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REMARKS ON A_p-REGULAR LATTICES OF MEASURABLE FUNCTIONS

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Abstract. A Banach lattice X of measurable functions on a space of homogeneous type is said to be A_p-regular if every $f \in X$ admits a majorant $g \geq |f|$ belonging to the Muckenhoupt class A_p with suitable control on the norm and the constant. It is well known that the A_p-regularity of the order dual X' of X implies the boundedness of the Hardy–Littlewood maximal operator on $X^{\frac{1}{p}}$ for $p > 1$ (equivalently, the A_1-regularity of this lattice), provided that X' is norming for X. This result admits a partial converse and an interesting characterization: the A_1-regularity of $X^{\frac{1}{p}}(\ell^p)$ implies the A_p-regularity of X', and for lattices X with the Fatou property these conditions are equivalent to the A_1-regularity of both $X^{\frac{1}{p}}$ and $\left(X^{\frac{1}{p}}\right)'$. As an application, an exact form of the self-duality of BMO-regularity is obtained, the A_q-regularity of the lattices $\mathrm{L}_\infty(\ell^p)$ for all $1 < p, q < \infty$ is established, and in many cases it is shown that the A_1-regularity of both Y and Y' yields the A_1-regularity of $Y(\ell^s)$ for all $1 < s < \infty$, which implies the boundedness of the Calderón–Zygmund operators in $Y(\ell^s)$.

Introduction

Let a quasimetric space S endowed with a measure ν be a space of homogeneous type, e.g., $S = \mathbb{R}^n$ or $S = \mathbb{T}^n$ with the Lebesgue measure, and let Ω be a σ-finite measurable space with measure μ. The generic point $\omega \in \Omega$ will be regarded as an additional variable. We consider quasinormed lattices X of measurable functions on $S \times \Omega$. For more details on lattices of measurable functions see, e.g., [11]; the definitions of most of the (standard) notions and properties can be found, e.g., in [14].

Let $p \geq 1$. A lattice X is said to be A_p-*regular* with constants (C, m) if for any $f \in X$ there exists a majorant $g \geq |f|$ such that $\|g\|_X \leq m\|f\|_X$ and $g(\,\cdot\,, \omega) \in \mathrm{A}_p$ with constant C for almost all $\omega \in \Omega$, where A_p is the Muckenhoupt class (see, e.g., [9, Chapter 5]).

As was demonstrated in [3], the mere existence of majorants of class A_1 already characterizes the natural ambient space $\bigcup_{p>1} \mathrm{L}_p(\mathbb{T}^n) = \bigcup_{w \in \mathrm{A}_2} \mathrm{L}_2(\mathbb{T}^n, w)$; there are also some generalizations of this result to spaces on $\mathbb{R}^n$ and also to the Hardy classes. The A_1-regularity property, which is equivalent to the boundedness of the Hardy–Littlewood maximal operator M (see, e.g., [14, Proposition 1]), was found to be useful in the study of some properties related to the Calderón–Zygmund operators (see [14, 7, 8, 15, 6]).

The A_p-regularity property was introduced as a refinement of the following notion, which is related to the interpolation of Hardy-type spaces (see, e.g., [2]): a lattice X is said to be BMO-*regular* with constants (C, m) if for any $f \in X$ there exists a majorant $g \geq |f|$ such that $\|g\|_X \leq m\|f\|_X$ and $\log g(\,\cdot\,, \omega) \in \mathrm{BMO}$ with norm of at most C for almost all $\omega \in \Omega$.

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An important feature of BMO-regularity is its *self-duality*: under suitable assumptions a lattice X is BMO-regular if and only if its order dual lattice X' is also BMO-regular. For the first time this property was proved, apparently, in [1] for the case of super-reflexive spaces on the circle (see the remarks in the proof of [1, Theorem 5.12]). Later, it was extended in [10] to the general case of Banach lattices on the circle satisfying the Fatou property, by using real interpolation of Hardy-type spaces with an additional variable, and this generalization yielded the BMO-regularity of the lattices $\mathrm{L}_\infty(\ell^q)$ (see also Corollary 6 below). Finally, in [14] an equivalent result on the divisibility of the BMO-regularity property was established by using only the real-variable techniques for lattices with the Fatou property on a space of homogeneous type.

We note that the proofs in both [10] and [14] are rather involved and rely on a fixed-point theorem. In §1 we give a simple and short proof of the self-duality of the BMO-regularity property for lattices on $\mathbb{R}^n$ and $\mathbb{T}$. This argument is based on well-known results of Rubio de Francia [5]; thus, in this case everything follows from the Hahn–Banach separation theorem and the Grothendieck theorem, without using fixed-point theorems or the divisibility property.

Furthermore, the results presented below yield an exact version for the self-duality of the BMO-regularity property, which can be stated in terms of the A_p-regularity property as follows. Recall that for every $1 < p < \infty$ the BMO-regularity of a lattice X is equivalent to the A_p-regularity of the lattice X^δ for a sufficiently small $\delta > 0$ with suitable estimates (see, e.g., the remarks after [14, Definition 1]).

Theorem 1. *Suppose that X is a Banach lattice of measurable functions on $S \times \Omega$ satisfying the Fatou property, and let $\alpha, \beta > 0$. The following conditions are equivalent*:

1. $X^{\frac{1}{\alpha+1}}$ *is* $\mathrm{A}_{\frac{\alpha+\beta+1}{\alpha+1}}$*-regular*;
2. $X'^{\frac{1}{\beta+1}}$ *is* $\mathrm{A}_{\frac{\alpha+\beta+1}{\beta+1}}$*-regular.*

We note that, in general, Theorem 1 fails if either α or β is zero; see the paragraph below. Theorem 1 is a natural reformulation of Theorem 14 given in §1 below, which expresses this result rather concisely in terms of an F^α_β-regularity property introduced therein. The proof of Theorem 1 taken in complete detail is quite elementary, and it is based only on the Hahn-Banach separation theorem, without any need for either the Grothendieck theorem or a fixed point theorem.

The main property of the Muckenhoupt weights shows at once that if X' is a norming space for a lattice X (e.g., if X has either the Fatou property or an order continuous norm), then the A_p-regularity of X' implies the A_1-regularity of $X^{\frac{1}{p}}$ (see, e.g., [14, Proposition 4]). The converse is false generally: for example, if $X = \mathrm{L}_\infty$, then $X^{\frac{1}{p}} = \mathrm{L}_\infty$ is an A_1-regular lattice for all $1 \le p < \infty$, but $X' = \mathrm{L}_1$ is not A_p-regular for any p if $S = \mathbb{T}$ or $S = \mathbb{R}^n$ (see, e.g., [14, Proposition 3]). Nevertheless, we establish the following characterization whose proof is given in §2 below.

Theorem 2. *Let X be a normed lattice of measurable functions on $S \times \Omega$ such that X' is a norming space for X. The following conditions are equivalent for all $1 < p < \infty$*:

1. $X^{\frac{1}{p}}(\ell^p) = [X(\ell^1)]^{\frac{1}{p}}$ *is* A_1*-regular*;
2. X' *is* A_p*-regular.*

If X has the Fatou property then these conditions are also equivalent to the following.

3. *Both* $X^{\frac{1}{p}}$ *and* $\left(X^{\frac{1}{p}}\right)' = X'^{\frac{1}{p}}\mathrm{L}_1^{1-\frac{1}{p}}$ *are* A_1*-regular.*

Thus A_p-regularity of lattices is closely related to the A_1-regularity of some derived lattices. The A_1-regularity of both Y and Y' implies (and often characterizes) the boundedness of the Calderón–Zygmund operators in Y and some other interesting properties (see [14, 7, 8, 15, 6]). In this regard, the following observations should be noted, which follow immediately from the equivalence of conditions 2 and 3 of Theorem 2; we also make use of the fact that the A_∞-regularity of a lattice is equivalent to its A_p-regularity for sufficiently large values of p.

Corollary 3. *Suppose that a normed lattice Y of measurable functions on $S\times\Omega$ satisfies the Fatou property and is p-convex for some (finite) $p>1$. The following conditions are equivalent.*

1. *Both Y and Y' are A_1-regular.*
2. *$(Y^p)'$ is A_p-regular.*

Corollary 4. *Let X be a lattice of measurable functions on $S\times\Omega$ with the Fatou property. The following conditions are equivalent.*

1. *X' is A_∞-regular.*
2. *Both X^δ and $(X^\delta)'=X'^\delta\mathrm{L}_1^{1-\delta}$ are A_1-regular for some $0<\delta<1$ (equivalently, for all sufficiently small δ).*

As an interesting example, consider the following question: for what weights w, is the lattice $X=\mathrm{L}_q(w)$ A_p-regular? We define (as in [14]) a weighted lattice $Z(w)$ to be the set $\{wf \mid f\in Z\}$ endowed with the norm $\|g\|_{Z(w)}=\|gw^{-1}\|_Z$. Thus, the weighted Lebesgue spaces with the "classical" weighted norm $\|f\|=\left(\int|f|^p w\right)^{\frac1p}$ look like $\mathrm{L}_p\left(w^{-\frac1p}\right)$ in our notation. It is well known that in the case of $p=1$ the necessary and sufficient condition is $w^{-q}\in\mathrm{A}_q$, and such lattices X and X' are A_1-regular only simultaneously. We have already noted that with $q=1$ there are no A_p-regularity in the typical cases; see, e.g., [14, Proposition 3]. The equivalence of conditions 1 and 3 in Theorem 2 yields (after a simple computation) the following characterization (see also Proposition 12 below).

Corollary 5. *Suppose that $1\le p\le\infty$, $1<q\le\infty$, and w is a weight. Then $X=\mathrm{L}_q(w)$ is A_p-regular if and only if $w^{q'}\in\mathrm{A}_{q'p}$.*

Theorem 2 allows us to refine the BMO-regularity property of the lattices $\mathrm{L}_\infty(\ell^q)$, which was first established, apparently, in [10] in the case of $S=\mathbb{T}$ by using the self-duality of the BMO-regularity.

Corollary 6. *The lattices $\mathrm{L}_\infty(\ell^q)$ on a measurable space $S\times\Omega\times\mathbb{Z}$ are A_p-regular for all $1<p,q<\infty$.*

It suffices to apply implication $3\Rightarrow2$ of Theorem 2 to $X=\mathrm{L}_1(\ell^{q'})$; the A_1-regularity of the lattices $X^{\frac1p}=\mathrm{L}_p(\ell^{q'p})$ and $(X^{\frac1p})'=\mathrm{L}_{p'}(\ell^{(q'p)'})$ is well known (see, e.g., [9, Chapter 2, §1.3.1] or Corollary 8 below).

Earlier, in [14, §1, Proposition 10] the A_p-regularity of the lattices $\mathrm{L}_\infty(\ell^q)$ was only proved for $q>1+\frac1p$. We mention that (at least for $S=\mathbb{R}^n$ or $S=\mathbb{T}$) the result of Proposition 6 is sharp in the sense of the admissible values of p and q: with $q=1$ the conclusion of Proposition 6 is false for all p (see [14, §1, Proposition 10]), and its falseness for $p=1$ and all q follows from the nonboundedness of the maximal operator on $\mathrm{L}_\infty(\ell^q)$ (see, e.g., [9, Chapter 2, §5.2]).

Theorem 2 also has some interesting applications concerning the boundedness of operators on lattices with an additional variable. The proof of the following result is given in §2 below.

Theorem 7. *Let Y be a normed lattice of measurable functions on $S\times\Omega$ satisfying the Fatou property. Suppose also that Y is p-convex with some $p>1$. If both Y and Y' are A_1-regular, then $Y(\ell^s)$ is also A_1-regular for all $1<s\le\infty$.*

It is unclear whether the p-convexity assumption is indispensable in the statement of Theorem 7; it might already follow from the assumed A_1-regularity of Y.

Combined with [7, Proposition 5], duality yields the following result. For the generalities concerning the Calderón–Zygmund operators, see, e.g., [9].

Corollary 8. *Let Y be a normed lattice of measurable functions on $S \times \Omega$ satisfying the Fatou property. Suppose also that Y is p-convex and q-concave with some $1 < p, q < \infty$. If both Y and Y' are A_1-regular, then $Y(\ell^s)$ and $Y'(\ell^{s'})$ are also A_1-regular for all $1 < s < \infty$, and, thus, any Calderón–Zygmund operator is bounded in $Y(\ell^s)$ for all $1 < s < \infty$.*

The main result of [7] and [8] yields yet another corollary. The definition of a nondegenerate operator can be found in [8]; we only note that the Hilbert transform on the circle and all Riesz transforms on $\mathbb{R}^n$ are nondegenerate.

Corollary 9. *Suppose that Y is a normed lattice of measurable functions on $\mathbb{R}^n \times \Omega$ or $\mathbb{T} \times \Omega$ such that Y is p-convex and q-concave with some $1 < p, q < \infty$ and Y satisfies the Fatou property. Then the boundedness of any nondegenerate Calderón–Zygmund operator T on Y implies the A_1-regularity of both Y and Y', and the boundedness of all Calderón–Zygmund operators on $Y(\ell^s)$ for all $1 < s < \infty$.*

It is not clear for what lattices E other than $E = \ell^s$ the results of Theorem 7 and its corollaries hold true. The proof suggests that this class of lattices probably includes all symmetric lattices on $\mathbb{Z}$ that are p-convex and q-concave with some $1 < p, q < \infty$ (because it is well known that such lattices are interpolation spaces for the couple (ℓ^p, ℓ^q)). Is this also true for fairly arbitrary UMD spaces E?

§1. Duality and factorisable weights

It is well known that the A_p weights are characterized in terms of the P. Jones factorization theorem: $w \in \mathrm{A}_p$ if and only if $w = w_0 w_1^{1-p}$ with some weights $w_0, w_1 \in \mathrm{A}_1$ and with some estimates for the constants; see, e.g., [9, Chapter 5, §5.3]. It is also well known that $\log w \in \mathrm{BMO}$ is equivalent to $w^\delta \in \mathrm{A}_2$ for some $\delta > 0$ with some estimates for the constants (see, e.g., [9, Chapter 5, §6.2]). These observations motivate the following notions, which appear to be quite convenient for studying BMO-regularity.

Definition 10. Let $\alpha, \beta \geq 0$. A weight w on $S \times \Omega$ is said to *belong to* F_β^α *with constant* C if there exist weights $\omega_0, \omega_1 \in \mathrm{A}_1$ with constant C such that $w = \frac{\omega_0^\alpha}{\omega_1^\beta}$.

Definition 11. Let $\alpha, \beta \geq 0$, and suppose that X is a quasinormed lattice of measurable functions on $S \times \Omega$. X is said to be F_β^α*-regular with constants* (C, m) if for any $f \in X$ there exists a majorant $w \in X$, $w \geq |f|$, such that $\|w\|_X \leq m\|f\|_X$ and $w \in \mathrm{F}_\beta^\alpha$ with constant C.

We note that, like the Muckenhoupt classes A_p, the weights belonging to F_β^α and the F_β^α-regularity condition have quite natural algebraic and order properties, and F_β^α-regularity admits an exact version of the divisibility theorem; see [8, §3].

In the present paper, however, we shall only need the following elementary properties. An application of the Hölder inequality shows that $w \in \mathrm{A}_1$ with the constant C implies $w^\delta \in \mathrm{A}_1$ with a constant C^δ for all $0 < \delta < 1$, so the classes are monotone in the parameters: $\mathrm{F}_{\beta_1}^{\alpha_1} \subset \mathrm{F}_\beta^\alpha$ for all $0 \leq \alpha_1 \leq \alpha$ and $0 \leq \beta_1 \leq \beta$ with some estimates for the constants, and the $\mathrm{F}_{\beta_1}^{\alpha_1}$-regularity of a lattice X implies its F_β^α-regularity. An example of the weights $w(t) = \frac{t^\gamma}{(t-1)^\delta}$ on the line (with suitable generalizations for the cases of

$\mathbb{R}^n$ and $\mathbb{T}$) shows that the classes F^α_β are distinct for distinct values of the parameters α and β.

For all $\alpha, \beta \geq 0$, $\delta > 0$, and weights w, the conditions $w \in \mathrm{F}^\alpha_\beta$, $w^\delta \in \mathrm{F}^{\delta\alpha}_{\delta\beta}$ and $w^{-\delta} \in \mathrm{F}^{\delta\beta}_{\delta\alpha}$ are equivalent. For the latter equivalence we also need to suitably clarify its meaning for the case of weights taking zero values on sets of positive measure; however, for simplicity we shall assume that all weights are nonnegative almost everywhere (we may always assume this in the F^α_β-property when majorizing nonzero functions at the expense of an arbitrarily small increase of the constant m). A lattice X is F^α_β-regular if and only if X^δ is $\mathrm{F}^{\delta\alpha}_{\delta\beta}$-regular with appropriate estimates for the constants. By the factorization theorem already mentioned above, $w \in \mathrm{A}_p$ with some $p > 1$ if and only if $w \in \mathrm{F}^1_{p-1}$ (in the case where $p = 1$ this equivalence is trivial), and a lattice X is A_p-regular if and only if X is F^1_{p-1}-regular. Accordingly, $\log w \in \mathrm{BMO}$ if and only if $w \in \mathrm{F}^\alpha_\beta$ for some some $\alpha, \beta > 0$ with suitable estimates for the constants, and a lattice X is BMO-regular if and only if it is F^α_β-regular for some $\alpha, \beta > 0$.

As a typical example, we consider the F^α_β-regularity property for weighted Lebesgue spaces.

Proposition 12. *Let $\alpha, \beta > 0$ and $1 < q < \infty$ be such that $\alpha q > 1$, and let w be a weight. The space $\mathrm{L}_q(w)$ is F^α_β-regular if and only if $w \in \mathrm{F}^{\alpha - \frac{1}{q}}_{\beta + \frac{1}{q}}$.*

Indeed, the F^α_β-regularity of $\mathrm{L}_q(w)$ is equivalent to the $\mathrm{F}^1_{\frac{\beta}{\alpha}}$-regularity of $\left[\mathrm{L}_q(w)\right]^{\frac{1}{\alpha}} = \mathrm{L}_{\alpha q}\big(w^{\frac{1}{\alpha}}\big)$, that is, to its $\mathrm{A}_{\frac{\beta}{\alpha}+1}$-regularity, which by Corollary 5 is equivalent to

$$w^{\frac{1}{\alpha}(\alpha q)'} \in \mathrm{A}_{(\alpha q)'(\frac{\beta}{\alpha}+1)} = \mathrm{A}_{1+\frac{\beta q+1}{\alpha q-1}} = \mathrm{F}^1_{\frac{\beta q+1}{\alpha q-1}}.$$

A simple computation shows that the latter is equivalent to $w \in \mathrm{F}^{\alpha - \frac{1}{q}}_{\beta + \frac{1}{q}}$.

Following [14, §2, Definition 2], we say that a mapping T is A_p-*bounded* with constants (C, m) if it is defined on a set Ω_T of measurable functions on $S \times \Omega$ such that the $(\nu \times \mu)$-closure of Ω_T (i.e., its closure with respect to convergence in measure on all sets of finite measure) contains L_∞, and for any weight $w \in \mathrm{A}_p$ with constant C we have

$$\|T(f)\|_{\mathrm{L}_p(w^{-\frac{1}{p}})} \leq m\|f\|_{\mathrm{L}_p(w^{-\frac{1}{p}})}$$

for all $f \in \Omega_T$. It is well known that the maximal operator and all Calderón–Zygmund operators are A_p-bounded for all $1 < p < \infty$. It is easy to show that (see, e.g., [14, §2, Proposition 13]) the A_p-regularity of X' implies (under suitable conditions) the boundedness of the A_p-bounded operators in $X^{\frac{1}{p}}$, and, in particular, it implies the A_1-regularity of $X^{\frac{1}{p}}$. Together with the divisibility property, this was is used in [14] in order to verify the self-duality of the BMO-regularity property.

However, a similar result can be established for the lattices $X\mathrm{L}_p$ by using the lattice product instead of duality (see also Proposition 18 below).

Proposition 13. *Suppose that Z is a quasi-normed lattice of measurable functions on $S \times \Omega$, $1 < p < \infty$, $\beta = \frac{1}{p}$, and Z is $\mathrm{F}^{1-\beta}_\beta$-regular with constants (C, m). Then all A_p-bounded operators T are bounded on $Z\mathrm{L}_p$.*

Indeed, due to order continuity, $Z\mathrm{L}_p \cap \Omega_T$ is dense in $Z\mathrm{L}_p$. Suppose that $f \in Z\mathrm{L}_p \cap \Omega_T$ with norm 1. Then there exist $g \in Z$, $h \in \mathrm{L}_p$ such that $f = gh$ and $\|g\|_Z \leq 2$, $\|h\|_{\mathrm{L}_p} \leq 1$. For simplicity we may assume (see, e.g., [14, §3, Proposition 14]) that $g > 0$ almost everywhere. The $\mathrm{F}^{1-\beta}_\beta$-regularity of Z implies that there exists a majorant $u \geq |g|$ such that $\|u\|_Z \le 2m$ and $u \in \mathrm{F}_{\beta}^{1-\beta}$ with constant C, whence

$$u^{-p} \in \mathrm{F}_{p(1-\beta)}^{p\beta} = \mathrm{F}_{p-1}^{1} = \mathrm{A}_p$$

with some constants independent of f. Thus,

$$\begin{aligned}\|Tf\|_{Z\mathrm{L}_p} &= \|u \cdot u^{-1}(Tf)\|_{Z\mathrm{L}_p} \le \|u\|_Z \|u^{-1}(Tf)\|_{\mathrm{L}_p} \\ &\le 2m\|Tf\|_{\mathrm{L}_p([u^{-p}]^{-\frac{1}{p}})} \le c\|f\|_{\mathrm{L}_p([u^{-p}]^{-\frac{1}{p}})} \\ &= c\|h \cdot gu^{-1}\|_{\mathrm{L}_p} \le c\|h\|_{\mathrm{L}_p} \le c\end{aligned}$$

with a constant c independent of f. We see that T is indeed bounded on $Z\mathrm{L}_p$.

Considering the case where $Z = \mathrm{L}_\infty(w)$ and $T = M$ shows that the conditions of Proposition 13 are sharp in the sense that the parameters $1-\beta$ and β cannot be replaced by larger numbers. With the help of Proposition 12 it is easy to check that in the case where $Z = \mathrm{L}_q(w)$ (with $\frac{1}{q} + \frac{1}{p} < 1$) and $T = M$ the converse to Proposition 13 is also true. In general, however, the A_1-regularity of $Z\mathrm{L}_p$ is weaker than the $\mathrm{F}_{\beta}^{1-\beta}$-regularity of Z. For example, should the equivalence be true for $Z = \mathrm{L}_\infty(\ell^q)$, this lattice would be $\mathrm{F}_{\frac{1}{p}}^{\frac{1}{p'}}$-regular for all $1 < p < \infty$ and (by raising to the power q) we would have the $\mathrm{F}_{\frac{q}{p}}^{\frac{q}{p'}}$-regularity of $\mathrm{L}_\infty(\ell^1)$, which is false for $\frac{q}{p'} \le 1$ (see [14, §1, Proposition 10]).

Now we are ready to state the main result concerning the self-duality of $\mathrm{F}_{\beta}^{\alpha}$-regularity.

Theorem 14. *Suppose that X is a Banach lattice of measurable functions on $S \times \Omega$ satisfying the Fatou property and $\alpha > 1$, $\beta > 0$. Then X is $\mathrm{F}_{\beta}^{\alpha}$-regular if and only if the lattice X' is $\mathrm{F}_{\alpha-1}^{\beta+1}$-regular.*

As an illustration to Theorem 14, now we deduce Corollary 6 from this result. Indeed, the A_1-regularity of $\mathrm{L}_t(\ell^s)$ for all $1 < t, s < \infty$ (see, e.g., [9, Chapter 2, §1.3.1], or Corollary 8) implies that under the assumptions of Corollary 6 the lattice $X = \mathrm{L}_1(\ell^{q'})$ is $\mathrm{F}_0^{1+\delta}$-regular for any $\delta > 0$, which by Theorem 14 yields the F_{δ}^{1}-regularity of $X' = \mathrm{L}_\infty(\ell^q)$, i.e., its A_p-regularity for all $p = \delta + 1 > 1$.

The proof of Theorem 14 is given in §2 below. For now we present a relatively simple argument (but with coarser estimates) that proves the self-duality of the BMO-regularity property for lattices X on spaces of homogeneous type S such that $\mathrm{L}_2(S)$ admits a linear operator T that is A_s-bounded for all $1 < s < \infty$ and A_2-nondegenerate (concerning A_2-nondegeneracy see, e.g., [14, Definition 3]). For example, in the case of $S = \mathbb{T}$ we can take the Hilbert transform $T = H$, and in the case where $S = \mathbb{R}^n$ any Riesz transform R_j will do for T.

We shall need the following known result (for the proof in the given form and some discussion see, e.g., [14, §6]).

Theorem 15. *Suppose that a Banach lattice Y of measurable functions on a measurable space $S \times \Omega$ has order continuous norm. If a linear operator T is bounded in $Y^{\frac{1}{2}}$, then for any $f \in Y'$ there exists a majorant $w \ge |f|$, $\|w\|_{Y'} \le 2\|f\|_{Y'}$, such that $\|T\|_{\mathrm{L}_2(w^{-\frac{1}{2}}) \to \mathrm{L}_2(w^{-\frac{1}{2}})} \le C$ with a constant C independent of f.*

To verify the self-duality of BMO-regularity, suppose that a Banach lattice X on $S \times \Omega$ satisfies the Fatou property and X is BMO-regular, so that it is $\mathrm{F}_{\beta}^{\alpha}$-regular with some $\alpha, \beta > 0$. We want to apply Theorem 15 to the lattice $Y = X^{\delta}\mathrm{L}_1^{1-\delta}$ and to the operator T with some sufficiently small $0 < \delta < 1$. If the conditions of Theorem 15 are satisfied in this case, then by the assumed A_2-nondegeneracy of T, the lattice $Y' = X'^{\delta}$ is A_2-regular, and so $X' = Y'^{\frac{1}{\delta}}$ is BMO-regular.

Thus, it suffices to prove that T is bounded on

$$Y^{\frac{1}{2}} = \left(X^{\delta}\mathrm{L}_1^{1-\delta}\right)^{\frac{1}{2}} = X^{\frac{\delta}{2}}\mathrm{L}_{\frac{2}{1-\delta}}.$$

For that, in its turn, it suffices to verify that $Z = X^{\frac{\delta}{2}}$ satisfies the conditions of Proposition 13 with $p = \frac{2}{1-\delta}$, i.e., that Z is $\mathrm{F}_{\beta}^{1-\beta}$-regular with $\beta = \frac{1-\delta}{2}$. The latter is equivalent to the $\mathrm{F}_{\frac{2}{\delta}\beta}^{\frac{2}{\delta}(1-\beta)}$-regularity of $X = Z^{\frac{2}{\delta}}$, which is the same as the $\mathrm{F}_{\frac{1}{\delta}-1}^{\frac{1}{\delta}+1}$-regularity of X. Choosing δ so small that $\frac{1}{\delta}+1 \geq \alpha$ and $\frac{1}{\delta}-1 \geq \beta$, we see that this assumption is satisfied.

The example of $X = \mathrm{L}_\infty$ shows that the conclusion of the "only if" part of Theorem 14 is false for $\beta = 0$ and any α, because $X' = \mathrm{L}_1$ is not A_p-regular with any $p > 1$ (see, e.g., [14, §1, Proposition 3]). It is not clear, however, whether the $\mathrm{F}_{\beta}^{\alpha}$-regularity of X with $\alpha \leq 1$ provides any additional information about the BMO-regularity of X'.

§2. Proof of the main results

The implication $2 \Rightarrow 1$ of Theorem 2 is established in the same way as [14, §1, Proposition 4]. To verify the other implications we introduce the following construction.

We fix some sequence $\{x_k\}_{k\in\mathbb{Z}}$ dense in S . For convenience, we enumerate all balls B_j, $j \in \mathbb{Z}$, of S with centers at these points and rational radii. Now we define a linear operator $\mathcal{M} = \{\mathcal{M}_j\}_{j\in\mathbb{Z}}$ on the functions $f = \{f_j\}_{j\in\mathbb{Z}}$ on $S \times \Omega \times \mathbb{Z}$ that are locally integrable in the first variable by

$$\mathcal{M}_j f_j(\,\cdot\,,\omega) = \left[\frac{1}{\nu(B_j)}\int_{B_j} f_j(t,\omega)\,dt\right]\chi_{B_j}(\,\cdot\,)$$

for all $j \in \mathbb{Z}$ and almost all $\omega \in \Omega$. $\mathcal{M}$ is a positive linear operator closely related to the Hardy–Littlewood maximal operator M: it is easily seen $\mathcal{M}f \leq \widetilde{M}f \leq cMf$ with a constant c, where $\widetilde{M}$ is the noncentered Hardy–Littlewood maximal operator, and $\|\mathcal{M}f(x,\omega,\,\cdot\,)\|_{\mathrm{l}^\infty} = \bigvee_j(\mathcal{M}_j f(x,\omega))$ is pointwise equivalent to $Mf(x,\omega)$ for almost all $x \in S$ and $\omega \in \Omega$ provided f is nonnegative.

We shall show that the conditions of Theorem 2 are equivalent to the following auxiliary condition.

4. *$\mathcal{M}$ is bounded on $X^{\frac{1}{p}}(\ell^p) = [X(\ell^1)]^{\frac{1}{p}}$.*

The implication $1 \Rightarrow 4$ follows at once from the estimate $\mathcal{M}f \leq cMf$. To establish $4 \Rightarrow 2$, we need the following known generalization [5, §3] of Theorem 15.

Theorem 16. *Suppose that a Banach lattice Y of measurable functions on $(S\times\Omega,\nu\times\mu)$ has order continuous norm, and let $1 < p < \infty$. If a linear operator $T : Y^{\frac{1}{p}} \to Y^{\frac{1}{p}}$ is bounded (as an operator acting in the first variable) on $Y^{\frac{1}{p}}(\ell^p) = [Y(\ell^1)]^{\frac{1}{p}}$, then for any $f \in Y'$ there exists a majorant $w \geq |f|$, $\|w\|_{Y'} \leq 2\|f\|_{Y'}$, such that*

$$\|T\|_{\mathrm{L}_p(w^{-\frac{1}{p}})\to\mathrm{L}_p(w^{-\frac{1}{p}})} \leq C$$

with a constant C independent of f.

The proof is essentially contained in the proof for the case of $p = 2$ ([14, §2, Theorem 6]), we only need to replace 2 with p in the arguments and make direct use of the assumption that T is bounded on $Y^{\frac{1}{p}}(\ell^p)$ rather than applying the Grothendieck theorem. We omit the details.

Now suppose that $\mathcal{M}$ is bounded on $X^{\frac{1}{p}}(\ell^p)$ under the assumptions of Theorem 2, and let $f \in X'$, $\|f\|_{X'} = 1$; we need to construct a suitable A_p-majorant for f. First, we additionally assume that X has order continuous norm. Let $Y = X(\ell^1)$, which is a lattice of measurable functions on $S \times \Omega \times \mathbb{Z}$. Since $\mathcal{M}$ is a positive operator, $\mathcal{M}$ is bounded on the lattice $Y^{\frac{1}{p}}(\ell^p)$ of measurable functions on $S \times \Omega \times \mathbb{Z} \times \mathbb{Z}$ as well as on $Y^{\frac{1}{p}}$ (see, e.g., [4, Volume 2, Proposition 1.d.9]). Then, by Theorem 16 applied to $\mathcal{M}$ and Y, for any function $g_k \in X'$ (to be exact, for the sequence $\{g_k\}_{j\in\mathbb{Z}}$; we construct the functions g_k inductively starting with $g_0 = f$), there exists a majorant $G_{k+1} = \{g_{k+1,j}\}_{j\in\mathbb{Z}} \in Y' = X'(\ell^\infty)$, $g_{k+1,j} \geq |g_k|$ for all j, such that $\left\| \bigvee_j g_{k+1,j} \right\|_{X'} = \|G_{k+1}\|_{Y'} \leq 2\|g_k\|_{X'}$ and

$$\|\mathcal{M}\|_{\mathrm{L}_p\left(G_{k+1}^{-\frac{1}{p}}\right) \to \mathrm{L}_p\left(G_{k+1}^{-\frac{1}{p}}\right)} \leq C. \tag{1}$$

We choose $g_0 = f$ and set inductively

$$g_{k+1} = \bigvee_j g_{k+1,j}.$$

Now let $w = \sum_{k\geq 0} 4^{-k} g_k$. It is easily seen that $w \geq |f|$ and

$$\|w\|_{X'} \leq \sum_{k\geq 0} 2^{-k} = 2.$$

Estimate (1) implies

$$\int |\mathcal{M}h|^p g_k \leq \int |\mathcal{M}h|^p G_{k+1} \leq C \int |h|^p G_{k+1} \leq C \int |h|^p g_{k+1} \tag{2}$$

for any $h \in \mathrm{L}_p(w^{-\frac{1}{p}})(\ell^p) \subset \mathrm{L}_p\bigl(G_{k+1}^{-\frac{1}{p}}\bigr)$. Multiplying inequalities (2) by 4^{-k} and summing yields

$$\|\mathcal{M}\|_{\mathrm{L}_p(w^{-\frac{1}{p}})(\ell^p) \to \mathrm{L}_p(w^{-\frac{1}{p}})(\ell^p)} \leq 4C. \tag{3}$$

Thus, by (3) we have

$$\|\mathcal{M}_j\|_{\mathrm{L}_p(w^{-\frac{1}{p}}) \to \mathrm{L}_p(w^{-\frac{1}{p}})} \leq 4C$$

for all $j \in \mathbb{Z}$. This implies that (see the proof of [14, §3, Proposition 19])

$$\|\mathcal{M}_j\|_{\mathrm{L}_p(w^{-\frac{1}{p}}(\,\cdot\,,\omega)) \to \mathrm{L}_p(w^{-\frac{1}{p}}(\,\cdot\,,\omega))} \leq c$$

for all $j \in \mathbb{Z}$ and almost all $\omega \in \Omega$ with a constant c independent of f. Fixing such $\omega \in \Omega$ and applying this norm estimate to the functions $\chi_{B_j} h(\,\cdot\,,\omega)$ for arbitrary nonnegative $h \in \mathrm{L}_p\bigl(w^{-\frac{1}{p}}(\,\cdot\,,\omega)\bigr)$ shows that

$$\left[\frac{1}{\nu(B_j)} \int_{B_j} h(\,\cdot\,,\omega)\, d\nu(\,\cdot\,)\right]^p \int_{B_j} w(\,\cdot\,,\omega) \leq c^p \int_{B_j} [h(\,\cdot\,,\omega)]^p w(\,\cdot\,,\omega) \tag{4}$$

for every $j \in \mathbb{Z}$. It is easy to check (using the local integrability of w in the first variable, which follows from the estimates) that (4) implies the same estimate for arbitrary balls B of S, which is equivalent to the fact that $w \in \mathrm{A}_p$ with constant c^p (see, e.g., [9, Chapter 5, §1.4]). Thus, w is a suitable A_p-majorant for f, which proves $4 \Rightarrow 2$ under an additional assumption.

Now we lift the assumption that the norm of X is order continuous. Suppose that $\mathcal{M}$ is bounded on $Z = \left[X(\ell^1)\right]^{\frac{1}{p}}$ under the assumptions of Theorem 2. The boundedness of M in L_p implies that $\mathcal{M}$ is also bounded on $\mathrm{L}_p(\ell^p) = \left[\mathrm{L}_1(\ell^1)\right]^{\frac{1}{p}}$. By complex interpolation (see, e.g., [12, Chapter 4, Theorem 1.14]), $\mathcal{M}$ is bounded on $Z^\theta\left[\mathrm{L}_p(\ell^p)\right]^{1-\theta} = \left[X_\theta(\ell^1)\right]^{\frac{1}{p}}$ uniformly in $0 < \theta < 1$, where $X_\theta = X^\theta \mathrm{L}_1^{1-\theta}$. The norm of X_θ is order continuous, and by the result already established we see that the lattices $X_\theta' = X'^\theta$ are A_p-regular uniformly in $0 < \theta < 1$. To deduce the A_p-regularity of X' from this, we use the following proposition, which will conclude the proof of the implication $4 \Rightarrow 2$ in Theorem 2.

Proposition 17. *Let X be a quasi-normed lattice of measurable functions on $S\times\Omega$ such that the lattices X^θ are A_p-regular uniformly on $0<\theta<1$. Then X is also A_p-regular.*

Indeed, suppose that $f\in X$, $f\ge 0$, and $\|f\|_X=1$; we need to show that f admits a suitable A_p-majorant. By assumption, for every $0<\theta<1$ there exists a majorant $g\ge f^\theta$, $\|g\|_{X^\theta}\le m$, such that $g\in \mathrm{A}_p$ with a constant C for some C and m independent of f. There exists $\rho>1$ such that $g^\rho\in\mathrm{A}_p$ with a constant C_1 independent of f and θ (see, e.g., [9, Chapter 5, §6.1]). Setting $\theta=\frac{1}{\rho}$, we see that the function $h=g^{\frac{1}{\theta}}\in X$, $\|g\|_X\le m^{\frac{1}{\theta}}$, is a suitable majorant for f.

Now we suppose that, under the assumptions of Theorem 2, the lattice X has the Fatou property. If condition 2 is satisfied, then we have the A_1-regularity of $X^{\frac1p}$ by $2\Rightarrow 1$, and the A_1-regularity of

$$\big(X^{\frac1p}\big)'=X'^{\frac1p}\mathrm{L}_1^{1-\frac1p}=X'^{\frac1p}\mathrm{L}_{p'}$$

follows from Proposition 13 because the A_p-regularity of X' is equivalent to its F^1_{p-1}-regularity and the $\mathrm{F}^{1-\beta}_\beta$-regularity of $X'^{\frac1p}$ with $\beta=\frac{1}{p'}$. Thus, the implication $2\Rightarrow 3$ is verified.

Finally, we establish the implication $3\Rightarrow 4$. The A_1-regularity of $X^{\frac1p}$ and $\big(X^{\frac1p}\big)'$ implies at once the A_1-regularity of the lattices $X^{\frac1p}(\ell^\infty)$ and $\big(X^{\frac1p}\big)'(\ell^\infty)$, and thus the boundedness of $\mathcal{M}$ on these lattices. Since $\mathcal{M}$ is a positive integral operator, its boundedness on an arbitrary lattice Z is equivalent to its boundedness on Z' if Z' is a norming lattice for Z; this follows at once from the Fubini theorem and the fact that it suffices to verify the boundedness on positive functions. Therefore, $\mathcal{M}$ is also bounded on $\big[\big(X^{\frac1p}\big)'(\ell^\infty)\big]'=X^{\frac1p}(\ell^1)$. The Calderón–Lozanovsky products are exact interpolation spaces for positive operators (see, e.g., [13]), so $\mathcal{M}$ is also bounded on

$$X^{\frac1p}(\ell^p)=\big[X^{\frac1p}(\ell^1)\big]^{\frac1p}\big[X^{\frac1p}(\ell^\infty)\big]^{1-\frac1p},$$

which means that condition 4 is satisfied as claimed. The proof of Theorem 2 is complete.

Now we prove Theorem 7. Suppose that a Banach lattice Y satisfies its assumptions: Y is p-convex with some $p>1$, Y satisfies the Fatou property, and both Y and Y' are A_1-regular. Then $Y=X^{\frac1p}$ with a Banach lattice $X=Y^p$. Since X satisfies condition 3 of Theorem 2, it also satisfies condition 1 of the same theorem, i.e., $X^{\frac1p}(\ell^p)=Y(\ell^p)$ is A_1-regular for all values of $p>1$ sufficiently close to 1. Since $Y(\ell^\infty)$ is also A_1-regular, the logarithmic convexity of the respective sets of A_1-majorants (or a direct application of the Hölder inequality; see, e.g., [14, §3, Proposition 16]) yields the A_1-regularity of $[Y(\ell^p)]^\delta[Y(\ell^\infty)]^{1-\delta}=Y(\ell^{\frac{p}{\delta}})$ for all values of p sufficiently close to 1 and any $0<\delta<1$, which implies that the lattices $Y(\ell^s)$ are A_1-regular for all $1<s<\infty$, as claimed.

Now it remains to prove Theorem 14. By symmetry, it suffices to verify the direct statement. First, we establish the following simple generalization of Proposition 13.

Proposition 18. *Suppose that Z is a quasinormed lattice of measurable functions on $S\times\Omega$, $1<p<\infty$, $\beta=\frac1p$, and Z is $\mathrm{F}^{1-\beta}_\beta$-regular with constants (C,m). Then $(Z\mathrm{L}_p)(\ell^s)$ is A_1-regular for all $1<s<\infty$.*

Compared to the proof of Proposition 13, it suffices to observe that, by Corollary 8, the lattices $\mathrm{L}_p\big(w^{-\frac1p}\big)(\ell^s)$ are A_1-regular for all $w\in\mathrm{A}_p$ and $1<p<\infty$, $1<s<\infty$. However, we give a complete proof for clarity.

Let $f\in(Z\mathrm{L}_p)(\ell^s)=Z(\ell^\infty)\mathrm{L}_p(\ell^s)$ with norm 1. Then there exist $g=\{g_j\}_{j\in\mathbb{Z}}\in Z(\ell^\infty)$ and $h=\{h_j\}_{j\in\mathbb{Z}}\in\mathrm{L}_p(\ell^s)$ such that $f=gh$ and $\|\bigvee_j g_j\|_Z=\|g\|_{Z(\ell^\infty)}\le 2$, $\|h\|_{\mathrm{L}_p(\ell^s)}\le 1$. For simplicity we may assume that $g>0$ almost everywhere. By replacing g with $\bigvee_j g_j$

and h with $\frac{g}{\bigvee_j g_j} h$ we may assume that g does not depend on the last variable while retaining all estimates on its norm. By the $\mathrm{F}_\beta^{1-\beta}$-regularity of Z, there exists a majorant $u \geq |g|$ such that $\|u\|_Z \leq 2m$ and $u \in \mathrm{F}_\beta^{1-\beta}$ with constant C, and thus

$$u^{-p} \in \mathrm{F}_{p(1-\beta)}^{p\beta} = \mathrm{F}_{p-1}^1 = \mathrm{A}_p$$

with some constants independent of f. Therefore,

$$\begin{aligned}
\|Mf\|_{(Z\mathrm{L}_p)(\ell^s)} &= \|u \cdot u^{-1}(Mf)\|_{Z(\ell^\infty)\mathrm{L}_p(\ell^s)} \\
&\leq \left\|\{u\}_{j\in\mathbb{Z}}\right\|_{Z(\ell^\infty)} \left\|u^{-1}(Mf)\right\|_{\mathrm{L}_p(\ell^s)} = \|u\|_Z \|Mf\|_{\mathrm{L}_p([u^{-p}]^{-\frac{1}{p}})(\ell^s)} \\
&\leq c\|f\|_{\mathrm{L}_p([u^{-p}]^{-\frac{1}{p}})(\ell^s)} = c\|h \cdot gu^{-1}\|_{\mathrm{L}_p(\ell^s)} \leq c\|h\|_{\mathrm{L}_p(\ell^s)} \leq c
\end{aligned}$$

with a constant c independent of f. Thus, the maximal operator M is bounded on $(Z\mathrm{L}_p)(\ell^s)$, and, hence, this lattice is A_1-regular, as claimed.

Now suppose that X is F_β^α-regular with some $\alpha > 1$ and $\beta > 0$ under the assumptions of Theorem 14. We want to invoke Proposition 18 to establish the A_1-regularity of $Z = Y^{\frac{1}{p}}(\ell^p)$ with $Y = X^\delta \mathrm{L}_1^{1-\delta}$ for some suitable $0 < \delta < 1$ and $1 < p < \infty$. Since $Y^{\frac{1}{p}} = X^{\frac{\delta}{p}} \mathrm{L}_1^{\frac{1-\delta}{p}}$, we need to check that $X^{\frac{\delta}{p}}$ is $\mathrm{F}_\beta^{1-\beta}$-regular with $\beta = \frac{1-\delta}{p}$, which is equivalent to the $\mathrm{F}_{\frac{1-\delta}{\delta}}^{\frac{p}{\delta} - \frac{1-\delta}{\delta}}$-regularity of X. Comparing this with the assumptions of the theorem yields the conditions $\alpha = \frac{p}{\delta} - \frac{1-\delta}{\delta}$ and $\beta = \frac{1-\delta}{\delta}$, which are satisfied with $\delta = \frac{1}{1+\beta}$ and $p = \delta\bigl(\alpha + \frac{1-\delta}{\delta}\bigr) = \frac{\alpha+\beta}{1+\beta}$. Proposition 18 gives the A_1-regularity of Z, which by implication $1 \Rightarrow 2$ of Theorem 2 implies the A_p-regularity of $Y' = X'^\delta$. Thus, Y' is F_{p-1}^1-regular, and so it is $\mathrm{F}_{\frac{\alpha-1}{1+\beta}}^1$-regular, and $X' = Y'^{\frac{1}{\delta}} = Y'^{1+\beta}$ is $\mathrm{F}_{\alpha-1}^{\beta+1}$-regular, as claimed.

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