spaces. The symbols $(\cdot,\cdot)_{\mathfrak{H}}$ and $\|\cdot\|_{\mathfrak{H}}$ stand for the inner product and the norm in $\mathfrak{H}$; the symbol $\|\cdot\|_{\mathfrak{H}\to\mathfrak{H}_*}$ stands for the norm of a continuous linear operator acting from $\mathfrak{H}$ to $\mathfrak{H}_*$. Sometimes, we omit the indices if this does not lead to confusion. If $A\colon\mathfrak{H}\to\mathfrak{H}_*$ is a linear operator, then $\operatorname{Dom} A$ denotes its domain. If $\mathfrak{N}$ is a subspace in $\mathfrak{H}$, then $\mathfrak{N}^\perp$ denotes its orthogonal complement.

The symbols $\langle\cdot,\cdot\rangle$ and $|\cdot|$ stand for the standard inner product and the norm in $\mathbb{C}^n$; $\mathbf{1}=\mathbf{1}_n$ is the identity $(n\times n)$-matrix. For $z\in\mathbb{C}$, by z^* we denote the complex conjugate number; we use this nonstandard notation because an overline is reserved to denote the mean value of a periodic function. If a is an $(m\times n)$-matrix, then $|a|$ denotes the norm of this matrix viewed as a linear operator from $\mathbb{C}^n$ to $\mathbb{C}^m$; a^t denotes the transpose matrix, and a^* stands for the adjoint $(n\times m)$-matrix.

The L_p-classes of $\mathbb{C}^n$-valued functions in a domain $\mathcal{O} \subset \mathbb{R}^d$ are denoted by $L_p(\mathcal{O};\mathbb{C}^n)$, $1 \le p \le \infty$. The Sobolev classes of $\mathbb{C}^n$-valued functions in a domain $\mathcal{O} \subset \mathbb{R}^d$ of order s and with integrability index p are denoted by $W_p^s(\mathcal{O};\mathbb{C}^n)$. For $p = 2$ we use the notation $H^s(\mathcal{O};\mathbb{C}^n)$. If $n = 1$, we write simply $L_p(\mathcal{O})$, $W_p^s(\mathcal{O})$, $H^s(\mathcal{O})$, etc., but often we use such abbreviated notation also for spaces of vector-valued or matrix-valued functions.

Next, $\mathbf{x} = (x^1, \dots, x^d) \in \mathbb{R}^d$, $iD_j = \partial_j = \partial/\partial x^j$, $j = 1, \dots, d$, $\nabla = (\partial_1, \dots, \partial_d)$, $\mathbf{D} = -i\nabla = (D_1, \dots, D_d)$.

Various constants in estimates are denoted by C, c, $\mathcal{C}$ (probably, with indices and marks).

§1. Abstract operator-theoretic method

Our approach to the study of homogenization problems is based on an abstract operator-theoretic method. In this section, we briefly describe the corresponding results obtained in [Su3].

We study the family of operators $B(t,\varepsilon)$ depending on two real-valued parameters $\varepsilon \in [0,1]$ and $t \in \mathbb{R}$.

1.1. The operators $X(t)$ and $A(t)$. Let $\mathfrak{H}$ and $\mathfrak{H}_*$ be complex separable Hilbert spaces. Suppose $X_0 \colon \mathfrak{H} \to \mathfrak{H}_*$ is a densely defined and closed operator and $X_1 \colon \mathfrak{H} \to \mathfrak{H}_*$ is a bounded operator. On the domain $\operatorname{Dom} X(t) = \operatorname{Dom} X_0$, we introduce the operator $X(t) := X_0 + tX_1$, $t \in \mathbb{R}$. Consider the family of positive selfadjoint operators $A(t) := X(t)^* X(t)$ in $\mathfrak{H}$. The operator $A(t)$ is generated in $\mathfrak{H}$ by the closed quadratic form $\|X(t)u\|^2_{\mathfrak{H}_*}$, $u \in \operatorname{Dom} X_0$. We put $A(0) = X_0^* X_0 =: A_0$ and $\mathfrak{N} := \operatorname{Ker} A_0 = \operatorname{Ker} X_0$. Suppose that the following condition is satisfied.

Condition 1.1. *The point $\lambda_0 = 0$ is an isolated point of the spectrum of A_0, and $0 < n := \dim \mathfrak{N} < \infty$.*

Let d^0 denote the *distance from the point $\lambda_0 = 0$ to the rest of the spectrum of A_0*. Denote $\mathfrak{N}_* := \operatorname{Ker} X_0^*$, $n_* := \dim \mathfrak{N}_*$. *Suppose that $n \le n_* \le \infty$.* Let P be the orthogonal projection of $\mathfrak{H}$ onto $\mathfrak{N}$, and let P_* be the orthogonal projection of $\mathfrak{H}_*$ onto $\mathfrak{N}_*$.

The operator family $A(t)$ was studied in detail in [BSu1, Chapter 1], [BSu2], [BSu4, Chapter 1].

1.2. The operators $Y(t)$ and Y_2. Let $\widetilde{\mathfrak{H}}$ be yet another Hilbert space. Suppose $Y_0 \colon \mathfrak{H} \to \widetilde{\mathfrak{H}}$ is a densely defined linear operator such that $\operatorname{Dom} X_0 \subset \operatorname{Dom} Y_0$ and $Y_1 \colon \mathfrak{H} \to \widetilde{\mathfrak{H}}$ is a bounded linear operator. We put $Y(t) = Y_0 + tY_1$, $\operatorname{Dom} Y(t) = \operatorname{Dom} Y_0$, and impose the following condition.

Condition 1.2. *There exists a constant $c_1 > 0$ such that*

$$\|Y(t)u\|_{\widetilde{\mathfrak{H}}} \le c_1 \|X(t)u\|_{\mathfrak{H}_*}, \quad u \in \operatorname{Dom} X_0, \quad t \in \mathbb{R}. \tag{1.1}$$

From (1.1) with $t = 0$ it follows that $\operatorname{Ker} X_0 \subset \operatorname{Ker} Y_0$, i.e., $Y_0 P = 0$.

Let $Y_2 \colon \mathfrak{H} \to \widetilde{\mathfrak{H}}$ be a densely defined operator such that $\operatorname{Dom} X_0 \subset \operatorname{Dom} Y_2$. We impose the following condition.

Condition 1.3. *For any $\nu > 0$ there exists a constant $C(\nu) > 0$ such that*

$$\|Y_2 u\|^2_{\widetilde{\mathfrak{H}}} \le \nu \|X(t)u\|^2_{\mathfrak{H}_*} + C(\nu)\|u\|^2_{\mathfrak{H}}, \quad u \in \operatorname{Dom} X_0, \quad t \in \mathbb{R}.$$

1.3. The form $\mathfrak{q}$. Let $\mathfrak{q}[u,v]$ be a densely defined Hermitian sesquilinear form in $\mathfrak{H}$. Assume that $\operatorname{Dom} X_0 \subset \operatorname{Dom} \mathfrak{q}$, and impose the following condition.

Condition 1.4. 1°. *There exist constants $c_2 \geq 0$ and $c_3 \geq 0$ such that*

$$|\mathfrak{q}[u,v]| \leq \left(c_2\|X(t)u\|^2_{\mathfrak{H}_*} + c_3\|u\|^2_{\mathfrak{H}}\right)^{1/2} \left(c_2\|X(t)v\|^2_{\mathfrak{H}_*} + c_3\|v\|^2_{\mathfrak{H}}\right)^{1/2}, \quad u,v \in \operatorname{Dom} X_0, \quad t \in \mathbb{R}.$$

2°. *There exist constants $0 < \kappa \leq 1$ and $c_0 \in \mathbb{R}$ such that*

$$\mathfrak{q}[u,u] \geq -(1-\kappa)\|X(t)u\|^2_{\mathfrak{H}_*} - c_0\|u\|^2_{\mathfrak{H}}, \quad u \in \operatorname{Dom} X_0, \quad t \in \mathbb{R}.$$

1.4. The operator $B(t,\varepsilon)$. In the space $\mathfrak{H}$, we consider the following Hermitian sesquilinear form:

$$b(t,\varepsilon)[u,v] = (X(t)u, X(t)v)_{\mathfrak{H}_*} + \varepsilon\left((Y(t)u, Y_2 v)_{\widetilde{\mathfrak{H}}} + (Y_2 u, Y(t)v)_{\widetilde{\mathfrak{H}}}\right) + \varepsilon^2 \mathfrak{q}[u,v], \quad u,v \in \operatorname{Dom} X_0. \tag{1.2}$$

Using Conditions 1.2, 1.3, and 1.4, it is easy to check the following inequalities (see [Su2, Subsection 1.4]):

$$b(t,\varepsilon)[u,u] \leq (2 + c_1^2 + c_2)\|X(t)u\|^2_{\mathfrak{H}_*} + (C(1) + c_3)\varepsilon^2\|u\|^2_{\mathfrak{H}}, \quad u \in \operatorname{Dom} X_0, \tag{1.3}$$

$$b(t,\varepsilon)[u,u] \geq \frac{\kappa}{2}\|X(t)u\|^2_{\mathfrak{H}_*} - (c_0 + c_4)\varepsilon^2\|u\|^2_{\mathfrak{H}}, \quad u \in \operatorname{Dom} X_0, \tag{1.4}$$

where $c_4 := 4\kappa^{-1}c_1^2 C(\nu)$ for $\nu = \kappa^2(16c_1^2)^{-1}$. By (1.3) and (1.4), the form $b(t,\varepsilon)[u,u]$ is closed and lower semibounded.

A selfadjoint operator in $\mathfrak{H}$ generated by the form (1.2) is denoted by $B(t,\varepsilon)$. Formally, we have

$$B(t,\varepsilon) = A(t) + \varepsilon(Y_2^* Y(t) + Y(t)^* Y_2) + \varepsilon^2 Q. \tag{1.5}$$

(Here Q is a formal object that corresponds to the form $\mathfrak{q}$.)

We consider the generalized resolvent $(B(t,\varepsilon) + \lambda\varepsilon^2 Q_0)^{-1}$ of the operator (1.5). Here $Q_0 \colon \mathfrak{H} \to \mathfrak{H}$ is a bounded and positive definite operator. We denote

$$\begin{aligned} B_\lambda(t,\varepsilon) &:= B(t,\varepsilon) + \lambda\varepsilon^2 Q_0, \\ b_\lambda(t,\varepsilon)[u,v] &:= b(t,\varepsilon)[u,v] + \lambda\varepsilon^2(Q_0 u, v)_{\mathfrak{H}}, \quad u,v \in \operatorname{Dom} X_0. \end{aligned} \tag{1.6}$$

The parameter λ is subject to the following restriction:

$$\begin{aligned} &\lambda > \|Q_0^{-1}\|(c_0 + c_4) \quad \text{if} \quad \lambda \geq 0, \\ &\lambda > \|Q_0\|^{-1}(c_0 + c_4) \quad \text{if} \quad \lambda < 0 \ \text{(and} \quad c_0 + c_4 < 0). \end{aligned} \tag{1.7}$$

Condition (1.7) implies that

$$b_\lambda(t,\varepsilon)[u,u] \geq \frac{\kappa}{2}\|X(t)u\|^2_{\mathfrak{H}_*} + \beta\varepsilon^2\|u\|^2_{\mathfrak{H}}, \quad u \in \operatorname{Dom} X_0, \tag{1.8}$$

where $\beta > 0$ is defined in terms of λ as follows:

$$\begin{aligned} &\beta = \lambda\|Q_0^{-1}\|^{-1} - c_0 - c_4 \quad \text{if} \quad \lambda \geq 0, \\ &\beta = \lambda\|Q_0\| - c_0 - c_4 \quad \text{if} \quad \lambda < 0 \ \text{(and} \quad c_0 + c_4 < 0). \end{aligned} \tag{1.9}$$

Thus, under the above assumptions the operator (1.6) is positive definite.

1.5. The operators Z, $\widetilde{Z}$, R, and S. Now we introduce some operators that arise when analytic perturbation theory is invoked. For the details, see [Su3, Subsections 1.6, 1.7]. Let $\omega \in \mathfrak{N}$, and let $\phi = \phi(\omega) \in \operatorname{Dom} X_0 \cap \mathfrak{N}^\perp$ be a (weak) solution of the equation $X_0^*(X_0\phi + X_1\omega) = 0$. We define a bounded operator $Z\colon \mathfrak{H} \to \mathfrak{H}$ by the relation $Zu = \phi(Pu)$, $u \in \mathfrak{H}$.

Similarly, let $\omega \in \mathfrak{N}$, and let $\psi = \psi(\omega) \in \operatorname{Dom} X_0 \cap \mathfrak{N}^\perp$ be a (weak) solution of the equation $X_0^* X_0 \psi + Y_0^* Y_2 \omega = 0$. We define a bounded operator $\widetilde{Z}\colon \mathfrak{H} \to \mathfrak{H}$ by the relation $\widetilde{Z}u = \psi(Pu)$, $u \in \mathfrak{H}$. Note that Z and $\widetilde{Z}$ take $\mathfrak{N}$ to $\mathfrak{N}^\perp$ and $\mathfrak{N}^\perp$ to $\{0\}$.

Next, we define an operator $R\colon \mathfrak{N} \to \mathfrak{N}_*$ by the formula

$$R\omega = X_0\phi(\omega) + X_1\omega = (X_0 Z + X_1)\omega, \quad \omega \in \mathfrak{N}. \tag{1.10}$$

Another description of R is given by $R = P_* X_1|_{\mathfrak{N}}$.

In accordance with [BSu1, Chapter 1, Subsection 1.3], the operator $S := R^*R\colon \mathfrak{N} \to \mathfrak{N}$ is called the *spectral germ of the operator family $A(t)$ at $t = 0$*. The germ S can be written as $S = PX_1^* P_* X_1|_{\mathfrak{N}}$.

1.6. Approximation of the operator $B_\lambda(t,\varepsilon)^{-1}$. *We fix a number $\delta \in (0, \kappa d^0/13)$, and next we fix a number $\tau_0 > 0$ such that*

$$\tau_0 \le \delta^{1/2} \left((2 + c_1^2 + c_2)\|X_1\|^2 + C(1) + c_3 + |\lambda|\|Q_0\| \right)^{-1/2}. \tag{1.11}$$

In the space $\mathfrak{N}$, consider the operator $L(t,\varepsilon)$ given by

$$\begin{aligned} L(t,\varepsilon) = t^2 S + t\varepsilon\big(-(X_0Z)^* X_0\widetilde{Z} - (X_0\widetilde{Z})^* X_0 Z + P(Y_2^* Y_1 + Y_1^* Y_2)\big)\big|_{\mathfrak{N}} \\ + \varepsilon^2\big(-(X_0\widetilde{Z})^* X_0\widetilde{Z}\big|_{\mathfrak{N}} + Q_{\mathfrak{N}} + \lambda Q_{0\mathfrak{N}}\big). \end{aligned} \tag{1.12}$$

Here $Q_{\mathfrak{N}}$ is the selfadjoint operator in $\mathfrak{N}$ generated by the form $\mathfrak{q}[\omega,\omega]$, $\omega \in \mathfrak{N}$, and $Q_{0\mathfrak{N}} := PQ_0|_{\mathfrak{N}}$.

Next, in the space $\mathfrak{H}$ we define the operator

$$N(t,\varepsilon) = t^3 N_{11} + t^2\varepsilon N_{12} + t\varepsilon^2 N_{21} + \varepsilon^3 N_{22}, \tag{1.13}$$

where

$$N_{11} = (X_1 Z)^* RP + (RP)^* X_1 Z, \tag{1.14}$$

$$\begin{aligned} N_{12} = (X_1\widetilde{Z})^* RP + (RP)^* X_1\widetilde{Z} + (X_1 Z)^* X_0 \widetilde{Z} + (X_0\widetilde{Z})^* X_1 Z \\ + (Y_2 Z)^* Y_0 Z + (Y_0 Z)^* Y_2 Z + (Y_2 Z)^* Y_1 P + (Y_1 P)^* Y_2 Z \\ + (Y_2 P)^* Y_1 Z + (Y_1 Z)^* Y_2 P, \end{aligned} \tag{1.15}$$

$$\begin{aligned} N_{21} = (X_0\widetilde{Z})^* X_1\widetilde{Z} + (X_1\widetilde{Z})^* X_0\widetilde{Z} + (Y_2 Z)^* Y_0\widetilde{Z} + (Y_0\widetilde{Z})^* Y_2 Z + (Y_2\widetilde{Z})^* Y_0 Z \\ + (Y_0 Z)^* Y_2\widetilde{Z} + (Y_2\widetilde{Z})^* Y_1 P + (Y_1 P)^* Y_2\widetilde{Z} + (Y_1\widetilde{Z})^* Y_2 P \\ + (Y_2 P)^* Y_1\widetilde{Z} + Z^* Q P + PQZ + \lambda(Z^* Q_0 P + PQ_0 Z), \end{aligned} \tag{1.16}$$

$$N_{22} = (Y_0\widetilde{Z})^* Y_2\widetilde{Z} + (Y_2\widetilde{Z})^* Y_0\widetilde{Z} + \widetilde{Z}^* QP + PQ\widetilde{Z} + \lambda(\widetilde{Z}^* Q_0 P + PQ_0\widetilde{Z}). \tag{1.17}$$

Note that in (1.16) the formal expression $Z^*QP + PQZ$ means the bounded selfadjoint operator in $\mathfrak{H}$ generated by the form $\mathfrak{q}[Pu, Zu] + \mathfrak{q}[Zu, Pu]$, $u \in \mathfrak{H}$. Similarly, in (1.17), the expression $\widetilde{Z}^*QP + PQ\widetilde{Z}$ is understood as the bounded selfadjoint operator in $\mathfrak{H}$ corresponding to the form $\mathfrak{q}[Pu, \widetilde{Z}u] + \mathfrak{q}[\widetilde{Z}u, Pu]$, $u \in \mathfrak{H}$. In [Su3], the following estimate for the operator (1.13) was proved:

$$\|N(t,\varepsilon)\|_{\mathfrak{H}\to\mathfrak{H}} \le \widehat{\mathcal{C}}(t^2 + \varepsilon^2)^{3/2}, \quad t \in \mathbb{R}, \quad 0 \le \varepsilon \le 1. \tag{1.18}$$

We assume that

$$A(t) \ge c_* t^2 I, \quad c_* > 0, \quad |t| \le \tau_0, \tag{1.19}$$

for some $c_* > 0$. Then, taking (1.8) into account, we have

$$B_\lambda(t,\varepsilon) \geq \check{c}_*(t^2+\varepsilon^2)I, \quad t^2+\varepsilon^2 \leq \tau_0^2, \quad \check{c}_* = \frac{1}{2}\min\{\kappa c_*, 2\beta\}, \tag{1.20}$$

where β is as in (1.9). As was checked in [Su3, §4], we have

$$L(t,\varepsilon) \geq \check{c}_*(t^2+\varepsilon^2)I_{\mathfrak{N}}, \quad t \in \mathbb{R}, \quad 0 \leq \varepsilon \leq 1. \tag{1.21}$$

Then the operator $L(t,\varepsilon)$ is invertible whenever $t^2+\varepsilon^2 \neq 0$.

Consider the operator

$$\begin{aligned} \mathcal{K}(t,\varepsilon) := (tZ+\varepsilon\widetilde{Z})L(t,\varepsilon)^{-1}P + L(t,\varepsilon)^{-1}P(tZ^*+\varepsilon\widetilde{Z}^*) \\ - L(t,\varepsilon)^{-1}N(t,\varepsilon)L(t,\varepsilon)^{-1}P, \ t \in \mathbb{R}, \ 0 < \varepsilon \leq 1, \end{aligned} \tag{1.22}$$

which is called the *corrector* for $B_\lambda(t,\varepsilon)^{-1}$.

The following result was obtained in [Su3, Theorem 4.3].

Theorem 1.5. *Suppose that the assumptions of Subsections* 1.1–1.4 *are satisfied. Let* $B_\lambda(t,\varepsilon)$ *be the operator defined in Subsection* 1.4. *Suppose that condition* (1.19) *is satisfied. Then for* $t^2+\varepsilon^2 \leq \tau_0^2$ *and* $0 < \varepsilon \leq 1$ *we have*

$$B_\lambda(t,\varepsilon)^{-1} = L(t,\varepsilon)^{-1}P + \mathcal{K}(t,\varepsilon) + J(t,\varepsilon).$$

Here P *is the orthogonal projection onto the subspace* $\mathfrak{N}$, *and* $L(t,\varepsilon)$ *is the operator defined by* (1.12). *The corrector* $\mathcal{K}(t,\varepsilon)$ *is defined by* (1.22), *where* $N(t,\varepsilon)$ *is defined as in* (1.13)–(1.17), *and* Z, $\widetilde{Z}$, R, *and* S *are the operators defined in Subsection* 1.5. *The reminder term* $J(t,\varepsilon)$ *satisfies*

$$\|J(t,\varepsilon)\|_{\mathfrak{H}\to\mathfrak{H}} \leq \mathcal{C}, \quad t^2+\varepsilon^2 \leq \tau_0^2, \quad 0 < \varepsilon \leq 1, \tag{1.23}$$

where τ_0 *is subject to* (1.11). *The corrector satisfies the estimate*

$$\|\mathcal{K}(t,\varepsilon)\|_{\mathfrak{H}\to\mathfrak{H}} \leq \mathcal{C}_0(t^2+\varepsilon^2)^{-1/2}, \quad t \in \mathbb{R}, \ 0 < \varepsilon \leq 1. \tag{1.24}$$

Remark 1.6. Certain (cumbersome) explicit expressions for the constants $\mathcal{C}$, $\widehat{\mathcal{C}}$, and $\mathcal{C}_0$ in estimates (1.23), (1.18), and (1.24) can be given; they were found in [Su3]. The character of dependence of these constants on the initial problem data is important for us. The constant $\mathcal{C}$ is a polynomial in the variables δ, δ^{-1}, $(\check{c}_*)^{-1}$, $\kappa^{-1/2}$, $\|X_1\|$, $\|Y_1\|$, c_1, $C(1)$, c_2, c_3, $|\lambda|\|Q_0\|$, and τ_0. The constant $\check{c}_*$ is defined by (1.20). The constant $\widehat{\mathcal{C}}$ is a polynomial of the variables $\delta^{-1/2}$, $\kappa^{1/2}$, $\|X_1\|$, $\|Y_1\|$, c_1, $C(1)^{1/2}$, $c_2^{1/2}$, $c_3^{1/2}$, and $|\lambda|\|Q_0\|$; the constant $\mathcal{C}_0$ depends polynomially on the same variables and on $(\check{c}_*)^{-1}$. The coefficients of the polynomials mentioned above are some positive absolute constants.

Chapter 1
Periodic differential operators in $L_2(\mathbb{R}^d;\mathbb{C}^n)$. Approximation of the generalized resolvent

§2. Preliminaries

2.1. Lattices in $\mathbb{R}^d$. Let Γ be a lattice in $\mathbb{R}^d$ generated by a basis $\mathbf{a}_1,\dots,\mathbf{a}_d$, i.e., $\Gamma = \{\mathbf{a} \in \mathbb{R}^d \,:\, \mathbf{a} = \sum_{j=1}^d n_j\mathbf{a}_j, \quad n_j \in \mathbb{Z}\}$. By Ω we denote the elementary cell of the lattice Γ: $\Omega = \{\mathbf{x} \in \mathbb{R}^d\colon\ \mathbf{x} = \sum_{j=1}^d \xi_j\mathbf{a}_j,\ 0 \leq \xi_j < 1\}$. The basis $\mathbf{b}_1,\dots,\mathbf{b}_d$ dual to $\mathbf{a}_1,\dots,\mathbf{a}_d$ is defined by the relations $\langle\mathbf{b}_l,\mathbf{a}_j\rangle = 2\pi\delta_{lj}$. This basis generates a *lattice* $\widetilde{\Gamma}$ *dual to the lattice* Γ: $\widetilde{\Gamma} = \{\mathbf{b} \in \mathbb{R}^d\colon\ \mathbf{b} = \sum_{j=1}^d m_j\mathbf{b}_j,\ m_j \in \mathbb{Z}\}$. By $\widetilde{\Omega}$ we denote the central *Brillouin zone* of the lattice $\widetilde{\Gamma}$:

$$\widetilde{\Omega} = \big\{\mathbf{k} \in \mathbb{R}^d \,:\, |\mathbf{k}| < |\mathbf{k}-\mathbf{b}|,\ 0 \neq \mathbf{b} \in \widetilde{\Gamma}\big\}. \tag{2.1}$$

Denote $|\Omega| = \operatorname{meas}\Omega$, $|\widetilde{\Omega}| = \operatorname{meas}\widetilde{\Omega}$. *Let* r_0 *denote the radius of the ball inscribed in* $\operatorname{clos}\widetilde{\Omega}$. We have $2r_0 = \min|\mathbf{b}|$, $0 \neq \mathbf{b} \in \widetilde{\Gamma}$.

2.2. The Fourier series. The following Fourier series expansion for a function $\mathbf{u}(\mathbf{x})$, $\mathbf{x} \in \Omega$,

$$\mathbf{u}(\mathbf{x}) = |\Omega|^{-1/2} \sum_{\mathbf{b}\in\widetilde{\Gamma}} \hat{\mathbf{u}}_{\mathbf{b}} \exp(i\langle \mathbf{b}, \mathbf{x}\rangle), \tag{2.2}$$

is associated with the lattice Γ. The corresponding discrete Fourier transformation $\mathbf{u} \mapsto \{\hat{\mathbf{u}}_{\mathbf{b}}\}$ is a unitary mapping of $L_2(\Omega;\mathbb{C}^n)$ onto $l_2(\widetilde{\Gamma};\mathbb{C}^n)$:

$$\|\mathbf{u}\|^2_{L_2(\Omega)} = \sum_{\mathbf{b}\in\widetilde{\Gamma}} |\hat{\mathbf{u}}_{\mathbf{b}}|^2. \tag{2.3}$$

By $\widetilde{W}^s_p(\Omega;\mathbb{C}^n)$ *we denote the subspace of functions in* $W^s_p(\Omega;\mathbb{C}^n)$ *whose* Γ*-periodic extension to* $\mathbb{R}^d$ *belongs to* $W^s_{p,\mathrm{loc}}(\mathbb{R}^d;\mathbb{C}^n)$. If $p=2$, we use the notation $\widetilde{H}^s(\Omega;\mathbb{C}^n)$.

We have

$$\int_\Omega |(\mathbf{D}+\mathbf{k})\mathbf{u}|^2\,d\mathbf{x} = \sum_{\mathbf{b}\in\widetilde{\Gamma}} |\mathbf{b}+\mathbf{k}|^2 |\hat{\mathbf{u}}_{\mathbf{b}}|^2, \quad \mathbf{u} \in \widetilde{H}^1(\Omega;\mathbb{C}^n), \quad \mathbf{k} \in \mathbb{R}^d, \tag{2.4}$$

and the convergence of the series on the right-hand side is equivalent to $\mathbf{u} \in \widetilde{H}^1(\Omega;\mathbb{C}^n)$. From (2.1), (2.3), and (2.4) it follows that

$$\int_\Omega |(\mathbf{D}+\mathbf{k})\mathbf{u}|^2\,d\mathbf{x} \geq \sum_{\mathbf{b}\in\widetilde{\Gamma}} |\mathbf{k}|^2 |\hat{\mathbf{u}}_{\mathbf{b}}|^2 = |\mathbf{k}|^2 \int_\Omega |\mathbf{u}|^2\,d\mathbf{x}, \quad \mathbf{u} \in \widetilde{H}^1(\Omega;\mathbb{C}^n), \quad \mathbf{k} \in \widetilde{\Omega}. \tag{2.5}$$

2.3. Initially, **the Gelfand transformation** $\mathcal{U}$ is defined on the functions $\mathbf{v} \in \mathcal{S}(\mathbb{R}^d;\mathbb{C}^n)$ in the Schwartz class by the formula

$$\tilde{\mathbf{v}}(\mathbf{k},\mathbf{x}) = (\mathcal{U}\mathbf{v})(\mathbf{k},\mathbf{x}) = |\widetilde{\Omega}|^{-1/2} \sum_{\mathbf{a}\in\Gamma} \exp(-i\langle \mathbf{k}, \mathbf{x}+\mathbf{a}\rangle)\mathbf{v}(\mathbf{x}+\mathbf{a}), \;\; \mathbf{x} \in \Omega, \;\; \mathbf{k} \in \widetilde{\Omega}.$$

Next, $\mathcal{U}$ extends by continuity up to a unitary mapping

$$\mathcal{U} \,:\, L_2(\mathbb{R}^d;\mathbb{C}^n) \to \int_{\widetilde{\Omega}} \oplus L_2(\Omega;\mathbb{C}^n)\,d\mathbf{k} =: \mathcal{H}. \tag{2.6}$$

The relation $\mathbf{v} \in H^1(\mathbb{R}^d;\mathbb{C}^n)$ is equivalent to the fact that $\tilde{\mathbf{v}}(\mathbf{k},\cdot) \in \widetilde{H}^1(\Omega;\mathbb{C}^n)$ for almost every $\mathbf{k} \in \widetilde{\Omega}$, and

$$\int_{\widetilde{\Omega}} \int_\Omega \left(|(\mathbf{D}+\mathbf{k})\tilde{\mathbf{v}}(\mathbf{k},\mathbf{x})|^2 + |\tilde{\mathbf{v}}(\mathbf{k},\mathbf{x})|^2 \right) d\mathbf{x}\,d\mathbf{k} < \infty.$$

Under the transformation $\mathcal{U}$, the operator in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ of multiplication by a bounded periodic function turns into multiplication by the same function on the fibers of the direct integral $\mathcal{H}$ occurring in (2.6). The action of the first order differential operator $b(\mathbf{D})$ on $\mathbf{v} \in H^1(\mathbb{R}^d;\mathbb{C}^n)$ turns into that of the operator $b(\mathbf{D}+\mathbf{k})$ on $\tilde{\mathbf{v}}(\mathbf{k},\cdot) \in \widetilde{H}^1(\Omega;\mathbb{C}^n)$.

2.4. Notation. The following notation is systematically used in what follows. Let $\varphi(\mathbf{x})$ be a Γ-periodic matrix-valued function such that $\varphi \in L_{1,\mathrm{loc}}(\mathbb{R}^d)$. Denote

$$\overline{\varphi} = |\Omega|^{-1} \int_\Omega \varphi(\mathbf{x})\,d\mathbf{x}. \tag{2.7}$$

If a Γ-periodic quadratic nondegenerate matrix-valued function $\varphi(\mathbf{x})$ is such that $\varphi^{-1} \in L_{1,\mathrm{loc}}(\mathbb{R}^d)$, we put

$$\underline{\varphi} = \left(|\Omega|^{-1} \int_\Omega \varphi(\mathbf{x})^{-1}\, d\mathbf{x} \right)^{-1}. \tag{2.8}$$

§3. Periodic differential operators. Direct integral expansion

3.1. Factorized second order operators $\mathcal{A}$. Let $b(\mathbf{D})\colon L_2(\mathbb{R}^d;\mathbb{C}^n) \to L_2(\mathbb{R}^d;\mathbb{C}^m)$ be a homogeneous first order DO with constant coefficients. *We assume that $m \geq n$.* The operator $b(\mathbf{D})$ can be written as $b(\mathbf{D}) = \sum_{l=1}^d b_l D_l$, where the b_l are constant $(m\times n)$-matrices. The *symbol* of the operator $b(\mathbf{D})$ is the $(m\times n)$-matrix $b(\boldsymbol{\xi}) = \sum_{l=1}^d b_l \xi_l$, $\boldsymbol{\xi} \in \mathbb{R}^d$. Suppose that

$$\operatorname{rank} b(\boldsymbol{\xi}) = n, \quad 0 \neq \boldsymbol{\xi} \in \mathbb{R}^d. \tag{3.1}$$

Relation (3.1) is equivalent to the inequalities

$$\alpha_0 \mathbf{1}_n \leq b(\boldsymbol{\theta})^* b(\boldsymbol{\theta}) \leq \alpha_1 \mathbf{1}_n, \quad |\boldsymbol{\theta}| = 1, \quad 0 < \alpha_0 \leq \alpha_1 < \infty, \tag{3.2}$$

with some positive constants α_0, α_1. Suppose that $h(\mathbf{x})$ is a Γ-periodic $(m \times m)$-matrix-valued function such that

$$h, h^{-1} \in L_\infty(\mathbb{R}^d). \tag{3.3}$$

We consider the DO

$$\mathcal{X} := hb(\mathbf{D})\colon L_2(\mathbb{R}^d;\mathbb{C}^n) \to L_2(\mathbb{R}^d;\mathbb{C}^m), \quad \operatorname{Dom} \mathcal{X} = H^1(\mathbb{R}^d;\mathbb{C}^n). \tag{3.4}$$

The operator (3.4) is closed. The selfadjoint operator $\mathcal{A} := \mathcal{X}^*\mathcal{X}$ in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ is generated by the closed quadratic form

$$\mathfrak{a}[\mathbf{u},\mathbf{u}] := \|\mathcal{X}\mathbf{u}\|^2_{L_2(\mathbb{R}^d;\mathbb{C}^m)}, \quad \mathbf{u} \in H^1(\mathbb{R}^d;\mathbb{C}^n). \tag{3.5}$$

Formally, $\mathcal{A}$ is given by

$$\mathcal{A} = b(\mathbf{D})^* g(\mathbf{x}) b(\mathbf{D}), \quad g(\mathbf{x}) := h(\mathbf{x})^* h(\mathbf{x}).$$

By using the Fourier transformation and (3.2), (3.3), it is easy to check that

$$\alpha_0 \|g^{-1}\|^{-1}_{L_\infty} \int_{\mathbb{R}^d} |\mathbf{D}\mathbf{u}|^2\, d\mathbf{x} \leq \mathfrak{a}[\mathbf{u},\mathbf{u}] \leq \alpha_1 \|g\|_{L_\infty} \int_{\mathbb{R}^d} |\mathbf{D}\mathbf{u}|^2\, d\mathbf{x}, \ \ \mathbf{u} \in H^1(\mathbb{R}^d;\mathbb{C}^n). \tag{3.6}$$

3.2. The operators $\mathcal{Y}$ and $\mathcal{Y}_2$. Consider the closed operator

$$\mathcal{Y}\colon L_2(\mathbb{R}^d;\mathbb{C}^n) \to L_2(\mathbb{R}^d;\mathbb{C}^{dn}),$$

defined by

$$\mathcal{Y}\mathbf{u} = \mathbf{D}\mathbf{u} = \operatorname{col}\{D_1\mathbf{u}, \dots, D_d\mathbf{u}\}, \quad \operatorname{Dom} \mathcal{Y} = H^1(\mathbb{R}^d;\mathbb{C}^n). \tag{3.7}$$

The lower estimate (3.6) means that

$$\|\mathcal{Y}\mathbf{u}\|_{L_2(\mathbb{R}^d)} \leq c_1 \|\mathcal{X}\mathbf{u}\|_{L_2(\mathbb{R}^d)}, \quad \mathbf{u} \in H^1(\mathbb{R}^d;\mathbb{C}^n), \tag{3.8}$$

$$c_1 = \alpha_0^{-1/2} \|g^{-1}\|^{1/2}_{L_\infty}. \tag{3.9}$$

Next, assume that $a_j(\mathbf{x})$, $j = 1, \dots, d$, are Γ-periodic $(n \times n)$-matrix-valued functions such that

$$a_j \in L_\rho(\Omega), \ \ \rho = 2 \text{ for } d = 1, \ \ \rho > d \text{ for } d \geq 2; \ \ j = 1, \dots, d. \tag{3.10}$$

Consider the operator $\mathcal{Y}_2\colon L_2(\mathbb{R}^d;\mathbb{C}^n) \to L_2(\mathbb{R}^d;\mathbb{C}^{dn})$ acting as multiplication by the $(dn \times n)$-matrix-valued function with the blocks $a_j(\mathbf{x})^*$, $j = 1, \dots, d$, i.e.,

$$\mathcal{Y}_2\mathbf{u} = \operatorname{col}\{a_1(\mathbf{x})^*\mathbf{u}, \dots, a_d(\mathbf{x})^*\mathbf{u}\}, \quad \operatorname{Dom} \mathcal{Y}_2 = H^1(\mathbb{R}^d;\mathbb{C}^n).$$

As was shown in [Su2, Subsection 5.2] (with the help of embedding theorems), for any $\nu > 0$ there exists a constant $C(\nu) > 0$ such that

$$\|\mathcal{Y}_2\mathbf{u}\|^2_{L_2(\mathbb{R}^d)} \le \nu\|\mathcal{X}\mathbf{u}\|^2_{L_2(\mathbb{R}^d)} + C(\nu)\|\mathbf{u}\|^2_{L_2(\mathbb{R}^d)}, \quad \mathbf{u} \in H^1(\mathbb{R}^d;\mathbb{C}^n). \tag{3.11}$$

Remark 3.1. For a fixed ν, the constant $C(\nu)$ in (3.11) depends only on $\|a_j\|_{L_\rho(\Omega)}$, $j = 1,\dots,d$, $\|g^{-1}\|_{L_\infty}$, α_0, d, ρ, and the parameters of the lattice Γ.

3.3. The form $q[\mathbf{u},\mathbf{u}]$. Suppose that $d\mu(\mathbf{x}) = \{d\mu_{jl}(\mathbf{x})\}$, $j,l = 1,\dots,n$, is a Γ-periodic σ-finite Borel measure in $\mathbb{R}^d$ with values in the class of Hermitian $(n\times n)$-matrices. In other words, $d\mu_{jl}(\mathbf{x})$ is a complex Γ-periodic measure in $\mathbb{R}^d$, and $d\mu_{jl} = d\mu^*_{lj}$. Suppose that the measure $d\mu$ is such that for any $u \in H^1(\mathbb{R}^d)$ the function $|u(\mathbf{x})|^2$ is integrable with respect to each measure $d\mu_{jl}$.

In $L_2(\mathbb{R}^d;\mathbb{C}^n)$, we consider the sesquilinear form

$$q[\mathbf{u},\mathbf{v}] = \int_{\mathbb{R}^d} \langle d\mu(\mathbf{x})\mathbf{u},\mathbf{v}\rangle = \sum_{j,l=1}^{n}\int_{\mathbb{R}^d} u_l v_j^*\, d\mu_{jl}(\mathbf{x}), \quad \mathbf{u},\mathbf{v} \in H^1(\mathbb{R}^d;\mathbb{C}^n).$$

We impose the following condition on the measure $d\mu$.

Condition 3.2. 1°. *There exist constants $\widetilde{c}_2 \ge 0$ and $c_3 \ge 0$ such that for any $\mathbf{u},\mathbf{v} \in H^1(\Omega;\mathbb{C}^n)$ we have*

$$\Bigl|\int_\Omega \langle d\mu(\mathbf{x})\mathbf{u},\mathbf{v}\rangle\Bigr| \le \bigl(\widetilde{c}_2\|\mathbf{D}\mathbf{u}\|^2_{L_2(\Omega)} + c_3\|\mathbf{u}\|^2_{L_2(\Omega)}\bigr)^{1/2}\bigl(\widetilde{c}_2\|\mathbf{D}\mathbf{v}\|^2_{L_2(\Omega)} + c_3\|\mathbf{v}\|^2_{L_2(\Omega)}\bigr)^{1/2}. \tag{3.12}$$

2°. *We have*

$$\int_\Omega \langle d\mu(\mathbf{x})\mathbf{u},\mathbf{u}\rangle \ge -\widetilde{c}\|\mathbf{D}\mathbf{u}\|^2_{L_2(\Omega)} - c_0\|\mathbf{u}\|^2_{L_2(\Omega)}, \quad \mathbf{u} \in H^1(\Omega;\mathbb{C}^n), \tag{3.13}$$

with some constants $c_0 \in \mathbb{R}$ and $\widetilde{c}$ such that $0 \le \widetilde{c} < \alpha_0\|g^{-1}\|^{-1}_{L_\infty}$.

For functions in $H^1(\mathbb{R}^d;\mathbb{C}^n)$, we write inequalities (3.12) and (3.13) over the shifted cells $\Omega + \mathbf{a}$, $\mathbf{a} \in \Gamma$, and sum up, obtaining similar inequalities with integration over $\mathbb{R}^d$. Together with (3.6), this yields

$$\begin{aligned} |q[\mathbf{u},\mathbf{v}]| \le \bigl(c_2\|\mathcal{X}\mathbf{u}\|^2_{L_2(\mathbb{R}^d)} + c_3\|\mathbf{u}\|^2_{L_2(\mathbb{R}^d)}\bigr)^{1/2}\bigl(c_2\|\mathcal{X}\mathbf{v}\|^2_{L_2(\mathbb{R}^d)} + c_3\|\mathbf{v}\|^2_{L_2(\mathbb{R}^d)}\bigr)^{1/2},& \\ \mathbf{u},\mathbf{v} \in H^1(\mathbb{R}^d;\mathbb{C}^n),& \end{aligned} \tag{3.14}$$

$$q[\mathbf{u},\mathbf{u}] \ge -(1-\kappa)\|\mathcal{X}\mathbf{u}\|^2_{L_2(\mathbb{R}^d)} - c_0\|\mathbf{u}\|^2_{L_2(\mathbb{R}^d)}, \quad \mathbf{u} \in H^1(\mathbb{R}^d;\mathbb{C}^n). \tag{3.15}$$

Here

$$c_2 = \widetilde{c}_2\alpha_0^{-1}\|g^{-1}\|_{L_\infty}, \quad \kappa = 1 - \widetilde{c}\alpha_0^{-1}\|g^{-1}\|_{L_\infty}, \quad 0 < \kappa \le 1. \tag{3.16}$$

Examples of the forms satisfying Condition 3.2 were given in [Su2, Subsection 5.5]. Here we discuss only one example.

Example 3.3. Suppose that the measure $d\mu$ is absolutely continuous with respect to Lebesgue measure, i.e., $d\mu(\mathbf{x}) = Q(\mathbf{x})d\mathbf{x}$, where $Q(\mathbf{x})$ is a Γ-periodic Hermitian $(n\times n)$-matrix-valued function in $\mathbb{R}^d$ such that

$$Q \in L_s(\Omega), \ \ s = 1 \text{ for } d = 1, \ \ s > \frac{d}{2} \text{ for } d \ge 2. \tag{3.17}$$

Then $q[\mathbf{u},\mathbf{v}] = \int_{\mathbb{R}^d}\langle Q(\mathbf{x})\mathbf{u},\mathbf{v}\rangle\, d\mathbf{x}$, $\mathbf{u},\mathbf{v} \in H^1(\mathbb{R}^d;\mathbb{C}^n)$. The embedding theorems show that, under condition (3.17), for any $\nu > 0$ there exists a positive constant $C_Q(\nu)$ such that

$$\int_\Omega |Q(\mathbf{x})||\mathbf{u}|^2\, d\mathbf{x} \le \nu\int_\Omega |\mathbf{D}\mathbf{u}|^2\, d\mathbf{x} + C_Q(\nu)\int_\Omega |\mathbf{u}|^2\, d\mathbf{x}, \quad \mathbf{u} \in H^1(\Omega;\mathbb{C}^n).$$

Then Condition 3.2 is satisfied with the constants $\widetilde{c}_2 = 1$, $c_3 = C_Q(1)$, $\widetilde{c} = \nu$, and $c_0 = C_Q(\nu)$, where $2\nu = \alpha_0 \|g^{-1}\|^{-1}_{L_\infty}$. It follows that inequalities (3.14) and (3.15) are fulfilled with $\kappa = 1/2$ and $c_2 = \alpha_0^{-1}\|g^{-1}\|_{L_\infty}$. The constant c_3 is controlled in terms of d, s, the norm $\|Q\|_{L_s(\Omega)}$, and the parameters of the lattice. The constant c_0 is controlled in terms of d, s, α_0, $\|g^{-1}\|_{L_\infty}$, $\|Q\|_{L_s(\Omega)}$, and the parameters of the lattice.

3.4. The operator pencil $\mathcal{B}(\varepsilon)$. In $L_2(\mathbb{R}^d;\mathbb{C}^n)$, we consider the quadratic form

$$\mathfrak{b}(\varepsilon)[\mathbf{u},\mathbf{u}] = \mathfrak{a}[\mathbf{u},\mathbf{u}] + 2\varepsilon\,\mathrm{Re}(\mathcal{Y}\mathbf{u},\mathcal{Y}_2\mathbf{u})_{L_2(\mathbb{R}^d)} + \varepsilon^2 q[\mathbf{u},\mathbf{u}], \quad \mathbf{u}\in H^1(\mathbb{R}^d;\mathbb{C}^n), \tag{3.18}$$

where $0 < \varepsilon \le 1$. By using (3.8), (3.11), (3.14), and (3.15), it is easy to check the following estimates (see [Su2, Subsection 5.4] for the details):

$$\mathfrak{b}(\varepsilon)[\mathbf{u},\mathbf{u}] \le (2 + c_1^2 + c_2)\mathfrak{a}[\mathbf{u},\mathbf{u}] + (C(1) + c_3)\varepsilon^2\|\mathbf{u}\|^2_{L_2(\mathbb{R}^d)}, \quad \mathbf{u}\in H^1(\mathbb{R}^d;\mathbb{C}^n), \tag{3.19}$$

$$\mathfrak{b}(\varepsilon)[\mathbf{u},\mathbf{u}] \ge \frac{\kappa}{2}\mathfrak{a}[\mathbf{u},\mathbf{u}] - (c_0 + c_4)\varepsilon^2\|\mathbf{u}\|^2_{L_2(\mathbb{R}^d)}, \quad \mathbf{u}\in H^1(\mathbb{R}^d;\mathbb{C}^n), \tag{3.20}$$

where

$$c_4 = 4\kappa^{-1}c_1^2 C(\nu) \text{ for } \nu = \kappa^2(16c_1^2)^{-1}. \tag{3.21}$$

Estimates (3.19) and (3.20) together with (3.6) show that the form (3.18) is closed and lower semibounded. *Let $\mathcal{B}(\varepsilon)$ denote the selfadjoint operator in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ generated by this form.* Formally, we can write

$$\mathcal{B}(\varepsilon) = b(\mathbf{D})^* g(\mathbf{x}) b(\mathbf{D}) + \varepsilon\sum_{j=1}^d (a_j(\mathbf{x})D_j + D_j a_j(\mathbf{x})^*) + \varepsilon^2 Q(\mathbf{x}),$$

where $Q(\mathbf{x})$ can be interpreted as a generalized matrix-valued potential generated by the measure $d\mu$ (in the context of Example 3.3, this is the matrix-valued function $Q(\mathbf{x})$).

In Chapter 1 *we study the generalized resolvent* $(\mathcal{B}(\varepsilon) + \lambda\varepsilon^2 Q_0)^{-1}$. Here Q_0 is the operator in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ acting as multiplication by a Γ-periodic, uniformly positive definite, and bounded $(n\times n)$-matrix-valued function $Q_0(\mathbf{x})$, and λ is a real-valued parameter. We put

$$\mathfrak{b}_\lambda(\varepsilon)[\mathbf{u},\mathbf{u}] = \mathfrak{b}(\varepsilon)[\mathbf{u},\mathbf{u}] + \lambda\varepsilon^2(Q_0\mathbf{u},\mathbf{u})_{L_2(\mathbb{R}^d)}, \quad \mathbf{u}\in H^1(\mathbb{R}^d;\mathbb{C}^n). \tag{3.22}$$

The operator corresponding to the form (3.22) is denoted by $\mathcal{B}_\lambda(\varepsilon)$. Then

$$\mathcal{B}_\lambda(\varepsilon) = \mathcal{B}(\varepsilon) + \lambda\varepsilon^2 Q_0. \tag{3.23}$$

The parameter λ is subject to the following restriction:

$$\begin{aligned} &\lambda > \|Q_0^{-1}\|_{L_\infty}(c_0 + c_4) \quad \text{if } \lambda \ge 0,\\ &\lambda > \|Q_0\|^{-1}_{L_\infty}(c_0 + c_4) \quad \text{if } \lambda < 0 \ \text{(and } \ c_0 + c_4 < 0). \end{aligned} \tag{3.24}$$

Condition (3.24) ensures that

$$\lambda(Q_0\mathbf{u},\mathbf{u})_{L_2(\mathbb{R}^d)} \ge (c_0 + c_4 + \beta)\|\mathbf{u}\|^2_{L_2(\mathbb{R}^d)}, \quad \mathbf{u}\in L_2(\mathbb{R}^d;\mathbb{C}^n), \tag{3.25}$$

where $\beta > 0$ is defined in terms of λ as follows:

$$\begin{aligned} &\beta = \lambda\|Q_0^{-1}\|^{-1}_{L_\infty} - c_0 - c_4 \quad \text{if } \lambda \ge 0,\\ &\beta = \lambda\|Q_0\|_{L_\infty} - c_0 - c_4 \quad \text{if } \lambda < 0 \ \text{(and } \ c_0 + c_4 < 0). \end{aligned} \tag{3.26}$$

From (3.20) and (3.25) it follows that

$$\mathfrak{b}_\lambda(\varepsilon)[\mathbf{u},\mathbf{u}] \ge \frac{\kappa}{2}\mathfrak{a}[\mathbf{u},\mathbf{u}] + \beta\varepsilon^2\|\mathbf{u}\|^2_{L_2(\mathbb{R}^d)}, \quad \mathbf{u}\in H^1(\mathbb{R}^d;\mathbb{C}^n), \ 0<\varepsilon\le 1.$$

Thus, under the above assumptions the operator (3.23) is positive definite.

For convenience of further references, by the "initial data" we shall mean the following set of parameters:

$$\begin{aligned} &d,\ m,\ n,\ \rho;\ \alpha_0,\ \alpha_1,\ \|g\|_{L_\infty},\ \|g^{-1}\|_{L_\infty},\ \|a_j\|_{L_\rho(\Omega)},\ j=1,\dots,d;\\ &\widetilde{c},\ c_0,\ \widetilde{c}_2,\ c_3 \text{ from Condition 3.2};\\ &\lambda,\ \|Q_0\|_{L_\infty},\ \|Q_0^{-1}\|_{L_\infty};\ \ \text{the parameters of the lattice } \Gamma. \end{aligned} \tag{3.27}$$

3.5. The operators $\mathcal{A}(\mathbf{k})$. We put

$$\mathfrak{H} = L_2(\Omega;\mathbb{C}^n), \quad \mathfrak{H}_* = L_2(\Omega;\mathbb{C}^m), \quad \widetilde{\mathfrak{H}} = L_2(\Omega;\mathbb{C}^{dn}) \tag{3.28}$$

and consider the closed operator $\mathcal{X}(\mathbf{k})\colon \mathfrak{H}\to\mathfrak{H}_*$, $\mathbf{k}\in\mathbb{R}^d$, defined by the relation

$$\mathcal{X}(\mathbf{k}) = hb(\mathbf{D}+\mathbf{k}), \quad \operatorname{Dom}\mathcal{X}(\mathbf{k}) = \widetilde{H}^1(\Omega;\mathbb{C}^n). \tag{3.29}$$

The selfadjoint operator $\mathcal{A}(\mathbf{k}) := \mathcal{X}(\mathbf{k})^*\mathcal{X}(\mathbf{k})$ in $\mathfrak{H}$ is generated by the closed quadratic form $\mathfrak{a}(\mathbf{k})[\mathbf{u},\mathbf{u}] := \|\mathcal{X}(\mathbf{k})\mathbf{u}\|^2_{\mathfrak{H}_*}$, $\mathbf{u}\in\widetilde{H}^1(\Omega;\mathbb{C}^n)$. By using the Fourier series expansion (2.2) for a function $\mathbf{u}$ and conditions (3.2), (3.3), it is easy to check that

$$\begin{gathered} \alpha_0\|g^{-1}\|^{-1}_{L_\infty}\|(\mathbf{D}+\mathbf{k})\mathbf{u}\|^2_{L_2(\Omega)} \le \mathfrak{a}(\mathbf{k})[\mathbf{u},\mathbf{u}] \le \alpha_1\|g\|_{L_\infty}\|(\mathbf{D}+\mathbf{k})\mathbf{u}\|^2_{L_2(\Omega)},\\ \mathbf{u}\in\widetilde{H}^1(\Omega;\mathbb{C}^n). \end{gathered} \tag{3.30}$$

From (2.5) and the lower estimate (3.30) it follows that

$$\mathcal{A}(\mathbf{k}) \ge c_*|\mathbf{k}|^2 I, \quad \mathbf{k}\in\widetilde{\Omega}, \quad c_* = \alpha_0\|g^{-1}\|^{-1}_{L_\infty}. \tag{3.31}$$

We put

$$\mathfrak{N} := \operatorname{Ker}\mathcal{A}(0) = \operatorname{Ker}\mathcal{X}(0). \tag{3.32}$$

Relation (3.30) with $\mathbf{k}=0$ shows that the kernel (3.32) consists of constant vector-valued functions:

$$\mathfrak{N} = \{\mathbf{u}\in L_2(\Omega;\mathbb{C}^n)\ :\ \mathbf{u}=\mathbf{c}\in\mathbb{C}^n\}, \quad \dim\mathfrak{N} = n. \tag{3.33}$$

Using the Fourier series, it is easy to estimate from below the distance d^0 from the point $\lambda_0 = 0$ to the rest of the spectrum of $\mathcal{A}(0)$ (for the details, see [Su2, Subsection 5.6]):

$$d^0 \ge 4c_* r_0^2. \tag{3.34}$$

3.6. The operators $\mathcal{Y}(\mathbf{k})$ and Y_2. Consider the operator $\mathcal{Y}(\mathbf{k})\colon \mathfrak{H}\to\widetilde{\mathfrak{H}}$ defined by the relations

$$\mathcal{Y}(\mathbf{k})\mathbf{u} = (\mathbf{D}+\mathbf{k})\mathbf{u} = \operatorname{col}\{(D_1+k_1)\mathbf{u},\dots,(D_d+k_d)\mathbf{u}\}, \ \operatorname{Dom}\mathcal{Y}(\mathbf{k}) = \widetilde{H}^1(\Omega;\mathbb{C}^n). \tag{3.35}$$

Using the lower estimate (3.30), we see that

$$\|\mathcal{Y}(\mathbf{k})\mathbf{u}\|_{\widetilde{\mathfrak{H}}} \le c_1\|\mathcal{X}(\mathbf{k})\mathbf{u}\|_{\mathfrak{H}_*}, \quad \mathbf{u}\in\widetilde{H}^1(\Omega;\mathbb{C}^n), \tag{3.36}$$

where the constant c_1 is defined by (3.9).

Consider the operator $Y_2\colon \mathfrak{H}\to\widetilde{\mathfrak{H}}$ acting as multiplication by the $(dn\times n)$-matrix-valued function composed of the blocks $a_j(\mathbf{x})^*$, $j=1,\dots,d$:

$$Y_2\mathbf{u} = \operatorname{col}\{a_1(\mathbf{x})^*\mathbf{u},\dots,a_d(\mathbf{x})^*\mathbf{u}\}, \quad \operatorname{Dom}Y_2 = \widetilde{H}^1(\Omega;\mathbb{C}^n). \tag{3.37}$$

As was shown in [Su2, Subsection 5.7], for any $\nu>0$ we have

$$\|Y_2\mathbf{u}\|^2_{\widetilde{\mathfrak{H}}} \le \nu\|\mathcal{X}(\mathbf{k})\mathbf{u}\|^2_{\mathfrak{H}_*} + C(\nu)\|\mathbf{u}\|^2_{\mathfrak{H}}, \quad \mathbf{u}\in\widetilde{H}^1(\Omega;\mathbb{C}^n), \quad \mathbf{k}\in\mathbb{R}^d, \tag{3.38}$$

where the constant $C(\nu)$ is the same as in (3.11).

3.7. The form $q_\Omega[\mathbf{u},\mathbf{u}]$. In the space $\mathfrak{H}$, we consider the form

$$q_\Omega[\mathbf{u},\mathbf{v}] = \int_\Omega \langle d\mu(\mathbf{x})\mathbf{u},\mathbf{v}\rangle = \sum_{j,l=1}^n \int_\Omega u_l v_j^* \, d\mu_{jl}(\mathbf{x}), \quad \mathbf{u},\mathbf{v} \in \widetilde{H}^1(\Omega;\mathbb{C}^n). \tag{3.39}$$

From Condition 3.2 it follows that

$$\begin{aligned} |q_\Omega[\mathbf{u},\mathbf{v}]| \le \left(c_2\|\mathcal{X}(\mathbf{k})\mathbf{u}\|^2_{\mathfrak{H}_*} + c_3\|\mathbf{u}\|^2_{\mathfrak{H}}\right)^{1/2} \left(c_2\|\mathcal{X}(\mathbf{k})\mathbf{v}\|^2_{\mathfrak{H}_*} + c_3\|\mathbf{v}\|^2_{\mathfrak{H}}\right)^{1/2}, \\ \mathbf{u},\mathbf{v} \in \widetilde{H}^1(\Omega;\mathbb{C}^n), \end{aligned} \tag{3.40}$$

$$q_\Omega[\mathbf{u},\mathbf{u}] \ge -(1-\kappa)\|\mathcal{X}(\mathbf{k})\mathbf{u}\|^2_{\mathfrak{H}_*} - c_0\|\mathbf{u}\|^2_{\mathfrak{H}}, \quad \mathbf{u},\mathbf{v} \in \widetilde{H}^1(\Omega;\mathbb{C}^n), \tag{3.41}$$

where the constants c_2, c_3, κ, and c_0 are the same as in (3.14), (3.15). Cf. [Su2, Subsection 5.8].

3.8. The operator pencil $\mathcal{B}(\mathbf{k},\varepsilon)$. In the space $\mathfrak{H}$, we consider the quadratic form

$$\mathfrak{b}(\mathbf{k},\varepsilon)[\mathbf{u},\mathbf{u}] = \mathfrak{a}(\mathbf{k})[\mathbf{u},\mathbf{u}] + 2\varepsilon\,\mathrm{Re}(\mathcal{Y}(\mathbf{k})\mathbf{u}, Y_2\mathbf{u})_{\widetilde{\mathfrak{H}}} + \varepsilon^2 q_\Omega[\mathbf{u},\mathbf{u}], \quad \mathbf{u} \in \widetilde{H}^1(\Omega;\mathbb{C}^n). \tag{3.42}$$

Using (3.36), (3.38), (3.40), and (3.41), we easily check the following estimates:

$$\begin{aligned} \mathfrak{b}(\mathbf{k},\varepsilon)[\mathbf{u},\mathbf{u}] &\le (2 + c_1^2 + c_2)\mathfrak{a}(\mathbf{k})[\mathbf{u},\mathbf{u}] + (C(1) + c_3)\varepsilon^2\|\mathbf{u}\|^2_{\mathfrak{H}}, \quad \mathbf{u} \in \widetilde{H}^1(\Omega;\mathbb{C}^n), \\ \mathfrak{b}(\mathbf{k},\varepsilon)[\mathbf{u},\mathbf{u}] &\ge \frac{\kappa}{2}\mathfrak{a}(\mathbf{k})[\mathbf{u},\mathbf{u}] - (c_0 + c_4)\varepsilon^2\|\mathbf{u}\|^2_{\mathfrak{H}}, \quad \mathbf{u} \in \widetilde{H}^1(\Omega;\mathbb{C}^n). \end{aligned} \tag{3.43}$$

Together with (3.30), these estimates show that the form (3.42) is closed and lower semibounded. The selfadjoint operator in $\mathfrak{H}$ generated by this form is denoted by $\mathcal{B}(\mathbf{k},\varepsilon)$. Formally, we have

$$\mathcal{B}(\mathbf{k},\varepsilon) = b(\mathbf{D}+\mathbf{k})^* g(\mathbf{x}) b(\mathbf{D}+\mathbf{k}) + \varepsilon\sum_{j=1}^d (a_j(\mathbf{x})(D_j + k_j) + (D_j + k_j)a_j(\mathbf{x})^*) + \varepsilon^2 Q(\mathbf{x}).$$

We put

$$\mathfrak{b}_\lambda(\mathbf{k},\varepsilon)[\mathbf{u},\mathbf{u}] = \mathfrak{b}(\mathbf{k},\varepsilon)[\mathbf{u},\mathbf{u}] + \lambda\varepsilon^2(Q_0\mathbf{u},\mathbf{u})_{\mathfrak{H}}, \quad \mathbf{u} \in \widetilde{H}^1(\Omega;\mathbb{C}^n). \tag{3.44}$$

Here Q_0 is the bounded operator in $L_2(\Omega;\mathbb{C}^n)$ acting as multiplication by the matrix-valued function $Q_0(\mathbf{x})$; the number λ is subject to (3.24). The selfadjoint operator

$$\mathcal{B}_\lambda(\mathbf{k},\varepsilon) = \mathcal{B}(\mathbf{k},\varepsilon) + \lambda\varepsilon^2 Q_0 \tag{3.45}$$

corresponds to the form (3.44). Taking (3.43) into account, it is easy to check that under condition (3.24) we have

$$\mathfrak{b}_\lambda(\mathbf{k},\varepsilon)[\mathbf{u},\mathbf{u}] \ge \frac{\kappa}{2}\mathfrak{a}(\mathbf{k})[\mathbf{u},\mathbf{u}] + \beta\varepsilon^2\|\mathbf{u}\|^2_{\mathfrak{H}}, \quad \mathbf{u} \in \widetilde{H}^1(\Omega;\mathbb{C}^n), \tag{3.46}$$

where β is defined by (3.26). Thus, the operator (3.45) is positive definite.

3.9. Direct integral expansion. Under the Gelfand transformation $\mathcal{U}$ (see Subsection 2.3), the operator (3.23) acting in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ expands in the direct integral of the operators (3.45) acting in $\mathfrak{H} = L_2(\Omega;\mathbb{C}^n)$:

$$\mathcal{U}\mathcal{B}_\lambda(\varepsilon)\mathcal{U}^{-1} = \int_{\widetilde{\Omega}} \oplus \mathcal{B}_\lambda(\mathbf{k},\varepsilon)\, d\mathbf{k}. \tag{3.47}$$

In more detail, (3.47) means that for $\mathbf{v} \in H^1(\mathbb{R}^d;\mathbb{C}^n)$ we have

$$\widetilde{\mathbf{v}}(\mathbf{k},\cdot) \in \widetilde{H}^1(\Omega;\mathbb{C}^n) \text{ for a. e. } \mathbf{k} \in \widetilde{\Omega}, \tag{3.48}$$

$$\mathfrak{b}_\lambda(\varepsilon)[\mathbf{v},\mathbf{v}] = \int_{\widetilde{\Omega}} \mathfrak{b}_\lambda(\mathbf{k},\varepsilon)[\widetilde{\mathbf{v}}(\mathbf{k},\cdot), \widetilde{\mathbf{v}}(\mathbf{k},\cdot)]\, d\mathbf{k}. \tag{3.49}$$

Conversely, if $\tilde{\mathbf{v}} \in \mathcal{H}$ satisfies (3.48) and the integral in (3.49) is finite, then $\mathbf{v} \in H^1(\mathbb{R}^d;\mathbb{C}^n)$ and (3.49) is valid.

§4. Incorporation of the operators $\mathcal{B}(\mathbf{k},\varepsilon)$ in the abstract pattern

4.1. For $d>1$, the operators $\mathcal{B}(\mathbf{k},\varepsilon)$ depend on the multidimensional parameter $\mathbf{k}$. As in [BSu1, Chapter 2], we distinguish a one-dimensional parameter t by putting $\mathbf{k}=t\boldsymbol{\theta}$, $t=|\mathbf{k}|$, $\boldsymbol{\theta}\in\mathbb{S}^{d-1}$. We apply the method of §1. Now, all the objects will depend on the additional parameter $\boldsymbol{\theta}$. We must make our constructions and estimates uniform in $\boldsymbol{\theta}$.

The spaces $\mathfrak{H}$, $\mathfrak{H}_*$, and $\widetilde{\mathfrak{H}}$ are defined by (3.28). The role of $X(t)$ is played by $X(t;\boldsymbol{\theta}) = \mathcal{X}(t\boldsymbol{\theta})$. By (3.29), we have $X(t;\boldsymbol{\theta}) = X_0 + tX_1(\boldsymbol{\theta})$, where

$$X_0 = hb(\mathbf{D}), \quad \operatorname{Dom} X_0 = \widetilde{H}^1(\Omega;\mathbb{C}^n); \quad X_1(\boldsymbol{\theta}) = hb(\boldsymbol{\theta}). \tag{4.1}$$

Next, the role of $A(t)$ is played by $A(t;\boldsymbol{\theta}) = \mathcal{A}(t\boldsymbol{\theta})$. We have $A(t;\boldsymbol{\theta}) = X(t;\boldsymbol{\theta})^*X(t;\boldsymbol{\theta})$. In accordance with (3.32) and (3.33), the kernel $\mathfrak{N} = \operatorname{Ker}\mathcal{A}(0) = \operatorname{Ker} X_0$ is n-dimensional and consists of constant vector-valued functions. Condition 1.1 is satisfied. The distance d^0 satisfies (3.34). As was shown in [BSu1, Chapter 2, §3], the condition $n \le n_* = \dim\operatorname{Ker} X_0^*$ is also fulfilled. Here, either $n_* = n$ (if $m=n$) or $n_*=\infty$ (if $m>n$). The orthogonal projection P of the space $\mathfrak{H} = L_2(\Omega;\mathbb{C}^n)$ onto the subspace $\mathfrak{N}$ is the operator of averaging over the cell:

$$P\mathbf{u} = |\Omega|^{-1}\int_\Omega \mathbf{u}(\mathbf{x})\,d\mathbf{x}. \tag{4.2}$$

Next, the role of $Y(t)$ is played by the operator $Y(t;\boldsymbol{\theta}) = \mathcal{Y}(t\boldsymbol{\theta})$. By (3.35), we have $Y(t;\boldsymbol{\theta}) = Y_0 + tY_1(\boldsymbol{\theta})$, where

$$\begin{gathered} Y_0\mathbf{u} = \mathbf{D}\mathbf{u} = \operatorname{col}\{D_1\mathbf{u},\dots,D_d\mathbf{u}\}, \quad \operatorname{Dom} Y_0 = \widetilde{H}^1(\Omega;\mathbb{C}^n); \\ Y_1(\boldsymbol{\theta})\mathbf{u} = \boldsymbol{\theta}\mathbf{u} = \operatorname{col}\{\theta_1\mathbf{u},\dots,\theta_d\mathbf{u}\}. \end{gathered} \tag{4.3}$$

Condition 1.2 is satisfied due to estimate (3.36). The operator Y_2 is defined by (3.37). Relation (3.38) shows that Condition 1.3 is also fulfilled. The role of the form $\mathfrak{q}$ of Subsection 1.3 is played by the form q_Ω defined by (3.39). Condition 1.4 is ensured by (3.40) and (3.41).

Finally, the role of the operator $B(t,\varepsilon)$ (see (1.5)) is played by the operator $B(t,\varepsilon;\boldsymbol{\theta}) = \mathcal{B}(t\boldsymbol{\theta},\varepsilon)$ corresponding to the form (3.42). Formally, we have

$$B(t,\varepsilon;\boldsymbol{\theta}) = A(t;\boldsymbol{\theta}) + \varepsilon(Y_2^*Y(t;\boldsymbol{\theta}) + Y(t;\boldsymbol{\theta})^*Y_2) + \varepsilon^2 Q.$$

The role of the operator Q_0 occurring in Subsection 1.4 is played by the operator of multiplication by the matrix-valued function $Q_0(\mathbf{x})$. The restriction (1.7) on the parameter λ is satisfied by condition (3.24). The operator (1.6) is realized as $B_\lambda(t,\varepsilon;\boldsymbol{\theta}) = B(t,\varepsilon;\boldsymbol{\theta}) + \lambda\varepsilon^2 Q_0$. Estimate (1.8) corresponds to (3.46).

Thus, now all the assumptions of the abstract method described in Subsections 1.1–1.4 are ensured.

4.2. In accordance with Subsection 1.6, we should fix a positive number δ such that $\delta < \kappa d^0/13$. Taking (3.34) into account, we fix δ as follows:

$$\delta = \frac{1}{4}\kappa c_* r_0^2 = \frac{1}{4}\kappa\alpha_0\|g^{-1}\|^{-1}_{L_\infty} r_0^2. \tag{4.4}$$

By (3.2), (3.3), and (4.1), we have

$$\|X_1(\boldsymbol{\theta})\| \le \alpha_1^{1/2}\|g\|^{1/2}_{L_\infty}, \quad \boldsymbol{\theta}\in\mathbb{S}^{d-1}. \tag{4.5}$$

Obviously, (4.3) implies that

$$\|Y_1(\boldsymbol{\theta})\| = 1, \quad \boldsymbol{\theta}\in\mathbb{S}^{d-1}. \tag{4.6}$$

Now we should fix a number τ_0 satisfying (1.11) (the right-hand side of (1.11) depends on $\boldsymbol{\theta}$, because involves $\|X_1(\boldsymbol{\theta})\|$). Using (4.5), we put

$$\tau_0 = \delta^{1/2}\left((2 + c_1^2 + c_2)\alpha_1 \|g\|_{L_\infty} + C(1) + c_3 + |\lambda| \|Q_0\|_{L_\infty}\right)^{-1/2}, \tag{4.7}$$

this number is suitable for all $\boldsymbol{\theta}$. Note that (4.7) implies that $\tau_0 < r_0/2$.

Formulas (3.16) and (4.4) show that the number δ is controlled in terms of α_0, $\|g^{-1}\|_{L_\infty}$, $\widetilde{c}$, and r_0. By (3.9) and Remark 3.1, τ_0 depends on the same parameters and also on c_2, c_3, λ, $\|Q_0\|_{L_\infty}$, $\|a_j\|_{L_\rho(\Omega)}$ $(j = 1, \dots, d)$, d, ρ, and the lattice Γ. Thus, δ and τ_0 are controlled in terms of the initial data (3.27).

Condition (1.19) is satisfied due to (3.31). Together with (3.46), this implies an estimate of the form (1.20):

$$B_\lambda(t, \varepsilon; \boldsymbol{\theta}) \geq \check{c}_*(t^2 + \varepsilon^2)I, \quad \mathbf{k} = t\boldsymbol{\theta} \in \widetilde{\Omega}, \ 0 < \varepsilon \leq 1, \tag{4.8}$$

$$\check{c}_* = \min\{\kappa c_*/2, \beta\}. \tag{4.9}$$

By (3.31), c_* is defined in terms of α_0 and $\|g^{-1}\|_{L_\infty}$ only. From (3.16), (3.21), (3.26), and Remark 3.1 we see that $\check{c}_*$ is controlled in terms of the initial data (3.27).

4.3. The operators $Z(\boldsymbol{\theta})$, $\widetilde{Z}$, and $R(\boldsymbol{\theta})$. Now the operator Z defined in Subsection 1.5 depends on $\boldsymbol{\theta}$. We introduce a Γ-periodic $(n \times m)$-matrix-valued function $\Lambda(\mathbf{x})$ as a weak solution of the equation

$$b(\mathbf{D})^* g(\mathbf{x})(b(\mathbf{D})\Lambda(\mathbf{x}) + \mathbf{1}_m) = 0, \quad \int_\Omega \Lambda(\mathbf{x})\, d\mathbf{x} = 0. \tag{4.10}$$

As was checked in [Su2, Subsection 6.3], the operator $Z(\boldsymbol{\theta})\colon \mathfrak{H} \to \mathfrak{H}$ is defined by the relation

$$Z(\boldsymbol{\theta}) = \Lambda b(\boldsymbol{\theta}) P, \tag{4.11}$$

where P is the projection (4.2).

Let $\widetilde{\Lambda}(\mathbf{x})$ be the Γ-periodic $(n \times n)$-matrix-valued function satisfying the equation

$$b(\mathbf{D})^* g(\mathbf{x}) b(\mathbf{D}) \widetilde{\Lambda}(\mathbf{x}) + \sum_{j=1}^d D_j a_j(\mathbf{x})^* = 0, \quad \int_\Omega \widetilde{\Lambda}(\mathbf{x})\, d\mathbf{x} = 0. \tag{4.12}$$

As was shown in [Su2, Subsection 6.3], the operator $\widetilde{Z}\colon \mathfrak{H} \to \mathfrak{H}$ (defined in Subsection 1.5) is now realized as

$$\widetilde{Z} = \widetilde{\Lambda} P. \tag{4.13}$$

Now the operator R defined by (1.10) depends on $\boldsymbol{\theta}$. In accordance with (4.1) and (4.11), the operator $R(\boldsymbol{\theta})\colon \mathfrak{N} \to \mathfrak{N}_*$ acts as multiplication by the matrix

$$R(\boldsymbol{\theta}) = h(b(\mathbf{D})\Lambda + \mathbf{1}_m) b(\boldsymbol{\theta}). \tag{4.14}$$

4.4. The operator $S(\boldsymbol{\theta})$. The effective matrix. The operator S defined in Subsection 1.5 depends on $\boldsymbol{\theta}$. It turns out that the operator $S(\boldsymbol{\theta}) = R(\boldsymbol{\theta})^* R(\boldsymbol{\theta})\colon \mathfrak{N} \to \mathfrak{N}$ acts as multiplication by the matrix

$$S(\boldsymbol{\theta}) = b(\boldsymbol{\theta})^* g^0 b(\boldsymbol{\theta}), \quad \boldsymbol{\theta} \in \mathbb{S}^{d-1}. \tag{4.15}$$

Here g^0 is a constant positive $(m \times m)$-matrix defined by

$$g^0 = |\Omega|^{-1} \int_\Omega \widetilde{g}(\mathbf{x})\, d\mathbf{x}, \tag{4.16}$$

where

$$\widetilde{g}(\mathbf{x}) := g(\mathbf{x})(b(\mathbf{D})\Lambda(\mathbf{x}) + \mathbf{1}_m). \tag{4.17}$$

All this follows from (4.14), the definition of $S(\boldsymbol{\theta})$, and equation (4.10) for Λ. See also [BSu1, Chapter 3, §1]. The matrix g^0 is called the *effective matrix*.

The effective matrix g^0 has the following properties (see [BSu1, Chapter 3, §1]).

Proposition 4.1. *We have* $\underline{g} \leq g^0 \leq \overline{g}$, *where* $\overline{g}$ *and* $\underline{g}$ *are defined in accordance with* (2.7) *and* (2.8). *If* $m = n$, *then* $g^0 = \underline{g}$.

Proposition 4.2. *The identity* $g^0 = \overline{g}$ *is equivalent to the relations*

$$b(\mathbf{D})^* \mathbf{g}_k(\mathbf{x}) = 0, \quad k = 1, \dots, m, \tag{4.18}$$

for the columns $\mathbf{g}_k(\mathbf{x})$ *of the matrix* $g(\mathbf{x})$.

Proposition 4.3. *The identity* $g^0 = \underline{g}$ *is equivalent to the representations*

$$\mathbf{l}_k(\mathbf{x}) = \mathbf{l}_k^0 + b(\mathbf{D})\mathbf{w}_k, \quad \mathbf{l}_k^0 \in \mathbb{C}^m, \quad \mathbf{w}_k \in \widetilde{H}^1(\Omega; \mathbb{C}^n), \quad k = 1, \dots, m, \tag{4.19}$$

for the columns $\mathbf{l}_k(\mathbf{x})$ *of the matrix* $g(\mathbf{x})^{-1}$.

§5. Approximation of the operator $\mathcal{B}_\lambda(\mathbf{k}, \varepsilon)^{-1}$

We are going to apply the abstract approach of Theorem 1.5 in order to approximate the operator $\mathcal{B}_\lambda(\mathbf{k}, \varepsilon)^{-1}$. For this, first we realize the operators $L(t, \varepsilon)$ and $N(t, \varepsilon)$ introduced in Subsection 1.6.

5.1. Now, the operator $L(t, \varepsilon)$ defined by (1.12) and acting in the space $\mathfrak{N}$ depends on $\boldsymbol{\theta}$ and is represented as

$$\begin{aligned} L(t, \varepsilon; \boldsymbol{\theta}) = t^2 S(\boldsymbol{\theta}) &- t\varepsilon\big((X_0 Z(\boldsymbol{\theta}))^* X_0 \widetilde{Z} + (X_0 \widetilde{Z})^* X_0 Z(\boldsymbol{\theta})\big)\big|_{\mathfrak{N}} \\ &+ t\varepsilon P\big(Y_2^* Y_1(\boldsymbol{\theta}) + Y_1(\boldsymbol{\theta})^* Y_2\big)\big|_{\mathfrak{N}} + \varepsilon^2\big(-(X_0\widetilde{Z})^* X_0 \widetilde{Z}\big|_{\mathfrak{N}} + Q_{\mathfrak{N}} + \lambda Q_{0\mathfrak{N}}\big). \end{aligned} \tag{5.1}$$

By (4.1), (4.11), and (4.13), we have $X_0 Z(\boldsymbol{\theta}) = h(b(\mathbf{D})\Lambda) b(\boldsymbol{\theta}) P$, $X_0 \widetilde{Z} = h(b(\mathbf{D})\widetilde{\Lambda})P$. Hence, the operator $(X_0 Z(\boldsymbol{\theta}))^* X_0 \widetilde{Z}|_{\mathfrak{N}}$ acts as multiplication by the matrix $b(\boldsymbol{\theta})^* V$, where the constant $(m \times n)$-matrix V is defined by

$$V = |\Omega|^{-1} \int_\Omega (b(\mathbf{D})\Lambda(\mathbf{x}))^* g(\mathbf{x}) (b(\mathbf{D})\widetilde{\Lambda}(\mathbf{x}))\, d\mathbf{x}. \tag{5.2}$$

The operator $(X_0 \widetilde{Z})^* X_0 Z(\boldsymbol{\theta})|_{\mathfrak{N}}$ corresponds to the matrix $V^* b(\boldsymbol{\theta})$. Similarly, the operator $(X_0 \widetilde{Z})^* X_0 \widetilde{Z}|_{\mathfrak{N}}$ acts as multiplication by the constant $(n \times n)$-matrix

$$W = |\Omega|^{-1} \int_\Omega (b(\mathbf{D})\widetilde{\Lambda}(\mathbf{x}))^* g(\mathbf{x}) (b(\mathbf{D})\widetilde{\Lambda}(\mathbf{x}))\, d\mathbf{x}. \tag{5.3}$$

By (3.37) and (4.3), the operator $P(Y_2^* Y_1(\boldsymbol{\theta}) + Y_1(\boldsymbol{\theta})^* Y_2)|_{\mathfrak{N}}$ acts as multiplication by the matrix $\sum_{j=1}^d (\overline{a_j + a_j^*})\theta_j$. Obviously, the operator $Q_{\mathfrak{N}}$ acts as multiplication by the matrix

$$\overline{Q} := |\Omega|^{-1} \int_\Omega d\mu(\mathbf{x}), \tag{5.4}$$

and $Q_{0\mathfrak{N}}$ corresponds to the matrix $\overline{Q_0}$.

As a result, taking (4.15) into account, we see that the operator (5.1) has the form

$$\begin{aligned} L(t, \varepsilon; \boldsymbol{\theta}) = t^2 b(\boldsymbol{\theta})^* g^0 b(\boldsymbol{\theta}) + t\varepsilon\Big(&- b(\boldsymbol{\theta})^* V - V^* b(\boldsymbol{\theta}) + \sum_{j=1}^d (\overline{a_j + a_j^*})\theta_j \Big) \\ &+ \varepsilon^2\left(-W + \overline{Q} + \lambda \overline{Q_0}\right). \end{aligned} \tag{5.5}$$

Since the symbol b is first order homogeneous, we can write the operator (5.5) in terms of the parameter $\mathbf{k} = t\boldsymbol{\theta}$:

$$(5.6)\quad \begin{aligned} L(t,\varepsilon;\boldsymbol{\theta}) =: L(\mathbf{k},\varepsilon) &= b(\mathbf{k})^* g^0 b(\mathbf{k}) - \varepsilon\left(b(\mathbf{k})^* V + V^* b(\mathbf{k})\right) \\ &+ \varepsilon \sum_{j=1}^{d} (\overline{a_j + a_j^*}) k_j + \varepsilon^2 \left(-W + \overline{Q} + \lambda \overline{Q_0}\right). \end{aligned}$$

By (1.21), the operator (5.6) satisfies the estimate

$$(5.7)\quad L(\mathbf{k},\varepsilon) \geq \check{c}_* (|\mathbf{k}|^2 + \varepsilon^2) I_{\mathfrak{N}}, \quad \mathbf{k} \in \widetilde{\Omega},\ \ 0 < \varepsilon \leq 1.$$

Note that, due to the presence of the projection P, in the expression for the operator $L(\mathbf{k},\varepsilon)P \colon \mathfrak{H} \to \mathfrak{H}$ we can replace $\mathbf{k}$ by $\mathbf{D} + \mathbf{k}$ (because $\mathbf{D}P = 0$). We put

$$(5.8)\quad \begin{aligned} \mathcal{B}_\lambda^0(\mathbf{k},\varepsilon) &= b(\mathbf{D}+\mathbf{k})^* g^0 b(\mathbf{D}+\mathbf{k}) - \varepsilon\left(b(\mathbf{D}+\mathbf{k})^* V + V^* b(\mathbf{D}+\mathbf{k})\right) \\ &+ \varepsilon \sum_{j=1}^{d} (\overline{a_j + a_j^*})(D_j + k_j) + \varepsilon^2 \left(-W + \overline{Q} + \lambda \overline{Q_0}\right). \end{aligned}$$

Then

$$(5.9)\quad L(\mathbf{k},\varepsilon)P = \mathcal{B}_\lambda^0(\mathbf{k},\varepsilon)P.$$

5.2. The operator $N(t,\varepsilon)$ defined by (1.13)–(1.17) depends on $\boldsymbol{\theta}$ and is represented as

$$(5.10)\quad N(t,\varepsilon;\boldsymbol{\theta}) = t^3 N_{11}(\boldsymbol{\theta}) + t^2 \varepsilon N_{12}(\boldsymbol{\theta}) + t\varepsilon^2 N_{21}(\boldsymbol{\theta}) + \varepsilon^3 N_{22},$$

where the operators $N_{jl}(\boldsymbol{\theta})$ are defined in accordance with (1.14)–(1.17) with the operators X_1, Z, R, Y_1 depending on $\boldsymbol{\theta}$. (Clearly, N_{22} does not depend on $\boldsymbol{\theta}$.)

By (4.1), (4.11), (4.14), and (4.17), we have

$$N_{11}(\boldsymbol{\theta}) = P b(\boldsymbol{\theta})^* \left(\Lambda^* b(\boldsymbol{\theta})^* \widetilde{g} + \widetilde{g}^* b(\boldsymbol{\theta}) \Lambda\right) b(\boldsymbol{\theta}) P.$$

We denote

$$(5.11)\quad M(\boldsymbol{\theta}) := \overline{\Lambda^* b(\boldsymbol{\theta})^* \widetilde{g}} + \overline{\widetilde{g}^* b(\boldsymbol{\theta}) \Lambda}.$$

Then $N_{11}(\boldsymbol{\theta}) = b(\boldsymbol{\theta})^* M(\boldsymbol{\theta}) b(\boldsymbol{\theta}) P$. Note that $M(\mathbf{k}) := tM(\boldsymbol{\theta})$ is a Hermitian $(m \times m)$-matrix-valued function first order homogeneous in $\mathbf{k}$. Thus, $M(\mathbf{k})$ is the symbol of the selfadjoint first order DO $M(\mathbf{D})$ with constant coefficients. As a result, the term $N_{11}(\mathbf{k}) := t^3 N_{11}(\boldsymbol{\theta})$ in (5.10) takes the form

$$(5.12)\quad N_{11}(\mathbf{k}) = b(\mathbf{k})^* M(\mathbf{k}) b(\mathbf{k}) P.$$

Consider the operator $N_{12}(\boldsymbol{\theta})$. From (4.1), (4.13), (4.14), and (4.17) it follows that

$$(5.13)\quad (X_1(\boldsymbol{\theta})\widetilde{Z})^* R(\boldsymbol{\theta}) P + (R(\boldsymbol{\theta})P)^* X_1(\boldsymbol{\theta}) \widetilde{Z} = \left(\widetilde{M}_1(\boldsymbol{\theta}) b(\boldsymbol{\theta}) + b(\boldsymbol{\theta})^* \widetilde{M}_1(\boldsymbol{\theta})^*\right) P,$$

where

$$(5.14)\quad \widetilde{M}_1(\boldsymbol{\theta}) := \overline{\widetilde{\Lambda}^* b(\boldsymbol{\theta})^* \widetilde{g}}.$$

Similarly, using (4.1), (4.11), and (4.13), we obtain

$$(5.15)\quad (X_1(\boldsymbol{\theta}) Z(\boldsymbol{\theta}))^* X_0 \widetilde{Z} + (X_0 \widetilde{Z})^* X_1(\boldsymbol{\theta}) Z(\boldsymbol{\theta}) = \left(\widehat{M}_1(\boldsymbol{\theta}) b(\boldsymbol{\theta}) + b(\boldsymbol{\theta})^* \widehat{M}_1(\boldsymbol{\theta})^*\right) P,$$

where $\widehat{M}_1(\boldsymbol{\theta}) := \overline{(b(\mathbf{D})\widetilde{\Lambda})^* g b(\boldsymbol{\theta}) \Lambda}$. Next, by (3.37), (4.3), and (4.11), we have

$$(5.16)\quad (Y_2 Z(\boldsymbol{\theta}))^* Y_0 Z(\boldsymbol{\theta}) + (Y_0 Z(\boldsymbol{\theta}))^* Y_2 Z(\boldsymbol{\theta}) = b(\boldsymbol{\theta})^* T_0 b(\boldsymbol{\theta}) P,$$

where T_0 is a Hermitian $(m \times m)$-matrix defined by

$$(5.17)\quad T_0 = 2 \sum_{j=1}^{d} \operatorname{Re} \overline{\Lambda^* a_j D_j \Lambda}.$$

Finally, from (3.37), (4.3), and (4.11) it follows that

$$(Y_2 Z(\boldsymbol{\theta}))^* Y_1(\boldsymbol{\theta}) P + (Y_1(\boldsymbol{\theta}) P)^* Y_2 Z(\boldsymbol{\theta}) + (Y_2 P)^* Y_1(\boldsymbol{\theta}) Z(\boldsymbol{\theta}) + (Y_1(\boldsymbol{\theta}) Z(\boldsymbol{\theta}))^* Y_2 P \\ = \big(\breve{M}_1(\boldsymbol{\theta}) b(\boldsymbol{\theta}) + b(\boldsymbol{\theta})^* \breve{M}_1(\boldsymbol{\theta})^*\big) P, \tag{5.18}$$

where $\breve{M}_1(\boldsymbol{\theta}) = \sum_{j=1}^d \overline{(a_j + a_j^*)\Lambda}\theta_j$. Taking (1.15) into account and summing up the terms (5.13), (5.15), (5.16), and (5.18), we arrive at the following representation:

$$N_{12}(\boldsymbol{\theta}) = (b(\boldsymbol{\theta})^* T_0 b(\boldsymbol{\theta}) + M_1(\boldsymbol{\theta}) b(\boldsymbol{\theta}) + b(\boldsymbol{\theta})^* M_1(\boldsymbol{\theta})^*)\, P,$$

where

$$M_1(\boldsymbol{\theta}) = \widetilde{M}_1(\boldsymbol{\theta}) + \widehat{M}_1(\boldsymbol{\theta}) + \breve{M}_1(\boldsymbol{\theta}) \\ = \overline{\widetilde{\Lambda}^* b(\boldsymbol{\theta})^* \widetilde{g}} + \overline{(b(\mathbf{D})\widetilde{\Lambda})^* g b(\boldsymbol{\theta}) \Lambda} + \sum_{j=1}^d \overline{(a_j + a_j^*)\Lambda}\theta_j. \tag{5.19}$$

Note that $M_1(\mathbf{k}) := t M_1(\boldsymbol{\theta})$ is an $(n \times m)$-matrix-valued function first order homogeneous in $\mathbf{k}$. Thus, $M_1(\mathbf{k})$ is the symbol of the first order DO $M_1(\mathbf{D})$ with constant coefficients. We put $N_{12}(\mathbf{k}) := t^2 N_{12}(\boldsymbol{\theta})$. It follows that the second term $\varepsilon N_{12}(\mathbf{k})$ in (5.10) can be written as

$$\varepsilon N_{12}(\mathbf{k}) = \varepsilon\, (b(\mathbf{k})^* T_0 b(\mathbf{k}) + M_1(\mathbf{k}) b(\mathbf{k}) + b(\mathbf{k})^* M_1(\mathbf{k})^*)\, P. \tag{5.20}$$

We proceed to the operator $N_{21}(\boldsymbol{\theta})$. By (4.1) and (4.13), we have

$$(X_0 \widetilde{Z})^* X_1(\boldsymbol{\theta}) \widetilde{Z} + (X_1(\boldsymbol{\theta}) \widetilde{Z})^* X_0 \widetilde{Z} = (M_2(\boldsymbol{\theta}) + M_2(\boldsymbol{\theta})^*)\, P, \tag{5.21}$$

where

$$M_2(\boldsymbol{\theta}) = \overline{(b(\mathbf{D})\widetilde{\Lambda})^* g b(\boldsymbol{\theta}) \widetilde{\Lambda}}. \tag{5.22}$$

Using (3.37), (4.3), (4.11), and (4.13), we get

$$(Y_2 Z(\boldsymbol{\theta}))^* Y_0 \widetilde{Z} + (Y_0 \widetilde{Z})^* Y_2 Z(\boldsymbol{\theta}) = (T_1^* b(\boldsymbol{\theta}) + b(\boldsymbol{\theta})^* T_1)\, P, \tag{5.23}$$

where $T_1 = \sum_{j=1}^d \overline{\Lambda^* a_j (D_j \widetilde{\Lambda})}$. Next, from (3.37), (4.3), (4.11), and (4.13) it follows that

$$(Y_2 \widetilde{Z})^* Y_0 Z(\boldsymbol{\theta}) + (Y_0 Z(\boldsymbol{\theta}))^* Y_2 \widetilde{Z} = (T_2 b(\boldsymbol{\theta}) + b(\boldsymbol{\theta})^* T_2^*)\, P, \tag{5.24}$$

where $T_2 = \sum_{j=1}^d \overline{\widetilde{\Lambda}^* a_j (D_j \Lambda)}$. Relations (3.37), (4.3), and (4.13) imply

$$(Y_2 \widetilde{Z})^* Y_1(\boldsymbol{\theta}) P + (Y_1(\boldsymbol{\theta}) P)^* Y_2 \widetilde{Z} + (Y_1(\boldsymbol{\theta}) \widetilde{Z})^* Y_2 P + (Y_2 P)^* Y_1(\boldsymbol{\theta}) \widetilde{Z} \\ = 2 \sum_{j=1}^d \operatorname{Re} \overline{(a_j + a_j^*)\widetilde{\Lambda}}\theta_j P. \tag{5.25}$$

In accordance with (4.11) and the definition of the form q, we have

$$Z(\boldsymbol{\theta})^* Q P + P Q Z(\boldsymbol{\theta}) = \big(b(\boldsymbol{\theta})^* \overline{\Lambda^* Q} + (\overline{\Lambda^* Q})^* b(\boldsymbol{\theta})\big)\, P, \tag{5.26}$$

where $\overline{\Lambda^* Q} = |\Omega|^{-1} \int_\Omega \Lambda(\mathbf{x})^*\, d\mu(\mathbf{x})$. Finally, by (4.11),

$$\lambda (Z(\boldsymbol{\theta})^* Q_0 P + P Q_0 Z(\boldsymbol{\theta})) = \lambda\, \big(b(\boldsymbol{\theta})^* \overline{\Lambda^* Q_0} + \overline{Q_0^* \Lambda} b(\boldsymbol{\theta})\big)\, P. \tag{5.27}$$

Using (1.16) and summing up the terms (5.21) and (5.23)–(5.27), we arrive at the following representation:

$$N_{21}(\boldsymbol{\theta}) = \left(M_2(\boldsymbol{\theta}) + M_2(\boldsymbol{\theta})^* + T^* b(\boldsymbol{\theta}) + b(\boldsymbol{\theta})^* T + 2 \sum_{j=1}^d \operatorname{Re} \overline{(a_j + a_j^*)\widetilde{\Lambda}}\theta_j \right) P,$$

where

$$
\begin{aligned}
T &= T_1 + T_2^* + \overline{\Lambda^* Q} + \lambda \overline{\Lambda^* Q_0} \\
&= \sum_{j=1}^{d} \Big(\overline{\Lambda^* a_j (D_j \widetilde{\Lambda})} + \overline{(D_j \Lambda)^* a_j^* \widetilde{\Lambda}} \Big) + \overline{\Lambda^* Q} + \lambda \overline{\Lambda^* Q_0}.
\end{aligned}
\tag{5.28}
$$

Note that the $(n \times n)$-matrix-valued function $M_2(\mathbf{k}) = t M_2(\boldsymbol{\theta})$ is first order homogeneous in $\mathbf{k}$. Therefore, $M_2(\mathbf{k})$ is the symbol of a matrix first order DO $M_2(\mathbf{D})$ with constant coefficients.

We put $N_{21}(\mathbf{k}) := t N_{21}(\boldsymbol{\theta})$. As a result, the third term $\varepsilon^2 N_{21}(\mathbf{k})$ in (5.10) can be written as

$$
\varepsilon^2 N_{21}(\mathbf{k}) = \varepsilon^2 \big(M_2(\mathbf{k}) + M_2(\mathbf{k})^* + T^* b(\mathbf{k}) + b(\mathbf{k})^* T \big) P + 2\varepsilon^2 \sum_{j=1}^{d} \operatorname{Re} \overline{(a_j + a_j^*) \widetilde{\Lambda}} k_j P.
\tag{5.29}
$$

It remains to consider the term N_{22}, which does not depend on $\boldsymbol{\theta}$. By (1.17), (3.37), (3.39), (4.3), and (4.13), we have

$$
N_{22} = \big(\widehat{T} + \widehat{T}^* \big) P,
\tag{5.30}
$$

where the matrix $\widehat{T}$ is given by

$$
\widehat{T} = \sum_{j=1}^{d} \overline{\widetilde{\Lambda}^* a_j (D_j \widetilde{\Lambda})} + \overline{\widetilde{\Lambda}^* Q} + \lambda \overline{\widetilde{\Lambda}^* Q_0}.
\tag{5.31}
$$

Here $\overline{\widetilde{\Lambda}^* Q} = |\Omega|^{-1} \int_\Omega \widetilde{\Lambda}(\mathbf{x})^* \, d\mu(\mathbf{x})$.

As a result, the operator $N(\mathbf{k}, \varepsilon) = N(t, \varepsilon; \boldsymbol{\theta})$ defined by (5.10) is represented as

$$
N(\mathbf{k}, \varepsilon) = N_{11}(\mathbf{k}) + \varepsilon N_{12}(\mathbf{k}) + \varepsilon^2 N_{21}(\mathbf{k}) + \varepsilon^3 N_{22},
\tag{5.32}
$$

where the summands on the right-hand side are defined by (5.12), (5.20), (5.29), and (5.30). The operator (5.32) satisfies an estimate of the form (1.18) with a constant $\widehat{\mathcal{C}}$, which, in general, depends on $\boldsymbol{\theta}$. By Remark 1.6, the constant $\widehat{\mathcal{C}}(\boldsymbol{\theta})$ is a polynomial with positive coefficients in the variables $\delta^{-1/2}$, $\kappa^{1/2}$, $\|X_1(\boldsymbol{\theta})\|$, $\|Y_1(\boldsymbol{\theta})\|$, c_1, $C(1)^{1/2}$, $c_2^{1/2}$, $c_3^{1/2}$, and $|\lambda| \|Q_0\|_{L_\infty}$. The parameters c_1, c_2, κ, $C(1)$, and δ do not depend on $\boldsymbol{\theta}$ and can be controlled in terms of the initial data (3.27) (see (3.9), (3.16), (4.4), and Remark 3.1). Using (4.5) and (4.6), we substitute $\|Y_1(\boldsymbol{\theta})\| = 1$ and replace $\|X_1(\boldsymbol{\theta})\|$ by $\alpha_1^{1/2} \|g\|_{L_\infty}^{1/2}$. The corresponding overstated constant is again denoted by $\widehat{\mathcal{C}}$; it fits for all $\boldsymbol{\theta}$ and depends only on the initial data (3.27). Thus, we have

$$
\|N(\mathbf{k}, \varepsilon)\|_{\mathfrak{H} \to \mathfrak{H}} \leq \widehat{\mathcal{C}} (|\mathbf{k}|^2 + \varepsilon^2)^{3/2}, \quad \mathbf{k} \in \mathbb{R}^d, \quad 0 < \varepsilon \leq 1.
\tag{5.33}
$$

Note that, due to the presence of the projection P, in the expressions for the operators $N_{11}(\mathbf{k})$, $N_{12}(\mathbf{k})$, and $N_{21}(\mathbf{k})$ we can replace $\mathbf{k}$ by $\mathbf{D} + \mathbf{k}$, cf. Subsection 5.1. Hence,

$$
N(\mathbf{k}, \varepsilon) = \mathcal{N}(\mathbf{k}, \varepsilon) P,
\tag{5.34}
$$

where $\mathcal{N}(\mathbf{k}, \varepsilon)$ is the selfadjoint third order DO given by

$$
\mathcal{N}(\mathbf{k}, \varepsilon) = \mathcal{N}_{11}(\mathbf{D} + \mathbf{k}) + \varepsilon \mathcal{N}_{12}(\mathbf{D} + \mathbf{k}) + \varepsilon^2 \mathcal{N}_{21}(\mathbf{D} + \mathbf{k}) + \varepsilon^3 \mathcal{N}_{22}.
\tag{5.35}
$$

The consecutive terms on the right-hand side are DOs of the third, second, first, and zeroth order, respectively, given by

$$\begin{aligned}
\mathcal{N}_{11}(\mathbf{D}+\mathbf{k}) &= b(\mathbf{D}+\mathbf{k})^* M(\mathbf{D}+\mathbf{k}) b(\mathbf{D}+\mathbf{k}),\\
\mathcal{N}_{12}(\mathbf{D}+\mathbf{k}) &= b(\mathbf{D}+\mathbf{k})^* T_0 b(\mathbf{D}+\mathbf{k}) + M_1(\mathbf{D}+\mathbf{k}) b(\mathbf{D}+\mathbf{k})\\
&\quad + b(\mathbf{D}+\mathbf{k})^* M_1(\mathbf{D}+\mathbf{k})^*,\\
\mathcal{N}_{21}(\mathbf{D}+\mathbf{k}) &= M_2(\mathbf{D}+\mathbf{k}) + M_2(\mathbf{D}+\mathbf{k})^* + T^* b(\mathbf{D}+\mathbf{k}) + b(\mathbf{D}+\mathbf{k})^* T\\
&\quad + 2\sum_{j=1}^{d} \operatorname{Re} \overline{(a_j + a_j^*)\widetilde{\Lambda}} (D_j + k_j),\\
\mathcal{N}_{22} &= \widehat{T} + \widehat{T}^*.
\end{aligned} \tag{5.36}$$

5.3. Approximation of the operator $\mathcal{B}_\lambda(\mathbf{k},\varepsilon)^{-1}$ for $|\mathbf{k}|^2+\varepsilon^2 \le \tau_0^2$. We apply Theorem 1.5 in order to approximate the operator $\mathcal{B}_\lambda(\mathbf{k},\varepsilon)^{-1}$ for $|\mathbf{k}|^2+\varepsilon^2 \le \tau_0^2$. (The set of the corresponding points $\mathbf{k}$ is a subset of $\widetilde{\Omega}$, because $\tau_0 < r_0/2$.) Now, the corrector (1.22) depends on $\boldsymbol{\theta}$; it is denoted by $\mathcal{K}(t,\varepsilon;\boldsymbol{\theta}) =: \mathcal{K}(\mathbf{k},\varepsilon)$. Using (4.11), (4.13), and (5.9), we transform the first term of the corrector:

$$\begin{aligned}
(tZ(\boldsymbol{\theta}) + \varepsilon\widetilde{Z}) L(t,\varepsilon;\boldsymbol{\theta})^{-1} P &= (\Lambda b(\mathbf{k}) + \varepsilon\widetilde{\Lambda}) L(\mathbf{k},\varepsilon)^{-1} P\\
&= (\Lambda b(\mathbf{D}+\mathbf{k}) + \varepsilon\widetilde{\Lambda}) \mathcal{B}_\lambda^0(\mathbf{k},\varepsilon)^{-1} P.
\end{aligned}$$

The second term of the corrector is adjoint to the first. Relations (5.9) and (5.34) allow us to represent the third term as

$$\begin{aligned}
-L(t,\varepsilon;\boldsymbol{\theta})^{-1} N(t,\varepsilon;\boldsymbol{\theta}) L(t,\varepsilon;\boldsymbol{\theta})^{-1} P &= -L(\mathbf{k},\varepsilon)^{-1} N(\mathbf{k},\varepsilon) L(\mathbf{k},\varepsilon)^{-1} P\\
&= -\mathcal{B}_\lambda^0(\mathbf{k},\varepsilon)^{-1} \mathcal{N}(\mathbf{k},\varepsilon) \mathcal{B}_\lambda^0(\mathbf{k},\varepsilon)^{-1} P.
\end{aligned}$$

As a result, the corrector for the operator $\mathcal{B}_\lambda(\mathbf{k},\varepsilon)^{-1}$ is given by

$$\begin{aligned}
\mathcal{K}(\mathbf{k},\varepsilon) &= (\Lambda b(\mathbf{D}+\mathbf{k}) + \varepsilon\widetilde{\Lambda}) \mathcal{B}_\lambda^0(\mathbf{k},\varepsilon)^{-1} P + \mathcal{B}_\lambda^0(\mathbf{k},\varepsilon)^{-1} P (b(\mathbf{D}+\mathbf{k})^* \Lambda^* + \varepsilon\widetilde{\Lambda}^*)\\
&\quad - \mathcal{B}_\lambda^0(\mathbf{k},\varepsilon)^{-1} \mathcal{N}(\mathbf{k},\varepsilon) \mathcal{B}_\lambda^0(\mathbf{k},\varepsilon)^{-1} P.
\end{aligned} \tag{5.37}$$

In general, the constants $\mathcal{C}$ and $\mathcal{C}_0$ occurring in (1.23) and (1.24) depend on $\boldsymbol{\theta}$. By Remark 1.6, the constant $\mathcal{C}(\boldsymbol{\theta})$ is a polynomial with positive coefficients in the variables δ, δ^{-1}, $(\check{c}_*)^{-1}$, $\kappa^{-1/2}$, $\|X_1(\boldsymbol{\theta})\|$, $\|Y_1(\boldsymbol{\theta})\|$, c_1, $C(1)$, c_2, c_3, $|\lambda|\|Q_0\|_{L_\infty}$, and τ_0. From (3.9), (3.16), (3.21), (3.26), (3.31), (4.4), (4.9), and Remark 3.1 it follows that the parameters c_1, c_2, κ, $C(1)$, δ, and $\check{c}_*$ are controlled in terms of the initial data (3.27) and do not depend on $\boldsymbol{\theta}$. The number τ_0 has already been chosen independent of $\boldsymbol{\theta}$ (see (4.7)); it also depends only on the initial data. Using (4.5) and (4.6), we substitute $\|Y_1(\boldsymbol{\theta})\| = 1$ and replace $\|X_1(\boldsymbol{\theta})\|$ by $\alpha_1^{1/2}\|g\|_{L_\infty}^{1/2}$, overstating the constant $\mathcal{C}(\boldsymbol{\theta})$. The resulting overstated constant is again denoted by $\mathcal{C}$; this constant fits for all values of $\boldsymbol{\theta}$ and is controlled in terms of the initial data (3.27) only. Similarly, after overstating, the constant $\mathcal{C}_0$ does not depend on $\boldsymbol{\theta}$ and is controlled in terms of the initial data (3.27).

Applying Theorem 1.5 to the operator $\mathcal{B}_\lambda(\mathbf{k},\varepsilon) = B_\lambda(t,\varepsilon;\boldsymbol{\theta})$, we obtain the following result.

Theorem 5.1. *Let $\mathcal{B}_\lambda(\mathbf{k},\varepsilon)$ be the operator* (3.45) *acting in the space $\mathfrak{H} = L_2(\Omega;\mathbb{C}^n)$, as defined in Subsection* 3.8. *Let $\mathcal{B}_\lambda^0(\mathbf{k},\varepsilon)$ be the operator* (5.8) *defined in Subsection* 5.1. *Let $\mathcal{K}(\mathbf{k},\varepsilon)$ be the corrector defined by* (5.37). *Then for $|\mathbf{k}|^2+\varepsilon^2 \le \tau_0^2$ and $0<\varepsilon\le 1$ we have*

$$\mathcal{B}_\lambda(\mathbf{k},\varepsilon)^{-1} = \mathcal{B}_\lambda^0(\mathbf{k},\varepsilon)^{-1} P + \mathcal{K}(\mathbf{k},\varepsilon) + J(\mathbf{k},\varepsilon), \tag{5.38}$$

where P is the projection (4.2). *The remainder term $J(\mathbf{k},\varepsilon)$ satisfies the following estimate*:

$$\|J(\mathbf{k},\varepsilon)\|_{\mathfrak{H}\to\mathfrak{H}} \le \mathcal{C}, \quad |\mathbf{k}|^2+\varepsilon^2 \le \tau_0^2, \quad 0<\varepsilon\le 1. \tag{5.39}$$

Suppose that τ_0 is defined by (4.7). *The corrector satisfies the estimate*

$$\|\mathcal{K}(\mathbf{k},\varepsilon)\|_{\mathfrak{H}\to\mathfrak{H}} \le \mathcal{C}_0(|\mathbf{k}|^2+\varepsilon^2)^{-1/2}, \quad \mathbf{k}\in\widetilde{\Omega}, \quad 0<\varepsilon\le 1. \tag{5.40}$$

The constants $\mathcal{C}$ and $\mathcal{C}_0$ depend only on the initial data (3.27).

5.4. Approximation of the operator $\mathcal{B}_\lambda(\mathbf{k},\varepsilon)^{-1}$ for $|\mathbf{k}|^2+\varepsilon^2>\tau_0^2$. If $\mathbf{k}\in\widetilde{\Omega}$, $0<\varepsilon\le 1$, and $|\mathbf{k}|^2+\varepsilon^2>\tau_0^2$, estimates are trivial: each term on the right-hand side of the identity $J(\mathbf{k},\varepsilon)=\mathcal{B}_\lambda(\mathbf{k},\varepsilon)^{-1}-\mathcal{B}^0_\lambda(\mathbf{k},\varepsilon)^{-1}P-\mathcal{K}(\mathbf{k},\varepsilon)$ is estimated separately. By (4.8), we have

$$\big\|\mathcal{B}_\lambda(\mathbf{k},\varepsilon)^{-1}\big\|_{\mathfrak{H}\to\mathfrak{H}} \le (\check{c}_*)^{-1}\tau_0^{-2}, \quad \mathbf{k}\in\widetilde{\Omega},\ 0<\varepsilon\le 1,\ |\mathbf{k}|^2+\varepsilon^2>\tau_0^2. \tag{5.41}$$

From (5.7) and (5.9) it follows that

$$\big\|\mathcal{B}^0_\lambda(\mathbf{k},\varepsilon)^{-1}P\big\|_{\mathfrak{H}\to\mathfrak{H}} \le (\check{c}_*)^{-1}\tau_0^{-2}, \quad \mathbf{k}\in\widetilde{\Omega},\ 0<\varepsilon\le 1,\ |\mathbf{k}|^2+\varepsilon^2>\tau_0^2. \tag{5.42}$$

The norm of the corrector satisfies (5.40) for all $\mathbf{k}\in\widetilde{\Omega}$ and $0<\varepsilon\le 1$, whence

$$\|\mathcal{K}(\mathbf{k},\varepsilon)\|_{\mathfrak{H}\to\mathfrak{H}} \le \mathcal{C}_0\tau_0^{-1}, \quad \mathbf{k}\in\widetilde{\Omega},\ 0<\varepsilon\le 1,\ |\mathbf{k}|^2+\varepsilon^2>\tau_0^2. \tag{5.43}$$

As a result, relations (5.41)–(5.43) imply the following estimate for the reminder term in (5.38) for $|\mathbf{k}|^2+\varepsilon^2>\tau_0^2$:

$$\|J(\mathbf{k},\varepsilon)\|_{\mathfrak{H}\to\mathfrak{H}} \le 2(\check{c}_*)^{-1}\tau_0^{-2}+\mathcal{C}_0\tau_0^{-1}, \quad \mathbf{k}\in\widetilde{\Omega},\ 0<\varepsilon\le 1,\ |\mathbf{k}|^2+\varepsilon^2>\tau_0^2.$$

Together with (5.39) this yields the following result.

Theorem 5.2. *Under the assumptions of Theorem* 5.1, *for $\mathbf{k}\in\widetilde{\Omega}$ and $0<\varepsilon\le 1$ formula* (5.38) *is valid, and the reminder term satisfies the estimate*

$$\|J(\mathbf{k},\varepsilon)\|_{\mathfrak{H}\to\mathfrak{H}} \le \mathcal{C}_1, \quad \mathbf{k}\in\widetilde{\Omega},\ 0<\varepsilon\le 1.$$

The constant $\mathcal{C}_1=\max\{\mathcal{C},2(\check{c}_*)^{-1}\tau_0^{-2}+\mathcal{C}_0\tau_0^{-1}\}$ depends only on the initial data (3.27).

§6. Approximation of the operator $\mathcal{B}_\lambda(\varepsilon)^{-1}$

6.1. Now we return to the operator $\mathcal{B}_\lambda(\varepsilon)$ acting in $L_2(\mathbb{R}^d;\mathbb{C}^n)$, as defined in Subsection 3.4. Approximation of the operator $\mathcal{B}_\lambda(\varepsilon)^{-1}$ is deduced from Theorem 5.2 with the help of the direct integral expansion.

We introduce the following effective operator with constant coefficients:

$$\begin{aligned}\mathcal{B}^0_\lambda(\varepsilon) := b(\mathbf{D})^*g^0b(\mathbf{D})+\varepsilon\Big(-b(\mathbf{D})^*V-V^*b(\mathbf{D})+\sum_{j=1}^d(\overline{a_j+a_j^*})D_j\Big)&\\ +\varepsilon^2(-W+\overline{Q}+\lambda\overline{Q_0}).&\end{aligned} \tag{6.1}$$

Here g^0 is the effective matrix defined in Subsection 4.4; the matrices V and W are defined by (5.2) and (5.3), respectively. The symbol of the operator (6.1) is the matrix $L(\mathbf{k},\varepsilon)$ (see (5.6)).

We introduce the following third order DO with constant coefficients:

$$\mathcal{N}(\varepsilon) = \mathcal{N}_{11}(\mathbf{D}) + \varepsilon \mathcal{N}_{12}(\mathbf{D}) + \varepsilon^2 \mathcal{N}_{21}(\mathbf{D}) + \varepsilon^3 \mathcal{N}_{22}, \tag{6.2}$$

$$\mathcal{N}_{11}(\mathbf{D}) = b(\mathbf{D})^* M(\mathbf{D}) b(\mathbf{D}), \tag{6.3}$$

$$\mathcal{N}_{12}(\mathbf{D}) = b(\mathbf{D})^* T_0 b(\mathbf{D}) + M_1(\mathbf{D}) b(\mathbf{D}) + b(\mathbf{D})^* M_1(\mathbf{D})^*, \tag{6.4}$$

$$\mathcal{N}_{21}(\mathbf{D}) = M_2(\mathbf{D}) + M_2(\mathbf{D})^* + T^* b(\mathbf{D}) + b(\mathbf{D})^* T + 2 \sum_{j=1}^{d} \operatorname{Re} \overline{(a_j + a_j^*) \widetilde{\Lambda}} D_j. \tag{6.5}$$

Recall that the matrices T_0 and T are defined by (5.17) and (5.28), and the homogeneous first order DOs $M(\mathbf{D})$, $M_1(\mathbf{D})$, $M_2(\mathbf{D})$ correspond to the symbols $M(\mathbf{k})$, $M_1(\mathbf{k})$, $M_2(\mathbf{k})$ (see (5.11), (5.19), (5.22)). The operator $\mathcal{N}_{22}$ is multiplication by the matrix $\widehat{T} + \widehat{T}^*$, where $\widehat{T}$ is the matrix given by (5.31).

We use the direct integral expansion (3.47) for the operator $\mathcal{B}_\lambda(\varepsilon)$. Then for the inverse operator we have

$$\mathcal{B}_\lambda(\varepsilon)^{-1} = \mathcal{U}^{-1} \Big(\int_{\widetilde{\Omega}} \oplus \mathcal{B}_\lambda(\mathbf{k}, \varepsilon)^{-1} \, d\mathbf{k} \Big) \mathcal{U}.$$

A similar expansion holds for the operator $\mathcal{B}^0_\lambda(\varepsilon)^{-1}$. The operator $\Lambda b(\mathbf{D}) + \widetilde{\Lambda}$ expands in the direct integral with the fibers $\Lambda b(\mathbf{D} + \mathbf{k}) + \widetilde{\Lambda}$. The operator $\mathcal{N}(\varepsilon)$ admits expansion in the direct integral of the operators $\mathcal{N}(\mathbf{k}, \varepsilon)$ (see (5.35)).

We define a bounded operator Π in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ by $\Pi = \mathcal{U}^{-1} [P] \mathcal{U}$, where $[P]$ is the operator in $\mathcal{H}$ (see (2.6)) acting on fibers as the operator P (the operator of averaging over the cell). As was checked in [BSu3, Subsection 6.1], the operator Π can be represented as

$$(\Pi \mathbf{u})(\mathbf{x}) = (2\pi)^{-d/2} \int_{\widetilde{\Omega}} e^{i \langle \mathbf{x}, \boldsymbol{\xi} \rangle} \widehat{\mathbf{u}}(\boldsymbol{\xi}) \, d\boldsymbol{\xi}, \tag{6.6}$$

where $\widehat{\mathbf{u}}$ is the Fourier image of the function $\mathbf{u}$. Thus, Π is the pseudodifferential operator whose symbol is the characteristic function $\chi_{\widetilde{\Omega}}(\boldsymbol{\xi})$ of the set $\widetilde{\Omega}$; this operator is smoothing. Note that Π is an orthogonal projection in $L_2(\mathbb{R}^d; \mathbb{C}^n)$, and it commutes with any DO with constant coefficients.

In $L_2(\mathbb{R}^d; \mathbb{C}^n)$, we define the operator

$$\begin{aligned} \mathcal{K}(\varepsilon) = (\Lambda b(\mathbf{D}) + \varepsilon \widetilde{\Lambda}) \mathcal{B}^0_\lambda(\varepsilon)^{-1} \Pi + \mathcal{B}^0_\lambda(\varepsilon)^{-1} \Pi (b(\mathbf{D})^* \Lambda^* + \varepsilon \widetilde{\Lambda}^*) \\ - \mathcal{B}^0_\lambda(\varepsilon)^{-1} \mathcal{N}(\varepsilon) \mathcal{B}^0_\lambda(\varepsilon)^{-1} \Pi, \end{aligned} \tag{6.7}$$

called the corrector for $\mathcal{B}_\lambda(\varepsilon)^{-1}$. From the above it follows that

$$\mathcal{K}(\varepsilon) = \mathcal{U}^{-1} \left(\int_{\widetilde{\Omega}} \oplus \mathcal{K}(\mathbf{k}, \varepsilon) \, d\mathbf{k} \right) \mathcal{U},$$

where $\mathcal{K}(\mathbf{k}, \varepsilon)$ is defined by (5.37).

Consider the operator $J(\varepsilon) := \mathcal{U}^{-1} \big(\int_{\widetilde{\Omega}} \oplus J(\mathbf{k}, \varepsilon) \, d\mathbf{k} \big) \mathcal{U}$ and note that

$$\| J(\varepsilon) \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} = \operatorname*{ess\,sup}_{\mathbf{k} \in \widetilde{\Omega}} \| J(\mathbf{k}, \varepsilon) \|_{\mathfrak{H} \to \mathfrak{H}}.$$

Now, the direct integral expansion and Theorem 5.2 allow us to deduce the following result from Theorem 5.2.

Theorem 6.1. *Suppose that $\mathcal{B}_\lambda(\varepsilon)$ is the operator in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ defined in Subsection 3.4, and $\mathcal{B}^0_\lambda(\varepsilon)$ is the operator defined by (6.1). Let $\mathcal{K}(\varepsilon)$ be the corrector defined by (6.7), where the operator Π is defined by (6.6). Then for $0 < \varepsilon \leq 1$ we have*

$$\mathcal{B}_\lambda(\varepsilon)^{-1} = \mathcal{B}^0_\lambda(\varepsilon)^{-1} \Pi + \mathcal{K}(\varepsilon) + J(\varepsilon), \tag{6.8}$$

where the reminder term satisfies

$$\|J(\varepsilon)\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le \mathcal{C}_1, \quad 0<\varepsilon\le 1.$$

The constant $\mathcal{C}_1$ *depends only on the initial data* (3.27).

6.2. Elimination of the operator Π. The approximation (6.8) involves the smoothing pseudodifferential operator Π. We shall try to get rid of Π. In the principal term, this is always possible. Indeed, relations (5.7) and (6.6) imply

$$\begin{aligned}
\|\mathcal{B}_\lambda^0(\varepsilon)^{-1}(I-\Pi)\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} &= \sup_{\boldsymbol{\xi}\in\mathbb{R}^d} \left|L(\boldsymbol{\xi},\varepsilon)^{-1}\right|(1-\chi_{\widetilde{\Omega}}(\boldsymbol{\xi}))\\
&\le \sup_{|\boldsymbol{\xi}|\ge r_0} (\check{c}_*)^{-1}(|\boldsymbol{\xi}|^2+\varepsilon^2)^{-1} \le (\check{c}_*)^{-1} r_0^{-2}.
\end{aligned} \tag{6.9}$$

Also, it is possible to replace Π by I in the third term of the corrector (6.7). Denote by $\mathfrak{n}(\mathbf{k},\varepsilon)$ the symbol of the differential operator $\mathcal{N}(\varepsilon)$. Clearly, the operator $N(\mathbf{k},\varepsilon)$ is the composition of the projection P and multiplication by this symbol: $N(\mathbf{k},\varepsilon)=\mathfrak{n}(\mathbf{k},\varepsilon)P$. Therefore, estimate (5.33) yields the same estimate for the matrix norm of the symbol $\mathfrak{n}(\mathbf{k},\varepsilon)$. Hence, by (5.7) and (6.6), we obtain

$$\begin{aligned}
&\|\mathcal{B}_\lambda^0(\varepsilon)^{-1}\mathcal{N}(\varepsilon)\mathcal{B}_\lambda^0(\varepsilon)^{-1}(1-\Pi)\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\\
&\quad= \sup_{\boldsymbol{\xi}\in\mathbb{R}^d}\left|L(\boldsymbol{\xi},\varepsilon)^{-1}\mathfrak{n}(\boldsymbol{\xi},\varepsilon)L(\boldsymbol{\xi},\varepsilon)^{-1}\right|(1-\chi_{\widetilde{\Omega}}(\boldsymbol{\xi}))\\
&\quad\le \sup_{|\boldsymbol{\xi}|\ge r_0} \widehat{\mathcal{C}}(\check{c}_*)^{-2}(|\boldsymbol{\xi}|^2+\varepsilon^2)^{-1/2} \le \widehat{\mathcal{C}}(\check{c}_*)^{-2} r_0^{-1}.
\end{aligned} \tag{6.10}$$

Theorem 6.1 and estimates (6.9), (6.10) imply the following result.

Theorem 6.2. *Under the assumptions of Theorem* 6.1*, put*

$$\begin{aligned}
\widetilde{\mathcal{K}}(\varepsilon) = (\Lambda b(\mathbf{D})+\varepsilon\widetilde{\Lambda})\mathcal{B}_\lambda^0(\varepsilon)^{-1}\Pi + \mathcal{B}_\lambda^0(\varepsilon)^{-1}\Pi(b(\mathbf{D})^*\Lambda^*+\varepsilon\widetilde{\Lambda}^*)&\\
-\mathcal{B}_\lambda^0(\varepsilon)^{-1}\mathcal{N}(\varepsilon)\mathcal{B}_\lambda^0(\varepsilon)^{-1}&.
\end{aligned} \tag{6.11}$$

Then for $0<\varepsilon\le 1$ *we have*

$$\left\|\mathcal{B}_\lambda(\varepsilon)^{-1}-\mathcal{B}_\lambda^0(\varepsilon)^{-1}-\widetilde{\mathcal{K}}(\varepsilon)\right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le \mathcal{C}_2.$$

The constant $\mathcal{C}_2=\mathcal{C}_1+(\check{c}_*)^{-1}r_0^{-2}+\widehat{\mathcal{C}}(\check{c}_*)^{-2}r_0^{-1}$ *depends only on the initial data* (3.27).

The first two terms of the corrector involve the oscillating factors Λ and $\widetilde{\Lambda}$. We are able to replace Π by I in these terms only under additional assumptions. In some cases, these assumptions are fulfilled automatically.

In the terms containing Λ, we can replace Π by I if the following condition is satisfied.

Condition 6.3. *Let* $\Lambda(\mathbf{x})$ *be the* Γ*-periodic solution of problem* (4.10)*. Suppose that the operator* $[\Lambda]$ *of multiplication by the matrix-valued function* $\Lambda(\mathbf{x})$ *acts continuously from* $H^1(\mathbb{R}^d;\mathbb{C}^m)$ *to* $L_2(\mathbb{R}^d;\mathbb{C}^n)$*.*

Proposition 6.4. *Under Condition* 6.3*, we have*

$$\left\|\Lambda b(\mathbf{D})\mathcal{B}_\lambda^0(\varepsilon)^{-1}(I-\Pi)\right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le \mathcal{C}_\Lambda, \quad 0<\varepsilon\le 1,$$

where the constant $\mathcal{C}_\Lambda=\alpha_1^{1/2}(\check{c}_*)^{-1}(1+r_0^{-2})^{1/2}\|[\Lambda]\|_{H^1(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}$ *depends on the initial data* (3.27) *and the* $(H^1\to L_2)$*-norm of the operator* $[\Lambda]$*.*

Proof. From (3.2), (5.7), and (6.6) it follows that

$$\begin{aligned}&\left\|b(\mathbf{D})\mathcal{B}^0_\lambda(\varepsilon)^{-1}(I-\Pi)\right\|_{L_2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)}\\ &\quad=\sup_{\boldsymbol{\xi}\in\mathbb{R}^d}(1+|\boldsymbol{\xi}|^2)^{1/2}|b(\boldsymbol{\xi})L(\boldsymbol{\xi},\varepsilon)^{-1}|(1-\chi_{\widetilde{\Omega}}(\boldsymbol{\xi}))\\ &\quad\le\alpha_1^{1/2}(\check{c}_*)^{-1}\sup_{|\boldsymbol{\xi}|\ge r_0}(1+|\boldsymbol{\xi}|^2)^{1/2}|\boldsymbol{\xi}|(|\boldsymbol{\xi}|^2+\varepsilon^2)^{-1}\le\alpha_1^{1/2}(\check{c}_*)^{-1}(1+r_0^{-2})^{1/2}.\end{aligned}$$

Using Condition 6.3, we arrive at the required inequality. □

In the following statement proved in [BSu3, §6], we distinguish some cases where Condition 6.3 is satisfied *a fortiori*.

Proposition 6.5. *Suppose that at least one of the following three assumptions is fulfilled*:

1) $d\le 4$;

2) *formulas* (4.19) *are valid* (*or, equivalently,* $g^0=\underline{g}$);

3) *the operator* $\mathcal{A}$ *acts in* $L_2(\mathbb{R}^d)$ *and is given by* $\mathcal{A}=\mathbf{D}^*g(\mathbf{x})\mathbf{D}$*, where* $g(\mathbf{x})$ *is a* Γ*-periodic matrix-valued function with real entries such that* $g,g^{-1}\in L_\infty$ *and* $g(\mathbf{x})>0$*. Then Condition* 6.3 *is satisfied, and the norm* $\|[\Lambda]\|_{H^1\to L_2}$ *depends only on* d*,* m*,* n*,* α_0*,* α_1*,* $\|g\|_{L_\infty}$*,* $\|g^{-1}\|_{L_\infty}$*, and the lattice* Γ.

As to the terms containing $\widetilde{\Lambda}$, here we can replace Π by I under the following assumption.

Condition 6.6. *Let* $\widetilde{\Lambda}(\mathbf{x})$ *be the* Γ*-periodic solution of problem* (4.12)*. Suppose that the operator* $[\widetilde{\Lambda}]$ *of multiplication by the matrix-valued function* $\widetilde{\Lambda}(\mathbf{x})$ *acts continuously from* $H^2(\mathbb{R}^d;\mathbb{C}^n)$ *to* $L_2(\mathbb{R}^d;\mathbb{C}^n)$.

Proposition 6.7. *Suppose that Condition* 6.6 *is satisfied. Then*

$$\left\|\varepsilon\widetilde{\Lambda}\mathcal{B}^0_\lambda(\varepsilon)^{-1}(I-\Pi)\right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\le\mathcal{C}_{\widetilde{\Lambda}},\quad 0<\varepsilon\le 1.\tag{6.12}$$

The constant $\mathcal{C}_{\widetilde{\Lambda}}=(\check{c}_*)^{-1}(1+r_0^{-2})\|[\widetilde{\Lambda}]\|_{H^2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}$ *depends on the initial data* (3.27) *and the* $(H^2\to L_2)$*-norm of the operator* $[\widetilde{\Lambda}]$.

Proof. From (5.7) and (6.6) it follows that

$$\begin{aligned}\left\|\mathcal{B}^0_\lambda(\varepsilon)^{-1}(I-\Pi)\right\|_{L_2(\mathbb{R}^d)\to H^2(\mathbb{R}^d)}&=\sup_{\boldsymbol{\xi}\in\mathbb{R}^d}(1+|\boldsymbol{\xi}|^2)|L(\boldsymbol{\xi},\varepsilon)^{-1}|(1-\chi_{\widetilde{\Omega}}(\boldsymbol{\xi}))\\ &\le(\check{c}_*)^{-1}\sup_{|\boldsymbol{\xi}|\ge r_0}(1+|\boldsymbol{\xi}|^2)(|\boldsymbol{\xi}|^2+\varepsilon^2)^{-1}\le(\check{c}_*)^{-1}(1+r_0^{-2}).\end{aligned}$$

Using Condition 6.6, we obtain (6.12). □

Now we distinguish some cases where Condition 6.6 is true *a fortiori*.

Lemma 6.8. *Let* $\Xi(\mathbf{x})$ *be a* Γ*-periodic* $(n\times n)$*-matrix-valued function in* $\mathbb{R}^d$ *such that* $\Xi\in L_p(\Omega)$*, where* $p=2$ *for* $d\le 3$*;* $p>2$ *for* $d=4$*;* $p=\frac{d}{2}$ *for* $d\ge 5$*. Then the operator* $[\Xi]$ *of multiplication by the matrix-valued function* $\Xi(\mathbf{x})$ *acts continuously from* $H^2(\mathbb{R}^d;\mathbb{C}^n)$ *to* $L_2(\mathbb{R}^d;\mathbb{C}^n)$*, and*

$$\|[\Xi]\|_{H^2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\le c\|\Xi\|_{L_p(\Omega)},\tag{6.13}$$

where $c=c(d,n,\Omega)$ *for* $d\ne 4$ *and* $c=c(p,n,\Omega)$ *for* $d=4$.

Proof. Let $\mathbf{a}\in\Gamma$ and $\mathbf{v}\in H^2(\mathbb{R}^d;\mathbb{C}^n)$. Then

$$\int_{\Omega+\mathbf{a}}|\Xi(\mathbf{x})\mathbf{v}(\mathbf{x})|^2\,d\mathbf{x}\le\|\Xi\|^2_{L_p(\Omega)}\|\mathbf{v}\|^2_{L_r(\Omega+\mathbf{a})},\tag{6.14}$$

where $p=2$ and $r=\infty$ for $d\le 3$; $p>2$ and $r=2p(p-2)^{-1}<\infty$ for $d=4$; $p=d/2$ and $r=2d(d-4)^{-1}$ for $d\ge 5$.

By the continuous embedding $H^2(\Omega)\subset L_r(\Omega)$, we have

$$\|\mathbf{v}\|_{L_r(\Omega+\mathbf{a})}\le c\|\mathbf{v}\|_{H^2(\Omega+\mathbf{a})}. \tag{6.15}$$

Substituting (6.15) in (6.14) and summing over $\mathbf{a}\in\Gamma$, we obtain

$$\int_{\mathbb{R}^d}|\Xi(\mathbf{x})\mathbf{v}(\mathbf{x})|^2\,d\mathbf{x}\le c^2\|\Xi\|^2_{L_p(\Omega)}\|\mathbf{v}\|^2_{H^2(\mathbb{R}^d)}. \qquad \square$$

Proposition 6.9. *Let $d\le 6$. Then Condition 6.6 is satisfied, and the norm $\|[\widetilde{\Lambda}]\|_{H^2\to L_2}$ is estimated by a constant depending only on m, n, d, α_0, $\|g^{-1}\|_{L_\infty}$, $\|a_j\|_{L_2(\Omega)}$, $j=1,\dots,d$, and the lattice Γ.*

Proof. The columns $\tilde{\mathbf{v}}_l(\mathbf{x})$, $l=1,\dots,n$, of the matrix $\widetilde{\Lambda}(\mathbf{x})$ belong to $\widetilde{H}^1(\Omega;\mathbb{C}^n)$, whence $\widetilde{\Lambda}\in L_\infty$ for $d=1$, $\widetilde{\Lambda}\in L_s(\Omega)$ with any $s<\infty$ for $d=2$, and $\widetilde{\Lambda}\in L_{2d/(d-2)}(\Omega)$ for $d\ge 3$. Thus, $\widetilde{\Lambda}$ satisfies the assumptions of Lemma 6.8 for $d\le 6$. We have $\|\widetilde{\Lambda}\|_{L_p(\Omega)}\le \breve{c}\|\widetilde{\Lambda}\|_{H^1(\Omega)}$, where $\breve{c}=\breve{c}(d,m,n,\Omega)$ for $d\ne 4$, $\breve{c}=\breve{c}(p,m,n,\Omega)$ for $d=4$. Then, taking (6.13) into account, we have

$$\|[\widetilde{\Lambda}]\|_{H^2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\le c\breve{c}\|\widetilde{\Lambda}\|_{H^1(\Omega)}. \tag{6.16}$$

In [Su2, (7.51), (7.52)], it was proved that

$$\|\widetilde{\Lambda}\|_{H^1(\Omega)}\le n^{1/2}(1+1/4r_0^2)^{1/2}\alpha_0^{-1}\|g^{-1}\|_{L_\infty}\left(\sum_{j=1}^d\|a_j\|^2_{L_2(\Omega)}\right)^{1/2}.$$

Combined with (6.16), this completes the proof. $\square$

Another case where Condition 6.6 is fulfilled is given by the following proposition.

Proposition 6.10. *Suppose $n=1$, $m=d$, and $\mathcal{A}=\mathbf{D}^*g(\mathbf{x})\mathbf{D}$, where $g(\mathbf{x})$ is a Γ-periodic matrix-valued function with real entries such that $g,g^{-1}\in L_\infty$ and $g(\mathbf{x})>0$. Then Condition 6.6 is satisfied, and the norm $\|[\widetilde{\Lambda}]\|_{H^2\to L_2}$ is estimated by a constant depending only on d, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|a_j\|_{L_\rho(\Omega)}$, $j=1,\dots,d$, and the lattice Γ.*

Proof. From [LaU, Chapter III, Theorem 13.1] it follows that under our assumptions the periodic solution of the problem (4.12) is bounded, i.e., $\widetilde{\Lambda}\in L_\infty$. The norm $\|\widetilde{\Lambda}\|_{L_\infty}$ is estimated by a constant depending only on d, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|a_j\|_{L_\rho(\Omega)}$, $j=1,\dots,d$, and the domain Ω. It remains to apply Lemma 6.8. $\square$

We summarize. Theorem 6.2 and Propositions 6.4, 6.7 imply the following result.

Theorem 6.11. *Suppose that the assumptions of Theorem 6.1 are satisfied.*

1°. *Assume Condition 6.3. Denote*

$$\mathcal{K}'(\varepsilon):=(\Lambda b(\mathbf{D})+\varepsilon\widetilde{\Lambda}\Pi)\mathcal{B}^0_\lambda(\varepsilon)^{-1}+\mathcal{B}^0_\lambda(\varepsilon)^{-1}(b(\mathbf{D})^*\Lambda^*+\varepsilon\Pi\widetilde{\Lambda}^*)-\mathcal{B}^0_\lambda(\varepsilon)^{-1}\mathcal{N}(\varepsilon)\mathcal{B}^0_\lambda(\varepsilon)^{-1}.$$

Then for $0<\varepsilon\le 1$ we have

$$\left\|\mathcal{B}_\lambda(\varepsilon)^{-1}-\mathcal{B}^0_\lambda(\varepsilon)^{-1}-\mathcal{K}'(\varepsilon)\right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\le\mathcal{C}', \tag{6.17}$$

where the constant $\mathcal{C}'=\mathcal{C}_2+2\mathcal{C}_\Lambda$ depends only on the initial data (3.27) and the norm $\|[\Lambda]\|_{H^1\to L_2}$.

2°. *Assume Condition 6.6. Denote*

$$\mathcal{K}''(\varepsilon):=(\Lambda b(\mathbf{D})\Pi+\varepsilon\widetilde{\Lambda})\mathcal{B}^0_\lambda(\varepsilon)^{-1}+\mathcal{B}^0_\lambda(\varepsilon)^{-1}(\Pi b(\mathbf{D})^*\Lambda^*+\varepsilon\widetilde{\Lambda}^*)-\mathcal{B}^0_\lambda(\varepsilon)^{-1}\mathcal{N}(\varepsilon)\mathcal{B}^0_\lambda(\varepsilon)^{-1}.$$

Then for $0 < \varepsilon \leq 1$ *we have*

$$\left\| \mathcal{B}_\lambda(\varepsilon)^{-1} - \mathcal{B}_\lambda^0(\varepsilon)^{-1} - \mathcal{K}''(\varepsilon) \right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leq \mathcal{C}'', \tag{6.18}$$

where the constant $\mathcal{C}'' = \mathcal{C}_2 + 2\mathcal{C}_{\widetilde{\Lambda}}$ *depends only on the initial data* (3.27) *and the norm* $\|[\widetilde{\Lambda}]\|_{H^2\to L_2}$.

3°. *Assume Conditions* 6.3 *and* 6.6. *Denote*

$$\mathcal{K}^0(\varepsilon) := (\Lambda b(\mathbf{D}) + \varepsilon\widetilde{\Lambda})\mathcal{B}_\lambda^0(\varepsilon)^{-1} + \mathcal{B}_\lambda^0(\varepsilon)^{-1}(b(\mathbf{D})^*\Lambda^* + \varepsilon\widetilde{\Lambda}^*) - \mathcal{B}_\lambda^0(\varepsilon)^{-1}\mathcal{N}(\varepsilon)\mathcal{B}_\lambda^0(\varepsilon)^{-1}.$$

Then for $0 < \varepsilon \leq 1$ *we have*

$$\left\| \mathcal{B}_\lambda(\varepsilon)^{-1} - \mathcal{B}_\lambda^0(\varepsilon)^{-1} - \mathcal{K}^0(\varepsilon) \right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leq \mathcal{C}^0, \tag{6.19}$$

where the constant $\mathcal{C}^0 = \mathcal{C}_2 + 2\mathcal{C}_\Lambda + 2\mathcal{C}_{\widetilde{\Lambda}}$ *depends only on the initial data* (3.27) *and the norms* $\|[\Lambda]\|_{H^1\to L_2}$, $\|[\widetilde{\Lambda}]\|_{H^2\to L_2}$.

Using Propositions 6.5, 6.9, and 6.10, we distinguish some cases where the assumptions of one of the items of Theorem 6.11 are true *a fortiori*.

Proposition 6.12. 1°. *Let* $d \leq 4$. *Then estimate* (6.19) *is valid with a constant* $\mathcal{C}^0$ *depending only on the initial data* (3.27).

2°. *Let* $d \leq 6$. *Then estimate* (6.18) *is valid with a constant* $\mathcal{C}''$ *depending only on the initial data* (3.27).

3°. *Suppose that we have the representations* (4.19), *i.e.,* $g^0 = \underline{g}$. *Then estimate* (6.17) *is true with a constant* $\mathcal{C}'$ *depending only on the initial data* (3.27).

4°. *Suppose* $n = 1$, $m = d$, *and* $\mathcal{A} = \mathbf{D}^* g(\mathbf{x})\mathbf{D}$, *where* $g(\mathbf{x})$ *is a* Γ*-periodic matrix-valued function with real entries such that* $g, g^{-1} \in L_\infty$, $g(\mathbf{x}) > 0$. *Then estimate* (6.19) *is valid with a constant* $\mathcal{C}^0$ *depending only on the initial data* (3.27).

6.3. Special cases. Assume that conditions (4.18) are satisfied (i.e., $g^0 = \overline{g}$). Then the solution of problem (4.10) is equal to zero: $\Lambda(\mathbf{x}) = 0$. Hence, by (5.2), $V = 0$. The effective operator (6.1) takes the form

$$\mathcal{B}_\lambda^0(\varepsilon) = b(\mathbf{D})^* g^0 b(\mathbf{D}) + \varepsilon \sum_{j=1}^d (\overline{a_j + a_j^*}) D_j + \varepsilon^2(-W + \overline{Q} + \lambda\overline{Q_0}). \tag{6.20}$$

The expression for the corrector (6.11) simplifies. By (5.11), (5.17), (5.19), (5.28), and (6.2)–(6.5), the expression for the operator $\mathcal{N}(\varepsilon)$ also simplifies. We arrive at the following statement.

Proposition 6.13. *Suppose that conditions* (4.18) *are satisfied* (*i.e.,* $g^0 = \overline{g}$). *Then the effective operator* $\mathcal{B}_\lambda^0(\varepsilon)$ *is given by* (6.20). *The corrector* (6.11) *takes the form*

$$\widetilde{\mathcal{K}}(\varepsilon) = \varepsilon\big(\widetilde{\Lambda}\mathcal{B}_\lambda^0(\varepsilon)^{-1}\Pi + \mathcal{B}_\lambda^0(\varepsilon)^{-1}\Pi\widetilde{\Lambda}^*\big) - \mathcal{B}_\lambda^0(\varepsilon)^{-1}\mathcal{N}(\varepsilon)\mathcal{B}_\lambda^0(\varepsilon)^{-1},$$

where

$$\begin{aligned} \mathcal{N}(\varepsilon) &= \varepsilon\mathcal{N}_{12}(\mathbf{D}) + \varepsilon^2\mathcal{N}_{21}(\mathbf{D}) + \varepsilon^3\mathcal{N}_{22}, \\ \mathcal{N}_{12}(\mathbf{D}) &= \widetilde{M}_1(\mathbf{D})b(\mathbf{D}) + b(\mathbf{D})^*\widetilde{M}_1(\mathbf{D})^*, \\ \mathcal{N}_{21}(\mathbf{D}) &= M_2(\mathbf{D}) + M_2(\mathbf{D})^* + 2\sum_{j=1}^d \operatorname{Re}\overline{(a_j + a_j^*)\widetilde{\Lambda}} D_j. \end{aligned} \tag{6.21}$$

Here $\widetilde{M}_1(\mathbf{D})$ *and* $M_2(\mathbf{D})$ *are first order DOs with the symbols* $\widetilde{M}_1(\mathbf{k})$ *and* $M_2(\mathbf{k})$ *defined as in* (5.14), (5.22), *and the matrix* $\mathcal{N}_{22}$ *is defined by* (5.31), (5.36).

Now we consider the case where the representations (4.19) hold true. Then $g^0 = \underline{g}$ and, as was mentioned in [BSu3, Remark 3.5], $\widetilde{g}(\mathbf{x}) = g^0 = \underline{g}$. In this case, the expression for the operator $\mathcal{N}(\varepsilon)$ simplifies because $M(\mathbf{k}) = 0$, by (5.11) and the condition $\overline{\Lambda} = 0$. Therefore, $\mathcal{N}_{11}(\mathbf{D}) = 0$ (see (6.3)). Moreover, from (5.14) and the condition $\overline{\widetilde{\Lambda}} = 0$ it follows that $\widetilde{M}_1(\mathbf{k}) = 0$, whence expression (5.19) takes the form

$$M_1(\mathbf{k}) = \overline{(b(\mathbf{D})\widetilde{\Lambda})^* g b(\mathbf{k})\Lambda} + \sum_{j=1}^{d} \overline{(a_j + a_j^*)\Lambda} k_j. \tag{6.22}$$

Proposition 6.14. *Suppose that conditions* (4.19) *are satisfied* (*i.e.*, $g^0 = \underline{g}$). *Then the operator* (6.2) *takes the form* $\mathcal{N}(\varepsilon) = \varepsilon\mathcal{N}_{12}(\mathbf{D}) + \varepsilon^2\mathcal{N}_{21}(\mathbf{D}) + \varepsilon^3\mathcal{N}_{22}$, *where the operator* $\mathcal{N}_{12}(\mathbf{D})$ *is defined by* (6.4), $M_1(\mathbf{D})$ *is the DO with the symbol* (6.22), *and the operators* $\mathcal{N}_{21}(\mathbf{D})$ *and* $\mathcal{N}_{22}$ *are defined by* (6.5) *and* (5.31), (5.36), *respectively.*

Now we consider the case where

$$\sum_{j=1}^{d} D_j a_j(\mathbf{x})^* = 0. \tag{6.23}$$

Then the periodic solution of problem (4.12) is equal to zero: $\widetilde{\Lambda}(\mathbf{x}) = 0$. Hence, by (5.2) and (5.3), we have $V = 0$ and $W = 0$. The effective operator (6.1) takes the form

$$\mathcal{B}_\lambda^0(\varepsilon) = b(\mathbf{D})^* g^0 b(\mathbf{D}) + \varepsilon \sum_{j=1}^{d} (\overline{a_j + a_j^*}) D_j + \varepsilon^2 (\overline{Q} + \lambda \overline{Q_0}). \tag{6.24}$$

The expressions for the corrector (6.11) and the operator $\mathcal{N}(\varepsilon)$ also simplify (by virtue of (5.19), (5.22), (5.28), (5.31), (5.36), and (6.2)–(6.5)).

Proposition 6.15. *Under condition* (6.23), *the effective operator is of the form* (6.24), *and the corrector* (6.11) *is given by*

$$\widetilde{\mathcal{K}}(\varepsilon) = \Lambda b(\mathbf{D}) \mathcal{B}_\lambda^0(\varepsilon)^{-1}\Pi + \mathcal{B}_\lambda^0(\varepsilon)^{-1}\Pi b(\mathbf{D})^*\Lambda^* - \mathcal{B}_\lambda^0(\varepsilon)^{-1}\mathcal{N}(\varepsilon)\mathcal{B}_\lambda^0(\varepsilon)^{-1},$$

where

$$\mathcal{N}(\varepsilon) = \mathcal{N}_{11}(\mathbf{D}) + \varepsilon\mathcal{N}_{12}(\mathbf{D}) + \varepsilon^2\mathcal{N}_{21}(\mathbf{D}).$$

Here, the operator $\mathcal{N}_{11}(\mathbf{D})$ *is defined by* (6.3); $\mathcal{N}_{12}(\mathbf{D})$ *is given by the expression* (6.4) *in which* $M_1(\mathbf{D})$ *is the DO with the symbol*

$$M_1(\mathbf{k}) = \sum_{j=1}^{d} \overline{(a_j + a_j^*)\Lambda} k_j, \tag{6.25}$$

and the operator $\mathcal{N}_{21}(\mathbf{D})$ *is given by*

$$\mathcal{N}_{21}(\mathbf{D}) = T^* b(\mathbf{D}) + b(\mathbf{D})^* T, \quad \text{and } T = \overline{\Lambda^* Q} + \lambda \overline{\Lambda^* Q_0}. \tag{6.26}$$

The case where the corrector is equal to zero is distinguished by conditions (4.18) and (6.23). Then $\Lambda = 0$ and $\widetilde{\Lambda} = 0$. In this case we also have $\mathcal{N}(\varepsilon) = 0$.

Proposition 6.16. *Suppose that conditions* (4.18) *and* (6.23) *are satisfied. Then the effective operator is of the form* (6.24), *and the corrector* (6.11) *is equal to zero. We have*

$$\left\| \mathcal{B}_\lambda(\varepsilon)^{-1} - \mathcal{B}_\lambda^0(\varepsilon)^{-1} \right\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \le \mathcal{C}_2, \quad 0 < \varepsilon \le 1.$$

The constant $\mathcal{C}_2$ *depends only on the initial data* (3.27).

Chapter 2
Homogenization of periodic differential operators

§7. Approximation of the generalized resolvent of the operator $\mathcal{B}_\varepsilon$

7.1. Statement of the problem. For any Γ-periodic function $\phi(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^d$, we denote $\phi^\varepsilon(\mathbf{x}) := \phi(\varepsilon^{-1}\mathbf{x})$, $\varepsilon > 0$. In $L_2(\mathbb{R}^d; \mathbb{C}^n)$, consider the operator

$$\mathcal{A}_\varepsilon = b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D}) \tag{7.1}$$

generated by the quadratic form

$$\mathfrak{a}_\varepsilon[\mathbf{u},\mathbf{u}] = \int_{\mathbb{R}^d} \langle g^\varepsilon(\mathbf{x}) b(\mathbf{D})\mathbf{u}, b(\mathbf{D})\mathbf{u}\rangle, \quad \mathbf{u} \in H^1(\mathbb{R}^d;\mathbb{C}^n). \tag{7.2}$$

The form (7.2) satisfies the following estimates similar to (3.6):

$$\alpha_0 \|g^{-1}\|^{-1}_{L_\infty} \|\mathbf{D}\mathbf{u}\|^2_{L_2(\mathbb{R}^d)} \le \mathfrak{a}_\varepsilon[\mathbf{u},\mathbf{u}] \le \alpha_1 \|g\|_{L_\infty} \|\mathbf{D}\mathbf{u}\|^2_{L_2(\mathbb{R}^d)}, \quad \mathbf{u} \in H^1(\mathbb{R}^d;\mathbb{C}^n). \tag{7.3}$$

Next, let $\mathcal{Y}\colon L_2(\mathbb{R}^d;\mathbb{C}^n) \to L_2(\mathbb{R}^d;\mathbb{C}^{dn})$ be the operator defined by (3.7), and let

$$\mathcal{Y}_{2,\varepsilon}\colon L_2(\mathbb{R}^d;\mathbb{C}^n) \to L_2(\mathbb{R}^d;\mathbb{C}^{dn})$$

be the operator of multiplication by the $(dn \times d)$-matrix consisting of the blocks $(a_j^\varepsilon(\mathbf{x}))^*$, $j = 1, \dots, d$, i.e.,

$$\mathcal{Y}_{2,\varepsilon}\mathbf{u} = \operatorname{col}\{(a_1^\varepsilon(\mathbf{x}))^*\mathbf{u}, \dots, (a_d^\varepsilon(\mathbf{x}))^*\mathbf{u}\}, \quad \mathbf{u} \in H^1(\mathbb{R}^d;\mathbb{C}^n).$$

Let $d\mu$ be a matrix-valued measure on $\mathbb{R}^d$, as defined in Subsection 3.3. We define the measure $d\mu^\varepsilon$ as follows. For any Borel set $\Delta \subset \mathbb{R}^d$, consider the set $\varepsilon^{-1}\Delta = \{\mathbf{y} = \varepsilon^{-1}\mathbf{x} : \mathbf{x} \in \Delta\}$ and put $\mu^\varepsilon(\Delta) = \varepsilon^d \mu(\varepsilon^{-1}\Delta)$. Consider the quadratic form q_ε defined by

$$q_\varepsilon[\mathbf{u},\mathbf{u}] = \int_{\mathbb{R}^d} \langle d\mu^\varepsilon(\mathbf{x})\mathbf{u}, \mathbf{u}\rangle, \quad \mathbf{u} \in H^1(\mathbb{R}^d;\mathbb{C}^n). \tag{7.4}$$

If q is the form given in Example 3.3, then $d\mu^\varepsilon(\mathbf{x}) = Q^\varepsilon(\mathbf{x})d\mathbf{x}$ and

$$q_\varepsilon[\mathbf{u},\mathbf{u}] = \int_{\mathbb{R}^d} \langle Q^\varepsilon(\mathbf{x})\mathbf{u}, \mathbf{u}\rangle\, d\mathbf{x}, \quad \mathbf{u} \in H^1(\mathbb{R}^d;\mathbb{C}^n).$$

Suppose that all the assumptions of Subsections 3.1–3.3 are satisfied. In $L_2(\mathbb{R}^d;\mathbb{C}^n)$, we consider the quadratic form

$$\mathfrak{b}_\varepsilon[\mathbf{u},\mathbf{u}] = \mathfrak{a}_\varepsilon[\mathbf{u},\mathbf{u}] + 2\operatorname{Re}(\mathcal{Y}\mathbf{u}, \mathcal{Y}_{2,\varepsilon}\mathbf{u})_{L_2(\mathbb{R}^d)} + q_\varepsilon[\mathbf{u},\mathbf{u}], \quad \mathbf{u} \in H^1(\mathbb{R}^d;\mathbb{C}^n). \tag{7.5}$$

Let T_ε be the unitary scaling transformation in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ defined by

$$(T_\varepsilon \mathbf{u})(\mathbf{y}) = \varepsilon^{d/2}\mathbf{u}(\varepsilon\mathbf{y}). \tag{7.6}$$

Let $\mathfrak{a}$ and $\mathfrak{b}(\varepsilon)$ be the quadratic forms defined by (3.5) and (3.18), respectively. Obviously, we have

$$\mathfrak{a}_\varepsilon[\mathbf{u},\mathbf{u}] = \varepsilon^{-2}\mathfrak{a}[T_\varepsilon\mathbf{u}, T_\varepsilon\mathbf{u}], \quad \mathbf{u} \in H^1(\mathbb{R}^d;\mathbb{C}^n),$$
$$\mathfrak{b}_\varepsilon[\mathbf{u},\mathbf{u}] = \varepsilon^{-2}\mathfrak{b}(\varepsilon)[T_\varepsilon\mathbf{u}, T_\varepsilon\mathbf{u}], \quad \mathbf{u} \in H^1(\mathbb{R}^d;\mathbb{C}^n).$$

Combining this with estimates (3.19) and (3.20), we obtain

$$\mathfrak{b}_\varepsilon[\mathbf{u},\mathbf{u}] \le (2 + c_1^2 + c_2)\mathfrak{a}_\varepsilon[\mathbf{u},\mathbf{u}] + (C(1) + c_3)\|\mathbf{u}\|^2_{L_2(\mathbb{R}^d)}, \quad \mathbf{u} \in H^1(\mathbb{R}^d;\mathbb{C}^n), \tag{7.7}$$

$$\mathfrak{b}_\varepsilon[\mathbf{u},\mathbf{u}] \ge \frac{\kappa}{2}\mathfrak{a}_\varepsilon[\mathbf{u},\mathbf{u}] - (c_0 + c_4)\|\mathbf{u}\|^2_{L_2(\mathbb{R}^d)}, \quad \mathbf{u} \in H^1(\mathbb{R}^d;\mathbb{C}^n). \tag{7.8}$$

From (7.3), (7.7), and (7.8) it follows that the form (7.5) is closed and lower semibounded. *The selfadjoint operator $\mathcal{B}_\varepsilon$ generated in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ by this form is our main object in Chapter* 2. Formally, we can write

$$\mathcal{B}_\varepsilon = b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D}) + \sum_{j=1}^d \left(a_j^\varepsilon(\mathbf{x}) D_j + D_j (a_j^\varepsilon(\mathbf{x}))^* \right) + Q^\varepsilon(\mathbf{x}), \tag{7.9}$$

where $Q^\varepsilon(\mathbf{x})$ is interpreted as the generalized matrix potential generated by the measure $d\mu^\varepsilon$. The coefficients of the operator (7.9) oscillate rapidly as $\varepsilon \to 0$.

The homogenization problem for the operator (7.9) *consists of approximating the resolvent* $(\mathcal{B}_\varepsilon + \lambda I)^{-1}$ *or the generalized resolvent* $(\mathcal{B}_\varepsilon + \lambda Q_0^\varepsilon)^{-1}$ *as* $\varepsilon \to 0$. Here Q_0^ε is the operator in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ acting as multiplication by the matrix-valued function $Q_0^\varepsilon(\mathbf{x})$. The matrix $Q_0(\mathbf{x})$ and the parameter λ are subject to the restrictions described in Subsection 3.4 (see (3.24)). We denote

$$\mathfrak{b}_{\lambda,\varepsilon}[\mathbf{u},\mathbf{u}] := \mathfrak{b}_\varepsilon[\mathbf{u},\mathbf{u}] + \lambda (Q_0^\varepsilon \mathbf{u}, \mathbf{u})_{L_2(\mathbb{R}^d)}, \quad \mathbf{u} \in H^1(\mathbb{R}^d;\mathbb{C}^n). \tag{7.10}$$

Then from (7.8), (3.24), and (3.26) it follows that

$$\mathfrak{b}_{\lambda,\varepsilon}[\mathbf{u},\mathbf{u}] \geq \frac{\kappa}{2} \mathfrak{a}_\varepsilon[\mathbf{u},\mathbf{u}] + \beta \|\mathbf{u}\|^2_{L_2(\mathbb{R}^d)}, \quad \mathbf{u} \in H^1(\mathbb{R}^d;\mathbb{C}^n).$$

Thus, the form (7.10) and the corresponding operator

$$\mathcal{B}_{\lambda,\varepsilon} = \mathcal{B}_\varepsilon + \lambda Q_0^\varepsilon \tag{7.11}$$

are positive definite.

7.2. The leading term of approximation for the operator $\mathcal{B}_{\lambda,\varepsilon}^{-1}$ was found in [Su2, Subsection 9.2]. In order to formulate the result, we should introduce the *effective operator* $\mathcal{B}_\lambda^0$. Let g^0 be the effective matrix defined in Subsection 4.4. Let V, W, and $\overline{Q}$ be the matrices defined by (5.2), (5.3), and (5.4), respectively. The effective operator for the operator (7.11) is given by

$$\mathcal{B}_\lambda^0 := b(\mathbf{D})^* g^0 b(\mathbf{D}) - b(\mathbf{D})^* V - V^* b(\mathbf{D}) + \sum_{j=1}^d \overline{(a_j + a_j^*)} D_j - W + \overline{Q} + \lambda \overline{Q_0}. \tag{7.12}$$

The operator (7.12) is an elliptic DO with constant coefficients; its symbol is the matrix $L(\boldsymbol{\xi}, 1)$ (see (5.6) with $\varepsilon = 1$). Then estimate (5.7) implies that the operator (7.12) is positive definite: $\mathcal{B}_\lambda^0 \geq \check{c}_* I$.

The following result was obtained in [Su2, Theorem 9.2].

Theorem 7.1. *Under the assumptions of Subsections* 3.1–3.4*, let $\mathcal{B}_{\lambda,\varepsilon}$ be the operator defined in Subsection* 7.1*. Let $\mathcal{B}_\lambda^0$ be the effective operator* (7.12)*. Then*

$$\left\| \mathcal{B}_{\lambda,\varepsilon}^{-1} - (\mathcal{B}_\lambda^0)^{-1} \right\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C_0 \varepsilon, \quad 0 < \varepsilon \leq 1. \tag{7.13}$$

The constant C_0 depends only on the initial data (3.27)*.*

7.3. Approximation with corrector for the operator $\mathcal{B}_{\lambda,\varepsilon}^{-1}$. Now we obtain a sharper approximation for the operator $\mathcal{B}_{\lambda,\varepsilon}^{-1}$, taking a corrector into account. For this, we use theorems from §6 and apply the scaling transformation. Let T_ε be the transformation defined by (7.6). The operators (7.11) and (3.23) obey the following identity:

$$\mathcal{B}_{\lambda,\varepsilon} = \varepsilon^{-2} T_\varepsilon^* \mathcal{B}_\lambda(\varepsilon) T_\varepsilon. \tag{7.14}$$

Similarly, for the operators (7.12) and (6.1) we have

$$\mathcal{B}_\lambda^0 = \varepsilon^{-2} T_\varepsilon^* \mathcal{B}_\lambda^0(\varepsilon) T_\varepsilon. \tag{7.15}$$

Next, let Π_ε denote the pseudodifferential operator in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ whose symbol is the characteristic function $\chi_{\widetilde{\Omega}/\varepsilon}(\boldsymbol{\xi})$ of the set $\widetilde{\Omega}/\varepsilon$:

$$(\Pi_\varepsilon \mathbf{f})(\mathbf{x}) = (2\pi)^{-d/2} \int_{\widetilde{\Omega}/\varepsilon} e^{i\langle \mathbf{x},\boldsymbol{\xi}\rangle} \widehat{\mathbf{f}}(\boldsymbol{\xi})\, d\boldsymbol{\xi}.$$

The operator Π_ε is related to the operator (6.6) by the identity

$$\Pi_\varepsilon = T_\varepsilon^* \Pi T_\varepsilon. \tag{7.16}$$

The corrector for the operator $\mathcal{B}_{\lambda,\varepsilon}^{-1}$ is introduced by

$$K_\varepsilon := \bigl(\Lambda^\varepsilon b(\mathbf{D}) + \widetilde{\Lambda}^\varepsilon\bigr)(\mathcal{B}_\lambda^0)^{-1}\Pi_\varepsilon + (\mathcal{B}_\lambda^0)^{-1}\Pi_\varepsilon\bigl(b(\mathbf{D})^*(\Lambda^\varepsilon)^* + (\widetilde{\Lambda}^\varepsilon)^*\bigr) - (\mathcal{B}_\lambda^0)^{-1}\mathcal{N}(\mathcal{B}_\lambda^0)^{-1}, \tag{7.17}$$

where

$$\mathcal{N} := \mathcal{N}_{11}(\mathbf{D}) + \mathcal{N}_{12}(\mathbf{D}) + \mathcal{N}_{21}(\mathbf{D}) + \mathcal{N}_{22}, \tag{7.18}$$

and the terms on the right-hand side of (7.18) are defined by (6.3), (6.4), (6.5), and (5.36), respectively. The operator $\mathcal{N}$ is a third order selfadjoint matrix DO with constant coefficients.

We mention the identities

$$[\Lambda^\varepsilon]b(\mathbf{D}) = \varepsilon^{-1} T_\varepsilon^*[\Lambda]b(\mathbf{D})T_\varepsilon, \quad [\widetilde{\Lambda}^\varepsilon] = T_\varepsilon^*[\widetilde{\Lambda}]T_\varepsilon, \quad \mathcal{N} = \varepsilon^{-3} T_\varepsilon^* \mathcal{N}(\varepsilon) T_\varepsilon, \tag{7.19}$$

where $\mathcal{N}(\varepsilon)$ is defined by (6.2). As a result, relations (7.15)–(7.17) and (7.19) imply that

$$K_\varepsilon = \varepsilon T_\varepsilon^* \widetilde{\mathcal{K}}(\varepsilon) T_\varepsilon, \tag{7.20}$$

where $\widetilde{\mathcal{K}}(\varepsilon)$ is the operator defined by (6.11).

Now, identities (7.14), (7.15), and (7.20) together with Theorem 6.2 directly imply our *main result.*

Theorem 7.2. *Under the assumptions of Theorem* 7.1, *let* K_ε *be the corrector defined by* (7.17). *Then*

$$\bigl\| \mathcal{B}_{\lambda,\varepsilon}^{-1} - (\mathcal{B}_\lambda^0)^{-1} - \varepsilon K_\varepsilon \bigr\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le \mathcal{C}_2 \varepsilon^2, \quad 0 < \varepsilon \le 1. \tag{7.21}$$

The constant $\mathcal{C}_2$ *depends only on the initial data* (3.27).

Estimate (7.21) is order sharp. The corrector (7.17) is a pseudodifferential operator of order -1, the first two terms of the corrector are mutually adjoint; they depend on ε, involving rapidly oscillating factors and the smoothing operator Π_ε. The third term of the corrector is a pseudodifferential operator with constant coefficients; it does not depend on ε.

Under some additional assumptions it is possible to get rid of Π_ε in some (sometimes, in all) terms of the corrector. The following theorem is deduced from Theorem 6.11 by the scaling transformation.

Theorem 7.3. *Suppose that the assumptions of Theorem* 7.1 *are satisfied.*

1°. *Suppose that Condition* 6.3 *is true. Denote*

$$K_\varepsilon' := \bigl(\Lambda^\varepsilon b(\mathbf{D}) + \widetilde{\Lambda}^\varepsilon \Pi_\varepsilon\bigr)(\mathcal{B}_\lambda^0)^{-1} + (\mathcal{B}_\lambda^0)^{-1}\bigl(b(\mathbf{D})^*(\Lambda^\varepsilon)^* + \Pi_\varepsilon(\widetilde{\Lambda}^\varepsilon)^*\bigr) - (\mathcal{B}_\lambda^0)^{-1}\mathcal{N}(\mathcal{B}_\lambda^0)^{-1}.$$

Then for $0 < \varepsilon \le 1$ *we have*

$$\bigl\| \mathcal{B}_{\lambda,\varepsilon}^{-1} - (\mathcal{B}_\lambda^0)^{-1} - \varepsilon K_\varepsilon' \bigr\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le \mathcal{C}' \varepsilon^2, \tag{7.22}$$

where $\mathcal{C}'$ *depends only on the initial data* (3.27) *and the norm* $\|[\Lambda]\|_{H^1\to L_2}$.

2°. *Suppose that Condition* 6.6 *is true. Denote*

$$K_\varepsilon'' := \bigl(\Lambda^\varepsilon b(\mathbf{D})\Pi_\varepsilon + \widetilde{\Lambda}^\varepsilon\bigr)(\mathcal{B}_\lambda^0)^{-1} + (\mathcal{B}_\lambda^0)^{-1}\bigl(\Pi_\varepsilon b(\mathbf{D})^*(\Lambda^\varepsilon)^* + (\widetilde{\Lambda}^\varepsilon)^*\bigr) - (\mathcal{B}_\lambda^0)^{-1}\mathcal{N}(\mathcal{B}_\lambda^0)^{-1}.$$

Then for $0 < \varepsilon \leq 1$ *we have*

$$\left\| \mathcal{B}_{\lambda,\varepsilon}^{-1} - (\mathcal{B}_\lambda^0)^{-1} - \varepsilon K_\varepsilon'' \right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leq \mathcal{C}''\varepsilon^2, \tag{7.23}$$

where $\mathcal{C}''$ *depends only on the initial data* (3.27) *and the norm* $\|[\widetilde{\Lambda}]\|_{H^2\to L_2}$.

3°. *Suppose that Conditions* 6.3 *and* 6.6 *are true. Denote*

$$K_\varepsilon^0 = \left(\Lambda^\varepsilon b(\mathbf{D}) + \widetilde{\Lambda}^\varepsilon\right)(\mathcal{B}_\lambda^0)^{-1} + (\mathcal{B}_\lambda^0)^{-1}(b(\mathbf{D})^*(\Lambda^\varepsilon)^* + (\widetilde{\Lambda}^\varepsilon)^*) - (\mathcal{B}_\lambda^0)^{-1}\mathcal{N}(\mathcal{B}_\lambda^0)^{-1}. \tag{7.24}$$

Then for $0 < \varepsilon \leq 1$ *we have*

$$\left\| \mathcal{B}_{\lambda,\varepsilon}^{-1} - (\mathcal{B}_\lambda^0)^{-1} - \varepsilon K_\varepsilon^0 \right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leq \mathcal{C}^0\varepsilon^2, \tag{7.25}$$

where the constant $\mathcal{C}^0$ *depends only on the initial data* (3.27) *and the norms* $\|[\Lambda]\|_{H^1\to L_2}$, $\|[\widetilde{\Lambda}]\|_{H^2\to L_2}$.

Using Propositions 6.5, 6.9, and 6.10, we distinguish some cases where the assumptions of one of the assertions of Theorem 7.3 are satisfied (cf. Proposition 6.12).

Proposition 7.4. 1°. *Let* $d \leq 4$. *Then estimate* (7.25) *is valid with a constant* $\mathcal{C}^0$ *depending only on the problem data* (3.27).

2°. *Let* $d \leq 6$. *Then estimate* (7.23) *is valid with a constant* $\mathcal{C}''$ *depending only on the initial data* (3.27).

3°. *Suppose that the representations* (4.19) *hold, i.e.,* $g^0 = \underline{g}$. *Then estimate* (7.22) *is valid with a constant* $\mathcal{C}'$ *depending only on the initial data* (3.27).

4°. *Suppose* $n = 1$, $m = d$, *and* $\mathcal{A} = \mathbf{D}^* g(\mathbf{x})\mathbf{D}$, *where* $g(\mathbf{x})$ *is a* Γ*-periodic matrix-valued function with real entries such that* $g, g^{-1} \in L_\infty$, $g(\mathbf{x}) > 0$. *Then estimate* (7.25) *is valid with a constant* $\mathcal{C}^0$ *depending only on the initial data* (3.27).

7.4. Special cases. By the scaling transformation, Proposition 6.13 implies the following statement.

Proposition 7.5. *Suppose that conditions* (4.18) *are satisfied* (*i.e.,* $g^0 = \overline{g}$). *Then the effective operator* $\mathcal{B}_\lambda^0$ *is given by*

$$\mathcal{B}_\lambda^0 = b(\mathbf{D})^* g^0 b(\mathbf{D}) + \sum_{j=1}^d (\overline{a_j + a_j^*}) D_j - W + \overline{Q} + \lambda\overline{Q_0}.$$

The corrector (7.17) *takes the form*

$$K_\varepsilon = \widetilde{\Lambda}^\varepsilon (\mathcal{B}_\lambda^0)^{-1}\Pi_\varepsilon + (\mathcal{B}_\lambda^0)^{-1}\Pi_\varepsilon(\widetilde{\Lambda}^\varepsilon)^* - (\mathcal{B}_\lambda^0)^{-1}\mathcal{N}(\mathcal{B}_\lambda^0)^{-1},$$

where $\mathcal{N} = \mathcal{N}_{12}(\mathbf{D}) + \mathcal{N}_{21}(\mathbf{D}) + \mathcal{N}_{22}$, *with the operators* $\mathcal{N}_{12}(\mathbf{D})$ *and* $\mathcal{N}_{21}(\mathbf{D})$ *as in* (6.21), *and with* $\mathcal{N}_{22}$ *defined by* (5.31), (5.36).

The following statement is a consequence of Proposition 6.14.

Proposition 7.6. *Suppose that conditions* (4.19) *are satisfied* (*i.e.,* $g^0 = \underline{g}$). *Then the operator* (7.18) *takes the form* $\mathcal{N} = \mathcal{N}_{12}(\mathbf{D}) + \mathcal{N}_{21}(\mathbf{D}) + \mathcal{N}_{22}$, *where* $\mathcal{N}_{12}(\mathbf{D})$ *is defined by* (6.4), $M_1(\mathbf{D})$ *is the DO with the symbol* (6.22), *and* $\mathcal{N}_{21}(\mathbf{D})$ *and* $\mathcal{N}_{22}$ *are defined by* (6.5) *and* (5.31), (5.36), *respectively.*

By the scaling transformation, Proposition 6.15 implies the following statement.

Proposition 7.7. *Suppose that condition* (6.23) *is satisfied. Then the effective operator takes the form*

$$\mathcal{B}_\lambda^0 = b(\mathbf{D})^* g^0 b(\mathbf{D}) + \sum_{j=1}^d (\overline{a_j + a_j^*}) D_j + \overline{Q} + \lambda\overline{Q_0}. \tag{7.26}$$

The corrector (7.17) *is given by*

$$K_\varepsilon = \Lambda^\varepsilon b(\mathbf{D})(\mathcal{B}^0_\lambda)^{-1}\Pi_\varepsilon + (\mathcal{B}^0_\lambda)^{-1}\Pi_\varepsilon b(\mathbf{D})^*(\Lambda^\varepsilon)^* - (\mathcal{B}^0_\lambda)^{-1}\mathcal{N}(\mathcal{B}^0_\lambda)^{-1},$$

where $\mathcal{N} = \mathcal{N}_{11}(\mathbf{D}) + \mathcal{N}_{12}(\mathbf{D}) + \mathcal{N}_{21}(\mathbf{D})$. *Here, the operator* $\mathcal{N}_{11}(\mathbf{D})$ *is defined by* (6.3), $\mathcal{N}_{12}(\mathbf{D})$ *is given by* (6.4), *where* $M_1(\mathbf{D})$ *is the DO with the symbol* (6.25), *and* $\mathcal{N}_{21}(\mathbf{D})$ *is defined by* (6.26).

The case where the corrector is equal to zero is distinguished by conditions (4.18) and (6.23); cf. Proposition 6.16.

Proposition 7.8. *Suppose that conditions* (4.18) *and* (6.23) *are satisfied. Then the effective operator is given by* (7.26), *and the corrector* (7.17) *is equal to zero. We have*

$$\left\| \mathcal{B}^{-1}_{\lambda,\varepsilon} - (\mathcal{B}^0_\lambda)^{-1} \right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le \mathcal{C}_2\varepsilon^2, \quad 0 < \varepsilon \le 1.$$

The constant $\mathcal{C}_2$ *depends only on the initial data* (3.27).

§8. Homogenization problems for the operator with a singular potential and the operator $\widetilde{\mathcal{B}}_\varepsilon$

8.1. Homogenization problem for the operator with a singular potential. Let $v(\mathbf{x})$ be a Hermitian Γ-periodic $(n \times n)$-matrix-valued function in $\mathbb{R}^d$ such that

$$v \in L_s(\Omega), \quad s = 1 \text{ for } d = 1, \quad s > \frac{d}{2} \text{ for } d \ge 2; \quad \int_\Omega v(\mathbf{x})\, d\mathbf{x} = 0.$$

We consider the homogenization problem for the operator $\mathcal{B}_\varepsilon$ generated by the quadratic form

$$\mathfrak{b}_\varepsilon[\mathbf{u},\mathbf{u}] = \mathfrak{a}_\varepsilon[\mathbf{u},\mathbf{u}] + \varepsilon^{-1}\int_{\mathbb{R}^d} \langle v^\varepsilon(\mathbf{x})\mathbf{u},\mathbf{u}\rangle\, d\mathbf{x} + q_\varepsilon[\mathbf{u},\mathbf{u}], \quad \mathbf{u} \in H^1(\mathbb{R}^d;\mathbb{C}^n),$$

where $\mathfrak{a}_\varepsilon$ and q_ε are the forms defined by (7.2) and (7.4). Formally, $\mathcal{B}_\varepsilon$ can be written as

$$\mathcal{B}_\varepsilon = \mathcal{A}_\varepsilon + \varepsilon^{-1}v^\varepsilon + Q^\varepsilon. \tag{8.1}$$

The operator (8.1) contains a “singular potential” $\varepsilon^{-1}v^\varepsilon$. It is easily seen that the operator (8.1) is a particular case of the operator (7.9) (for the details, see [Su2, §11]). Indeed, the matrix-valued function $v(\mathbf{x})$ can be represented as

$$v(\mathbf{x}) = -\sum_{j=1}^d D_j a_j(\mathbf{x}), \tag{8.2}$$

where the $a_j(\mathbf{x})$, $j = 1,\dots,d$, are Γ-periodic matrix-valued functions such that

$$a_j(\mathbf{x})^* = -a_j(\mathbf{x}), \quad a_j \in L_\rho(\Omega), \quad \overline{a_j} = 0, \quad j = 1,\dots,d. \tag{8.3}$$

Here $\rho = \infty$ for $d = 1$; $\rho = ds(d-s)^{-1}$ for $d \ge 2$ and $s < d$; $d < \rho < \infty$ for $d \ge 2$ and $s = d$; $\rho = \infty$ for $d \ge 2$ and $s > d$. In any case, ρ is such that the a_j satisfy the required condition (3.10). This representation can be found as follows. Let $\Phi(\mathbf{x})$ be a Γ-periodic matrix-valued solution of the equation $\Delta\Phi(\mathbf{x}) = v(\mathbf{x})$. We put $a_j(\mathbf{x}) = D_j\Phi(\mathbf{x})$. Then the required conditions hold true, and

$$\|a_j\|_{L_\rho(\Omega)} \le C_s\|v\|_{L_s(\Omega)}, \quad j = 1,\dots,d.$$

The constant C_s depends only on s, d, and Ω, and for $s = d \ge 2$ it depends also on ρ.

By (8.2) and (8.3), $\sum_{j=1}^d \left(a_j^\varepsilon D_j + D_j(a_j^\varepsilon)^*\right) = \varepsilon^{-1}v^\varepsilon$. Thus, the operator (8.1) takes the form (7.9). We can apply the general results to approximate the inverse of the operator

$$\mathcal{B}_{\lambda,\varepsilon} = \mathcal{B}_\varepsilon + \lambda Q_0^\varepsilon. \tag{8.4}$$

Let us construct the effective operator. The matrix $\Lambda(\mathbf{x})$ and the effective matrix g^0 are defined by (4.10) and (4.16) as before. By (8.2) and (8.3), equation (4.12) for $\widetilde{\Lambda}$ can be written as

$$b(\mathbf{D})^* g(\mathbf{x}) b(\mathbf{D}) \widetilde{\Lambda}(\mathbf{x}) + v(\mathbf{x}) = 0, \qquad \int_\Omega \widetilde{\Lambda}(\mathbf{x})\, d\mathbf{x} = 0. \tag{8.5}$$

The constant matrices V and W are defined as in (5.2) and (5.3). Since $\overline{a_j} = 0$, the effective operator (7.12) takes the form

$$\mathcal{B}^0_\lambda = b(\mathbf{D})^* g^0 b(\mathbf{D}) - b(\mathbf{D})^* V - V^* b(\mathbf{D}) - W + \overline{Q} + \lambda \overline{Q_0}. \tag{8.6}$$

The corrector K_ε for the operator (8.1) is given by (7.17); compared with the general case, the calculation of the third term of the corrector is simpler because $a_j + a_j^* = 0$ and (8.2) is true. The operator $\mathcal{N}$ is defined by (7.18). In this formula, the first term $\mathcal{N}_{11}(\mathbf{D})$ is defined by (6.3); the second term $\mathcal{N}_{12}(\mathbf{D})$ is given by (6.4), where $T_0 = \overline{\Lambda^* v \Lambda}$ and $M_1(\mathbf{D})$ is the DO with the symbol $M_1(\boldsymbol{\xi}) = \overline{\widetilde{\Lambda}^* b(\boldsymbol{\xi})^* \widetilde{g}} + \overline{(b(\mathbf{D})\widetilde{\Lambda})^* g b(\boldsymbol{\xi}) \Lambda}$; the third term $\mathcal{N}_{21}(\mathbf{D})$ is given by $\mathcal{N}_{21}(\mathbf{D}) = M_2(\mathbf{D}) + M_2(\mathbf{D})^* + T^* b(\mathbf{D}) + b(\mathbf{D})^* T$, where $T = \overline{\Lambda^* v \widetilde{\Lambda}} + \overline{\Lambda^* Q} + \lambda \overline{\Lambda^* Q_0}$, and $M_2(\mathbf{D})$ is the DO with the symbol (5.22); and the last term $\mathcal{N}_{22}$ is given by $\mathcal{N}_{22} = \overline{\widetilde{\Lambda}^* v \widetilde{\Lambda}} + 2\,\mathrm{Re}\,\big(\overline{\widetilde{\Lambda}^* Q} + \lambda \overline{\widetilde{\Lambda}^* Q_0}\big)$. Thus, the corrector does not involve the functions a_j directly, but contains v.

Applying Theorems 7.1 and 7.2, we obtain the following result.

Theorem 8.1. *Under the above assumptions, suppose that the operator $\mathcal{B}_{\lambda,\varepsilon}$ is defined by* (8.1), (8.4). *Let $\Lambda(\mathbf{x})$ be the periodic solution of problem* (4.10), *and let $\widetilde{\Lambda}(\mathbf{x})$ be the periodic solution of problem* (8.5). *Let $\mathcal{B}^0_\lambda$ be the effective operator* (8.6). *Suppose that the corrector K_ε is defined as in* (7.17) (*with the simplifications mentioned above*). *Then estimates* (7.13) *and* (7.21) *are valid with constants depending only on d, m, n, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|v\|_{L_s(\Omega)}$, $\widetilde{c}$, c_0, $\widetilde{c}_2$, c_3, λ, $\|Q_0\|_{L_\infty}$, $\|Q_0^{-1}\|_{L_\infty}$, and the parameters of the lattice Γ.*

Similarly, we can apply Theorem 7.3 and Propositions 7.4, 7.5, and 7.6 to the operator (8.1). In formulations, one should take into account that $a_j + a_j^* = 0$ and that $\widetilde{\Lambda}$ is the periodic solution of problem (8.5). We shall not enter into details.

8.2. Homogenization problem for the operator $\widetilde{\mathcal{B}}_\varepsilon$. For applications, it is of interest to consider the operator $\widetilde{\mathcal{B}}_\varepsilon$ acting in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ and related to the operator (7.9) by the formula

$$\widetilde{\mathcal{B}}_\varepsilon = (f^\varepsilon)^* \mathcal{B}_\varepsilon f^\varepsilon. \tag{8.7}$$

Here $f(\mathbf{x})$ is a Γ-periodic $(n \times n)$-matrix-valued function in $\mathbb{R}^d$ such that $f, f^{-1} \in L_\infty$. More precisely, $\widetilde{\mathcal{B}}_\varepsilon$ is the selfadjoint operator corresponding to the quadratic form

$$\widetilde{\mathfrak{b}}_\varepsilon[\mathbf{u}, \mathbf{u}] = \mathfrak{b}_\varepsilon[f^\varepsilon \mathbf{u}, f^\varepsilon \mathbf{u}], \qquad f^\varepsilon \mathbf{u} \in H^1(\mathbb{R}^d; \mathbb{C}^n). \tag{8.8}$$

Let $\widetilde{Q}_0(\mathbf{x})$ be a Γ-periodic positive $(n \times n)$-matrix-valued function such that $\widetilde{Q}_0, \widetilde{Q}_0^{-1} \in L_\infty$. We define the matrix-valued function $Q_0(\mathbf{x})$ by

$$Q_0(\mathbf{x}) = (f(\mathbf{x})^*)^{-1} \widetilde{Q}_0(\mathbf{x}) f(\mathbf{x})^{-1}. \tag{8.9}$$

The problem is to approximate the generalized resolvent $(\widetilde{\mathcal{B}}_\varepsilon + \lambda \widetilde{Q}_0^\varepsilon)^{-1}$ for small ε. Denote

$$\widetilde{\mathcal{B}}_{\lambda,\varepsilon} := \widetilde{\mathcal{B}}_\varepsilon + \lambda \widetilde{Q}_0^\varepsilon. \tag{8.10}$$

Now, the "initial data" is the following set of parameters:

$$
\begin{aligned}
&d,\ m,\ n,\ \rho;\ \alpha_0,\ \alpha_1,\ \|g\|_{L_\infty},\ \|g^{-1}\|_{L_\infty},\ \|a_j\|_{L_\rho(\Omega)},\ j=1,\dots,d;\\
&\widetilde{c},\ c_0,\ \widetilde{c}_2,\ c_3 \text{ from Condition 3.2; } \|f\|_{L_\infty},\ \|f^{-1}\|_{L_\infty};\\
&\lambda,\ \|\widetilde{Q}_0\|_{L_\infty},\ \|\widetilde{Q}_0^{-1}\|_{L_\infty};\ \text{the parameters of the lattice } \Gamma.
\end{aligned}
\tag{8.11}
$$

The parameter λ is subject to the following restriction:

$$
\begin{aligned}
&\lambda > \|f\|^2_{L_\infty}\|\widetilde{Q}_0^{-1}\|_{L_\infty}(c_0+c_4) \ \text{ if } \lambda \ge 0;\\
&\lambda > \|f^{-1}\|^{-2}_{L_\infty}\|\widetilde{Q}_0\|^{-1}_{L_\infty}(c_0+c_4) \ \text{ if } \lambda < 0 \ \text{ (and } c_0+c_4<0),
\end{aligned}
\tag{8.12}
$$

which ensures (3.24). Now, it is convenient to define the number β as follows:

$$
\begin{aligned}
&\beta = \lambda\|\widetilde{Q}_0^{-1}\|^{-1}_{L_\infty}\|f\|^{-2}_{L_\infty} - c_0 - c_4 \ \text{ if } \lambda \ge 0;\\
&\beta = \lambda\|\widetilde{Q}_0\|_{L_\infty}\|f^{-1}\|^{2}_{L_\infty} - c_0 - c_4 \ \text{ if } \lambda < 0 \ \text{ (and } c_0+c_4<0).
\end{aligned}
$$

This can only reduce the number defined by (3.26). (All the rest remains valid with the new definition of β.)

Relations (8.7), (8.9), and (8.10) imply that for the operators (8.10) and (7.11) we have $\widetilde{\mathcal{B}}_{\lambda,\varepsilon}^{-1} = (f^\varepsilon)^{-1}\mathcal{B}_{\lambda,\varepsilon}^{-1}((f^\varepsilon)^*)^{-1}$. Then Theorems 7.1 and 7.2 directly imply the following result.

Theorem 8.2. *Suppose that $\widetilde{\mathcal{B}}_\varepsilon$ is the operator generated by the quadratic form* (8.8), *and let $\widetilde{\mathcal{B}}_{\lambda,\varepsilon}$ be the operator* (8.10). *Let $\mathcal{B}_{\lambda,\varepsilon}$ be the operator defined by* (7.9) *and* (7.11), *and assume* (8.9). *Let $\mathcal{B}^0_\lambda$ be the effective operator* (7.12), *and let K_ε be the corrector* (7.17). *Then for $0<\varepsilon\le 1$ we have*

$$
\big\|\widetilde{\mathcal{B}}_{\lambda,\varepsilon}^{-1} - (f^\varepsilon)^{-1}(\mathcal{B}^0_\lambda)^{-1}((f^\varepsilon)^*)^{-1}\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le \widetilde{C}_0\varepsilon,
$$

$$
\big\|\widetilde{\mathcal{B}}_{\lambda,\varepsilon}^{-1} - (f^\varepsilon)^{-1}\left((\mathcal{B}^0_\lambda)^{-1} + \varepsilon K_\varepsilon\right)((f^\varepsilon)^*)^{-1}\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le \widetilde{\mathcal{C}}_2\varepsilon^2.
$$

The constants $\widetilde{C}_0 = C_0\|f^{-1}\|^2_{L_\infty}$ and $\widetilde{\mathcal{C}}_2 = \mathcal{C}_2\|f^{-1}\|^2_{L_\infty}$ depend only on the initial data (8.11).

Similarly, it is possible to deduce some consequences for the operator $\widetilde{\mathcal{B}}_{\lambda,\varepsilon}$ from Theorem 7.3 and Propositions 7.4, 7.8. We shall not dwell on this.

§9. Application of the general pattern: the Schrödinger operator

9.1. The scalar elliptic operator. Consider the case where $n=1$, $m=d$, $b(\mathbf{D})=\mathbf{D}$, and $g(\mathbf{x})$ is a Γ-periodic symmetric $(d\times d)$-matrix-valued function *with real entries*; $g(\mathbf{x})$ is assumed to be bounded and positive definite. Then the operator $\mathcal{A}_\varepsilon$ takes the form $\mathcal{A}_\varepsilon = \mathbf{D}^* g^\varepsilon(\mathbf{x})\mathbf{D} = -\operatorname{div} g^\varepsilon(\mathbf{x})\nabla$. Obviously, in this case $\alpha_0=\alpha_1=1$; see (3.2).

Next, let $\mathbf{A}(\mathbf{x}) = \operatorname{col}\{A_1(\mathbf{x}),\dots,A_d(\mathbf{x})\}$, where the $A_j(\mathbf{x})$ are Γ-periodic real-valued functions such that

$$
A_j \in L_\rho(\Omega), \quad \rho = 2 \ \text{ for } d=1, \quad \rho > d \ \text{ for } d\ge 2; \quad j=1,\dots,d.
\tag{9.1}
$$

Let $v(\mathbf{x})$ and $\mathcal{V}(\mathbf{x})$ be real-valued Γ-periodic functions such that

$$
\begin{aligned}
&v,\mathcal{V}\in L_s(\Omega), \quad \int_\Omega v(\mathbf{x})\,d\mathbf{x} = 0,\\
&s=1 \ \text{ for } d=1, \quad s>\frac{d}{2} \ \text{ for } d\ge 2.
\end{aligned}
\tag{9.2}
$$

Consider the operator $\mathcal{B}_\varepsilon$ in $L_2(\mathbb{R}^d)$ given formally by the differential expression

$$\mathcal{B}_\varepsilon = (\mathbf{D} - \mathbf{A}^\varepsilon(\mathbf{x}))^* g^\varepsilon(\mathbf{x})(\mathbf{D} - \mathbf{A}^\varepsilon(\mathbf{x})) + \varepsilon^{-1} v^\varepsilon(\mathbf{x}) + \mathcal{V}^\varepsilon(\mathbf{x}). \tag{9.3}$$

The precise definition of the operator $\mathcal{B}_\varepsilon$ is in terms of the quadratic form

$$\mathfrak{b}_\varepsilon[u,u] = \int_{\mathbb{R}^d} \left(\langle g^\varepsilon (\mathbf{D} - \mathbf{A}^\varepsilon)u, (\mathbf{D} - \mathbf{A}^\varepsilon)u \rangle + \left(\varepsilon^{-1} v^\varepsilon + \mathcal{V}^\varepsilon\right) |u|^2 \right) d\mathbf{x}, \quad u \in H^1(\mathbb{R}^d).$$

The operator (9.3) can be treated as the periodic Schrödinger operator with metric g^ε, magnetic potential $\mathbf{A}^\varepsilon$, and electric potential $\varepsilon^{-1} v^\varepsilon + \mathcal{V}^\varepsilon$ containing a "singular" first term.

It is easily seen (for the details, see [Su2, Subsection 13.1]) that the operator (9.3) can be written in the required form:

$$\mathcal{B}_\varepsilon = \mathbf{D}^* g^\varepsilon(\mathbf{x}) \mathbf{D} + \sum_{j=1}^d \left(a_j^\varepsilon(\mathbf{x}) D_j + D_j (a_j^\varepsilon(\mathbf{x}))^* \right) + Q^\varepsilon(\mathbf{x}).$$

The real-valued function $Q(\mathbf{x})$ is defined as follows:

$$Q(\mathbf{x}) = \mathcal{V}(\mathbf{x}) + \langle g(\mathbf{x})\mathbf{A}(\mathbf{x}), \mathbf{A}(\mathbf{x}) \rangle. \tag{9.4}$$

The complex-valued functions $a_j(\mathbf{x})$ are given by

$$a_j(\mathbf{x}) = -\eta_j(\mathbf{x}) + i\zeta_j(\mathbf{x}), \tag{9.5}$$

where the $\eta_j(\mathbf{x})$ are the components of the vector-valued function $\boldsymbol{\eta}(\mathbf{x}) = g(\mathbf{x})\mathbf{A}(\mathbf{x})$, and the $\zeta_j(\mathbf{x})$ are defined in terms of the Γ-periodic solution of the equation $\Delta\Phi(\mathbf{x}) = v(\mathbf{x})$ by $\zeta_j(\mathbf{x}) = -\partial_j \Phi(\mathbf{x})$ (cf. Subsection 8.1). We have

$$v(\mathbf{x}) = -\sum_{j=1}^d \partial_j \zeta_j(\mathbf{x}). \tag{9.6}$$

It is easily seen that the functions (9.5) satisfy condition (3.10) with a suitable ρ' (depending on ρ and s); here, the norms $\|a_j\|_{L_{\rho'}(\Omega)}$ are controlled in terms of $\|g\|_{L_\infty}$, $\|\mathbf{A}\|_{L_\rho(\Omega)}$, $\|v\|_{L_s(\Omega)}$, and the parameters of the lattice Γ. (For the details, see [Su2, Subsection 13.1].) The function (9.4) satisfies condition (3.17) with a suitable $s' = \min\{s, \rho/2\}$. Thus, now Example 3.3 is realized.

As usual, we denote

$$\mathcal{B}_{\lambda,\varepsilon} = \mathcal{B}_\varepsilon + \lambda Q_0^\varepsilon, \tag{9.7}$$

where $Q_0(\mathbf{x})$ is a positive definite and bounded Γ-periodic function, and the parameter λ is subject to the restriction (3.24) with c_0 and c_4 corresponding to the operator (9.3). We are interested in approximation of the inverse operator $\mathcal{B}_{\lambda,\varepsilon}^{-1}$. In the case under consideration, the initial data (3.27) reduces to the following set of parameters:

$$\begin{aligned} &d,\ \rho,\ s;\ \|g\|_{L_\infty},\ \|g^{-1}\|_{L_\infty},\ \|\mathbf{A}\|_{L_\rho(\Omega)},\ \|v\|_{L_s(\Omega)},\ \|\mathcal{V}\|_{L_s(\Omega)}, \\ &\lambda,\ \|Q_0\|_{L_\infty},\ \|Q_0^{-1}\|_{L_\infty};\ \text{the parameters of the lattice } \Gamma. \end{aligned} \tag{9.8}$$

We calculate the effective operator. The matrix $\Lambda(\mathbf{x})$ is the row $\Lambda(\mathbf{x}) = i\Psi(\mathbf{x})$, $\Psi(\mathbf{x}) = (\psi_1(\mathbf{x}), \dots, \psi_d(\mathbf{x}))$, where $\psi_j \in \widetilde{H}^1(\Omega)$ is the (weak) solution of the problem

$$\operatorname{div} g(\mathbf{x})(\nabla \psi_j(\mathbf{x}) + \mathbf{e}_j) = 0, \qquad \int_\Omega \psi_j(\mathbf{x})\, d\mathbf{x} = 0.$$

Here $\mathbf{e}_1, \dots, \mathbf{e}_d$ is the standard orthonormal basis in $\mathbb{R}^d$. The solutions $\psi_j(\mathbf{x})$ are real-valued, and the row $\Lambda(\mathbf{x})$ has purely imaginary entries. The matrix $\widetilde{g}(\mathbf{x})$ is the $(d \times d)$-matrix with the columns $g(\mathbf{x})(\nabla \psi_j(\mathbf{x}) + \mathbf{e}_j)$. The effective matrix g^0 is defined in a standard way: $g^0 = |\Omega|^{-1} \int_\Omega \widetilde{g}(\mathbf{x})\, d\mathbf{x}$. Clearly, $\widetilde{g}(\mathbf{x})$ and g^0 have real entries.

By (9.5) and (9.6), the periodic solution of problem (4.12) is represented as $\widetilde{\Lambda}(\mathbf{x}) = \widetilde{\Lambda}_1(\mathbf{x}) + i\widetilde{\Lambda}_2(\mathbf{x})$, where the real-valued Γ-periodic functions $\widetilde{\Lambda}_1(\mathbf{x})$, $\widetilde{\Lambda}_2(\mathbf{x})$ are solutions of the following problems:

$$
\begin{aligned}
-\operatorname{div} g(\mathbf{x})\nabla\widetilde{\Lambda}_1(\mathbf{x}) + v(\mathbf{x}) = 0, &\quad \int_\Omega \widetilde{\Lambda}_1(\mathbf{x})\, d\mathbf{x} = 0,\\
-\operatorname{div} g(\mathbf{x})\nabla\widetilde{\Lambda}_2(\mathbf{x}) + \operatorname{div} g(\mathbf{x})\mathbf{A}(\mathbf{x}) = 0, &\quad \int_\Omega \widetilde{\Lambda}_2(\mathbf{x})\, d\mathbf{x} = 0.
\end{aligned}
$$

The column V (see (5.2)) can be written as $V = V_1 + iV_2$, where the columns V_1 and V_2 with real entries are given by

$$
V_1 = \overline{(\nabla\Psi)^t g\nabla\widetilde{\Lambda}_2}, \quad V_2 = -\overline{(\nabla\Psi)^t g\nabla\widetilde{\Lambda}_1}.
$$

In accordance with (5.3), the constant W is defined by the relation

$$
W = \overline{\langle g(\nabla\widetilde{\Lambda}_2 - i\nabla\widetilde{\Lambda}_1), \nabla\widetilde{\Lambda}_2 - i\nabla\widetilde{\Lambda}_1\rangle} = \overline{\langle g\nabla\widetilde{\Lambda}_1, \nabla\widetilde{\Lambda}_1\rangle} + \overline{\langle g\nabla\widetilde{\Lambda}_2, \nabla\widetilde{\Lambda}_2\rangle}.
$$

Now, formula (7.12) for $\mathcal{B}^0_\lambda$ takes the form

$$
\mathcal{B}^0_\lambda u = -\operatorname{div} g^0\nabla u + 2i\langle V_1 + \overline{\boldsymbol{\eta}}, \nabla u\rangle + (-W + \overline{Q} + \lambda\overline{Q_0})u.
$$

In other words,

$$
\mathcal{B}^0 = (\mathbf{D} - \mathbf{A}^0)^* g^0 (\mathbf{D} - \mathbf{A}^0) + \mathcal{V}^0, \tag{9.9}
$$

$$
\mathcal{B}^0_\lambda = \mathcal{B}^0 + \lambda\overline{Q_0}, \tag{9.10}
$$

where

$$
\mathbf{A}^0 = (g^0)^{-1}(V_1 + \overline{g\mathbf{A}}), \quad \mathcal{V}^0 = \overline{\mathcal{V}} + \overline{\langle g\mathbf{A}, \mathbf{A}\rangle} - \langle g^0\mathbf{A}^0, \mathbf{A}^0\rangle - W.
$$

The effective operator $\mathcal{B}^0$ can be viewed as the Schrödinger operator with constant effective coefficients: the metric g^0, magnetic potential $\mathbf{A}^0$, and electric potential $\mathcal{V}^0$.

By Proposition 7.4(4°), we can use the corrector K^0_ε that does not contain the smoothing operator (see (7.24)). The first term of this corrector takes the form

$$
(\Lambda^\varepsilon\mathbf{D} + \widetilde{\Lambda}^\varepsilon)(\mathcal{B}^0_\lambda)^{-1} = (\Psi^\varepsilon\nabla + \widetilde{\Lambda}^\varepsilon)(\mathcal{B}^0_\lambda)^{-1}. \tag{9.11}
$$

The second term is adjoint to the first.

In order to find the third term of the corrector, we need to calculate the operator $\mathcal{N}$ (see (7.18)). We show that the first summand in (7.18) is equal to zero. Indeed, $M(\boldsymbol{\theta})$ (see (5.11)) is a Hermitian matrix with purely imaginary entries, because $\Lambda(\mathbf{x})$ has purely imaginary entries, while $\widetilde{g}(\mathbf{x})$ and $b(\boldsymbol{\theta}) = \boldsymbol{\theta}$ have real entries. Then $b(\boldsymbol{\theta})^* M(\boldsymbol{\theta}) b(\boldsymbol{\theta}) = 0$ (as a Hermitian purely imaginary (1×1)-matrix). By (6.3), it follows that $\mathcal{N}_{11}(\mathbf{D}) = 0$.

Using (6.4), (5.17), (5.19), and (9.6), we check that the symbol of the operator $\mathcal{N}_{12}(\mathbf{D})$ is given by

$$
\mathcal{N}_{12}(\boldsymbol{\xi}) = 2\overline{\widetilde{\Lambda}_1\langle\widetilde{g}\boldsymbol{\xi}, \boldsymbol{\xi}\rangle} - 2\overline{\langle g\boldsymbol{\xi}, \nabla\widetilde{\Lambda}_1\rangle\Psi\boldsymbol{\xi}} + \overline{v(\Psi\boldsymbol{\xi})^2}.
$$

Hence,

$$
\begin{aligned}
\mathcal{N}_{12}(\mathbf{D}) &= \sum_{k,l=1}^d \mathcal{N}_{12,kl} D_k D_l,\\
\mathcal{N}_{12,kl} &= 2\overline{\widetilde{\Lambda}_1\widetilde{g}_{kl}} + \overline{v\psi_k\psi_l} - \sum_{j=1}^d \overline{(g_{jl}\psi_k + g_{jk}\psi_l)\,\partial_j\widetilde{\Lambda}_1}.
\end{aligned} \tag{9.12}
$$

Next, applying (6.5), (5.22), (5.28), and (9.6), we see that the symbol of the operator $\mathcal{N}_{21}(\mathbf{D})$ is given by

$$\mathcal{N}_{21}(\boldsymbol{\xi}) = 2\overline{\widetilde{\Lambda}_1\langle g\boldsymbol{\xi}, \nabla\widetilde{\Lambda}_2\rangle} - 2\overline{\widetilde{\Lambda}_2\langle g\boldsymbol{\xi}, \nabla\widetilde{\Lambda}_1\rangle} + 2\overline{(\Psi\boldsymbol{\xi})\langle\boldsymbol{\eta}, \nabla\widetilde{\Lambda}_1\rangle} - 2\overline{\widetilde{\Lambda}_1\langle\boldsymbol{\eta}, \nabla(\Psi\boldsymbol{\xi})\rangle} + 2\overline{v\widetilde{\Lambda}_2(\Psi\boldsymbol{\xi})} - 4\overline{\widetilde{\Lambda}_1\langle\boldsymbol{\eta}, \boldsymbol{\xi}\rangle}.$$

Hence,

$$\begin{aligned} \mathcal{N}_{21}(\mathbf{D}) &= \sum_{k=1}^{d} \mathcal{N}_{21,k} D_k, \\ \mathcal{N}_{21,k} &= 2\sum_{j=1}^{d} \overline{g_{jk}\big(\widetilde{\Lambda}_1\partial_j\widetilde{\Lambda}_2 - \widetilde{\Lambda}_2\partial_j\widetilde{\Lambda}_1\big)} + 2\overline{\psi_k\langle\boldsymbol{\eta}, \nabla\widetilde{\Lambda}_1\rangle} - 2\overline{\widetilde{\Lambda}_1\langle\boldsymbol{\eta}, \nabla\psi_k\rangle} \\ &\quad + 2\overline{v\widetilde{\Lambda}_2\psi_k} - 4\overline{\eta_k\widetilde{\Lambda}_1}. \end{aligned} \tag{9.13}$$

Finally, by (5.31), (5.36), and (9.6),

$$\mathcal{N}_{22} = 2\overline{\widetilde{\Lambda}_2\langle\boldsymbol{\eta}, \nabla\widetilde{\Lambda}_1\rangle} - 2\overline{\widetilde{\Lambda}_1\langle\boldsymbol{\eta}, \nabla\widetilde{\Lambda}_2\rangle} + \overline{v\big(\widetilde{\Lambda}_1^2 + \widetilde{\Lambda}_2^2\big)} + 2\overline{\widetilde{\Lambda}_1(Q + \lambda Q_0)}. \tag{9.14}$$

As a result, we obtain

$$\mathcal{N} = \sum_{k,l=1}^{d} \mathcal{N}_{12,kl} D_k D_l + \sum_{k=1}^{d} \mathcal{N}_{21,k} D_k + \mathcal{N}_{22}, \tag{9.15}$$

where the coefficients are defined by (9.12), (9.13), and (9.14).

By (7.24) and (9.11), the corrector is given by

$$K_\varepsilon^0 = (\Psi^\varepsilon\nabla + \widetilde{\Lambda}^\varepsilon)(\mathcal{B}_\lambda^0)^{-1} + (\mathcal{B}_\lambda^0)^{-1}(\Psi^\varepsilon\nabla + \widetilde{\Lambda}^\varepsilon)^* - (\mathcal{B}_\lambda^0)^{-1}\mathcal{N}(\mathcal{B}_\lambda^0)^{-1}, \tag{9.16}$$

where $\mathcal{N}$ is the operator (9.15).

Theorem 7.1 and Proposition 7.4(4°) imply the following result.

Proposition 9.1. *Let $\mathcal{B}_{\lambda,\varepsilon}$ be the operator defined by* (9.3) *and* (9.7) *whose coefficients satisfy conditions of Subsection* 9.1. *Let $\mathcal{B}_\lambda^0$ be the operator given by* (9.9) *and* (9.10) *with the effective coefficients described in Subsection* 9.1. *Suppose that K_ε^0 is the corrector* (9.16). *Then*

$$\big\|\mathcal{B}_{\lambda,\varepsilon}^{-1} - (\mathcal{B}_\lambda^0)^{-1}\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le C_0\varepsilon, \quad 0 < \varepsilon \le 1,$$
$$\big\|\mathcal{B}_{\lambda,\varepsilon}^{-1} - (\mathcal{B}_\lambda^0)^{-1} - \varepsilon K_\varepsilon^0\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le \mathcal{C}^0\varepsilon^2, \quad 0 < \varepsilon \le 1.$$

The constants C_0 and $\mathcal{C}^0$ depend only on the initial data (9.8).

Remark 9.2. 1) If the magnetic potential $\mathbf{A}(\mathbf{x})$ is subject to the gauge conditions

$$\operatorname{div} g(\mathbf{x})\mathbf{A}(\mathbf{x}) = 0, \quad \overline{g\mathbf{A}} = 0, \tag{9.17}$$

then $\widetilde{\Lambda}_2(\mathbf{x}) = 0$ and $V_1 = 0$. Then the expression (9.9) simplifies: $\mathcal{B}^0 = \mathbf{D}^* g^0\mathbf{D} + \mathcal{V}^0$, where $\mathcal{V}^0 = \overline{\mathcal{V}} + \overline{\langle g\mathbf{A}, \mathbf{A}\rangle} - \overline{\langle g\nabla\widetilde{\Lambda}_1, \nabla\widetilde{\Lambda}_1\rangle}$. The expressions (9.13) and (9.14) for the coefficients of $\mathcal{N}$ also simplify.

2) If the gauge conditions (9.17) are fulfilled and $v(\mathbf{x}) = 0$, then $\widetilde{\Lambda}_1(\mathbf{x}) = 0$, $\widetilde{\Lambda}_2(\mathbf{x}) = 0$, and $V = 0$, $W = 0$. Then the operator (9.9) is given by $\mathcal{B}^0 = \mathbf{D}^* g^0\mathbf{D} + \mathcal{V}^0$, where $\mathcal{V}^0 = \overline{\mathcal{V}} + \overline{\langle g\mathbf{A}, \mathbf{A}\rangle}$, the third term of the corrector is equal to zero, and the corrector (9.16) simplifies: $K_\varepsilon^0 = \Psi^\varepsilon\nabla(\mathcal{B}_\lambda^0)^{-1} + (\mathcal{B}_\lambda^0)^{-1}(\Psi^\varepsilon\nabla)^*$.

3) If $\mathbf{A} = 0$, then $\widetilde{\Lambda}_2 = 0$. In this case, the operator (9.9) takes the form $\mathcal{B}^0 = \mathbf{D}^* g^0\mathbf{D} + \mathcal{V}^0$, where $\mathcal{V}^0 = \overline{\mathcal{V}} - \overline{\langle g\nabla\widetilde{\Lambda}_1, \nabla\widetilde{\Lambda}_1\rangle}$. Next, in this case the operator (9.13) is

equal to zero: $\mathcal{N}_{21}(\mathbf{D}) = 0$, and expression (9.14) for $\mathcal{N}_{22}$ simplifies. The corrector can be written as

$$K_\varepsilon^0 = (\Psi^\varepsilon \nabla + \widetilde{\Lambda}_1^\varepsilon)(\mathcal{B}_\lambda^0)^{-1} + (\mathcal{B}_\lambda^0)^{-1}(\Psi^\varepsilon \nabla + \widetilde{\Lambda}_1^\varepsilon)^* - (\mathcal{B}_\lambda^0)^{-1}\mathcal{N}(\mathcal{B}_\lambda^0)^{-1},$$

where $\mathcal{N} = \mathcal{N}_{12}(\mathbf{D}) + \mathcal{N}_{22}$.

Example. In the one-dimensional case, consider the operator

$$\mathcal{B}_{\lambda,\varepsilon} = -\frac{d}{dx} g^\varepsilon(x) \frac{d}{dx} + \varepsilon^{-1} v^\varepsilon(x) + \lambda I,$$

where $g(x)$ and $v(x)$ are 1-periodic real-valued functions such that $g(x)$ is bounded and positive definite, and $v \in L_1(0,1)$, $\overline{v} = 0$. In this case $\Psi(x)$ is the 1-periodic solution of the problem

$$\frac{d}{dx} g(x) \Big(\frac{d\Psi(x)}{dx} + 1 \Big) = 0, \qquad \int_0^1 \Psi(x)\, dx = 0.$$

This implies that $g^0 = \underline{g}$ and $\widetilde{g}(x) = g(x)(\Psi'(x) + 1) = g^0$. The function $\widetilde{\Lambda}(x) = \widetilde{\Lambda}_1(x)$ is the 1-periodic solution of the problem

$$-\frac{d}{dx} g(x) \frac{d\widetilde{\Lambda}(x)}{dx} + v(x) = 0, \qquad \int_0^1 \widetilde{\Lambda}(x)\, dx = 0.$$

Let $a(x)$ be a 1-periodic function such that $a'(x) = v(x)$. Then $g(x)\widetilde{\Lambda}'(x) = a(x) - g^0 \overline{a g^{-1}}$. The effective operator is given by

$$\mathcal{B}_\lambda^0 = -g^0 \frac{d^2}{dx^2} - W + \lambda I, \quad W = \int_0^1 g(x)(\widetilde{\Lambda}'(x))^2\, dx.$$

Calculation of the operator $\mathcal{N}$ gives

$$\mathcal{N} = -2g^0 (\overline{a \Psi g^{-1}}) \frac{d^2}{dx^2} + \overline{v \widetilde{\Lambda}^2}.$$

The corrector takes the form

$$K_\varepsilon^0 = \Big(\Psi^\varepsilon \frac{d}{dx} + \widetilde{\Lambda}^\varepsilon \Big)(\mathcal{B}_\lambda^0)^{-1} + (\mathcal{B}_\lambda^0)^{-1} \Big(-\frac{d}{dx} \Psi^\varepsilon + \widetilde{\Lambda}^\varepsilon \Big) - (\mathcal{B}_\lambda^0)^{-1} \mathcal{N} (\mathcal{B}_\lambda^0)^{-1}.$$

Note that if the metric $g(x)$ is nonconstant and the singular potential is nonzero, then in general the third term of the corrector is nonzero. The function Ψ is defined only via g. One can choose a periodic function $a(x)$ not orthogonal to Ψg^{-1} in $L_2(0,1)$ and define $v(x) = a'(x)$. Then $\overline{a \Psi g^{-1}} \neq 0$ and, trivially, $\mathcal{N} \neq 0$.

9.2. The periodic Schrödinger operator. In $L_2(\mathbb{R}^d)$, consider the operator $\widetilde{\mathcal{A}} = \mathbf{D}^* \breve{g}(\mathbf{x}) \mathbf{D} + \breve{v}(\mathbf{x})$, where $\breve{g}(\mathbf{x})$ is a Γ-periodic symmetric $(d \times d)$-matrix-valued function with real entries, assumed to be bounded and positive definite; $\breve{v}(\mathbf{x})$ is a real-valued Γ-periodic function such that

$$\breve{v} \in L_s(\Omega), \quad s = 1 \text{ for } d = 1, \quad s > \frac{d}{2} \text{ for } d \geq 2.$$

The precise definition of the operator $\widetilde{\mathcal{A}}$ is in terms of the quadratic form

$$\widetilde{\mathfrak{a}}[u,u] = \int_{\mathbb{R}^d} \big(\langle \breve{g}(\mathbf{x}) \mathbf{D}u, \mathbf{D}u \rangle + \breve{v}(\mathbf{x}) |u|^2 \big)\, d\mathbf{x}, \quad u \in H^1(\mathbb{R}^d). \tag{9.18}$$

By adding an appropriate constant to $\breve{v}$, *we may assume that the bottom of the spectrum of $\widetilde{\mathcal{A}}$ is the point $\lambda_0 = 0$.* Under this condition, the operator $\widetilde{\mathcal{A}}$ admits a convenient factorization (see, e.g., [BSu1, Chapter 6, Subsection 1.1]). To describe this factorization, consider the equation

$$\mathbf{D}^* \breve{g}(\mathbf{x}) \mathbf{D} \omega(\mathbf{x}) + \breve{v}(\mathbf{x}) \omega(\mathbf{x}) = 0, \tag{9.19}$$

which admits a Γ-periodic solution $\omega \in \widetilde{H}^1(\Omega)$ defined up to a constant factor. This factor can be fixed so that $\omega(\mathbf{x}) > 0$ and

$$\int_\Omega \omega^2(\mathbf{x})\, d\mathbf{x} = |\Omega|. \tag{9.20}$$

It turns out that the solution $\omega(\mathbf{x})$ is positive definite and bounded, and the norms $\|\omega\|_{L_\infty}$, $\|\omega^{-1}\|_{L_\infty}$ are controlled in terms of $\|\breve{g}\|_{L_\infty}$, $\|\breve{g}^{-1}\|_{L_\infty}$, and $\|\breve{v}\|_{L_s(\Omega)}$. The function ω is a multiplier both in $H^1(\mathbb{R}^d)$ and in $\widetilde{H}^1(\Omega)$. Substituting $u = \omega\varphi$, we write (9.18) as

$$\widetilde{\mathfrak{a}}[u,u] = \int_{\mathbb{R}^d} \omega^2(\mathbf{x})\langle \breve{g}(\mathbf{x})\mathbf{D}\varphi, \mathbf{D}\varphi\rangle\, d\mathbf{x}, \quad u = \omega\varphi, \;\; \varphi \in H^1(\mathbb{R}^d).$$

This yields the factorization

$$\widetilde{\mathcal{A}} = \omega^{-1}\mathbf{D}^* g \mathbf{D}\omega^{-1}, \quad g = \omega^2 \breve{g}. \tag{9.21}$$

Now, we consider the operator

$$\widetilde{\mathcal{A}}_\varepsilon = (\omega^\varepsilon)^{-1}\mathbf{D}^* g^\varepsilon \mathbf{D}(\omega^\varepsilon)^{-1}. \tag{9.22}$$

In the initial terms, (9.22) can be written as

$$\widetilde{\mathcal{A}}_\varepsilon = \mathbf{D}^* \breve{g}^\varepsilon \mathbf{D} + \varepsilon^{-2}\breve{v}^\varepsilon, \tag{9.23}$$

which can be treated as the Schrödinger operator with rapidly oscillating metric $\breve{g}^\varepsilon$ and strongly singular potential $\varepsilon^{-2}\breve{v}^\varepsilon$.

Next, let $\mathbf{A}(\mathbf{x}) = \operatorname{col}\{A_1(\mathbf{x}), \dots, A_d(\mathbf{x})\}$, where the $A_j(\mathbf{x})$ are Γ-periodic real-valued functions satisfying (9.1). Let $\widetilde{v}(\mathbf{x})$ and $\widetilde{\mathcal{V}}(\mathbf{x})$ be Γ-periodic real-valued functions such that

$$\begin{aligned} &\widetilde{v}, \widetilde{\mathcal{V}} \in L_s(\Omega), \quad s = 1 \text{ for } d = 1, \quad s > \frac{d}{2} \text{ for } d \geq 2; \\ &\int_\Omega \widetilde{v}(\mathbf{x})\omega^2(\mathbf{x})\, d\mathbf{x} = 0. \end{aligned} \tag{9.24}$$

Consider the operator $\widetilde{\mathcal{B}}_\varepsilon$ defined formally by the expression

$$\widetilde{\mathcal{B}}_\varepsilon = (\mathbf{D} - \mathbf{A}^\varepsilon)^* \breve{g}^\varepsilon (\mathbf{D} - \mathbf{A}^\varepsilon) + \varepsilon^{-2}\breve{v}^\varepsilon + \varepsilon^{-1}\widetilde{v}^\varepsilon + \widetilde{\mathcal{V}}^\varepsilon. \tag{9.25}$$

(The precise definition is given in terms of quadratic forms.) The operator (9.25) can be treated as the Schrödinger operator with the metric $\breve{g}^\varepsilon$, the magnetic potential $\mathbf{A}^\varepsilon$, and the electric potential $\varepsilon^{-2}\breve{v}^\varepsilon + \varepsilon^{-1}\widetilde{v}^\varepsilon + \widetilde{\mathcal{V}}^\varepsilon$ containing the singular terms $\varepsilon^{-2}\breve{v}^\varepsilon$ and $\varepsilon^{-1}\widetilde{v}^\varepsilon$.

We put

$$v(\mathbf{x}) := \widetilde{v}(\mathbf{x})\omega^2(\mathbf{x}), \quad \mathcal{V}(\mathbf{x}) := \widetilde{\mathcal{V}}(\mathbf{x})\omega^2(\mathbf{x}). \tag{9.26}$$

Taking (9.22) and (9.23) into account, we check that $\widetilde{\mathcal{B}}_\varepsilon = (\omega^\varepsilon)^{-1}\mathcal{B}_\varepsilon(\omega^\varepsilon)^{-1}$, where the operator $\mathcal{B}_\varepsilon$ is given by (9.3) with g defined by (9.21) and v, $\mathcal{V}$ defined by (9.26). By (9.24), the coefficients v and $\mathcal{V}$ satisfy the required conditions (9.2).

Let $\widetilde{Q}_0(\mathbf{x})$ be a Γ-periodic real-valued function; it is assumed to be positive definite and bounded. We put

$$\widetilde{\mathcal{B}}_{\lambda,\varepsilon} = \widetilde{\mathcal{B}}_\varepsilon + \lambda \widetilde{Q}_0^\varepsilon. \tag{9.27}$$

Denote $Q_0(\mathbf{x}) = \widetilde{Q}_0(\mathbf{x})\omega^2(\mathbf{x})$. The operator (9.27) is related to the operator $\mathcal{B}_{\lambda,\varepsilon} = \mathcal{B}_\varepsilon + \lambda Q_0^\varepsilon$ by the formula

$$\widetilde{\mathcal{B}}_{\lambda,\varepsilon} = (\omega^\varepsilon)^{-1}\mathcal{B}_{\lambda,\varepsilon}(\omega^\varepsilon)^{-1}. \tag{9.28}$$

Now the assumptions of Subsection 8.2 are realized with $f = \omega^{-1}$. The parameter λ is subject to the restriction (8.12) with $f = \omega^{-1}$ and with constants c_0, c_4 corresponding to the operator $\mathcal{B}_\varepsilon$. Now the initial data reduce to the following set of parameters:

$$
\begin{aligned}
&d, \rho, s;\ \|\breve{g}\|_{L_\infty},\ \|\breve{g}^{-1}\|_{L_\infty}, \|\mathbf{A}\|_{L_\rho(\Omega)},\ \|\breve{v}\|_{L_s(\Omega)},\ \|\widetilde{v}\|_{L_s(\Omega)},\ \|\widetilde{\mathcal{V}}\|_{L_s(\Omega)}, \\
&\lambda,\ \|\widetilde{Q}_0\|_{L_\infty},\ \|\widetilde{Q}_0^{-1}\|_{L_\infty};\ \text{the parameters of the lattice } \Gamma.
\end{aligned} \tag{9.29}
$$

Proposition 9.1 and the representation (9.28) directly imply the following result (cf. Theorem 8.2).

Proposition 9.3. *Suppose that $\widetilde{\mathcal{B}}_{\lambda,\varepsilon}$ is the operator defined by* (9.25), (9.27), *whose coefficients $\breve{g}^\varepsilon$, $\mathbf{A}^\varepsilon$, $\breve{v}^\varepsilon$, $\widetilde{v}^\varepsilon$, $\widetilde{\mathcal{V}}^\varepsilon$, and $\widetilde{Q}_0^\varepsilon$ satisfy the conditions of Subsection* 9.2. *Let $\omega(\mathbf{x})$ be the Γ-periodic positive solution of equation* (9.19) *satisfying* (9.20). *Suppose that $\mathcal{B}_{\lambda,\varepsilon}$ is the operator defined by* (9.3), (9.7) *with the coefficients $g^\varepsilon = \breve{g}^\varepsilon(\omega^\varepsilon)^2$, $\mathbf{A}^\varepsilon$, $v^\varepsilon = \widetilde{v}^\varepsilon(\omega^\varepsilon)^2$, $\mathcal{V}^\varepsilon = \widetilde{\mathcal{V}}^\varepsilon(\omega^\varepsilon)^2$, and $Q_0^\varepsilon = \widetilde{Q}_0^\varepsilon(\omega^\varepsilon)^2$. Let $\mathcal{B}_\lambda^0$ be the effective operator for $\mathcal{B}_{\lambda,\varepsilon}$ defined by* (9.9), (9.10). *Let K_ε^0 be the corrector* (9.16) *for the operator $\mathcal{B}_{\lambda,\varepsilon}$. Then*

$$
\big\|\widetilde{\mathcal{B}}_{\lambda,\varepsilon}^{-1} - \omega^\varepsilon(\mathcal{B}_\lambda^0)^{-1}\omega^\varepsilon\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le \widetilde{C}_0\varepsilon, \quad 0<\varepsilon\le 1,
$$
$$
\big\|\widetilde{\mathcal{B}}_{\lambda,\varepsilon}^{-1} - \omega^\varepsilon\big((\mathcal{B}_\lambda^0)^{-1} + \varepsilon K_\varepsilon^0\big)\omega^\varepsilon\big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le \widetilde{\mathcal{C}}^0\varepsilon^2, \quad 0<\varepsilon\le 1.
$$

The constants $\widetilde{C}_0$ and $\widetilde{\mathcal{C}}^0$ depend only on the initial data (9.29).

Remark 9.4. By Remark 9.2, the expressions for the effective operator and the corrector simplify if the magnetic potential is subject to (9.17), i.e., $\operatorname{div}\breve{g}(\mathbf{x})\omega^2(\mathbf{x})\mathbf{A}(\mathbf{x}) = 0$, $\overline{\breve{g}\omega^2\mathbf{A}} = 0$; but now these conditions involve the function $\omega(\mathbf{x})$.

§10. Application of the general pattern: the two-dimensional Pauli operator

10.1. Model operators. We start with the model examples considered before in [Su2, Subsection 14.1]. Let $d=2$, and let $\omega_\pm(\mathbf{x})$ be Γ-periodic functions in $\mathbb{R}^2$ such that

$$
\omega_\pm(\mathbf{x}) > 0, \quad \omega_\pm, \omega_\pm^{-1} \in L_\infty. \tag{10.1$_\pm$}
$$

Denote $\partial_+ := D_1 + iD_2$, $\partial_- := D_1 - iD_2$. In $L_2(\mathbb{R}^2)$, we consider the pair of operators

$$
\mathcal{A}_{+,\varepsilon} = \partial_+(\omega_+^\varepsilon)^2\partial_-, \quad \mathcal{A}_{-,\varepsilon} = \partial_-(\omega_-^\varepsilon)^2\partial_+. \tag{10.2$_\pm$}
$$

The operator $\mathcal{A}_{\pm,\varepsilon}$ is of the form (7.1) with $m=n=1$, $b(\mathbf{D}) = D_1 \mp iD_2$, and $g = g_\pm = \omega_\pm^2$. Obviously, $\alpha_0 = \alpha_1 = 1$ (see (3.2)).

Next, let $\boldsymbol{\eta}_\pm(\mathbf{x}) = \operatorname{col}\{\eta_{1,\pm}(\mathbf{x}), \eta_{2,\pm}(\mathbf{x})\}$, where the $\eta_{j,\pm}(\mathbf{x})$, $j=1,2$, are Γ-periodic real-valued functions such that

$$
\eta_{j,\pm} \in L_\rho(\Omega), \quad \rho > 2,\ \ j=1,2.
$$

Let $v_\pm(\mathbf{x})$ and $Q_\pm(\mathbf{x})$ be real-valued Γ-periodic functions such that

$$
v_\pm, Q_\pm \in L_s(\Omega),\ \ s>1; \quad \int_\Omega v_\pm(\mathbf{x})\,d\mathbf{x} = 0.
$$

Consider the operator $\mathcal{B}_{\pm,\varepsilon}$ given formally by

$$
\mathcal{B}_{\pm,\varepsilon}u = \mathcal{A}_{\pm,\varepsilon}u + i\sum_{j=1}^{2}\big(\eta_{j,\pm}^\varepsilon\partial_j u + \partial_j(\eta_{j,\pm}^\varepsilon u)\big) + \varepsilon^{-1}v_\pm^\varepsilon u + Q_\pm^\varepsilon u. \tag{10.3$_\pm$}
$$

(The precise definition is in terms of the corresponding quadratic forms.)

In order to represent this operator in the form (7.9) (cf. Subsection 8.1), we introduce the periodic solution $\Phi_\pm$ of the equation $\Delta\Phi_\pm = v_\pm$ and put $\zeta_{j,\pm} = -\partial_j \Phi_\pm$. Then

$$(10.4)_\pm \qquad v_\pm(\mathbf{x}) = -\sum_{j=1}^{2} \partial_j \zeta_{j,\pm}(\mathbf{x}).$$

Denote $a_{j,\pm}(\mathbf{x}) = -\eta_{j,\pm}(\mathbf{x}) + i\zeta_{j,\pm}(\mathbf{x})$, $j = 1, 2$. It is easily seen that $a_{j,\pm} \in L_{\rho'}(\Omega)$ with some $\rho' > 2$ (ρ' depends on ρ and s), and that $\|a_{j,\pm}\|_{L_{\rho'}(\Omega)}$ is controlled in terms of $\|\eta_{j,\pm}\|_{L_\rho(\Omega)}$, $\|v_\pm\|_{L_s(\Omega)}$, and the parameters of Γ; for the details, see [Su2, Subsection 14.1]; cf. Subsection 8.1.

The operator $(10.3)_\pm$ can be written as

$$\mathcal{B}_{\pm,\varepsilon} = \mathcal{A}_{\pm,\varepsilon} + \sum_{j=1}^{2} \left(a^\varepsilon_{j,\pm} D_j + D_j (a^\varepsilon_{j,\pm})^* \right) + Q^\varepsilon_\pm,$$

which corresponds to the general form (7.9). All the required assumptions on the coefficients are fulfilled. Let $Q_{0,\pm}$ be a Γ-periodic, bounded, and positive definite function. We put

$$(10.5)_\pm \qquad \mathcal{B}_{\pm,\lambda,\varepsilon} = \mathcal{B}_{\pm,\varepsilon} + \lambda Q^\varepsilon_{0,\pm}.$$

Suppose that λ is subject to the restriction (3.24) with $Q_0 = Q_{0,\pm}$ and c_0, c_4 corresponding to the operator $\mathcal{B}_{\pm,\varepsilon}$. The initial data reduce to the following set of parameters:

$$(10.6)_\pm \qquad \begin{aligned} &\rho, s;\ \|\omega_\pm\|_{L_\infty},\ \|\omega_\pm^{-1}\|_{L_\infty},\ \|\boldsymbol{\eta}_\pm\|_{L_\rho(\Omega)},\ \|v_\pm\|_{L_s(\Omega)},\ \|Q_\pm\|_{L_s(\Omega)}, \\ &\lambda,\ \|Q_{0,\pm}\|_{L_\infty}\ \|Q_{0,\pm}^{-1}\|_{L_\infty};\ \text{ the parameters of the lattice } \Gamma. \end{aligned}$$

Now we describe the effective operator. Since $m = n = 1$, Proposition 4.1 shows that the effective constant $g^0_\pm$ coincides with $\underline{g_\pm}$, i.e.,

$$g^0_\pm = \left(|\Omega|^{-1} \int_\Omega (\omega_\pm(\mathbf{x}))^{-2}\, d\mathbf{x} \right)^{-1}.$$

The effective operator for $\mathcal{A}_{\pm,\varepsilon}$ is given by $\mathcal{A}^0_\pm = -g^0_\pm \Delta$.

Now the role of $\Lambda(\mathbf{x})$ is played by the function $\Lambda_\pm(\mathbf{x})$ that is a Γ-periodic solution of the problem

$$(10.7)_\pm \qquad \partial_\pm \omega^2_\pm(\mathbf{x})(\partial_\mp \Lambda_\pm(\mathbf{x}) + 1) = 0, \qquad \int_\Omega \Lambda_\pm(\mathbf{x})\, d\mathbf{x} = 0.$$

The solution of problem $(10.7)_\pm$ satisfies $\omega^2_\pm(\mathbf{x})(\partial_\mp \Lambda_\pm(\mathbf{x}) + 1) = g^0_\pm$, whence $\Lambda_\pm$ is also a periodic solution of the problem

$$-\Delta\Lambda_\pm(\mathbf{x}) = g^0_\pm \partial_\pm (\omega_\pm(\mathbf{x}))^{-2}, \qquad \int_\Omega \Lambda_\pm(\mathbf{x})\, d\mathbf{x} = 0.$$

The role of $\widetilde{\Lambda}(\mathbf{x})$ is played by the Γ-periodic solution $\widetilde{\Lambda}_\pm(\mathbf{x})$ of the problem

$$\partial_\pm \omega^2_\pm(\mathbf{x}) \partial_\mp \widetilde{\Lambda}_\pm(\mathbf{x}) + v_\pm(\mathbf{x}) + i \operatorname{div} \boldsymbol{\eta}_\pm(\mathbf{x}) = 0, \qquad \int_\Omega \widetilde{\Lambda}_\pm(\mathbf{x})\, d\mathbf{x} = 0.$$

The numbers (5.2) and (5.3) for the operators $\mathcal{B}_{\pm,\varepsilon}$ take the form

$$\begin{aligned} V_\pm &= |\Omega|^{-1} \int_\Omega (\partial_\mp \Lambda_\pm(\mathbf{x}))^* \omega^2_\pm(\mathbf{x}) (\partial_\mp \widetilde{\Lambda}_\pm(\mathbf{x}))\, d\mathbf{x}, \\ W_\pm &= |\Omega|^{-1} \int_\Omega \omega^2_\pm(\mathbf{x}) |\partial_\mp \widetilde{\Lambda}_\pm(\mathbf{x})|^2\, d\mathbf{x}. \end{aligned}$$

Calculating the effective operator by the general rule (7.12), we obtain

$$(10.8)_\pm \qquad \mathcal{B}^0_{\pm,\lambda} = -g^0_\pm \Delta + 2i(\eta^0_{1,\pm}\partial_1 + \eta^0_{2,\pm}\partial_2) + \overline{Q_\pm} - W_\pm + \lambda\overline{Q_{0,\pm}},$$

where $\eta^0_{1,\pm} = \operatorname{Re} V_\pm + \overline{\eta_{1,\pm}}$, $\eta^0_{2,\pm} = \mp \operatorname{Im} V_\pm + \overline{\eta_{2,\pm}}$.

Now we proceed to the description of the corrector. By Proposition 7.4(1°), we can use the corrector (7.24), which does not involve the smoothing operator. The first term of the corrector $K^0_{\pm,\varepsilon}$ is given by $(\Lambda^\varepsilon_\pm \partial_\mp + \widetilde{\Lambda}^\varepsilon_\pm)(\mathcal{B}^0_{\pm,\lambda})^{-1}$; the second term is adjoint to the first. In order to find the third term, we need to calculate the operator $\mathcal{N}_\pm$ of the form (7.18). By Proposition 7.6, we have

$$(10.9)_\pm \qquad \mathcal{N}_\pm = \mathcal{N}_{12,\pm}(\mathbf{D}) + \mathcal{N}_{21,\pm}(\mathbf{D}) + \mathcal{N}_{22,\pm}.$$

In accordance with (6.4), (5.17), (6.22), and $(10.4)_\pm$, the first summand in $(10.9)_\pm$ takes the form

$$(10.10)_\pm \qquad \mathcal{N}_{12,\pm}(\mathbf{D}) = M_{1,\pm}(\mathbf{D})\partial_\mp + \partial_\pm M_{1,\pm}(\mathbf{D})^* - T_{0,\pm}\Delta,$$

where

$$(10.11)_\pm \qquad T_{0,\pm} = \sum_{j=1}^{2} i\, \overline{\eta_{j,\pm}\left(\Lambda^*_\pm \partial_j \Lambda_\pm - \Lambda_\pm \partial_j \Lambda^*_\pm\right)} + \overline{v_\pm |\Lambda_\pm|^2},$$

$$(10.12)_\pm \qquad M_{1,\pm}(\mathbf{D}) = \overline{(\partial_\mp \widetilde{\Lambda}_\pm)^* \omega^2_\pm \Lambda_\pm}(D_1 \mp iD_2) - 2\sum_{j=1}^{2} \overline{\eta_{j,\pm}\Lambda_\pm} D_j.$$

By (6.5), (5.22), (5.28), and $(10.4)_\pm$, the second term in $(10.9)_\pm$ is given by

$$(10.13)_\pm \qquad \mathcal{N}_{21,\pm}(\mathbf{D}) = M_{2,\pm}(\mathbf{D}) + M_{2,\pm}(\mathbf{D})^* + T^*_\pm \partial_\mp + \partial_\pm T_\pm - 4\sum_{j=1}^{2} \operatorname{Re} \overline{\eta_{j,\pm}\widetilde{\Lambda}_\pm} D_j,$$

where

$$(10.14)_\pm \qquad M_{2,\pm}(\mathbf{D}) = \overline{(\partial_\mp \widetilde{\Lambda}_\pm)^* \omega^2_\pm \widetilde{\Lambda}_\pm}(D_1 \mp iD_2),$$

$$(10.15)_\pm \qquad T_\pm = \sum_{j=1}^{2} i\, \overline{\eta_{j,\pm}\left(\Lambda^*_\pm \partial_j \widetilde{\Lambda}_\pm - \widetilde{\Lambda}_\pm \partial_j \Lambda^*_\pm\right)} + \overline{v_\pm \Lambda^*_\pm \widetilde{\Lambda}_\pm} + \overline{\Lambda^*_\pm (Q_\pm + \lambda Q_{0,\pm})}.$$

Finally, the third term in $(10.9)_\pm$ is calculated with the help of (5.31), (5.36), and $(10.4)_\pm$:

$$(10.16)_\pm \quad \mathcal{N}_{22,\pm} = \sum_{j=1}^{2} i\, \overline{\eta_{j,\pm}\left(\widetilde{\Lambda}^*_\pm \partial_j \widetilde{\Lambda}_\pm - \widetilde{\Lambda}_\pm \partial_j \widetilde{\Lambda}^*_\pm\right)} + \overline{v_\pm |\widetilde{\Lambda}_\pm|^2} + \overline{2(\operatorname{Re}\widetilde{\Lambda}_\pm)(Q_\pm + \lambda Q_{0,\pm})}.$$

As a result, the corrector for the operator $(10.5)_\pm$ takes the form

$$(10.17)_\pm \qquad \begin{aligned} K^0_{\pm,\varepsilon} = \left(\Lambda^\varepsilon_\pm \partial_\mp + \widetilde{\Lambda}^\varepsilon_\pm\right)(\mathcal{B}^0_{\pm,\lambda})^{-1} + (\mathcal{B}^0_{\pm,\lambda})^{-1}\left(\Lambda^\varepsilon_\pm \partial_\mp + \widetilde{\Lambda}^\varepsilon_\pm\right)^* \\ - (\mathcal{B}^0_{\pm,\lambda})^{-1}\mathcal{N}_\pm(\mathcal{B}^0_{\pm,\lambda})^{-1}, \end{aligned}$$

where the operator $\mathcal{N}_\pm$ is defined by $(10.9)_\pm$–$(10.16)_\pm$. Applying Theorem 7.1 and Proposition 7.4(1°), we obtain the following result.

Proposition 10.1($\pm$). *Let $\mathcal{B}_{\pm,\lambda,\varepsilon}$ be the operator defined by* $(10.2)_\pm$, $(10.3)_\pm$, $(10.5)_\pm$, *with the coefficients satisfying the assumptions of Subsection* 10.1. *Suppose that $\mathcal{B}^0_{\pm,\lambda}$ is* *the operator defined by* $(10.8)_\pm$ *with the effective coefficients described in Subsection* 10.1. *Let* $K^0_{\pm,\varepsilon}$ *be the corrector* $(10.17)_\pm$. *Then*

$$\big\|\mathcal{B}^{-1}_{\pm,\lambda,\varepsilon} - (\mathcal{B}^0_{\pm,\lambda})^{-1}\big\|_{L_2(\mathbb{R}^2)\to L_2(\mathbb{R}^2)} \le C_{0,\pm}\varepsilon, \quad 0<\varepsilon\le 1,$$
$$\big\|\mathcal{B}^{-1}_{\pm,\lambda,\varepsilon} - (\mathcal{B}^0_{\pm,\lambda})^{-1} - \varepsilon K^0_{\pm,\varepsilon}\big\|_{L_2(\mathbb{R}^2)\to L_2(\mathbb{R}^2)} \le \mathcal{C}^0_{\pm}\varepsilon^2, \quad 0<\varepsilon\le 1.$$

The constants $C_{0,\pm}$ *and* $\mathcal{C}^0_\pm$ *depend only on the initial data* $(10.6)_\pm$.

10.2. The matrix model operator. Now we consider the matrix model operator studied before in [Su2, Subsection 14.2]. Recall the standard notation for the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Consider a pair of Γ-periodic functions $\omega_+(\mathbf{x})$, $\omega_-(\mathbf{x})$ in $\mathbb{R}^2$ satisfying the previous conditions $(10.1)_\pm$. Let $\mathcal{A}_{+,\varepsilon}$ and $\mathcal{A}_{-,\varepsilon}$ be the operators defined by $(10.2)_\pm$. In $L_2(\mathbb{R}^2;\mathbb{C}^2)$, we consider the matrix operator

$$\mathcal{A}_{\times,\varepsilon} = \operatorname{diag}\{\mathcal{A}_{-,\varepsilon}, \mathcal{A}_{+,\varepsilon}\}. \tag{10.18}$$

The operator (10.18) can be written in a factorized form

$$\mathcal{A}_{\times,\varepsilon} = b_\times(\mathbf{D}) g^\varepsilon_\times b_\times(\mathbf{D}), \tag{10.19}$$

where

$$\begin{gathered} b_\times(\mathbf{D}) = b_\times(\mathbf{D})^* = D_1\sigma_1 + D_2\sigma_2 = \begin{pmatrix} 0 & \partial_- \\ \partial_+ & 0 \end{pmatrix}, \\ h_\times = \operatorname{diag}\{\omega_+, \omega_-\}, \quad g_\times = h_\times^2 = \operatorname{diag}\{\omega_+^2, \omega_-^2\}. \end{gathered} \tag{10.20}$$

The operator (10.19) is of the form (7.1) with $m=n=2$, $g=g_\times$, and $b(\mathbf{D}) = b_\times(\mathbf{D})$.

Next, let $\eta_{j,\times}(\mathbf{x})$, $j=1,2$, $v_\times(\mathbf{x})$, and $Q_\times(\mathbf{x})$ be Γ-periodic Hermitian (2×2)-matrix-valued functions in $\mathbb{R}^2$ such that

$$\eta_{j,\times} \in L_\rho(\Omega), \quad \rho>2, \ \ j=1,2;$$
$$v_\times, Q_\times \in L_s(\Omega), \quad s>1; \quad \int_\Omega v_\times(\mathbf{x})\,d\mathbf{x} = 0.$$

Consider the operator $\mathcal{B}_{\times,\varepsilon}$ given formally by

$$\mathcal{B}_{\times,\varepsilon}\mathbf{u} = \mathcal{A}_{\times,\varepsilon}\mathbf{u} + i\sum_{j=1}^2 (\eta^\varepsilon_{j,\times}\partial_j \mathbf{u} + \partial_j(\eta^\varepsilon_{j,\times}\mathbf{u})) + \varepsilon^{-1} v^\varepsilon_\times \mathbf{u} + Q^\varepsilon_\times \mathbf{u}. \tag{10.21}$$

(The precise definition is in terms of the corresponding quadratic form.) In order to represent the operator (10.21) in the required form (7.9), we consider a Γ-periodic solution $\Phi_\times(\mathbf{x})$ of the equation $\Delta\Phi_\times = v_\times$ and put $\zeta_{j,\times} = -\partial_j\Phi_\times$. Then

$$v_\times = -\sum_{j=1}^2 \partial_j \zeta_{j,\times}. \tag{10.22}$$

We put $a_{j,\times} = -\eta_{j,\times} + i\zeta_{j,\times}$, $j=1,2$. It is easily seen that $a_{j,\times} \in L_{\rho'}(\Omega)$ with some $\rho'>2$ (ρ' depends on ρ and s), and that $\|a_{j,\times}\|_{L_{\rho'}(\Omega)}$ is controlled in terms of $\|\eta_{j,\times}\|_{L_\rho(\Omega)}$, $\|v_\times\|_{L_s(\Omega)}$, and the parameters of Γ; for the details, see [Su2, Subsection 14.2]; cf. Subsection 8.1. The operator (10.21) can be written as

$$\mathcal{B}_{\times,\varepsilon} = \mathcal{A}_{\times,\varepsilon} + \sum_{j=1}^2 (a^\varepsilon_{j,\times} D_j + D_j (a^\varepsilon_{j,\times})^*) + Q^\varepsilon_\times, \tag{10.23}$$

which corresponds to (7.9). All the required conditions on the coefficients are satisfied. Let $Q_{0,\times}(\mathbf{x})$ be a Γ-periodic, bounded, and positive definite (2×2)-matrix-valued function. Denote

$$\mathcal{B}_{\times,\lambda,\varepsilon} := \mathcal{B}_{\times,\varepsilon} + \lambda Q^{\varepsilon}_{0,\times}. \tag{10.24}$$

Suppose that λ satisfies condition (3.24) with $Q_0 = Q_{0,\times}$ and c_0, c_4 corresponding to the operator $\mathcal{B}_{\times,\varepsilon}$. The initial data for the operator (10.24) reduce to the set of parameters

$$\begin{gathered}\rho, s;\ \|\omega_+\|_{L_\infty},\ \|\omega_+^{-1}\|_{L_\infty},\ \|\omega_-\|_{L_\infty},\ \|\omega_-^{-1}\|_{L_\infty},\ \|\eta_{j,\times}\|_{L_\rho(\Omega)},\ j=1,2;\\ \|v_\times\|_{L_s(\Omega)},\ \|Q_\times\|_{L_s(\Omega)},\ \lambda,\ \|Q_{0,\times}\|_{L_\infty},\ \|Q_{0,\times}^{-1}\|_{L_\infty};\\ \text{the parameters of the lattice } \Gamma.\end{gathered} \tag{10.25}$$

We describe the effective operator. Since $m = n = 2$, Proposition 4.1 shows that the effective matrix $g^0_\times$ coincides with $\underline{g_\times}$, i.e.,

$$g^0_\times = \underline{g_\times} = \operatorname{diag}\{g^0_+, g^0_-\}.$$

The effective operator for $\mathcal{A}_{\times,\varepsilon}$ is given by $\mathcal{A}^0_\times = -g^0_\times \Delta$.

The role of Λ is played by the Γ-periodic solution $\Lambda_\times$ of the problem

$$b_\times(\mathbf{D}) g_\times(\mathbf{x})(b_\times(\mathbf{D})\Lambda_\times(\mathbf{x}) + \mathbf{1}_2) = 0, \quad \int_\Omega \Lambda_\times(\mathbf{x})\, d\mathbf{x} = 0.$$

The matrix-valued function $\Lambda_\times$ can be expressed in terms of the solutions $\Lambda_\pm$ of problems $(10.7)_\pm$:

$$\Lambda_\times(\mathbf{x}) = \begin{pmatrix} 0 & \Lambda_-(\mathbf{x}) \\ \Lambda_+(\mathbf{x}) & 0 \end{pmatrix}.$$

The role of $\widetilde{\Lambda}$ is played by the Γ-periodic solution $\widetilde{\Lambda}_\times$ of the problem

$$b_\times(\mathbf{D}) g_\times(\mathbf{x}) b_\times(\mathbf{D})\widetilde{\Lambda}_\times(\mathbf{x}) + v_\times(\mathbf{x}) + i\sum_{j=1}^{2} \partial_j \eta_{j,\times}(\mathbf{x}) = 0, \quad \int_\Omega \widetilde{\Lambda}_\times(\mathbf{x})\, d\mathbf{x} = 0.$$

The matrices (5.2) and (5.3) take the form

$$\begin{aligned} V_\times &= |\Omega|^{-1} \int_\Omega (b_\times(\mathbf{D})\Lambda_\times(\mathbf{x}))^* g_\times(\mathbf{x})(b_\times(\mathbf{D})\widetilde{\Lambda}_\times(\mathbf{x}))\, d\mathbf{x},\\ W_\times &= |\Omega|^{-1} \int_\Omega (b_\times(\mathbf{D})\widetilde{\Lambda}_\times(\mathbf{x}))^* g_\times(\mathbf{x})(b_\times(\mathbf{D})\widetilde{\Lambda}_\times(\mathbf{x}))\, d\mathbf{x}.\end{aligned}$$

Calculating the effective operator by the general rule (7.12), we obtain

$$\mathcal{B}^0_{\times,\lambda} = -g^0_\times \Delta + 2i(\eta^0_{1,\times}\partial_1 + \eta^0_{2,\times}\partial_2) + \overline{Q_\times} - W_\times + \lambda\overline{Q_{0,\times}}, \tag{10.26}$$

where $2\eta^0_{1,\times} = 2\overline{\eta_{1,\times}} + \sigma_1 V_\times + V^*_\times \sigma_1$, $2\eta^0_{2,\times} = 2\overline{\eta_{2,\times}} + \sigma_2 V_\times + V^*_\times\sigma_2$.

By Proposition 7.4(1°), we can use the corrector $K^0_{\times,\varepsilon}$ of the form (7.24). The first term of this corrector is given by $(\Lambda^\varepsilon_\times b_\times(\mathbf{D}) + \widetilde{\Lambda}^\varepsilon_\times)(\mathcal{B}^0_{\times,\lambda})^{-1}$, the second term is adjoint to the first. In order to find the third term of the corrector, we need to calculate the operator $\mathcal{N}_\times$ of the form (7.18). By Proposition 7.6,

$$\mathcal{N}_\times = \mathcal{N}_{12,\times}(\mathbf{D}) + \mathcal{N}_{21,\times}(\mathbf{D}) + \mathcal{N}_{22,\times}. \tag{10.27}$$

In accordance with (6.4), (5.17), (6.22), and (10.22), the first term in (10.27) is represented as

$$\mathcal{N}_{12,\times}(\mathbf{D}) = M_{1,\times}(\mathbf{D}) b_\times(\mathbf{D}) + b_\times(\mathbf{D}) M_{1,\times}(\mathbf{D})^* + b_\times(\mathbf{D}) T_{0,\times} b_\times(\mathbf{D}), \tag{10.28}$$

where

$$T_{0,\times} = \sum_{j=1}^{2} i \left(\overline{\Lambda_\times^* \eta_{j,\times} \partial_j \Lambda_\times} - \overline{(\partial_j \Lambda_\times^*) \eta_{j,\times} \Lambda_\times} \right) + \overline{\Lambda_\times^* v_\times \Lambda_\times}, \tag{10.29}$$

$$M_{1,\times}(\mathbf{D}) = \overline{(b_\times(\mathbf{D})\widetilde{\Lambda}_\times)^* \operatorname{diag}\{\omega_+^2 \Lambda_+, \omega_-^2 \Lambda_-\}} \begin{pmatrix} \partial_- & 0 \\ 0 & \partial_+ \end{pmatrix} - 2 \sum_{j=1}^{2} \overline{\eta_{j,\times} \Lambda_\times} D_j. \tag{10.30}$$

By (6.5), (5.22), (5.28), and (10.22), the second term in (10.27) is given by

$$\mathcal{N}_{21,\times}(\mathbf{D}) = M_{2,\times}(\mathbf{D}) + M_{2,\times}(\mathbf{D})^* + T_\times^* b_\times(\mathbf{D}) + b_\times(\mathbf{D}) T_\times - 4 \sum_{j=1}^{2} \operatorname{Re} \overline{\eta_{j,\times} \widetilde{\Lambda}_\times} D_j, \tag{10.31}$$

where

$$T_\times = \sum_{j=1}^{2} i \left(\overline{\Lambda_\times^* \eta_{j,\times} \partial_j \widetilde{\Lambda}_\times} - \overline{(\partial_j \Lambda_\times^*) \eta_{j,\times} \widetilde{\Lambda}_\times} \right) + \overline{\Lambda_\times^* v_\times \widetilde{\Lambda}_\times} + \overline{\Lambda_\times^* (Q_\times + \lambda Q_{0,\times})}, \tag{10.32}$$

and $M_{2,\times}(\mathbf{D})$ is the DO with the symbol

$$M_{2,\times}(\boldsymbol{\xi}) = \overline{(b_\times(\mathbf{D})\widetilde{\Lambda}_\times)^* g_\times b_\times(\boldsymbol{\xi}) \widetilde{\Lambda}_\times}. \tag{10.33}$$

The third term in (10.27) can be found with the help of (5.31), (5.36), and (10.22):

$$\begin{aligned} \mathcal{N}_{22,\times} = \sum_{j=1}^{2} i \left(\overline{\widetilde{\Lambda}_\times^* \eta_{j,\times} \partial_j \widetilde{\Lambda}_\times} - \overline{(\partial_j \widetilde{\Lambda}_\times^*) \eta_{j,\times} \widetilde{\Lambda}_\times} \right) + \overline{\widetilde{\Lambda}_\times^* v_\times \widetilde{\Lambda}_\times} \\ + 2 \operatorname{Re} \overline{\widetilde{\Lambda}_\times^* (Q_\times + \lambda Q_{0,\times})}. \end{aligned} \tag{10.34}$$

As a result, the corrector for the operator (10.24) takes the form

$$\begin{aligned} K_{\times,\varepsilon}^0 = \left(\Lambda_\times^\varepsilon b_\times(\mathbf{D}) + \widetilde{\Lambda}_\times^\varepsilon \right) \left(\mathcal{B}_{\times,\lambda}^0 \right)^{-1} + \left(\mathcal{B}_{\times,\lambda}^0 \right)^{-1} \left(\Lambda_\times^\varepsilon b_\times(\mathbf{D}) + \widetilde{\Lambda}_\times^\varepsilon \right)^* \\ - \left(\mathcal{B}_{\times,\lambda}^0 \right)^{-1} \mathcal{N}_\times \left(\mathcal{B}_{\times,\lambda}^0 \right)^{-1}, \end{aligned} \tag{10.35}$$

where the operator $\mathcal{N}_\times$ is defined as in (10.27)–(10.34). Theorem 7.1 and Proposition 7.4(1°) lead to the following statement.

Proposition 10.2. *Suppose that $\mathcal{B}_{\times,\lambda,\varepsilon}$ is the operator defined by* (10.19), (10.21), (10.24), *with the coefficients satisfying the assumptions of Subsection* 10.2. *Let $\mathcal{B}_{\times,\lambda}^0$ be the operator* (10.26) *with the effective coefficients described above in Subsection* 10.2. *Let $K_{\times,\varepsilon}^0$ be the corrector* (10.35). *Then*

$$\left\| \mathcal{B}_{\times,\lambda,\varepsilon}^{-1} - (\mathcal{B}_{\times,\lambda}^0)^{-1} \right\|_{L_2(\mathbb{R}^2) \to L_2(\mathbb{R}^2)} \le C_{0,\times} \varepsilon, \quad 0 < \varepsilon \le 1,$$

$$\left\| \mathcal{B}_{\times,\lambda,\varepsilon}^{-1} - (\mathcal{B}_{\times,\lambda}^0)^{-1} - \varepsilon K_{\times,\varepsilon}^0 \right\|_{L_2(\mathbb{R}^2) \to L_2(\mathbb{R}^2)} \le \mathcal{C}_\times^0 \varepsilon^2, \quad 0 < \varepsilon \le 1.$$

The constants $C_{0,\times}$ and $\mathcal{C}_\times^0$ depend only on the initial data (10.25).

10.3. Definition and factorization of the Pauli operator. (See, e.g., [BSu1, Chapter 6, §2].) Suppose we are given a magnetic potential $\breve{\mathbf{A}}(\mathbf{x}) = \operatorname{col}\{\breve{A}_1(\mathbf{x}), \breve{A}_2(\mathbf{x})\}$, where the $\breve{A}_j(\mathbf{x})$ are Γ-periodic real-valued functions in $\mathbb{R}^2$ such that

$$\breve{A}_j \in L_\rho(\Omega), \quad \rho > 2, \;\; j = 1, 2. \tag{10.36}$$

In $L_2(\mathbb{R}^2; \mathbb{C}^2)$, we consider the selfadjoint operator

$$\mathcal{D} = (D_1 - \breve{A}_1)\sigma_1 + (D_2 - \breve{A}_2)\sigma_2, \quad \operatorname{Dom} \mathcal{D} = H^1(\mathbb{R}^2; \mathbb{C}^2). \tag{10.37}$$

The operator (10.37) is the zero-mass Dirac operator. By definition, the Pauli operator $\mathcal{P}$ is the square of $\mathcal{D}$:

$$\mathcal{P} = \mathcal{D}^2 = \begin{pmatrix} P_- & 0 \\ 0 & P_+ \end{pmatrix}. \tag{10.38}$$

The precise definition of $\mathcal{P}$ is given in terms of the closed quadratic form $\|\mathcal{D}\mathbf{u}\|^2_{L_2(\mathbb{R}^2)}$, $\mathbf{u} \in H^1(\mathbb{R}^2; \mathbb{C}^2)$. If the potential $\breve{\mathbf{A}}(\mathbf{x})$ is a Lipschitz function, then the blocks of the operator (10.38) look like this:

$$P_\pm = (\mathbf{D} - \breve{\mathbf{A}}(\mathbf{x}))^2 \pm \breve{B}(\mathbf{x}), \quad \breve{B}(\mathbf{x}) := \partial_1 \breve{A}_2(\mathbf{x}) - \partial_2 \breve{A}_1(\mathbf{x}).$$

Here $\breve{B}(\mathbf{x})$ has the meaning of a magnetic field.

The operator $\mathcal{P}$ admits a convenient factorization. A gauge transformation allows us to assume that the potential $\breve{\mathbf{A}}(\mathbf{x})$ is subject to the conditions

$$\operatorname{div} \breve{\mathbf{A}}(\mathbf{x}) = 0, \quad \int_\Omega \breve{\mathbf{A}}(\mathbf{x})\, d\mathbf{x} = 0, \tag{10.39}$$

and still satisfies (10.36). (The first equation in (10.39) is understood in the sense of distributions.) Then there exists a Γ-periodic real-valued function $\varphi(\mathbf{x})$ such that

$$\nabla \varphi(\mathbf{x}) = \operatorname{col}\{\breve{A}_2(\mathbf{x}), -\breve{A}_1(\mathbf{x})\}, \quad \int_\Omega \varphi(\mathbf{x})\, d\mathbf{x} = 0. \tag{10.40}$$

Here, we have $\varphi \in \widetilde{W}^1_\rho(\Omega) \subset C^\alpha$, $\alpha = 1 - 2\rho^{-1}$. Denote

$$\omega_+(\mathbf{x}) = e^{\varphi(\mathbf{x})}, \quad \omega_-(\mathbf{x}) = e^{-\varphi(\mathbf{x})}. \tag{10.41}$$

The functions (10.41) satisfy conditions $(10.1)_\pm$ automatically, and $\omega_+(\mathbf{x})\omega_-(\mathbf{x}) = 1$. The norms $\|\omega_+\|_{L_\infty}$ and $\|\omega_-\|_{L_\infty}$ are controlled in terms of $\|\breve{\mathbf{A}}\|_{L_\rho(\Omega)}$ and the parameters of the lattice Γ. Moreover, the functions $\omega_\pm$ are multipliers both in $H^1(\mathbb{R}^d)$ and in $\widetilde{H}^1(\Omega)$.

Let matrices $h_\times(\mathbf{x})$ and $g_\times(\mathbf{x})$ and the operator $b_\times(\mathbf{D})$ be defined as in (10.20). The operator (10.38) admits the factorization

$$\mathcal{P} = h_\times b_\times(\mathbf{D}) g_\times b_\times(\mathbf{D}) h_\times. \tag{10.42}$$

The blocks $P_\pm$ of $\mathcal{P}$ can be represented as

$$P_+ = \omega_- \partial_+ \omega_+^2 \partial_- \omega_-, \quad P_- = \omega_+ \partial_- \omega_-^2 \partial_+ \omega_+.$$

10.4. The operators $\widetilde{\mathcal{B}}_{\pm,\varepsilon}$. (Before they were studied in [Su2, Subsection 14.4].) Now we consider the operators

$$P_{\pm,\varepsilon} = \omega_\mp^\varepsilon \partial_\pm (\omega_\pm^\varepsilon)^2 \partial_\mp \omega_\mp^\varepsilon = \omega_\mp^\varepsilon \mathcal{A}_{\pm,\varepsilon} \omega_\mp^\varepsilon. \tag{10.43$_\pm$}$$

(We remind the reader that the operators $\mathcal{A}_{\pm,\varepsilon}$ were defined in $(10.2)_\pm$.) If the magnetic potential is a Lipschitz function, then the operators $(10.43)_\pm$ can be written as $P_{\pm,\varepsilon} = (\mathbf{D} - \varepsilon^{-1}\breve{\mathbf{A}}^\varepsilon)^2 \pm \varepsilon^{-2}\breve{B}^\varepsilon$; these operators are the blocks of the Pauli operator with the singular magnetic potential $\varepsilon^{-1}\breve{\mathbf{A}}^\varepsilon$.

Next, let $\mathbf{A}(\mathbf{x}) = \operatorname{col}\{A_1(\mathbf{x}), A_2(\mathbf{x})\}$ be a vector-valued potential, where the $A_j(\mathbf{x})$ are Γ-periodic real-valued functions such that $A_j \in L_\rho(\Omega)$, $\rho > 2$. Let $\breve{v}_\pm(\mathbf{x})$ and $\breve{Q}(\mathbf{x})$ be real-valued Γ-periodic functions such that

$$\breve{v}_\pm, \breve{Q} \in L_s(\Omega), \quad s > 1; \quad \int_\Omega \omega_\pm^2(\mathbf{x}) \breve{v}_\pm(\mathbf{x})\, d\mathbf{x} = 0.$$

We consider the operator $\widetilde{\mathcal{B}}_{\pm,\varepsilon}$ given formally by the expression

$$(10.44)_\pm \qquad \widetilde{\mathcal{B}}_{\pm,\varepsilon} u = P_{\pm,\varepsilon} u + i \sum_{j=1}^{2} \big(A_j^\varepsilon(\mathbf{x}) \partial_j u + \partial_j (A_j^\varepsilon(\mathbf{x}) u) \big) + \varepsilon^{-1} \breve{v}_\pm^\varepsilon(\mathbf{x}) u + \breve{Q}^\varepsilon(\mathbf{x}) u.$$

(The precise definition is in terms of the corresponding quadratic form.)

Remark 10.3. Assuming that the potentials $\mathbf{A}(\mathbf{x})$ and $\breve{\mathbf{A}}(\mathbf{x})$ are Lipschitz functions, we consider the operators

$$(10.45)_\pm \qquad \mathfrak{P}_{\pm,\varepsilon} = (\mathbf{D} - \varepsilon^{-1}\breve{\mathbf{A}}^\varepsilon - \mathbf{A}^\varepsilon)^2 \pm (\varepsilon^{-2}\breve{B}^\varepsilon + \varepsilon^{-1} B^\varepsilon),$$

where $B(\mathbf{x}) = \partial_1 A_2(\mathbf{x}) - \partial_2 A_1(x)$, and

$$(10.46)_\pm \qquad \widetilde{\mathcal{B}}_{\pm,\varepsilon} = \mathfrak{P}_{\pm,\varepsilon} + \varepsilon^{-1}\widetilde{v}^\varepsilon + \mathcal{V}^\varepsilon.$$

Here $\widetilde{v}(\mathbf{x})$ and $\mathcal{V}(\mathbf{x})$ are Γ-periodic real-valued functions such that $\widetilde{v}, \mathcal{V} \in L_s(\Omega)$, $s > 1$. We also impose the additional normalization condition on $\widetilde{v}$:

$$\int_\Omega \omega_\pm^2(\mathbf{x}) \big(\widetilde{v}(\mathbf{x}) + 2\langle \breve{\mathbf{A}}(\mathbf{x}), \mathbf{A}(\mathbf{x}) \rangle \pm B(\mathbf{x}) \big)\, d\mathbf{x} = 0.$$

Then the operator $(10.46)_\pm$ can be written as $(10.44)_\pm$ with $\breve{v}_\pm = \widetilde{v} + 2\langle \breve{\mathbf{A}}, \mathbf{A} \rangle \pm B$ and $\breve{Q} = \mathcal{V} + |\mathbf{A}|^2$. The operators $(10.45)_\pm$ are the blocks of the Pauli operator with the magnetic potential $\varepsilon^{-1}\breve{\mathbf{A}}^\varepsilon + \mathbf{A}^\varepsilon$ (which contains a singular term), and the operators $(10.46)_\pm$ are perturbations of the previous ones by the electric potential $\varepsilon^{-1}\widetilde{v}^\varepsilon + \mathcal{V}^\varepsilon$ (which also contains a singular term).

Taking $(10.3)_\pm$, $(10.43)_\pm$, and $(10.44)_\pm$ into account, we see that the operators $\widetilde{\mathcal{B}}_{\pm,\varepsilon}$ and $\mathcal{B}_{\pm,\varepsilon}$ satisfy relations like (8.7):

$$(10.47)_\pm \qquad \widetilde{\mathcal{B}}_{+,\varepsilon} = \omega_-^\varepsilon \mathcal{B}_{+,\varepsilon} \omega_-^\varepsilon, \quad \widetilde{\mathcal{B}}_{-,\varepsilon} = \omega_+^\varepsilon \mathcal{B}_{-,\varepsilon} \omega_+^\varepsilon.$$

Now, in (10.3$\pm$) we should put

$$(10.48)_\pm \qquad \eta_{j,\pm} = \omega_\pm^2 A_j, \;\; j = 1, 2; \quad v_\pm = \omega_\pm^2 \breve{v}_\pm, \;\; Q_\pm = \omega_\pm^2 \breve{Q}.$$

Let $\widetilde{Q}_0(\mathbf{x})$ be a Γ-periodic real-valued function such that $\widetilde{Q}_0(\mathbf{x}) > 0$; $\widetilde{Q}_0, \widetilde{Q}_0^{-1} \in L_\infty$. We put

$$(10.49)_\pm \qquad \widetilde{\mathcal{B}}_{\pm,\lambda,\varepsilon} = \widetilde{\mathcal{B}}_{\pm,\varepsilon} + \lambda \widetilde{Q}_0^\varepsilon.$$

Assume that the parameter λ is subject to the restriction (8.12) with $f = \omega_\mp$ and c_0, c_4 corresponding to the operator $\mathcal{B}_{\pm,\varepsilon}$. The initial data reduce to the following set of parameters:

$$(10.50)_\pm \qquad \begin{gathered} \rho, s; \; \|\breve{\mathbf{A}}\|_{L_\rho(\Omega)}, \; \|\mathbf{A}\|_{L_\rho(\Omega)}, \; \|\breve{v}_\pm\|_{L_s(\Omega)}, \; \|\breve{Q}\|_{L_s(\Omega)}, \\ \lambda, \; \|\widetilde{Q}_0\|_{L_\infty}, \; \|\widetilde{Q}_0^{-1}\|_{L_\infty}; \text{ the parameters of the lattice } \Gamma. \end{gathered}$$

From $(10.47)_\pm$, $(10.49)_\pm$, and $(10.5)_\pm$ it follows that

$$(10.51)_\pm \qquad \widetilde{\mathcal{B}}_{\pm,\lambda,\varepsilon} = \omega_\mp^\varepsilon \mathcal{B}_{\pm,\lambda,\varepsilon} \omega_\mp^\varepsilon,$$

and, now, in $(10.5)_\pm$ we put

$$(10.52)_\pm \qquad Q_{0,\pm} = \omega_\pm^2 \widetilde{Q}_0.$$

Proposition 10.1($\pm$) and representation $(10.51)_\pm$ imply the following result (cf. Theorem 8.2).

Proposition 10.4($\pm$). *Let $\widetilde{\mathcal{B}}_{\pm,\lambda,\varepsilon}$ be the operator defined in accordance with* (10.43)$_\pm$, (10.44)$_\pm$, *and* (10.49)$_\pm$, *with the coefficients $\breve{\mathbf{A}}$, $\mathbf{A}$, $\breve{v}_\pm$, $\breve{Q}$, and $\widetilde{Q}_0$ satisfying the assumptions of Subsections* 10.3, 10.4. *Suppose that ω_+, ω_- are the functions defined by* (10.40), (10.41). *Let $\mathcal{B}_{\pm,\lambda,\varepsilon}$ be the operator defined as in* (10.2)$_\pm$, (10.3)$_\pm$, *and* (10.5)$_\pm$, *with the coefficients related to the coefficients of the operator $\widetilde{\mathcal{B}}_{\pm,\lambda,\varepsilon}$ by formulas* (10.48)$_\pm$, (10.52)$_\pm$. *Let $\mathcal{B}^0_{\pm,\lambda}$ be the effective operator for $\mathcal{B}_{\pm,\lambda,\varepsilon}$ defined by* (10.8)$_\pm$. *Let $K^0_{\pm,\varepsilon}$ be the corrector* (10.17)$_\pm$ *for the operator $\mathcal{B}_{\pm,\lambda,\varepsilon}$. Then*

$$\|\widetilde{\mathcal{B}}^{-1}_{\pm,\lambda,\varepsilon} - \omega^\varepsilon_\pm(\mathcal{B}^0_{\pm,\lambda})^{-1}\omega^\varepsilon_\pm\|_{L_2(\mathbb{R}^2)\to L_2(\mathbb{R}^2)} \le \widetilde{C}_{0,\pm}\varepsilon, \quad 0<\varepsilon\le 1,$$

$$\|\widetilde{\mathcal{B}}^{-1}_{\pm,\lambda,\varepsilon} - \omega^\varepsilon_\pm\big((\mathcal{B}^0_{\pm,\lambda})^{-1} + \varepsilon K^0_{\pm,\varepsilon}\big)\omega^\varepsilon_\pm\|_{L_2(\mathbb{R}^2)\to L_2(\mathbb{R}^2)} \le \widetilde{\mathcal{C}}^0_\pm\varepsilon^2, \quad 0<\varepsilon\le 1.$$

The constants $\widetilde{C}_{0,\pm}$ and $\widetilde{\mathcal{C}}^0_\pm$ depend only on the initial data (10.50)$_\pm$.

10.5. The operator $\widetilde{\mathcal{B}}_{\times,\varepsilon}$. (Before, this operator was studied in [Su2, Subsection 14.5].) We return to the matrix Pauli operator (10.42) and consider the corresponding operator

$$\mathcal{P}_\varepsilon = h^\varepsilon_\times b_\times(\mathbf{D}) g^\varepsilon_\times b_\times(\mathbf{D}) h^\varepsilon_\times. \tag{10.53}$$

Then

$$\mathcal{P}_\varepsilon = \begin{pmatrix} P_{-,\varepsilon} & 0 \\ 0 & P_{+,\varepsilon}\end{pmatrix},$$

where the blocks $P_{\pm,\varepsilon}$ are as in (10.43)$_\pm$.

Next, let $\breve{\eta}_{j,\times}$, $j=1,2$, be Γ-periodic *diagonal* (2×2)-matrix-valued functions with real entries and such that $\breve{\eta}_{j,\times}\in L_\rho(\Omega)$, $\rho>2$. Let $\breve{v}_\times(\mathbf{x})$ and $\breve{Q}_\times(\mathbf{x})$ be Γ-periodic Hermitian (2×2)-matrix-valued functions in $\mathbb{R}^2$ such that

$$\breve{v}_\times, \breve{Q}_\times \in L_s(\Omega), \ \ s>1; \quad \int_\Omega h_\times(\mathbf{x})^{-1}\breve{v}_\times(\mathbf{x}) h_\times(\mathbf{x})^{-1}\,d\mathbf{x} = 0.$$

Consider the operator $\widetilde{\mathcal{B}}_{\times,\varepsilon}$ given formally by the expression

$$\widetilde{\mathcal{B}}_{\times,\varepsilon}\mathbf{u} = \mathcal{P}_\varepsilon\mathbf{u} + i\sum_{j=1}^2\big(\breve{\eta}^\varepsilon_{j,\times}(\mathbf{x})\partial_j\mathbf{u} + \partial_j(\breve{\eta}^\varepsilon_{j,\times}(\mathbf{x})\mathbf{u})\big) + \varepsilon^{-1}\breve{v}^\varepsilon_\times(\mathbf{x})\mathbf{u} + \breve{Q}^\varepsilon_\times(\mathbf{x})\mathbf{u}. \tag{10.54}$$

The precise definition is in terms of the corresponding quadratic form.

Remark 10.5. Let $\mathfrak{P}_{\pm,\varepsilon}$ be the operators (10.45)$_\pm$, and let the potentials $\breve{\mathbf{A}}$, $\mathbf{A}$ be Lischitz functions. Consider the operator $\mathfrak{P}_\varepsilon = \operatorname{diag}\{\mathfrak{P}_{-,\varepsilon}, \mathfrak{P}_{+,\varepsilon}\}$; this is the matrix Pauli operator with the magnetic potential $\varepsilon^{-1}\breve{\mathbf{A}}^\varepsilon + \mathbf{A}^\varepsilon$ (which contains a singular term). Let $\widetilde{v}_\times(\mathbf{x})$ and $\mathcal{V}_\times(\mathbf{x})$ be Γ-periodic Hermitian (2×2)-matrix-valued functions such that $\widetilde{v}_\times, \mathcal{V}_\times \in L_s(\Omega)$, $s>1$. Consider the perturbed Pauli operator

$$\widetilde{\mathcal{B}}_{\times,\varepsilon} = \mathfrak{P}_\varepsilon + \varepsilon^{-1}\widetilde{v}^\varepsilon_\times + \mathcal{V}^\varepsilon_\times. \tag{10.55}$$

We impose the following normalization condition on $\widetilde{v}_\times$:

$$\int_\Omega h_\times(\mathbf{x})^{-1}\big(\widetilde{v}_\times(\mathbf{x}) + 2\langle\breve{\mathbf{A}}(\mathbf{x}), \mathbf{A}(\mathbf{x})\rangle\mathbf{1} - B(\mathbf{x})\sigma_3\big)h_\times(\mathbf{x})^{-1}\,d\mathbf{x} = 0.$$

Then the operator (10.55) can be written in the form (10.54) with

$$\breve{\eta}_{j,\times} = A_j\mathbf{1}, \ \ j=1,2; \quad \breve{v}_\times = \widetilde{v}_\times + 2\langle\breve{\mathbf{A}}, \mathbf{A}\rangle\mathbf{1} - B\sigma_3, \quad \breve{Q}_\times = \mathcal{V}_\times + |\mathbf{A}|^2\mathbf{1}.$$

By (10.19) and (10.53), we see that the operators (10.54) and (10.21) satisfy the identity

$$\widetilde{\mathcal{B}}_{\times,\varepsilon} = h^\varepsilon_\times\mathcal{B}_{\times,\varepsilon}h^\varepsilon_\times. \tag{10.56}$$

Now, in (10.21) we should put

$$\eta_{j,\times} = h_\times^{-1}\check{\eta}_{j,\times}h_\times^{-1}, \ \ j=1,2; \quad v_\times = h_\times^{-1}\check{v}_\times h_\times^{-1}, \quad Q_\times = h_\times^{-1}\check{Q}_\times h_\times^{-1}. \tag{10.57}$$

In order to check identity (10.56), we use the fact that the matrices $\check{\eta}_{j,\times}$ and $h_\times$ are diagonal, and therefore, they commute.

Let $\widetilde{Q}_{0,\times}(\mathbf{x})$ be a Γ-periodic, positive definite, and bounded (2×2)-matrix-valued function on $\mathbb{R}^2$. We put

$$\widetilde{\mathcal{B}}_{\times,\lambda,\varepsilon} = \widetilde{\mathcal{B}}_{\times,\varepsilon} + \lambda \widetilde{Q}^\varepsilon_{0,\times}. \tag{10.58}$$

Assume that λ is subject to the restriction (8.12) with $f = h_\times$ and c_0, c_4 corresponding to the operator $\mathcal{B}_{\times,\varepsilon}$. The initial data reduce to the set of parameters

$$\begin{aligned} &\rho, s;\ \|\check{\mathbf{A}}\|_{L_\rho(\Omega)},\ \|\check{\eta}_{j,\times}\|_{L_\rho(\Omega)},\ j=1,2;\ \|\check{v}_\times\|_{L_s(\Omega)},\ \|\check{Q}_\times\|_{L_s(\Omega)},\\ &\lambda,\ \|\widetilde{Q}_{0,\times}\|_{L_\infty},\ \|\widetilde{Q}^{-1}_{0,\times}\|_{L_\infty};\ \text{the parameters of the lattice } \Gamma. \end{aligned} \tag{10.59}$$

From (10.56), (10.58), and (10.24) it follows that

$$\widetilde{\mathcal{B}}_{\times,\lambda,\varepsilon} = h^\varepsilon_\times \mathcal{B}_{\times,\lambda,\varepsilon} h^\varepsilon_\times, \tag{10.60}$$

and, now, in (10.24) we put

$$Q_{0,\times} = h_\times^{-1}\widetilde{Q}_{0,\times}h_\times^{-1}. \tag{10.61}$$

Proposition 10.2 and the representation (10.60) imply the following result (cf. Theorem 8.2).

Proposition 10.6. *Suppose that the operator $\widetilde{\mathcal{B}}_{\times,\lambda,\varepsilon}$ is defined in accordance with* (10.53), (10.54)*, and* (10.58)*, and that its coefficients $\check{\mathbf{A}}$, $\check{\eta}_{j,\times}$ $(j=1,2)$, $\check{v}_\times$, $\check{Q}_\times$, and $\widetilde{Q}_{0,\times}$ satisfy the assumptions of Subsections* 10.3, 10.5*. Let ω_+ and ω_- be the functions defined by* (10.40), (10.41)*, and let $h_\times$ and $g_\times$ be the matrix-valued functions defined by* (10.20)*. Suppose that $\mathcal{B}_{\times,\lambda,\varepsilon}$ is the operator defined as in* (10.19), (10.21), (10.24)*, and that its coefficients are related to the coefficients of the operator $\widetilde{\mathcal{B}}_{\times,\lambda,\varepsilon}$ by formulas* (10.57), (10.61)*. Let $\mathcal{B}^0_{\times,\lambda}$ be the effective operator for $\mathcal{B}_{\times,\lambda,\varepsilon}$ defined by* (10.26)*. Let $K^0_{\times,\varepsilon}$ be the corrector* (10.35) *for the operator $\mathcal{B}_{\times,\lambda,\varepsilon}$. Then for $0<\varepsilon\le 1$ we have*

$$\left\|\widetilde{\mathcal{B}}^{-1}_{\times,\lambda,\varepsilon} - (h^\varepsilon_\times)^{-1}(\mathcal{B}^0_{\times,\lambda})^{-1}(h^\varepsilon_\times)^{-1}\right\|_{L_2(\mathbb{R}^2)\to L_2(\mathbb{R}^2)} \le \widetilde{C}_{0,\times}\varepsilon,$$

$$\left\|\widetilde{\mathcal{B}}^{-1}_{\times,\lambda,\varepsilon} - (h^\varepsilon_\times)^{-1}\left((\mathcal{B}^0_{\times,\lambda})^{-1} + \varepsilon K^0_{\times,\varepsilon}\right)(h^\varepsilon_\times)^{-1}\right\|_{L_2(\mathbb{R}^2)\to L_2(\mathbb{R}^2)} \le \widetilde{\mathcal{C}}^0_\times \varepsilon^2.$$

The constants $\widetilde{C}_{0,\times}$ and $\widetilde{\mathcal{C}}^0_\times$ depend only on the initial data (10.59).

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