

APPROXIMATE COMMUTATIVITY FOR A DECAYING POTENTIAL AND A FUNCTION OF AN ELLIPTIC OPERATOR

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ABSTRACT. For a continuous function $\varphi(\lambda)$, $\lambda \in \mathbb{R}$, with compact support, a bounded function $W(x)$, $x \in \mathbb{R}^d$, with power-like asymptotics at infinity, and a suitable selfadjoint operator H in $L_2(\mathbb{R}^d)$, estimates for the singular values of the operator $\varphi(H)W - W\varphi(H)$ are considered. It is proved that the singular values of $\varphi(H)W - W\varphi(H)$ decay faster than those of $\varphi(H)W$. A relationship between the singular values asymptotics for the operators $\varphi(H)W$ and $\varphi^n(H)W^n$ is also established.

§0. INTRODUCTION. THE MAIN RESULT

1. In the present paper, we consider singular value estimates for operators in $L_2(\mathbb{R}^d)$ of the form $\varphi(H)W(x) - W(x)\varphi(H)$ with appropriate functions $\varphi(\lambda)$, $\lambda \in \mathbb{R}$, $W(x)$, $x \in \mathbb{R}^d$, and a selfadjoint operator H in $L_2(\mathbb{R}^d)$. It is assumed that the semigroup e^{-tH} satisfies the so-called Nash–Aronson estimate (see Subsection 2 of the Introduction), the function W has power-like asymptotics at infinity, and φ is a continuous function with compact support on the line. Also, we establish a relationship between the singular values asymptotics for the operators $\varphi(H)W$ and $\varphi^n(H)W^n$. Results of this type are of certain interest in the spectral theory of differential operators.

In this paper we continue the research started in [1], where singular values estimates for the operators of the form $\varphi(H)W$ were studied (and some technical difficulties were taken care of “beforehand”). For an arbitrary compact operator \mathbb{T} , we denote the singular values of \mathbb{T} by $s_m(\mathbb{T})$. If the function $W \in L_\infty(\mathbb{R}^d)$ has a power-like asymptotics $W(x) \sim \omega(x/|x|)|x|^{-d/p}$, $|x| \rightarrow +\infty$, then by the results of [1], the singular values of the operator $\varphi(H)W$ satisfy the estimate $s_m(\varphi(H)W) \leq Cm^{-1/p}$, $m \in \mathbb{N}$. The main result of the present paper (see Theorem 0.4) reads as follows: *the singular values of the operator $\varphi(H)W - W\varphi(H)$ obey the relation*

$$(0.1) \quad s_m(\varphi(H)W - W\varphi(H)) = o(m^{-1/p}), \quad m \rightarrow +\infty.$$

This estimate allows us to relate the singular values asymptotics of the operators $\varphi(H)W$ and $\varphi^n(H)W^n$. Namely, for an arbitrary $n \in \mathbb{N}$ the following identities hold true (see Theorem 0.5):

$$\begin{aligned} \liminf_{m \rightarrow +\infty} s_m(\varphi(H)W)m^{1/p} &= \left(\liminf_{m \rightarrow +\infty} s_m(\varphi^n(H)W^n)m^{n/p} \right)^{1/n}; \\ \limsup_{m \rightarrow +\infty} s_m(\varphi(H)W)m^{1/p} &= \left(\limsup_{m \rightarrow +\infty} s_m(\varphi^n(H)W^n)m^{n/p} \right)^{1/n}. \end{aligned}$$

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Our argument is based on the singular value estimates for the operator $\varphi(H)W$ obtained in [1] (see Propositions 0.3 and 1.2) and on the “pseudolocal” property of the operator $\varphi(H)$ (see Corollary 1.7), which is a consequence of the Nash–Aronson estimate for the kernel of the operator e^{-tH} , $t > 0$. In more detail, for us it is important that $\varphi(H)$ is an integral operator with a kernel $K_\varphi(x, y)$ satisfying inequality (1.6). Moreover, we employ a certain version of the Cwikel estimate for operators with “nonnegative” kernels (see Proposition 1.1). The results obtained can be used in the study of the discrete spectrum of an elliptic differential operator perturbed by a decaying potential. A separate publication will be devoted to these applications.

Considerations “equivalent” to estimate (0.1) can be found in the papers [2, 3, 4] etc. In fact, properties similar to (1.6) were employed in many papers dealing with spectral asymptotics (for example, we point out the paper [5], where the spectral asymptotics was studied for a pseudodifferential operator having an anisotropic-homogeneous symbol).

The paper consists of the Introduction and two sections. In the Introduction, the Nash–Aronson estimate is discussed, the required functional and operator classes are introduced, and the main results (Theorems 0.4, 0.5) are formulated; §1 contains preliminary material necessary for the proof of main results. These proofs are presented in §2.

For a measurable function $f: \mathcal{X} \rightarrow \mathbb{C}$, the symbol $[f(x)]$ denotes the operator of multiplication by f . The norm in a (quasi)normed space X is denoted by $\|\cdot\|_X$; if spaces X, Y are (quasi)normed, the standard norm of a bounded linear operator $T: X \rightarrow Y$ is denoted either by $\|T\|_{X \rightarrow Y}$ or by $\|T\|$ (without the subscript), provided that this does not cause confusion. The class of compact operators is denoted by \mathfrak{S}_∞ . For the class of Hilbert–Schmidt operators we use the notation \mathfrak{S}_2 . For a separable measurable space with a σ -finite measure $(\mathcal{Z}, d\nu)$, we write $L_p(\mathcal{Z}, d\nu)$ to denote the standard class L_p ; if \mathcal{Z} is a countable set, the notation $L_p(\mathcal{Z}, d\nu) =: l_p(\mathcal{Z}, d\nu)$ is utilized.

2. Nash–Aronson estimate. Throughout the paper, the following is assumed (cf. [1]).

Condition 1. Let H be a selfadjoint and lower semibounded operator in $L_2(\mathbb{R}^d)$, $d \geq 1$. Assume that the operator e^{-tH} , $t > 0$, is an integral operator in $L_2(\mathbb{R}^d)$ with the kernel $K(t, x, y)$, $x, y \in \mathbb{R}^d$, $t > 0$, and that

$$(0.2) \quad |K(t, x, y)| \leq M_1 t^{-d/2} e^{M_2 t - \frac{|x-y|^2}{M_3 t}}, \quad x, y \in \mathbb{R}^d, \quad t > 0,$$

where the $M_i > 0$, $i = 1, 2, 3$, are independent of $x, y \in \mathbb{R}^d$, $t > 0$.

Inequality (0.2) is called the Nash–Aronson estimate. It was obtained in [6] for the kernel of the semigroup generated by an elliptic second order differential operator. In more detail, assume that a real $(d \times d)$ -matrix-valued function $a(x)$ and a real potential $b(x)$ satisfy the following conditions:

$$(0.3) \quad a = a^t \in L_\infty(\mathbb{R}^d, \text{Matr}(\mathbb{R}, d \times d)), \quad b = \bar{b} \in L_\infty(\mathbb{R}^d), \quad c_0 \mathbf{1} \leq a(x), \quad 0 < c_0.$$

It is easily seen that, under condition (0.3), the operator $H = -\text{div } a(x) \text{ grad} + b(x)$ is well defined by its quadratic form, is selfadjoint in $L_2(\mathbb{R}^d)$, and is lower semibounded. The results of [6] (see also [7, Chapter 3, §2]) imply the following statement.

Theorem 0.1. Under condition (0.3), the operator H satisfies Condition 1.

3. Now we introduce the required function spaces and classes of compact operators.

Function spaces. Let $(\mathcal{Z}, d\nu)$ be a separable measurable space with a σ -finite measure. Alongside the standard classes $L_p(\mathcal{Z}, d\nu)$, we consider “weak L_p classes” (see, e.g., [8] for the details; see also [9, 1]). Namely, for a ν -measurable function $f: \mathcal{Z} \rightarrow \mathbb{C}$ put $O_f(s) := \{z \in \mathcal{Z} : |f(z)| > s\}$, $\nu_f(s) := \nu(O_f(s))$, $s > 0$. The “weak L_p -class” $L_{p,\infty}(\mathcal{Z}, d\nu)$

is singled out by the condition that the functional $\|f\|_{L_{p,\infty}} := \sup_{s>0} s\nu_f^{1/p}(s)$ is finite. The space $L_{p,\infty}$ is complete with respect to the quasinorm $\|\cdot\|_{L_{p,\infty}}$; it is in general nonseparable and contains the separable subspace $L_{p,\infty}^0 := \{f \in L_{p,\infty} : \nu_f(s) = o(s^{-p}), s \rightarrow +0, s \rightarrow +\infty\}$.

Classes of operators. For an arbitrary compact operator \mathbb{T} acting from a Hilbert space \mathfrak{H}_1 into a Hilbert space \mathfrak{H}_2 , we denote the singular values of the operator \mathbb{T} (i.e., the consecutive eigenvalues of the operator $(\mathbb{T}^*\mathbb{T})^{1/2}$) by $s_m(\mathbb{T})$, $m \in \mathbb{N}$; let $n(s, \mathbb{T}) := \#\{m \in \mathbb{N} : s_m(\mathbb{T}) > s\}$ be the distribution function of singular values. The class $\mathfrak{S}_{p,\infty}(\mathfrak{H}_1, \mathfrak{H}_2)$ (see, e.g., [8]), which is an operator analog of the “weak” L_p -class, is singled out by the condition that the functional $\|\mathbb{T}\|_{\mathfrak{S}_{p,\infty}} := \sup_{s>0} sn^{1/p}(s, \mathbb{T})$ is finite. The space $\mathfrak{S}_{p,\infty}$ is complete with respect to the quasinorm $\|\cdot\|_{\mathfrak{S}_{p,\infty}}$, it is in general nonseparable and contains the separable subspace $\mathfrak{S}_{p,\infty}^0 := \{\mathbb{T} \in \mathfrak{S}_{p,\infty} : n(s, \mathbb{T}) = o(s^{-p}), s \rightarrow +0\}$, in which the set of finite-rank operators is dense. Two continuous (see, e.g., [10]) functionals $\Delta_p(\mathbb{T}) := \limsup_{s \rightarrow 0} s^p n(s, \mathbb{T})$, $\delta_p(\mathbb{T}) := \liminf_{s \rightarrow 0} s^p n(s, \mathbb{T})$ are defined on the space $\mathfrak{S}_{p,\infty}$. It is not hard to check that, for an operator $\mathbb{T} \in \mathfrak{S}_{p,\infty}$, we have $\mathbb{T} \in \mathfrak{S}_{p,\infty}^0$ if and only if $\Delta_p(\mathbb{T}) = 0$. For an arbitrary compact operator \mathbb{T} , the fact that $\mathbb{T} \in \mathfrak{S}_{p,\infty}$ is equivalent to the condition $s_m(\mathbb{T}) = O(m^{-1/p})$, $m \rightarrow +\infty$, and the relations $\|\mathbb{T}\|_{\mathfrak{S}_{p,\infty}} = \sup_{m \in \mathbb{N}} m^{1/p} s_m(\mathbb{T})$, $\Delta_p(\mathbb{T}) = \limsup_{m \rightarrow +\infty} m s_m^p(\mathbb{T})$, $\delta_p(\mathbb{T}) = \liminf_{m \rightarrow +\infty} m s_m^p(\mathbb{T})$ hold true. An operator \mathbb{T} belongs to $\mathfrak{S}_{p,\infty}^0$ if and only if $s_m(\mathbb{T}) = o(m^{-1/p})$, $m \rightarrow +\infty$. We have $\mathfrak{S}_{p_1,\infty} \subset \mathfrak{S}_{p_2,\infty}^0$ whenever $p_1 < p_2$ (see, e.g., [8]). The Hilbert–Schmidt operators \mathfrak{S}_2 are contained in each class $\mathfrak{S}_{p,\infty}^0$, $p > 2$. Note further that for any nonnegative compact operator $\mathbb{T} \in \mathfrak{S}_{p,\infty}$ we have $D_p(\mathbb{T}) = D_{p/n}(\mathbb{T}^n)$, $D = \Delta, \delta$, $n \in \mathbb{N}$.

We shall need the following statement (see, e.g., [11, item 3 of Subsection 11.6]).

Proposition 0.2. *If $T_1 \in \mathfrak{S}_{p,\infty}$ and $T_2 \in \mathfrak{S}_{q,\infty}$, then $T_1 T_2 \in \mathfrak{S}_{r,\infty}$, where $r^{-1} = p^{-1} + q^{-1}$, and*

$$\|T_1 T_2\|_{\mathfrak{S}_{r,\infty}} \leq C(p, q) \|T_1\|_{\mathfrak{S}_{p,\infty}} \|T_2\|_{\mathfrak{S}_{q,\infty}}, \quad r^{-1} = p^{-1} + q^{-1}.$$

If, moreover, either $T_1 \in \mathfrak{S}_{p,\infty}^0$ or $T_2 \in \mathfrak{S}_{q,\infty}^0$, then $T_1 T_2 \in \mathfrak{S}_{r,\infty}^0$.

Next, we state a condition (of Cwikel type) ensuring that some special operators belong to $\mathfrak{S}_{p,\infty}$. Let H be an operator satisfying Condition 1, and let $f(\lambda)$, $\lambda \in \mathbb{R}$, be a bounded Borel function, and let $g \in L_\infty(\mathbb{R}^d)$. Assume that, for some $s > 0$ and $\varepsilon > 0$, we have

$$(0.4) \quad f(\lambda) = O(\lambda^{-\frac{d}{2s}-\varepsilon}), \quad \lambda \rightarrow +\infty; \quad g(x) = O(|x|^{-d/s}), \quad |x| \rightarrow +\infty.$$

The following statement is true (see [1, Proposition 3.2]).

Proposition 0.3. *If conditions (0.4) are fulfilled for some $s > 0$ and $\varepsilon > 0$, then $f(H)[g(x)] \in \mathfrak{S}_{s,\infty}$. If, moreover, $g(x) = o(|x|^{-d/s})$, $|x| \rightarrow +\infty$, then $f(H)[g(x)] \in \mathfrak{S}_{s,\infty}^0$.*

4. The main result. Suppose H is an operator satisfying Condition 1. Assume that two functions $V, W \in L_\infty(\mathbb{R}^d)$ obey the following conditions:

$$(0.5) \quad W(x) = \omega(x/|x|)|x|^{-d/p} + o(|x|^{-d/p}), \quad |x| \rightarrow +\infty, \quad p \in (0, +\infty);$$

$$(0.6) \quad V(x) = O(|x|^{-d/q}), \quad |x| \rightarrow +\infty, \quad q \in (0, +\infty].$$

Let $\varphi(\lambda)$, $\lambda \in \mathbb{R}$, be a bounded compactly supported Borel function. By Proposition 0.3, we have $\varphi(H)WV \in \mathfrak{S}_{r,\infty}$, $r^{-1} = p^{-1} + q^{-1}$. Pick a bounded compactly supported Borel function $\psi(\lambda)$, $\lambda \in \mathbb{R}$, such that $\varphi \cdot \psi = \varphi$. Then $\psi(H)V \in \mathfrak{S}_{q,\infty}$, $W\varphi(H) \in \mathfrak{S}_{p,\infty}$ by Proposition 0.3. Together with the identity $W\varphi(H)V = W\varphi(H)\psi(H)V$ and

Proposition 0.2, this yields the relation $W\varphi(H)V \in \mathfrak{S}_{r,\infty}$, $r^{-1} = p^{-1} + q^{-1}$. The main result of the paper is as follows.

Theorem 0.4. *Let H be an operator satisfying Condition 1, let two functions $V, W \in L_\infty(\mathbb{R}^d)$ be such that conditions (0.5), (0.6) are fulfilled, and let $\varphi(\lambda)$, $\lambda \in \mathbb{R}$, be a continuous compactly supported function. Then*

$$(0.7) \quad \varphi(H)WV - W\varphi(H)V \in \mathfrak{S}_{r,\infty}^0, \quad r^{-1} = p^{-1} + q^{-1}.$$

The proof of Theorem 0.4 is contained in §2. Theorem 0.4 implies the following statement (see item 3 of §2).

Theorem 0.5. *Let H be an operator satisfying Condition 1, let $W \in L_\infty(\mathbb{R}^d)$ be a function satisfying (0.5), and let $\varphi(\lambda)$, $\lambda \in \mathbb{R}$, be a continuous compactly supported function. Then*

$$(0.8) \quad D_p(\varphi(H)W) = D_{p/n}(\varphi^n(H)W^n), \quad n \in \mathbb{N}, \quad D = \Delta, \delta.$$

§1. PRELIMINARY RESULTS

1. One special case of Cwikel’s estimate. Let $(\mathcal{X}, d\rho)$, $(\mathcal{Y}, d\tau)$ be separable measurable spaces with σ -finite measures; let $t(\cdot, \cdot) : \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{C}$ be the kernel of a bounded integral operator $T : L_2(\mathcal{Y}, d\tau) \rightarrow L_2(\mathcal{X}, d\rho)$ (cf. [12, Subsection 1.3]). Assume that the function $|t(x, y)|$ is also the kernel of a bounded integral operator $T_0 : L_2(\mathcal{Y}, d\tau) \rightarrow L_2(\mathcal{X}, d\rho)$. Denote by $d\nu(x, y)$ the measure $|t(x, y)|^2 d\rho(x) d\tau(y)$ on $\mathcal{X} \times \mathcal{Y}$. It is easily seen that for any function $a \in L_2(\mathcal{X} \times \mathcal{Y}, d\nu)$ the product $a(x, y)t(x, y)$ is the kernel of a Hilbert–Schmidt operator $\mathbb{T}_a : L_2(\mathcal{Y}, d\tau) \rightarrow L_2(\mathcal{X}, d\rho)$, and

$$(1.1) \quad \|\mathbb{T}_a\|_{\mathfrak{S}_2} = \|a\|_{L_2}$$

It is not hard to check that, for any function $a \in L_\infty(\mathcal{X} \times \mathcal{Y}, d\nu)$, the product $a(x, y)t(x, y)$ is the kernel of a bounded integral operator $\mathbb{T}_a : L_2(\mathcal{Y}, d\tau) \rightarrow L_2(\mathcal{X}, d\rho)$, and

$$(1.2) \quad \|\mathbb{T}_a\| \leq \|T_0\| \|a\|_{L_\infty}.$$

By linear interpolation, estimates (1.1) and (1.2) yield the following statement.

Proposition 1.1. *Suppose that $a \in L_{p,\infty}(\mathcal{X} \times \mathcal{Y}, d\nu)$ with $p \in (2, +\infty)$. Then the kernel $a(x, y)t(x, y)$ gives rise to a bounded integral operator $\mathbb{T}_a \in \mathfrak{S}_{p,\infty}$, and we have $\|\mathbb{T}_a\|_{\mathfrak{S}_{p,\infty}} \leq C(p) \|T_0\|^{1-\frac{2}{p}} \|a\|_{L_{p,\infty}}$. If, moreover, $\nu_a(s) = o(s^{-p})$, $s \rightarrow +0$, then $\mathbb{T}_a \in \mathfrak{S}_{p,\infty}^0$.*

Proof. The fact that $\mathbb{T}_a \in \mathfrak{S}_{p,\infty}$ and the quasinorm estimate for $\|\mathbb{T}_a\|_{\mathfrak{S}_{p,\infty}}$ follow from, e.g., Theorem 3.1 in [8, §3, item 2] (see also Theorem 3.2 in [12]). The last claim of Proposition 1.1 is justified as follows (cf. [9, Subsection 4.8]). Consider the “truncated” function a_ε , $\varepsilon > 0$. Namely, put $a_\varepsilon(x) = a(x)$ if $|a(x)| > \varepsilon$ and $a_\varepsilon(x) = 0$ if $|a(x)| \leq \varepsilon$. Then $a_\varepsilon \in L_2$, whence $\mathbb{T}_{a_\varepsilon} \in \mathfrak{S}_2 \subset \mathfrak{S}_{p,\infty}^0$. Under the condition $\nu_a(s) = o(s^{-p})$, $s \rightarrow +0$, a_ε converges to a as $\varepsilon \rightarrow +0$ in the metrics of $L_{p,\infty}$, so that $\mathbb{T}_{a_\varepsilon} \rightarrow \mathbb{T}_a$ in $\mathfrak{S}_{p,\infty}$. Thus, $\mathbb{T}_a \in \mathfrak{S}_{p,\infty}^0$. \square

2. Let H be an operator satisfying Condition 1. Next, let $f(\lambda)$, $\lambda \in \mathbb{R}$, be a bounded Borel function. Let a function $g(x)$, $x \in \mathbb{R}^d$, be defined by the formula

$$(1.3) \quad g(x) := \zeta(|x|)|x|^{-d/s}\omega(x/|x|), \quad x \in \mathbb{R}^d, \quad s > 0.$$

Here $\omega \in L_\infty(\mathbb{S}^{d-1})$, ζ is a continuous “cut-off at zero”, i.e., $\zeta \in C(\mathbb{R})$ is monotone nondecreasing and $\zeta(t) = 0$ if $t < 1$, $\zeta(t) = 1$ if $t > 2$. For brevity, we denote $a_0 := \min\{\inf \sigma(H), 0\}$, $a_j := 2^j$, $j \in \mathbb{N}$. With the function $f : \mathbb{R} \rightarrow \mathbb{C}$ we associate the sequence

$$u(f) := \{u_j(f)\}_{j=0}^\infty, \quad u_j(f) := \sup_\lambda \{|f(\lambda)|, \lambda \in [a_j, a_{j+1}]\}, \quad j \in \mathbb{Z}_+.$$

Finally, we introduce the measure $d\nu := a_{j+1}^{d/2} dj$ on the set \mathbb{Z}_+ , where dj is the counting measure on \mathbb{Z}_+ . The following statement is true (see [1, Proposition 3.5]).

Proposition 1.2. *Let H be an operator satisfying Condition 1, let $f(\lambda)$, $\lambda \in \mathbb{R}$, be a bounded Borel function, and let the function $g \in L_\infty(\mathbb{R}^d)$ be defined by (1.3). Assume that one of the following conditions is fulfilled:*

- a) *if $s \neq 2$, then $u(f) \in l_s(\mathbb{Z}_+, d\nu)$;*
- b) *if $s = 2$, then $u(f) \in l_\alpha(\mathbb{Z}_+, d\nu)$ for some $\alpha \in (1, 2)$.*

Then $f(H)[g(x)] \in \mathfrak{S}_{s,\infty}$ and the estimate $\|f(H)[g(x)]\|_{\mathfrak{S}_{s,\infty}} \leq C\|u(f)\|\|\omega\|$ holds true. Here it is assumed that

$$\begin{aligned} \|u(f)\| &:= \|u(f)\|_{l_s}, & \|\omega\| &= \|\omega\|_{L_s}, & C &:= C(M_1, M_2, M_3, d, s), & s > 2; \\ \|u(f)\| &:= \|u(f)\|_{l_s}, & \|\omega\| &= \|\omega\|_{L_2}, & C &:= C(M_1, M_2, M_3, d, s, \zeta), & s \in (0, 2); \\ \|u(f)\| &:= \|u(f)\|_{l_\alpha}, & \|\omega\| &= \|\omega\|_{L_{4/\alpha}}, & C &:= C(M_1, M_2, M_3, d, \alpha, \zeta), & s = 2. \end{aligned}$$

If $f(\lambda)$, $\lambda \in \mathbb{R}$, is a bounded compactly supported Borel function, then the following corollary can easily be deduced from Proposition 1.2.

Corollary 1.3. *Let H be an operator satisfying Condition 1, let $f(\lambda)$, $\lambda \in \mathbb{R}$, be a bounded compactly supported Borel function with $\text{supp } f \subset (-A, A)$, and let the function $g \in L_\infty(\mathbb{R}^d)$ be defined by (1.3). Then $f(H)[g(x)] \in \mathfrak{S}_{s,\infty}$, and the estimate*

$$\|f(H)[g(x)]\|_{\mathfrak{S}_{s,\infty}} \leq C \cdot \max_{\lambda \in \mathbb{R}} |f(\lambda)| \cdot \|\omega\|_{L_\theta}$$

holds true, where $\theta = \theta_s := \max\{2, s + 1\}$, $C := C(M_1, M_2, M_3, d, s, A, \zeta)$.

Comment. Clearly, here we have noticeably roughened the result of Proposition 1.2.

Now, let φ be a bounded Borel function with compact support, $\text{supp } \varphi \subset (-A, A)$, let $V \in L_\infty(\mathbb{R}^d)$ satisfy estimate (0.6), and let $W \in L_\infty(\mathbb{R}^d)$ be defined by the formula

$$(1.4) \quad W(x) := \zeta(|x|)\omega(x/|x|)|x|^{-d/p}, \quad x \in \mathbb{R}^d, \quad p \in (0, +\infty),$$

where $\omega \in L_\infty(\mathbb{S}^{d-1})$ and ζ is a continuous cut-off at zero. We shall use Proposition 0.3 and Corollary 1.3 in order to estimate the singular values of the operators $\varphi(H)VW$ and $W\varphi(H)V$.

Pick a smooth function $\psi(\lambda)$, $\lambda \in \mathbb{R}$, with compact support so that $\psi(\lambda) = 1$, $\lambda \in (-A - 1, A + 1)$, $\text{supp } \psi \in (-A - 2, A + 2)$. The identity $W\varphi(H)V = W\varphi(H)\psi(H)V$, Propositions 0.2 and 0.3, and Corollary 1.3 yield the following statement.

Proposition 1.4. *Let H be an operator satisfying Condition 1, let $\varphi(\lambda)$, $\lambda \in \mathbb{R}$, be a bounded compactly supported Borel function with $\text{supp } \varphi \subset (-A, A)$, let $V \in L_\infty(\mathbb{R}^d)$ satisfy estimate (0.6), and let $W \in L_\infty(\mathbb{R}^d)$ be defined by formula (1.4), where $\omega \in L_\infty(\mathbb{S}^{d-1})$ and ζ is a continuous cut-off at zero. Then $W\varphi(H)V \in \mathfrak{S}_{r,\infty}$, $r^{-1} = p^{-1} + q^{-1}$, and we have $\|W\varphi(H)V\|_{\mathfrak{S}_{r,\infty}} \leq C \max_{\mathbb{R}} |\varphi| \|\omega\|_{L_\theta}$, where $\theta = \theta_p = \max\{2, p + 1\}$, $C = C(M_1, M_2, M_3, d, q, p, A, V, \psi, \zeta)$.*

Finally, since $n(s, \mathbb{T}) = n(s^2, \mathbb{T}\mathbb{T}^*)$, Proposition 0.2 shows that

$$\begin{aligned} \|\varphi(H)VW\|_{\mathfrak{S}_{r,\infty}} &= \|\varphi(H)|W|^2|V|^2\psi(H)\bar{\varphi}(H)\|_{\mathfrak{S}_{r/2,\infty}}^{1/2} \\ &\leq \left(C(p, q) \|\varphi(H)|W|^2\|_{\mathfrak{S}_{p/2,\infty}} \| |V|^2\psi(H) \|_{\mathfrak{S}_{q/2,\infty}} \max_{\mathbb{R}} |\varphi| \right)^{1/2}. \end{aligned}$$

Together with Corollary 1.3, these estimates yield the next statement.

Proposition 1.5. *Let H be an operator satisfying Condition 1, let $\varphi(\lambda)$, $\lambda \in \mathbb{R}$, be a bounded compactly supported Borel function with $\text{supp } \varphi \subset (-A, A)$, let $V \in L_\infty(\mathbb{R}^d)$ satisfy (0.6), and let $W \in L_\infty(\mathbb{R}^d)$ be defined by formula (1.4), where $\omega \in L_\infty(\mathbb{S}^{d-1})$*

and ζ is a continuous cut-off at zero. Then $\varphi(H)WV \in \mathfrak{S}_{r,\infty}$, $r^{-1} = p^{-1} + q^{-1}$, and we have $\|\varphi(H)VW\|_{\mathfrak{S}_{r,\infty}} \leq C \max_{\mathbb{R}} |\varphi| \|\omega\|_{L_\eta}$, where $\eta = \eta_p := \max\{4, p + 2\}$, $C = C(M_1, M_2, M_3, d, q, p, A, V, \psi, \zeta)$.

Comment. In the proof of our main result (see item 2 in §2) we shall need to know the dependence of upper estimates for the quasinorms $\|W\varphi(H)V\|_{\mathfrak{S}_{r,\infty}}$, $\|\varphi(H)VW\|_{\mathfrak{S}_{r,\infty}}$ on the norms of φ and ω in the corresponding classes; the parameters p , q , and A and the functions V , ψ , and ζ will be fixed.

3. Nash–Aronson estimate in the complex plane. Estimate (0.2) can be extended from the set of positive times to the entire right half-plane $\mathbb{C}_r := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. Repeating the argument of [7, Chapter 3, §3.4], we arrive at the following statement.

Proposition 1.6. *Let H be an operator satisfying Condition 1. Then the operator e^{-zH} , $\operatorname{Re} z > 0$, is an integral operator in $L_2(\mathbb{R}^d)$ with a kernel $K(z, x, y)$, $x, y \in \mathbb{R}^d$, $\operatorname{Re} z > 0$, such that*

$$(1.5) \quad |K(z, x, y)| \leq \widetilde{M}_1 (\operatorname{Re} z)^{-d/2} \exp\left(\widetilde{M}_2 \operatorname{Re} z - \frac{|x-y|^2}{\widetilde{M}_3} \operatorname{Re}(z^{-1})\right), \quad x, y \in \mathbb{R}^d, \operatorname{Re} z > 0,$$

where the $\widetilde{M}_i > 0$, $i = 1, 2, 3$, are independent of $x, y \in \mathbb{R}^d$ and $z \in \mathbb{C}_r$.

Repeating the argument of [7, Chapter 3, §3.4] and using Proposition 1.6, we can show the following.

Corollary 1.7. *Let H be an operator satisfying Condition 1. Then for any function $\varphi \in C_0^\infty(\mathbb{R})$ the operator $\varphi(H)$ is an integral operator in $L_2(\mathbb{R}^d)$ with a kernel $K_\varphi(x, y)$, $x, y \in \mathbb{R}^d$, such that*

$$(1.6) \quad |K_\varphi(x, y)| \leq C(\varphi, N)(1 + |x - y|^2)^{-N}, \quad x, y \in \mathbb{R}^d, \quad N \in \mathbb{N}.$$

Remark 1.8. Observe that, by Schur’s lemma, the function $|K_\varphi(x, y)|$ is the kernel of a bounded integral operator.

§2. PROOF OF THE MAIN RESULT

1. Proof of Theorem 0.4 in the “smooth” case. 1) Here we verify the claim of Theorem 0.4 under the following conditions. Let H be an operator satisfying Condition 1; let $\varphi(\lambda)$, $\lambda \in \mathbb{R}$, be a smooth function with compact support; let $V \in L_\infty(\mathbb{R}^d)$ satisfy (0.6); let $W(x)$, $x \in \mathbb{R}^d$, be defined by (1.4), where $\omega \in C^\infty(\mathbb{S}^{d-1})$; finally, let ζ be a smooth cut-off at zero, i.e., $\zeta \in C^\infty(\mathbb{R})$ is monotone nondecreasing and $\zeta(t) = 0$ if $t < 1$, $\zeta(t) = 1$ if $t > 2$.

2) The properties of the kernel $K_\varphi(x, y)$, $x, y \in \mathbb{R}^d$ of the integral operator $\varphi(H)$ have some consequences to be described below. We introduce the measure $d\nu_\varphi(x, y) := |K_\varphi(x, y)|^2 dx dy$ on $\mathbb{R}^d \times \mathbb{R}^d$. For brevity, denote $\langle x \rangle := (1 + |x|^2)^{1/2}$, $x \in \mathbb{R}^d$. Estimate (1.6) yields the following.

Proposition 2.1. *Suppose that a measurable function $a: \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{C}$ satisfies the condition*

$$(2.1) \quad |a(x, y)| \leq C \langle x - y \rangle^\gamma \langle y \rangle^{-d/\varrho}, \quad x, y \in \mathbb{R}^d.$$

for some $\varrho \in (0, +\infty)$ and $\gamma \geq 0$. Then $a \in L_{\varrho,\infty}(\mathbb{R}^d \times \mathbb{R}^d, d\nu_\varphi)$.

Applying Proposition 2.1 and Proposition 1.1, we arrive at the next statement.

Proposition 2.2. *Suppose that a measurable function $a: \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{C}$ satisfies condition (2.1) with some $\varrho \in (2, +\infty)$. Then the kernel $a(x, y)K_\varphi(x, y)$, $x, y \in \mathbb{R}^d$, gives rise to a bounded integral operator $\varphi_a(H) \in \mathfrak{S}_{\varrho,\infty}$.*

3) Now we state a suitable property of a function $W(x)$ of the form (1.4).

Proposition 2.3. *Let a function $W(x)$, $x \in \mathbb{R}^d$, be defined by formula (1.4), where ζ is a smooth cut-off at zero, $\omega \in C^\infty(\mathbb{S}^{d-1})$. Then*

$$(2.2) \quad |W(y) - W(x)| \leq C(p, d) \langle y \rangle^{-d/p-1} \langle x - y \rangle^{d/p+2}, \quad x, y \in \mathbb{R}^d.$$

Proof. Inequality (2.2) follows from the relation

$$W(y) - W(x) = \int_0^1 (\nabla W(x + t(y - x)), y - x)_{\mathbb{C}^d} dt, \quad x, y \in \mathbb{R}^d,$$

and the estimate $|\nabla W(x)| = O(|x|^{-d/p-1})$, $|x| \rightarrow +\infty$. □

Now we can prove Theorem 0.4 in the “smooth case”.

Proposition 2.4. *Let $V \in L_\infty(\mathbb{R}^d)$ satisfy condition (0.6), let $W \in L_\infty(\mathbb{R}^d)$ be defined by (1.4), where ζ is a smooth cut-off at zero and $\omega \in C^\infty(\mathbb{S}^{d-1})$, and let $\varphi(\lambda)$, $\lambda \in \mathbb{R}$, be a smooth function with compact support. Then (0.7) is true.*

Proof. Pick a smooth compactly supported function $\psi(\lambda)$, $\lambda \in \mathbb{R}$, subject to the condition $\varphi(\lambda) \cdot \psi(\lambda) \equiv \varphi(\lambda)$. For arbitrary $m_1, m_2 \in (0, +\infty]$, we have

$$(2.3) \quad \begin{aligned} \varphi(H)VW - W\varphi(H)V &= \psi(H)\langle x \rangle^{-d/m_1} (\langle x \rangle^{d/m_1} \varphi(H)VW - \langle x \rangle^{d/m_1} W\varphi(H)V) \\ &\quad + (\psi(H)W\langle x \rangle^{d/m_2} - W\psi(H)\langle x \rangle^{d/m_2}) \langle x \rangle^{-d/m_2} \varphi(H)V \\ &= \psi(H)\langle x \rangle^{-d/m_1} S_1 + S_2 \langle x \rangle^{-d/m_2} \varphi(H)V, \end{aligned}$$

where

$$S_1 := \langle x \rangle^{d/m_1} \varphi(H)VW - \langle x \rangle^{d/m_1} W\varphi(H)V, \quad S_2 := \psi(H)W\langle x \rangle^{d/m_2} - W\psi(H)\langle x \rangle^{d/m_2}.$$

Now Propositions 0.2 and 0.3 show that

$$(2.4) \quad \begin{aligned} \psi(H)\langle x \rangle^{-d/m_1} &\in \mathfrak{S}_{m_1, \infty}, \\ \langle x \rangle^{-d/m_2} \varphi(H)V &= \langle x \rangle^{-d/m_2} \psi(H)\varphi(H)V \in \mathfrak{S}_{l, \infty}, \quad l^{-1} = m_2^{-1} + q^{-1}. \end{aligned}$$

The operator S_1 has kernel $K_\varphi(x, y)a_1(x, y)$, where

$$a_1(x, y) = \langle x \rangle^{d/m_1} (W(y) - W(x))V(y).$$

By Proposition 2.3 and the trivial inequality $\langle x + y \rangle^\alpha \leq 2^{|\alpha|/2} \langle x \rangle^\alpha \langle y \rangle^{|\alpha|}$, where $x, y \in \mathbb{R}^d$ and $\alpha \in \mathbb{R}$, we have

$$(2.5) \quad |a_1(x, y)| \leq C(p, m_1, d) \langle y \rangle^{-d(1/r+1/d-1/m_1)} \langle x - y \rangle^{d(1/p+1/m_1)+2}, \quad x, y \in \mathbb{R}^d.$$

The operator S_2 has kernel $K_\psi(x, y)a_2(x, y)$, where $a_2(x, y) = (W(y) - W(x))\langle y \rangle^{d/m_2}$. Proposition 2.3 yields the estimate

$$(2.6) \quad |a_2(x, y)| \leq C(p, d) \langle y \rangle^{-d(1/p+1/d-1/m_2)} \langle x - y \rangle^{d/p+2}, \quad x, y \in \mathbb{R}^d.$$

If m_1, m_2 are chosen so that $1/r + 1/d - 1/m_1 \in (0, 1/2)$ and $1/p + 1/d - 1/m_2 \in (0, 1/2)$, then by Proposition 2.2 we get $S_1 \in \mathfrak{S}_{\varrho_1, \infty}$, where $\varrho_1^{-1} = r^{-1} + d^{-1} - m_1^{-1}$, and $S_2 \in \mathfrak{S}_{\varrho_2, \infty}$, where $\varrho_2^{-1} = p^{-1} + d^{-1} - m_2^{-1}$. Combining this with (2.3), (2.4), and Proposition 0.2, we see that

$$\varphi(H)VW - W\varphi(H)V \in \mathfrak{S}_{n, \infty} \subset \mathfrak{S}_{r, \infty}^0, \quad n^{-1} = r^{-1} + d^{-1}. \quad \square$$

2. Proof of Theorem 0.4 in the general case. Propositions 1.4 and 1.5 allow us to “pass to a closure” in the claim of Proposition 2.4 and to prove Theorem 0.4 under the following conditions: φ is a continuous function with compact support on the line; the function $V \in L_\infty(\mathbb{R}^d)$ satisfies estimate (0.6); the function $W \in L_\infty(\mathbb{R}^d)$ is defined by formula (1.4), where $\omega \in L_\infty(\mathbb{S}^{d-1})$ and ζ is a smooth cut-off at zero. By Proposition 0.3, the addition to the function $W(x)$ of a bounded summand that admits the estimate $o(|x|^{-d/p})$, $|x| \rightarrow \infty$, leads to adding summands of class $\mathfrak{S}_{r,\infty}^0$ to the operators $W\varphi(H)V$ and $\varphi(H)VW$, which completes the proof of Theorem 0.4. \square

3. Proof of Theorem 0.5. Note that the operators $\varphi(H)$ and $[W(x)]$ only differ from the operators $|\varphi|(H)$ and $[|W(x)|]$ by unitary factors. Therefore, it suffices to check the claim of Theorem 0.5 under the additional conditions $\varphi \geq 0$, $W \geq 0$. Next, we use the following two obvious identities (on the right-hand sides all terms cancel out, except for the first and the last ones)

$$\begin{aligned} & \varphi^n(H)W^n - (\varphi(H)W)^n \\ (2.7) \quad &= \sum_{l=1}^n (\varphi(H)W)^{l-1} (\varphi(H)(\varphi^{n-l}(H)W - W\varphi^{n-l}(H))W^{n-l}); \\ & (\varphi(H)W)^n - (W^{1/2}\varphi(H)W^{1/2})^n \\ (2.8) \quad &= \sum_{l=1}^n (W^{1/2}\varphi(H)W^{1/2})^{l-1} (\varphi(H)W - W^{1/2}\varphi(H)W^{1/2})(\varphi(H)W)^{n-l}. \end{aligned}$$

Using formulas (2.7) and (2.8), Propositions 0.2 and 0.3, and Theorem 0.4, we conclude that

$$\varphi^n(H)W^n - (W^{1/2}\varphi(H)W^{1/2})^n \in \mathfrak{S}_{p/n,\infty}^0.$$

Therefore,

$$(2.9) \quad D_{p/n}(\varphi^n(H)W^n) = D_{p/n}((W^{1/2}\varphi(H)W^{1/2})^n) = D_p(W^{1/2}\varphi(H)W^{1/2}), \quad D = \Delta, \delta.$$

It remains to observe that, by Theorem 0.4, $W^{1/2}\varphi(H)W^{1/2} - \varphi(H)W \in \mathfrak{S}_{p,\infty}^0$; combined with (2.9), this gives (0.8). \square

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