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# MARTINGALE TRANSFORMS OF THE RADEMACHER SEQUENCE IN REARRANGEMENT INVARIANT SPACES

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Abstract. Let $v_k = c_k\chi_{\{\tau\ge k\}}$, where $\tau$ is a stopping time with respect to the Rademacher system $\{r_k\}$ and $c_k \in \mathbb{R}$, $k = 1, 2, \dots$. Then $\big\|\sum_{k=1}^n v_k r_k\big\|_X \asymp \big\|(\sum_{k=1}^n v_k^2)^{1/2}\big\|_X$ if and only if the rearrangement invariant Banach function space $X$ has nontrivial Boyd indices. If the $v_k$ are the vectors $\sum_{i=0}^{k-1} a_k^i r_i$, $k = 1, 2, \dots$, the same relation is fulfilled if and only if $X$ contains the closure of $L_\infty$ in the Orlicz space $\exp L_1$. In the second part of the paper, a new unconditionality criterion for the Haar system in a rearrangement invariant space is obtained in terms of a decoupling version of the transforms $f_n = \sum_{k=1}^n v_k r_k$.

## §1. Introduction

In 1966, Burkholder proved that for every $1 < p < \infty$ there exists a constant $C_p > 0$ such that

$$C_p^{-1}\|f_\infty\|_p \le \|P(f)\|_p \le C_p\|f_\infty\|_p \tag{1}$$

for all uniformly integrable martingales $f = \{f_n\}_{n=0}^\infty$, where $f_\infty = \lim_{n\to\infty} f_n$ a.e. and $P(f) := \big(f_0^2 + \sum_{n=1}^\infty (f_n - f_{n-1})^2\big)^{1/2}$, see [11]. At the same time, (1) fails if $p = 1$ or $p = \infty$. Considering a similar question for an arbitrary rearrangement invariant (r.i.) space $X$, Kikuchi proved in [17] that

$$C^{-1}\|f_\infty\|_X \le \|P(f)\|_X \le C\|f_\infty\|_X \tag{2}$$

for all uniformly integrable martingales $f = \{f_n\}_{n=0}^\infty \subset X$ if and only if the Boyd indices of $X$ are nontrivial, i.e.,

$$0 < \alpha_X \le \beta_X < 1. \tag{3}$$

The present paper is devoted to the study of martingales of the form

$$f_n := \sum_{k=1}^n v_k r_k, \quad n = 1, 2, \dots, \tag{4}$$

where the $r_k$ are the Rademacher functions, i.e., $r_k(t) = \operatorname{sgn}\big[\sin(2^k\pi t)\big]$, $0 \le t \le 1$, and $\{v_k\}_{k=1}^\infty$ is an arbitrary predictable sequence with respect to the filtration $\{\Sigma_k\}_{k=0}^\infty$ of $\sigma$-algebras generated by the system $\{r_k\}_{k=0}^\infty$, where $r_0 \equiv 1$. In the simplest case, where the $v_k$ are constants, by the classical Rodin–Semenov theorem (see [25]), the inequalities

$$C^{-1}\bigg(\sum_{k=1}^\infty c_k^2\bigg)^{1/2} \le \bigg\|\sum_{k=1}^\infty c_k r_k\bigg\|_X \le C\bigg(\sum_{k=1}^\infty c_k^2\bigg)^{1/2}$$

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are fulfilled for arbitrary $c_k \in \mathbb{R}$ if and only if the r.i. space $X$ contains $G$, where $G$ is the closure of $L_\infty$ in the Orlicz space $\exp L_2$. If we talk about sequences of independent functions (that are also independent from the Rademacher system), the result is different: the inequalities

$$C^{-1}\left\|\left(\sum_{k=1}^{\infty} v_k^2\right)^{1/2}\right\|_X \le \left\|\sum_{k=1}^{\infty} v_k(s)r_k(t)\right\|_{X([0,1]\times[0,1])} \le C\left\|\left(\sum_{k=1}^{\infty} v_k^2\right)^{1/2}\right\|_X$$

are valid for an arbitrary sequence $\{v_k\}_{k=1}^{\infty} \subset X$ of independent functions if and only if $X$ possesses the so-called Kruglov property (in symbols: $X \in \mathbb{K}$), see [2, Theorem 2]. Here $X([0,1]\times[0,1])$ is the r.i. space of functions on the square $[0,1]\times[0,1]$ with the norm $\|f\|_{X([0,1]\times[0,1])} = \|f^*\|_X$, where $f^*$ is the nonincreasing rearrangement of $|f|$ (see the next section for the definitions). We refer to [10, 8, 9] for more information concerning the Kruglov property. Here we only note that the condition $\alpha_X > 0$ implies the Kruglov property and that the Orlicz space $\exp L_p$ belongs to $\mathbb{K}$ if and only if $p \ge 1$.

In the first part of the paper, the behavior of the martingale transforms (4) is studied for other classes of $\{\Sigma_k\}$-predictable sequences. Refining the result of Kikuchi [17] cited above, we prove that inequalities (3) must be true already in the case where (2) is fulfilled for the martingales (4) with the coefficients $v_k$ of the form $v_k = c_k\chi_{\{\tau\ge k\}}$, $k = 1, 2, \dots$, where the stopping time $\tau$ with respect to the filtration $\{\Sigma_k\}$ and the numbers $c_k \in \mathbb{R}$ are arbitrary (Theorem 1). Moreover, it is shown that the following two stopping times:

$$\tau_1 := \inf\left\{k = 2, 3, \dots : r_k \ne r_{k-1}\right\} \quad \text{and} \quad \tau_2 := \inf\left\{k = 2, 3, \dots : r_k = r_{k-1}\right\}$$

are "critical": if

$$C^{-1}\left\|\sum_{i=1}^{\infty} c_i\chi_{\{\tau_j\ge i\}}r_i\right\|_X \le \left\|\left(\sum_{i=1}^{\infty} c_i^2\chi_{\{\tau_j\ge i\}}\right)^{1/2}\right\|_X \le C\left\|\sum_{i=1}^{\infty} c_i\chi_{\{\tau_j\ge i\}}r_i\right\|_X$$

for some $C > 0$, $j = 1, 2$, and all $c_i \in \mathbb{R}$, then the Boyd indices of $X$ satisfy (3).

At the same time, if the $v_k$ are arbitrary linear combinations $\sum_{i=0}^{k-1} a_k^i r_i$, $k = 1, 2, \dots$, then inequalities (2) are true if and only if $X$ contains the closure of $L_\infty$ in the Orlicz space $\exp L_1$ (Theorem 2).

A natural decoupling version of the transforms (4) is given by the martingales

$$g_n(s,t) := \sum_{k=1}^{n} v_k(s)r_k(t), \quad 0 \le s,\ t \le 1.$$

On the basis of Hitczenko's results [13] about comparison of these martingales with the transforms (4) in $L_p$-spaces and also on Talagrand's example exposed in the same paper, in the second part of this article we discuss a similar question for general r.i. spaces. Also in the second part of the paper, we prove a new criterion for the unconditionality of the Haar system in an r.i. space in terms of the transforms $g_n$, $n = 1, 2, \dots$ (see Theorem 3).

## §2. Definitions and preliminaries

(a) *Rearrangement invariant spaces.* A Banach space $X$ of measurable functions on $[0,1]$ is said to be *rearrangement invariant* (or symmetric) if 1) it is an ideal space, i.e., the conditions $|x(t)| \le |y(t)|$ for a.e. $t \in [0,1]$ and $y \in X$ imply that $x \in X$ and $\|x\|_X \le \|y\|_X$; 2) whenever $y \in X$ and $x$ is a function such that $x$ and $y$ are *equimeasurable*, i.e.,

$$m\{t \in [0,1] : |y(t)| > u\} = m\{t \in [0,1] : |x(t)| > u\} \quad (u > 0),$$

where $m(a)$ is the Lebesgue measure of a set $a \subset \mathbb{R}$, we have $x \in X$ and $\|x\|_X = \|y\|_X$. In particular, an arbitrary measurable function $x(t)$ on $[0,1]$ is equimeasurable with its

*nonincreasing rearrangement* continuous from the left:

$$x^*(t) := \inf\big\{u \geq 0 \ : \ m\{s \in [0,1] \ : \ |x(s)| > u\} < t\big\}, \quad 0 \leq t \leq 1.$$

If $X$ is an r.i. space on $[0,1]$, its dual $X'$ is defined to consist of all $y$ such that

$$\|y\|_{X'} = \sup\left\{\int_0^1 x(t)y(t)\,dt \ : \ \|x\|_X \leq 1\right\} < \infty.$$

The space $X'$ is also rearrangement invariant; it embeds isometrically to the conjugate space $X^*$, moreover, $X' = X^*$ if and only if $X$ is separable. The space $X$ is said to be *maximal* if $X = X''$. This property is equivalent to the following one: given $x_n \in X$ $(n = 1, 2, \dots)$ with $\sup_{n=1,2,\dots} \|x_n\|_X < \infty$ and $x_n \to x$ a.e. on $[0,1]$, we have $x \in X$ and $\|x\|_X \leq \liminf_{n\to\infty} \|x_n\|_X$. Throughout the paper, we shall assume that an r.i. space is either separable or maximal. Recall that any such space has the following property: if $x \in X$ and

$$\int_0^t y^*(s)\,ds \leq \int_0^t x^*(s)\,ds, \quad 0 \leq t \leq 1,$$

then $y \in X$ and $\|y\|_X \leq \|x\|_X$. Furthermore, the norm in any such r.i. space is *order semicontinuous*, that is, $X$ is isometrically embedded in $X''$. We also assume the normalization condition $\|\chi_{[0,1]}\|_X = 1$, where $\chi_e(t)$ is the characteristic function of a measurable set $e \subset [0,1]$. Then $L_\infty \subset X \subset L_1$ and $\|x\|_{L_1} \leq \|x\|_X \leq \|x\|_{L_\infty}$ for every $x \in L_\infty$. If $X \neq L_\infty$, then the closure of $L_\infty$ in $X$ is called the *separable part* of $X$ and is denoted by $X_0$; this is a separable r.i. space. In particular, the norm of an arbitrary function in $X_0$ is absolutely continuous. This means that $\lim_{k\to\infty} \|x\chi_{e_k}\|_X = 0$ whenever $\{e_k\}_{k=1}^\infty$ is a monotone decreasing sequence of subsets of $[0,1]$ with $\lim_{k\to\infty} m(e_k) = 0$.

The dilation operator $\sigma_\tau x(t) = x(t/\tau)\chi_{[0,1]}(t/\tau)$ $(0 \leq t \leq 1)$, where $\tau > 0$, acts boundedly in every r.i. space $X$. Moreover, $\|\sigma_\tau\|_{X\to X} \leq \max(1,\tau)$ (see [18, Theorems 2.4.4 and 2.4.5]). The quantities

$$\alpha_X = \lim_{\tau\to 0} \frac{\ln \|\sigma_\tau\|_X}{\ln \tau} \quad \text{and} \quad \beta_X = \lim_{\tau\to\infty} \frac{\ln \|\sigma_\tau\|_X}{\ln \tau}$$

are called the Boyd indices of the r.i. space $X$. We always have $0 \leq \alpha_X \leq \beta_X \leq 1$. For example, $\alpha_{L_p} = \beta_{L_p} = 1/p$ for all $p \in [1,\infty]$.

The *Orlicz spaces* (a natural generalization of $L_p$) constitute an important class of r.i. spaces. Let $M(u)$ be an *Orlicz function*, i.e., a strictly monotone increasing convex function on $[0,\infty)$, $M(0) = 0$. The Orlicz space $L_M$ consists of all measurable functions $x(t)$ on $[0,1]$ such that

$$\int_0^1 M\left(|x(t)|/\lambda\right)\,dt \leq 1$$

for sufficiently large $\lambda > 0$. The norm on $L_M$ is defined as follows: $\|x\|_{L_M} = \inf \lambda$, where the infimum is taken over all $\lambda > 0$ for which the preceding inequality is true. The classical Orlicz space built on the basis of an Orlicz function equivalent to $e^{u^p} - 1$ is usually denoted by $\exp L_p$. We quote a well-known description of the separable part of this space (see [21] or [3, Lemma 3.2]): $x \in (\exp L_p)_0$ if and only if

$$\lim_{t\to+0} \frac{x^*(t)}{\log_2^{1/p}(2/t)} = 0.$$

(b) *Martingale transforms and the Haar system.* Let $\{\mathcal{A}_k\}_{k=0}^\infty$ be a filtration of $\sigma$-algebras on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, i.e., $\mathcal{A}_0 \subset \mathcal{A}_1 \subset \cdots \subset \mathcal{A}_k \subset \cdots \subset \mathcal{A}$. A sequence of random variables $\{v_k\}_{k=1}^\infty$ is said to be *predictable* with respect to this filtration (or $\{\mathcal{A}_k\}$-*predictable*) if $v_k$ is $\mathcal{A}_{k-1}$-measurable for all $k = 0, 1, \dots$.

Suppose we are given a sequence $\{\xi_k\}_{k=0}^\infty$ of independent random variables on $(\Omega, \mathcal{A}, \mathbb{P})$ with $\int_\Omega \xi_k \, d\mathbb{P} = 0$. Let $\mathcal{A}_k$ be the $\sigma$-algebra generated by the collection $\{\xi_0, \xi_1, \dots, \xi_k\}$, $k = 0, 1, \dots$. Then for every $\{\mathcal{A}_k\}$-predictable sequence $\{v_k\}_{k=1}^\infty$, the random variables

$$\eta_n := \sum_{k=1}^n v_k \xi_k, \quad n = 1, 2, \dots,$$

form a martingale (see [23, 26] or [28] for the definition and properties of martingales). For this reason, the sequence $\{\eta_n\}_{n=1}^\infty$ is called a martingale transform of the sequence $\{\xi_k\}_{k=0}^\infty$. We remind the reader that a random variable $\tau = \tau(\omega)$ with values in $\mathbb{N} \cup \{0\}$ is called a *stopping time* relative to the filtration $\{\mathcal{A}_k\}_{k=0}^\infty$ of $\sigma$-algebras if $\{\omega \in \Omega : \tau(\omega) \le k\} \in \mathcal{A}_k$ $(k = 0, 1, \dots)$.

In this paper, we study martingale transforms of the simplest sequence of independent functions, namely, the Rademacher system $\{r_k\}_{k=0}^\infty$, where $r_0 \equiv 1$. In this case, $\mathcal{A}_k$ is the $\sigma$-algebra $\Sigma_k$ generated by the dyadic intervals of rank $k$, i.e., by the intervals $\Delta_k^i = ((i-1)2^{-k}, i2^{-k})$, $i = 1, \dots, 2^k$, $k = 0, 1, \dots$. By the Rademacher–Kolmogorov theorem, the series $\sum_{k=0}^\infty a_k r_k(t)$ converges a.e. if and only if $a = (a_k)_{k=1}^\infty \in \ell_2$ (see [16, §4.5] or [3, Theorems 1.1 and 1.2]).

Recall that the *Haar system* is formed by the functions $h_{0,0}(t) = 1$,

$$h_{n,k}(t) = \begin{cases} 1, & t \in \Delta_{n+1}^{2k-1}, \\ -1, & t \in \Delta_{n+1}^{2k}, \\ 0 & \text{for all other } t \in [0,1], \end{cases}$$

where $n = 0, 1, \dots$, $k = 1, \dots, 2^n$. See the monographs [15, 12, 24] and [22, Chapter 1] for more details. The complete orthonormal system $\{2^{n/2} h_n^k\}$ is a monotone basis in every separable r.i. space. The Haar system is an unconditional basis in a separable r.i. space $X$ if and only if $0 < \alpha_X \le \beta_X < 1$ (see [20, 2.c.6] or [18, 2.9.6]). Another equivalent condition is $\|x\|_X \asymp \|Px\|_X$, where

$$x = \sum_{k=1}^\infty \sum_{i=1}^{2^{k-1}} c_{k,i} h_{k-1,i} \quad \text{and} \quad Px = \left( \sum_{k=1}^\infty \sum_{i=1}^{2^{k-1}} c_{k,i}^2 h_{k-1,i}^2 \right)^{1/2}, \tag{5}$$

see [15, Theorem 1.11]. Throughout, when we write $f \asymp g$ we mean that $f$ and $g$ are nonnegative functions (in particular, they may be quasinorms) satisfying $C^{-1} \le g/f \le C$ for some constant $C$ uniformly on the domain of $f$ and $g$. The function $Px(t)$ is called the *Paley function*. The Fourier–Haar coefficients $c_{n,k} = c_{n,k}(x)$ of a function $x \in L_1 = L_1[0,1]$ are defined by the formula

$$c_{n,k} = 2^n \int_0^1 x(s) h_{n,k}(s) \, ds, \quad n = 0, 1, \dots, \ \ k = 1, \dots, 2^n.$$

Let $\{v_k\}_{k=1}^\infty$ be a $\{\Sigma_k\}$-predictable sequence. By Doob's classical theorem (see [28, Theorem 3.14] or [26, Theorem 7.4.1]), the condition

$$\sup_{n=1,2,\dots} \int_0^1 \Big| \sum_{k=1}^n v_k(t) r_k(t) \Big| \, dt < \infty$$

implies the a.e. convergence of the series

$$Rv := \sum_{k=1}^\infty v_k r_k \tag{6}$$

on $[0,1]$ to an integrable function. Observe that the terms $v_k$, $k = 1, 2, \dots$, of any $\{\Sigma_k\}$-predictable sequence can be represented in the form

$$v_k := \sum_{i=1}^{2^{k-1}} c_{k,i}\chi_{\Delta_{k-1}^i}, \quad k = 1, 2, \dots, \tag{7}$$

with some $c_{k,i} \in \mathbb{R}$. Therefore, since

$$r_{n+1}(t) = \sum_{k=1}^{2^n} h_{n,k}(t), \quad n = 0, 1, 2, \dots,$$

we have

$$Rv = \sum_{k=1}^{\infty}\sum_{i=1}^{2^{k-1}} c_{k,i}h_{k-1,i}. \tag{8}$$

This formula shows that the $c_{k,i}$ are the Fourier–Haar coefficients of the function $Rv$. In the same terms, the Paley function acquires the following form:

$$Pv = \left(\sum_{k=1}^{\infty} v_k^2\right)^{1/2}. \tag{9}$$

The last formulas, when combined with Theorem 2 in [7] about Haar series in r.i. spaces, yield the following result.

**Proposition 1.** *Let $X$ be an r.i. space on $[0,1]$.*

1. *The relation*

$$\|Rv\|_X \le C\|Pv\|_X$$

*is fulfilled for every $\{\Sigma_k\}$-predictable sequence $\{v_k\}$ with $C > 0$ independent of the sequence if and only if $\alpha_X > 0$.*

2. *The following statements are equivalent.*

(a) *There exists $C > 0$ such that for every $\{\Sigma_k\}$-predictable sequence $\{v_k\}$ we have*

$$\|Pv\|_X \le C\|Rv\|_X;$$

(b) $0 < \alpha_X \le \beta_X < 1$.

In the sequel, we shall also consider a decoupling version of the series $Rv$, specifically, the (formal) series

$$R'v(s,t) := \sum_{k=1}^{\infty} v_k(s)r_k(t), \quad 0 \le s,\ t \le 1.$$

If the coefficients $v_k$ are defined by (7), we have

$$R'v(s,t) = \sum_{k=1}^{\infty}\sum_{i=1}^{2^{k-1}} c_k^i\chi_{\Delta_{k-1}^i}(s)r_k(t). \tag{10}$$

In distinction with $\{v_k r_k\}_{k=1}^{\infty}$, the sequence $\{v_k(s)r_k(t)\}_{k=1}^{\infty}$ is unconditional with constant 1 in every r.i. space on $[0,1]\times[0,1]$. Moreover, the following slightly stronger statement holds true (we shall prove it for completeness).

**Lemma 1.** *The family of functions*

$$\left\{\chi_{\Delta_{k-1}^i}(s)r_k(t) :\ k = 1, 2, \dots;\ \ i = 1, 2, \dots, 2^{k-1}\right\},$$

*where the $\Delta_j^i$ are the dyadic subintervals of $[0,1]$, is 1-unconditional in every r.i. space $X$ on $([0,1]\times[0,1])$.*

*Proof.* For arbitrary $c_{k,i} \in \mathbb{R}$, $\varepsilon_{k,i} = \pm 1$, and $\tau > 0$ we have

$$m\Big\{(s,t) \in [0,1] \times [0,1] : \Big|\sum_{k=1}^{\infty}\sum_{i=1}^{2^{k-1}} \varepsilon_{k,i} c_{k,i} \chi_{\Delta_{k-1}^i}(s) r_k(t)\Big| > \tau\Big\}$$
$$= \int_0^1 m\Big\{t \in [0,1] : \Big|\sum_{k=1}^{\infty}\sum_{i=1}^{2^{k-1}} \varepsilon_{k,i} c_{k,i} \chi_{\Delta_{k-1}^i}(s) r_k(t)\Big| > \tau\Big\} ds$$
$$= \int_0^1 m\Big\{t \in [0,1] : \Big|\sum_{k=1}^{\infty} \varepsilon_{k,i_k(s)} c_{k,i_k(s)} r_k(t)\Big| > \tau\Big\} ds,$$

where the numbers $i_k(s) = 1, \dots, 2^{k-1}$ are chosen so that $s \in \Delta_{k-1}^{i_k(s)}$ for arbitrary $s \in [0,1]$ and $k = 1, 2, \dots$. Elementary properties of the Rademacher system allow us to rewrite the last quantity as follows:

$$\int_0^1 m\Big\{t \in [0,1] : \Big|\sum_{k=1}^{\infty} c_{k,i_k(s)} r_k(t)\Big| > \tau\Big\} ds$$
$$= m\Big\{(s,t) \in [0,1] \times [0,1] : \Big|\sum_{k=1}^{\infty}\sum_{i=1}^{2^{k-1}} c_{k,i} \chi_{\Delta_{k-1}^i}(s) r_k(t)\Big| > \tau\Big\}.$$

By the definition of r.i. spaces, it follows that

$$\Big\|\sum_{k=1}^{\infty}\sum_{i=1}^{2^{k-1}} \varepsilon_{k,i} c_{k,i} \chi_{\Delta_{k-1}^i}(s) r_k(t)\Big\|_{X([0,1]\times[0,1])} = \Big\|\sum_{k=1}^{\infty}\sum_{i=1}^{2^{k-1}} c_{k,i} \chi_{\Delta_{k-1}^i}(s) r_k(t)\Big\|_{X([0,1]\times[0,1])} \tag{11}$$

for every $\varepsilon_{k,i} = \pm 1$. This completes the proof by [15, Theorem 1.10]. □

## §3. Martingale transforms generated by a stopping time

In this section, we consider the martingale transforms (4) in the case where $v_k = c_k \chi_{\{\tau \ge k\}}$, where $\tau$ is a stopping time with respect to the filtration $\{\Sigma_k\}$ of $\sigma$-algebras and $c_k \in \mathbb{R}$ $(k = 1, 2, \dots)$. We refine Proposition 1 by showing that the Boyd indices must be nontrivial provided relations (2) are fulfilled for this special class of $\{\Sigma_k\}$-predictable sequences.

**Theorem 1.** *For an arbitrary r.i. space $X$ on $[0,1]$, the following properties are equivalent.*

(a) *We have*

$$\Big\|\sum_{i=1}^{\infty} c_i \chi_{\{\tau \ge i\}} r_i\Big\|_X \asymp \Big\|\Big(\sum_{i=1}^{\infty} c_i^2 \chi_{\{\tau \ge i\}}\Big)^{1/2}\Big\|_X \tag{12}$$

*with constants independent of the stopping time $\tau$ and $c_i \in \mathbb{R}$.*

(b) $0 < \alpha_X \le \beta_X < 1$.

*Proof.* By Proposition 1, it suffices to show that $(a) \Rightarrow (b)$. Consider the stopping time

$$\tau_1 := \inf\{k = 2, 3, \cdots : r_k \ne r_{k-1}\}.$$

Putting

$$R_1 = R_1(c) := \sum_{i=1}^{\infty} c_i \chi_{\{\tau_1 \ge i\}} r_i,$$

where $c_i \ge 0$, $c_{2i} = c_{2i-1}$, $i = 1, 2, \dots$, we obtain

$$|R_1| = \sum_{k=2}^{\infty} \Big| \sum_{i=1}^{k} c_i r_i \Big| \chi_{\{\tau_1 = k\}} = |c_1 + c_2 - c_3| \chi_{\{\tau_1 = 3\}} + (c_1 + c_2) \chi_{\{\tau_1 = 4\}}$$
$$+ \dots \Big| \sum_{i=1}^{2k-2} c_i - c_{2k-1} \Big| \chi_{\{\tau_1 = 2k-1\}} + \Big( \sum_{i=1}^{2k-2} c_i \Big) \chi_{\{\tau_1 = 2k\}} + \dots$$
$$\ge \sum_{k=1}^{\infty} \Big( \sum_{i=1}^{k} c_{2i} \Big) \chi_{\{\tau_1 = 2k+2\}}.$$

The sets $\{\tau_1 = 2k+2\}$, $k = 1, 2, \dots$, are mutually disjoint and $m\{\tau_1 = 2k+2\} = 2^{-2k-1}$. Thus, the last relation implies

$$R_1^*(t) \ge \sum_{k=1}^{\infty} \Big( \sum_{i=1}^{k} a_i \Big) \chi_{[2^{-2k-1}, 2^{-2k})}(t), \quad 0 < t \le 1,$$

where $a_k = c_{2k}$. Denoting by $x_a$ the right-hand side of this inequality, we observe that

$$x_a + \sigma_{1/2} x_a = \sum_{k=1}^{\infty} \Big( \sum_{i=1}^{k} a_i \Big) \chi_{[2^{-2k-2}, 2^{-2k})}.$$

Since $\|\sigma_{1/2}\|_{X \to X} \le 1$, it follows that

$$\Big\| \sum_{k=1}^{\infty} \Big( \sum_{i=1}^{k} a_i \Big) \chi_{[4^{-k-1}, 4^{-k})} \Big\|_X \le 2 \|R_1\|_X. \tag{13}$$

On the other hand, putting

$$\tau_2 := \inf\{k = 2, 3, \cdots : r_k = r_{k-1}\}, \quad R_2 = R_2(c) := \sum_{i=1}^{\infty} c_i \chi_{\{\tau_2 \ge i\}} r_i$$

and assuming as before that $c_i \ge 0$, $c_{2i} = c_{2i-1}$, $i = 1, 2, \dots$, we obtain

$$|R_2| = \sum_{k=2}^{\infty} \Big| \sum_{i=1}^{k} c_i r_i \Big| \chi_{\{\tau_2 = k\}} = (c_1 + c_2) \chi_{\{\tau_2 = 2\}} + c_3 \chi_{\{\tau_2 = 3\}}$$
$$+ \dots (c_{2k-1} + c_{2k}) \chi_{\{\tau_2 = 2k\}} + c_{2k+1} \chi_{\{\tau_2 = 2k+1\}} + \dots \tag{14}$$
$$= 2 \sum_{k=1}^{\infty} c_{2k} \chi_{\{\tau_2 = 2k\}} + \sum_{k=2}^{\infty} c_{2k} \chi_{\{\tau_2 = 2k-1\}}.$$

Since the sets $\{\tau_2 = j\}$, $j = 2, 3, \dots$, are mutually disjoint and $m\{\tau_2 = j\} = 2^{-j+1}$, we see that

$$\Big( \sum_{k=2}^{\infty} c_{2k} \chi_{\{\tau_2 = 2k-1\}} \Big)^*(t) = \sigma_2 \Big( \sum_{k=2}^{\infty} c_{2k} \chi_{\{\tau_2 = 2k\}} \Big)^*(t), \quad 0 \le t \le 1.$$

Since $\|\sigma_2\|_{X \to X} \le 2$, it follows that

$$\Big\| 2 \sum_{k=1}^{\infty} c_{2k} \chi_{\{\tau_2 = 2k\}} + \sum_{k=2}^{\infty} c_{2k} \chi_{\{\tau_2 = 2k-1\}} \Big\|_X \le 4 \Big\| \sum_{k=1}^{\infty} c_{2k} \chi_{[4^{-k}, 4^{-k+1})} \Big\|_X$$
$$= 4 \Big\| \sigma_2 \Big( \sum_{k=1}^{\infty} c_{2k} \chi_{[4^{-k-1}, 4^{-k})} \Big) \Big\|_X \le 8 \Big\| \sum_{k=1}^{\infty} c_{2k} \chi_{[4^{-k-1}, 4^{-k})} \Big\|_X.$$

Together with (14), this implies the inequality

$$\|R_2\|_X \le 8\Big\|\sum_{k=1}^{\infty} a_k \chi_{[4^{-k-1},4^{-k})}\Big\|_X, \tag{15}$$

where $a_k = c_{2k}$.

Next, since $m\{\tau_1 = k\} = m\{\tau_2 = k\}$, $k = 1, 2, \dots$, our assumptions imply

$$\begin{aligned}\|R_1(c)\|_X &\asymp \Big\|\Big(\sum_{i=1}^{\infty} c_i^2 \chi_{\{\tau_1 \ge i\}}\Big)^{1/2}\Big\|_X = \Big\|\sum_{k=1}^{\infty}\Big(\sum_{i=1}^{k} c_i^2 \chi_{\{\tau_1 = k\}}\Big)^{1/2}\Big\|_X \\ &= \Big\|\sum_{k=1}^{\infty}\Big(\sum_{i=1}^{k} c_i^2 \chi_{\{\tau_2 = k\}}\Big)^{1/2}\Big\|_X = \Big\|\Big(\sum_{i=1}^{\infty} c_i^2 \chi_{\{\tau_2 \ge i\}}\Big)^{1/2}\Big\|_X \asymp \|R_2(c)\|_X.\end{aligned}$$

As a result, by (13) and (15) it follows that

$$\Big\|\sum_{k=1}^{\infty}\Big(\sum_{i=1}^{k} a_i\Big)\chi_{[4^{-k-1},4^{-k})}\Big\|_X \le K\Big\|\sum_{k=1}^{\infty} a_k \chi_{[4^{-k-1},4^{-k})}\Big\|_X \tag{16}$$

for some $K > 0$ and an arbitrary sequence $(a_k)_{k=1}^{\infty}$ of nonnegative numbers.

We show that for an arbitrary function $y \in L_1[0,1]$ with $y = y^*$ and for every $m \in \mathbb{N}$ we have

$$\int_0^t \sigma_{4^{-m}} y(s)\, ds \le \frac{1}{m}\int_0^t \sum_{k=0}^{m} \sigma_{4^{-k}} y(s)\, ds, \quad 0 \le t \le 1. \tag{17}$$

Indeed, there is no loss of generality in assuming that $0 \le t \le 4^{-m}$. Then, since the function $\frac{1}{t}\int_0^t y(s)\, ds$ is monotone decreasing, we see that

$$\begin{aligned}\int_0^t \sum_{k=0}^{m} \sigma_{4^{-k}} y(s)\, ds &= \sum_{k=0}^{m} \int_0^t y(4^k s)\, ds = \sum_{k=0}^{m} 4^{-k} \int_0^{4^k t} y(s)\, ds \\ &\ge m4^{-m} \int_0^{4^m t} y(s)\, ds = m\int_0^t \sigma_{4^{-m}} y(s)\, ds,\end{aligned}$$

and (17) follows. By the assumption (see §2) imposed on the r.i. space $X$, we obtain

$$\|\sigma_{4^{-m}} y\|_X \le \frac{1}{m}\Big\|\sum_{k=0}^{m} \sigma_{4^{-k}} y\Big\|_X \tag{18}$$

if the function $y$ is nonnegative and monotone nonincreasing.

Put

$$v_a := \sum_{k=1}^{\infty} a_k \chi_{[4^{-k-1},4^{-k})}, \tag{19}$$

where $a_k \ge 0$, $k = 1, 2, \dots$, and the sequence $(a_k)_{k=1}^{\infty}$ is monotone nondecreasing. Since

$$\sum_{k=1}^{\infty}\Big(\sum_{i=1}^{k} a_i\Big)\chi_{[4^{-k-1},4^{-k})} = \sum_{k=0}^{\infty} \sigma_{4^{-k}} v_a,$$

we can rewrite (16) as follows:

$$\Big\|\sum_{k=0}^{m} \sigma_{4^{-k}} v_a\Big\|_X \le K\|v_a\|_X, \quad m \in \mathbb{N}.$$

Since the function $v_a$ is monotone nonincreasing, this and formula (18) imply

$$\|\sigma_{4^{-m}} v_a\|_X \le \frac{K}{m}\|v_a\|_X, \quad m \in \mathbb{N}. \tag{20}$$

Now, assume that $x \in X$ is arbitrary. Putting

$$v := \sum_{k=1}^{\infty} x^*(4^{-k})\chi_{[4^{-k-1},4^{-k})}, \tag{21}$$

we see that

$$\sigma_{1/4}x^*(t) \le v(t) \le x^*(t), \quad 0 < t \le 1.$$

Observe that $v = v_a$ if $a_k = x^*(4^{-k})$, $k = 1, 2, \ldots$. Thus, combining the last inequality with (20), we arrive at

$$\|\sigma_{4^{-m}}\left(\sigma_{1/4}x^*\right)\|_X \le \|\sigma_{4^{-m}}v\|_X \le \frac{K}{m}\|v\|_X \le \frac{K}{m}\|x\|_X, \quad m \in \mathbb{N}.$$

Consequently,

$$\|\sigma_{4^{-m-1}}x\|_X \le \frac{K}{m}\|x\|_X,$$

and therefore, $\lim_{s\to+0}\|\sigma_s\|_{X\to X} = 0$, i.e., $\alpha_X > 0$.

To show that $\beta_X < 1$, it suffices to prove that the adjoint $Q$,

$$Qx(t) := \int_t^1 \frac{x(s)}{s}\,ds, \quad 0 < t \le 1,$$

to the Hardy operator is bounded on $X$ (see [18, Theorem 2.6.8]).

Take $x \in X$ with $x = x^*$ and define $v$ by (21). Since $\operatorname{supp} v \subset [0, 1/4]$, we have

$$Q(\sigma_4 v)(t) = \int_t^1 \frac{v(s/4)}{s}\,ds = \int_{t/4}^1 \frac{v(s)}{s}\,ds = \sigma_4(Qv)(t).$$

Since

$$v(t) \le x(t) \le \sigma_4 v(t), \ \ 0 < t \le 1,$$

it follows that

$$\|Qx\|_X \le \|Q(\sigma_4 v)\|_X = \|\sigma_4(Qv)\|_X \le 4\|Qv\|_X.$$

It can easily be shown that

$$\|Q\|_{X\to X} = \sup_{\substack{x=x^*\\ \|x\|_X\le 1}} \|Q_x\}_X$$

(see also [19, Proof of Theorem 1]). Thus, the above relations imply that

$$\|Q\|_{X\to X} \le 4\sup\|Qv_a\|_X, \tag{22}$$

where the supremum is taken over all functions $v_a$, $\|v_a\|_X \le 1$, of the form (19), where the $a_k \ge 0$ are nonnegative and monotone nondecreasing. To estimate the supremum in (22), we observe that

$$Qv_a(4^{-k-1}) = \ln 4 \cdot \sum_{i=1}^{k} a_i,$$

for all $k \in \mathbb{N}$; thus, since $Qv_a$ is monotone decreasing, we have

$$Qv_a(t) \le \ln 4 \cdot \sum_{k=1}^{\infty}\left(\sum_{i=1}^{k} a_i\right)\chi_{[4^{-k-1},4^{-k})}(t).$$

By (16), this implies the inequality $\|Qv_a\|_X \le K\ln 4\|v_a\|_X \le K\ln 4$. So, $Q$ is bounded on $X$ by (22). This completes the proof of the theorem. □

## §4. Martingale transforms generated by linear combinations of Rademacher functions

In the case where the $v_k$ are linear combinations of Rademacher functions, the martingale transforms of the Rademacher sequence behave not quite so as in Theorem 1.

We remind the reader that the symbol $X_0$ stands for the closure of $L_\infty$ in an r.i. space $X$.

**Theorem 2.** *Suppose $X$ is an r.i. space on $[0,1]$. Let $v_k = \sum_{i=0}^{k-1} a_k^i r_i$, $k = 1, 2, \dots$, where $r_0(t) = 1$. Then the inequalities*

$$C^{-1}\|Pv\|_X \le \|Rv\|_X \le C\|Pv\|_X \tag{23}$$

*are fulfilled with some $C$ independent of the $a_k^i \in \mathbb{R}$ if and only if $X \supset (\exp L_1)_0$.*

*Proof.* First, we recall that every r.i. space $X$ on $[0,1]$ embeds in $L_1$ with constant 1 (see §2). Thus, applying the Khinchin and Minkowski inequalities, we obtain

$$\begin{aligned}\|Pv\|_X &= \left\|\left(\sum_{k=1}^{\infty}\left(\sum_{i=0}^{k-1} a_k^i r_i\right)^2\right)^{1/2}\right\|_X \ge \left\|\left(\sum_{k=1}^{\infty}\left(\sum_{i=0}^{k-1} a_k^i r_i\right)^2\right)^{1/2}\right\|_{L_1} \\ &\ge \left(\sum_{k=1}^{\infty}\left\|\sum_{i=0}^{k-1} a_k^i r_i\right\|_{L_1}^2\right)^{1/2} \ge \frac{1}{\sqrt{2}}\left(\sum_{k=1}^{\infty}\sum_{i=0}^{k-1}(a_k^i)^2\right)^{1/2}.\end{aligned} \tag{24}$$

We show that the reverse inequality is true whenever $X \supset (\exp L_1)_0$. After squaring, we have

$$\|Pv\|_X = \left\|\left(\sum_{k=1}^{\infty}\left(\sum_{i=0}^{k-1} a_k^i r_i\right)^2\right)^{1/2}\right\|_X = \left\|\left(\sum_{k=1}^{\infty}\sum_{i=0}^{k-1}(a_k^i)^2 + 2\sum_{k=1}^{\infty}\sum_{0\le i<j<k} a_k^i a_k^j r_i r_j\right)^{1/2}\right\|_X.$$

So, putting

$$A^2 := \sum_{k=1}^{\infty}\sum_{i=0}^{k-1}(a_k^i)^2 \quad \text{and} \quad U := \sum_{k=1}^{\infty}\sum_{0\le i<j<k} a_k^i a_k^j r_i r_j,$$

we arrive at

$$\|Pv\|_X = \left\|(A^2 + 2U)^{1/2}\right\|_X \le \|A^2 + 2U\|_X^{1/2} \le (A^2 + 2\|U\|_X)^{1/2}. \tag{25}$$

In [1] it was proved that the system $\{r_i r_k\}_{1\le i<k<\infty}$ is equivalent in an r.i. space $X$ to the standard basis of $\ell_2$ if and only if $X \supset (\exp L_1)_0$. So, if this embedding does occur, we use the identity

$$U = \sum_{0\le i<j<\infty}\left(\sum_{k=j+1}^{\infty} a_k^i a_k^j\right) r_i r_j$$

and the Minkowski inequality to obtain

$$\begin{aligned}\|U\|_X \asymp \left(\sum_{0\le i<j<\infty}\left(\sum_{k=j+1}^{\infty} a_k^i a_k^j\right)^2\right)^{1/2} &\le \sum_{k=1}^{\infty}\left(\sum_{0\le i<j<k}(a_k^i a_k^j)^2\right)^{1/2} \\ &= \sum_{k=1}^{\infty}\left(\sum_{j=1}^{k-1}(a_k^j)^2\sum_{i=0}^{j-1}(a_k^i)^2\right)^{1/2} \le A^2.\end{aligned}$$

By (25), it follows that

$$\|Pv\|_X \le \sqrt{3}A = \sqrt{3}\left(\sum_{k=1}^{\infty}\sum_{i=0}^{k-1}(a_k^i)^2\right)^{1/2}. \tag{26}$$

On the other hand, since $X \supset (\exp L_1)_0$ and

$$Rv = \sum_{k=1}^{\infty} v_k r_k = \sum_{k=1}^{\infty} \sum_{i=0}^{k-1} a_k^i r_i r_k,$$

we can apply the result from [1] already used to conclude that

$$\|Rv\|_X \asymp \left( \sum_{k=1}^{\infty} \sum_{i=0}^{k-1} (a_k^i)^2 \right)^{1/2}. \tag{27}$$

Together with (24) and (26), this implies (23).

Now, conversely, let (23) be fulfilled. Then, in particular,

$$\left\| \sum_{k=1}^{\infty} a_k r_k \right\|_X \asymp \left( \sum_{k=1}^{\infty} a_k^2 \right)^{1/2},$$

with constants independent of $a_k \in \mathbb{R}$, $k = 1, 2, \ldots$. Therefore, $X \supset G$, where $G$ is the closure of $L_\infty$ in the Orlicz space $\exp L_2$ (see [25] and also §1). Since $G$ is 2-convex (see, e.g., [14]), we obtain

$$\|Pv\|_X \le \|Pv\|_G \le \left( \sum_{k=1}^{\infty} \left\| \sum_{i=0}^{k-1} a_k^i r_i \right\|_G^2 \right)^{1/2} \le C \left( \sum_{k=1}^{\infty} \sum_{i=0}^{k-1} (a_k^i)^2 \right)^{1/2}.$$

Together with (24) and (23), this implies (27), i.e., the system $\{r_i r_k\}_{1 \le i < k < \infty}$ in the r.i. space $X$ is equivalent to the standard basis of $\ell_2$. But then, again, $X \supset (\exp L_1)_0$ by the results of [1], and the theorem follows. □

## §5. The unconditionality of the Haar system in an r.i. space

It has already been mentioned that the Haar system is an unconditional basis in a separable r.i. space $X$ if and only if $\|x\|_X \asymp \|Px\|_X$, where $Px$ is the Paley function defined by (5) (see [15, Theorem 1.11]). We prove a similar unconditionality criterion in terms of the decoupling sum (10).

We start with a statement about comparison of $Pv = \left( \sum_{k=1}^{\infty} v_k^2 \right)^{1/2}$ and the sum

$$R'v(s,t) := \sum_{k=1}^{\infty} v_k(s) r_k(t),$$

which is of independent interest.

**Proposition 2.** *For every r.i. space $X$ on $I = [0,1]$, the following conditions are equivalent.*

(a) *There exists $M > 0$ such that for every $\{\Sigma_k\}$-predictable sequence $\{v_k\}$ we have*

$$\|R'v\|_{X(I \times I)} \le M \|Pv\|_X. \tag{28}$$

(b) *The lower Boyd index $\alpha_X$ is strictly positive.*

To prove this statement, we need the following lemma (see [4] or [3, Lemma 7.2(a)]) about the boundedness of the tensor product operator in an r.i. space.

**Lemma 2.** *Let $\varphi$ be an unbounded measurable function on $I$. If the operator $T_\varphi x(s,t) = x(s)\varphi(t)$ is bounded from $X$ to $X(I \times I)$, then $\alpha_X > 0$.*

*Proof of Proposition* 2. The implication $(b) \Rightarrow (a)$ is well known (see, e.g., [20, Proposition 2.d.1]). So, we only need to prove that (b) is a consequence of (a).

So, let (28) be fulfilled in $X$. Let $\varphi$ be the mapping inverse to the function

$$\Psi(u) = \frac{2}{\sqrt{2\pi}} \int_u^\infty e^{-\frac{v^2}{2}} dv, \quad 0 \le u < \infty.$$

Since $\Psi(0) = 1$, we see that $\varphi$ is a monotone decreasing function defined on $[0,1]$ with $\lim_{t\to+0} \varphi(t) = +\infty$. Let $z_n := n^{-\frac{1}{2}} \sum_{i=1}^n r_i$. The central limit theorem (see [26, Theorem 3.3.3]) shows that

$$\lim_{n\to\infty} m\{t \in [0,1] : |z_n(t)| > u\} = \Psi(u), \quad u \ge 0,$$

whence $\lim_{n\to\infty} z_n^*(t) = \varphi(t)$ $(0 < t \le 1)$ by [3, Lemma 3.1].

For fixed $k$, $l \in \mathbb{N}$ with $l > k$ and an arbitrary $\Sigma_k$-measurable function $v$, we put $v_i = 0$ if $1 \le i \le k$ or $i > l$ and $v_i = \frac{1}{\sqrt{l-k}} v$ if $i = k+1, \dots, l$. Clearly, the sequence $\{v_i\}$ is $\{\Sigma_i\}$-predictable and, moreover,

$$\sum_{i=1}^\infty v_i(s) r_i(t) = v(s) z_{k,l}(t), \quad \text{where} \quad z_{k,l} := \frac{1}{\sqrt{l-k}} \sum_{i=k+1}^l r_i.$$

By (28), we obtain

$$\|v(s) z_{k,l}(t)\|_{X(I\times I)} \le M \|v\|_X, \tag{29}$$

where $M$ is independent of $v$, $k$, and $l$. Next, since $z_{k,l}^* = z_{l-k}^*$, the above discussion ensures the relation $\lim_{l\to\infty} z_{k,l}^*(t) = \varphi(t)$ $(0 < t \le 1)$ for every $k \in \mathbb{N}$. Since the norm of $X$ is order semicontinuous, the properties of $X''$ and (29) show that

$$\|T_\varphi v\|_{X''(I\times I)} = \|v(s)\varphi(t)\|_{X''(I\times I)} \le M \|v\|_{X''}. \tag{30}$$

Clearly, any $\Sigma_k$-measurable function (with some $k \in \mathbb{N}$) can play the role of $v$ here. Thus, in particular, $X \ne L_\infty$. Therefore, the norm of $X''$ is absolutely continuous on $L_\infty$.

Let $x \in L_\infty$. It is easily seen that there exists a sequence $\{v_k\}$, $v_k \ge 0$, of $\Sigma_k$-measurable functions with $v_k \le \|x\|_\infty$ and $v_k \to |x|$ a.e. on $[0,1]$. Then $\|v_k\|_{X''} \to \|x\|_{X''}$ and, by (30),

$$\|v_k(s)\varphi(t)\|_{X''(I\times I)} \le M \|v_k\|_{X''}, \quad k = 1, 2, \dots.$$

Consequently, by the maximality of $X''$, the limit passage as $k \to \infty$ in the last inequality yields

$$\|T_\varphi x\|_{X''(I\times I)} = \|x(s)\varphi(t)\|_{X''(I\times I)} \le M \|x\|_{X''}$$

for all $x \in L_\infty$. Now, if $x \in X''$ is arbitrary, there exists a monotone increasing sequence of functions $\{x_k\} \subset L_\infty$, $x_k \ge 0$, with $x_k \to |x|$ a.e. on $[0,1]$. As before, this allows us to extend the preceding inequality to the entire space $X''$. As a result, we obtain

$$\|T_\varphi x\|_{X''(I\times I)} \le M \|x\|_{X''},$$

whence we see that $T_\varphi$ is bounded from $X''$ to $X''(I \times I)$. Then $\alpha_{X''} > 0$ by Lemma 2. Since $\alpha_X = \alpha_{X''}$ (see [18, Theorem 2.4.11]), the proposition is proved. □

*Remark* 1. An inequality reverse to (28), more precisely, the estimate

$$\|Pv\|_X \le \sqrt{2} \|R'v\|_{X(I\times I)},$$

holds true in an arbitrary r.i. space $X$ and for arbitrary $v_k \in X$, $k = 1, 2, \dots$ (see, e.g., [20, Proposition 2.d.1]).

**Theorem 3.** *For an arbitrary r.i. space $X$, the following conditions are equivalent.*
(i) *The Haar system is unconditional in $X$.*
(ii) *We have*

$$\left\|\sum_{k=1}^{\infty}\sum_{i=1}^{2^{k-1}} c_{k,i}h_{k-1,i}\right\|_X \asymp \left\|\sum_{k=1}^{\infty}\sum_{i=1}^{2^{k-1}} c_{k,i}\chi_{\Delta_{k-1}^i}(s)r_k(t)\right\|_{X(I\times I)} \tag{31}$$

with constants independent of $c_{k,i}\in\mathbb{R}$.

*Proof.* If (i) is fulfilled, then $0<\alpha_X\le\beta_X<1$ by [20, 2.c.6] or [18, 2.9.6]). But then, by [15, Theorem 1.11] and Proposition 2 of the present paper, each of the two sides of (31) is equivalent to the $X$-norm of the Paley function, i.e., to the quantity

$$\left\|\left(\sum_{k=1}^{\infty}\sum_{i=1}^{2^{k-1}}(c_k^i h_{k-1}^i)^2\right)^{1/2}\right\|_X,$$

So, we obtain (31).

The reverse implication is a direct consequence of the fact that, by Lemma 1, the family of functions $\{\chi_{\Delta_{k-1}^i}(s)r_k(t) : k=1,2,\dots;i=1,2,\dots,2^{k-1}\}$ is unconditional in an arbitrary r.i. space. □

## §6. Decoupling-inequality for martingale transforms in r.i. spaces

Now, we want to discuss the r.i. spaces $X$ in which the inequality

$$\left\|\sum_{k=1}^{\infty}\sum_{i=1}^{2^{k-1}} c_{k,i}h_{k-1,i}\right\|_X \le C\left\|\sum_{k=1}^{\infty}\sum_{i=1}^{2^{k-1}} c_{k,i}\chi_{\Delta_{k-1}^i}(s)r_k(t)\right\|_{X(I\times I)} \tag{32}$$

is fulfilled with some constant $C>0$ depending only on $X$ or, equivalently, for every $\{\Sigma_k\}$-predictable sequence $\{v_k\}$ we have

$$\left\|\sum_{k=1}^{\infty} v_k r_k\right\|_X \le C\left\|\sum_{k=1}^{\infty} v_k(s)r_k(t)\right\|_{X(I\times I)}. \tag{33}$$

By the first statement of Proposition 1, Proposition 2, and Remark 1, these inequalities are true if $\alpha_X>0$. However, (33) is also true in many r.i. spaces with zero lower Boyd index. Indeed, the main result of [13] shows that this inequality is fulfilled for $X=L_p$, $1\le p<\infty$, with a constant $C$ independent of $p$. Thus, (33) is fulfilled in an arbitrary r.i. space that is extrapolatory with respect to the scale $L_p$ (see [5, 6]), in particular, in all exponential Oricz spaces $\exp L_p$, $p>0$. At the same time, an example of Talagrand (see [13]) shows that the corresponding estimate for the distributions of the sums in (33) fails. More precisely, there is no constant $C$ such that the inequality

$$m\left\{u\in I : \left|\sum_{k=1}^{\infty} v_k(u)r_k(u)\right|>C\tau\right\} \le Cm\left\{(s,t)\in I\times I : \left|\sum_{k=1}^{\infty} v_k(s)r_k(t)\right|>\tau\right\}$$

is true for every $\{\Sigma_k\}$-predictable sequence $\{v_k\}$ and all $\tau>0$. By using that example, we can prove an even stronger statement.

**Proposition 3.** *There is no constant $C>0$ such that for every $\{\Sigma_k\}$-predictable sequence $\{v_k\}$ and all $0<t\le 1$ we have*

$$\int_0^t\left(\sum_{k=1}^{\infty} v_k r_k\right)^*(u)\,du \le C\int_0^t\left(\sum_{k=1}^{\infty} v_k(s)r_k(t)\right)^*(u)\,du.$$

*Proof.* As in Talagrand's example (see [13, Theorem 6.1]), for every $k \in \mathbb{N}$, $k \geq 4$, we put $N_1 = 2^{2k}$ and

$$N_i - N_{i-1} = 2^{1-i}N_1, \quad i = 2, \dots, k.$$

Next, we define the sets

$$A_1 = \{t \in [0,1] : r_1(t) = \dots = r_{N_1}(t) = 1\},$$
$$A_i = A_{i-1} \cap \{t \in [0,1] : r_{N_{i-1}+1}(t) = \dots = r_{N_i}(t)\}, \quad i = 2, \dots, k,$$

and introduce a $\{\Sigma_k\}$-predictable sequence $\{v_k\}$ by setting

$$v_1 = \dots = v_{N_1} = 1,$$
$$v_{N_1+1} = \dots = v_{N_2} = 2\chi_{A_1},$$
$$v_{N_{k-1}+1} = \dots = v_{N_k} = 2^{k-1}\chi_{A_{k-1}}.$$

By the definitions, we obtain

$$\left|\sum_{j=1}^{N_k} v_j(u)r_j(u)\right| \leq \sum_{i=1}^{k}\sum_{j=N_{i-1}+1}^{N_i} |v_j(u)r_j(u)| \leq \sum_{i=1}^{k} 2^{i-1}2^{1-i}N_1 = kN_1, \quad 0 \leq u \leq 1,$$

and, similarly,

$$\left|\sum_{j=1}^{N_k} v_j(s)r_j(t)\right| \leq kN_1, \quad 0 \leq s,\ t \leq 1. \tag{34}$$

Consequently,

$$m\left\{u \in I : \left|\sum_{j=1}^{N_k} v_j(u)r_j(u)\right| \geq kN_1\right\} = m\left\{u \in I : \left|\sum_{j=1}^{N_k} v_j(u)r_j(u)\right| = kN_1\right\}$$
$$= 2m\{r_j = 1, j = 1, 2, \dots, N_k\} = 2 \cdot 2^{-N_1(1+2+\dots+2^{1-k})} = 2^{-2N_1(1-2^{-k})},$$

whence for $\alpha_k := 2^{-2N_1(1-2^{-k})}$ we obtain

$$\int_0^{\alpha_k} \left(\sum_{j=1}^{N_k} v_j r_j\right)^*(u)\, du = kN_1\alpha_k. \tag{35}$$

On the other hand, as was shown in the proof of Theorem 6.1 in [13], we have

$$m\left\{(s,t) \in I \times I : \left|\sum_{j=1}^{N_k} v_j(s)r_j(t)\right| \geq 4N_1\right\} \leq k \cdot 2^{-N_1 2^{2-k}} \cdot \alpha_k,$$

and, therefore, from (34) and (35) it follows that

$$\int_0^{\alpha_k} \left(\sum_{j=1}^{N_k} v_j(s)r_j(t)\right)^*(u)\, du \leq kN_1 \cdot k \cdot 2^{-N_1 2^{2-k}} \cdot \alpha_k + 4N_1 \cdot \alpha_k$$
$$= kN_1\alpha_k \cdot \left(k2^{-2^{k+2}} + \frac{4}{k}\right) = \left(k2^{-2^{k+2}} + \frac{4}{k}\right)\int_0^{\alpha_k} \left(\sum_{j=1}^{N_k} v_j r_j\right)^*(u)\, du.$$

Since $k \in \mathbb{N}$ may be taken arbitrarily large, the statement follows. □

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