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BOUNDEDNESS OF A VARIATION OF THE POSITIVE HARMONIC FUNCTION ALONG THE NORMALS TO THE BOUNDARY

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Abstract. Let u be a positive harmonic function on the unit disk. Bourgain showed that the radial variation

$$\operatorname{var}(u|_{[0,re^{i\theta}]}) = \int_0^1 |u'(re^{i\theta})|\,dr$$

of u is finite for many points θ, and moreover, that the set

$$\mathcal{V}(u) = \{e^{i\theta} : \operatorname{var}(u|_{[0,re^{i\theta}]}) < +\infty\}$$

is dense in the unit circle $\mathbb{T}$ and its Hausdorff dimension equals one. In the paper, this result is generalized to a class of smooth domains in $\mathbb{R}^d$, $d \geq 3$.

§1. Introduction

1.1. Let O be a domain in the d-dimensional real Euclidean space $\mathcal{E}^d$, $d \geq 2$, and let S be its boundary, which we assume to be C^2-smooth (see Subsection 9.1 below; we believe that this condition can be relaxed considerably). Let $\vec{N}(p)$ denote the inward normal to S at $p \in S$. The interval $\{ta + (1-t)b : 0 \leq t < 1\}$, where $a, b \in \mathcal{E}^d$, is denoted by $(a, b]$. Let r be a positive function on S such that $(p, p + r(p)\vec{N}(p)] \subset O$ for $p \in S$, and let u be a real-valued function on O. The *normal variation of u at $p \in S$* is

(1.1) $$(\vec{\mathrm{N}}\operatorname{var} u)(p) := \operatorname{var}\left(u|_{(p,p+r(p)\vec{N}(p)]}\right).$$

We are only concerned with the question as to whether this quantity is finite, and u is assumed to be smooth on O. Therefore, the explicit choice of r is of no importance.

Let $E \subset S_1 \subset S$. We say that the set E is *ultradense* in S_1 if for any $p \in S_1$ and $\rho > 0$ we have

$$\dim(E \cap \mathbb{B}^d(p, \rho)) = d - 1,$$

where $\mathbb{B}^d(p, \rho)$ is the unit ball $\mathcal{E}^d$ of radius ρ (we usually drop the superscript), and dim is the Hausdorff dimension.

Put

(1.2) $$\mathcal{V}(u) := \{p \in S : (\vec{\mathrm{N}}\operatorname{var} u)(p) < +\infty\}.$$

1.2.

Theorem 1. *If u is harmonic and positive on O, then $\mathcal{V}(u)$ is ultradense in S.*

It is well known (see [PK, HW1]) that any function u positive and harmonic on O has finite boundary values along *almost all* (with respect to the $(d-1)$-dimensional

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Hausdorff measure) normal vectors $\vec{N}(p)$, $p \in S$. Theorem 1 states that for many p a stronger version of this property is true: the variation $(\vec{\mathrm{N}}\operatorname{var} u)(p)$ is finite.

In the case where $d = 2$ and O is the unit disk, Theorem 1 was proved by Bourgain, see [B1, B2]. The case of $d \geq 3$ and O being a half-space in $\mathcal{E}^d$ is due to O'Neill [ON]. All these special cases are contained in our Theorem 1. For $d = 2$ this theorem can easily be deduced from [B1] via conformal mapping. However, when we study the boundary behavior of harmonic functions on more or less arbitrary d-dimensional domains, the transition from $d = 2$ to $d \geq 3$ is usually quite complicated (see, e.g., [C, pp. 48–49] and [HW1, HW2]). Moreover, the problem we are interested in complicates further by the fact that the remarkable results of [B1, B2] relied heavily upon the use of *Fourier transform*, which is well suited for the case where S is a group of some sort, but can hardly be adapted to deal with domains in Theorem 1 (the paper [ON] employed the spherical harmonics method). While we still follow the general approach of [B1, B2] (introduction of B-points and B-measures ν_ε, see Subsections 2.4 and §4 below), it was necessary to modify the arguments used in the construction of these objects and to avoid methods of harmonic analysis. Here we use the theory of linear differential equations in Banach spaces ("multiplicative integral", see [DK]). It is possible that this approach can be used in more general situations (harmonic functions on Riemannian manifolds, solutions of elliptic equations).

1.3. In the next subsection we formulate the main result of our paper (Theorem 2). We deduce it in §9 from Theorem 3, the proof of which occupies the most part of the paper (§§2–8). In Theorem 3 we consider a particular case of the domain O (an "almost half-space"), and u is subject to some additional conditions. We reduce the general case (Theorem 2) to this special one in §9. Thus, it can be said that Theorem 3 is only a step on the way to prove Theorem 2. However, we believe that §§2–8 devoted to Theorem 3 actually form the most substantial part of our paper, and the method of constructing the Bourgain measures ν_ε (see Subsection 2.4) is interesting on its own right.

1.4. Now we return to Theorem 1. Under its assumptions, we see that

$$\operatorname{var}(u|(p, p + r(p)\vec{N}(p)]) = \int_0^{r(p)} \left| \frac{\partial}{\partial y} \left(u \left(p + y\vec{N}(p) \right) \right) \right| dy \tag{1.3}$$

and Theorem 1 follows from Theorem 2, where we replace the integral in (1.3) by an integral that involves the derivatives of u *along all directions* (rather than along the normal vector $\vec{\mathcal{N}}(p)$)). Put

$$\mathcal{V}_{\mathrm{grad}}(u) := \left\{ p \in S \,:\, \int_0^{r(p)} \left| \nabla u(p + y\vec{N}(p)) \right| dy < +\infty \right\}. \tag{1.4}$$

Theorem 2. *Under the assumptions of Theorem* 1*, the set* $\mathcal{V}_{\mathrm{grad}}(u)$ *is ultradense in* S.

1.5. Remarks.

1. The arguments employed in [B1] and [M] can be used to study the so-called strong convergence of approximate identities (its special case for the Poisson integral for $\mathbb{R}^d$ is directly related to the results in our paper). This topic was discussed in [B2, M].

2. In Theorems 1 and 2 (and also in Theorem 3, see below), we may assume u to be bounded (it suffices to consider the function $u + C$, where C is a sufficiently large constant). Moreover, even the special case where u is continuous up to the boundary S of O is interesting. However, the fact that u is bounded from below (or from above) *is essential.* We cannot drop this condition, because there exists a harmonic function u on the unit disk with boundary values in BMO and such that $\mathcal{V}(u)$ (on the unit circle) is empty (see [J]).

3. The set $\mathcal{V}(u)$ occurring in Theorem 1 and having maximal Hausdorff dimension everywhere on S can still be of $(d-1)$-dimensional measure zero, even if O is the unit disk, and u is harmonic and continuous up to the boundary (see [R, M]).

4. In the special case where $d = 2$ and O is the unit disk, Theorem 2 was proved by Bourgain [B2]: *if f is analytic in the unit disk and has positive real part, then the set of endpoints of the radii along which the variation of f is finite* (*so that the image of such a radius is rectifiable*) *is ultradense in the unit circle*. Indeed,

$$\operatorname{var}_{[0,e^{it}]} f = \int_0^1 \left|\nabla(\operatorname{Re} f)(re^{it})\right| dr, \quad t \in (-\pi, \pi].$$

Now, putting $u := \operatorname{Re} f$ and applying Theorem 2, we obtain the result.

§2. A special case: O is a near half-space. Statement of Theorem 3

2.1. In this section we take $\mathcal{E}^d = \mathbb{R}^d = \mathbb{R}^{d-1} \times \mathbb{R}$, and the points in $\mathbb{R}^d$ are written as (x, y) where $x \in \mathbb{R}^{d-1}$, $y \in \mathbb{R}$. The domain $O = O_\Phi$ is the epigraph of a function $\Phi \in C^2(\mathbb{R}^{d-1})$:

$$O = \left\{(x, y) \in \mathbb{R}^d \,:\, x \in \mathbb{R}^{d-1},\ y > \Phi(x)\right\}. \tag{2.1}$$

We assume that

$$\begin{aligned} &\Phi(0),\ \nabla\Phi(0) = 0; \\ &\text{there exists}\ \ r = r(0) \in (0,1)\ \ \text{such that for any}\ \ x \in \mathbb{R}^{d-1} \setminus \mathbb{B}^{d-1}(0,r) \\ &\text{we have}\ \ \Phi(x) = 0, \ \ \text{and also}\ \ |\nabla\Phi(x)| < \tfrac{1}{100}\ \ \text{for any}\ \ x \in \mathbb{R}^{d-1}. \end{aligned} \tag{2.2}$$

The boundary S of O is the graph of Φ, and the domain O itself can be viewed as a small perturbation of the upper half-space $\mathbb{R}^d_+ = \mathbb{R}^{d-1} \times (0, +\infty)$.

Our positive harmonic function u is further assumed to satisfy the following conditions:

$$\lim_{(x^*,y)\to(x_0^*,0)} u(x^*, y) = \lim_{|x^*|+y\to+\infty} u(x^*, y) = 0, \tag{2.3}$$

for $x_0^* \in \mathbb{R}^{d-1}$, $|x_0^*| > r$.

Such a function u admits harmonic extension from O to the exterior of $\mathbb{B}^d(0, r)$, and we have $u|_{(S\setminus\mathbb{B}^d(0,r))} = 0$.

2.2. Let $\vec{e}_1, \dots, \vec{e}_d$ be the standard basis in $\mathbb{R}^d$. For $\phi \in \mathbb{R}^d$ and $y \in \mathbb{R}$, we denote by ϕ_y the point $\phi + y\vec{e}_d$. Given $y \in \mathbb{R}$ and a function v on $E \subset \mathbb{R}^d$, we define v_y on $E_{-y} := E - y\vec{e}_d$ by $v_y(\phi) = v(\phi_y)$.

The advantage of our special case (2.1) as compared to the general one is due to the fact that there is a semigroup of shifts $\phi \mapsto \phi_y$, $y \geq 0$, acting on O. The space $\mathcal{H}(O)$ of functions harmonic on O is preserved by this semigroup. For any $v \in \mathcal{H}(O)$ and $y > 0$, we see that v_y is harmonic on $O_{-y} \supset \bar{O} = O \cup S$.

2.3. The Poisson kernel for O. The *Poisson kernel* for O is the function

$$p^O(z, \zeta) := \frac{\partial G^O}{\partial \vec{N}(\zeta)}(z, \zeta), \quad z \in O,\ \ \zeta \in S, \tag{2.4}$$

where G^O is the Green function for O, and $\vec{N}(\zeta)$ is the inward normal to S at $\zeta \in S$. If φ is bounded and continuous on S, then $P^O(\varphi)$ defined by

$$P^O(\varphi)(z) = \int_S \varphi(\zeta) p^O(z, \zeta)\, ds(\zeta), \quad z \in O, \tag{2.5}$$

is harmonic on O, and extends continuously to $O \cup S$; moreover, this extension coincides with φ on S. In (2.5), s is the $(d-1)$-dimensional Hausdorff measure in $\mathbb{R}^d$.

Let $\bar{S} = S \cup \{\infty\}$, and let $C(\bar{S})$ denote the set of functions $\varphi \in C(S)$ that are finite at infinity and satisfy the relation $+\infty > \varphi(\infty) := \lim_{|\zeta|\to\infty} \varphi(\zeta)$. If $\varphi \in C(\bar{S})$, then $\lim_{z\in O,\, z\to\infty} P^O(\varphi)(z) = \varphi(\infty)$, so that $P^O(\phi)$ is continuous in $O \cup \bar{S}$ and coincides with φ on $\bar{S}$.

2.4. Vertical and mean vertical variation of a function defined on a near half-space. Let $v \in C^1(O_\Phi)$, where $O_\Phi(= O)$ is a "near half-space", see Subsection 2.1. The *vertical variation of* v *at* $x \in S$ is defined by

$$\mathrm{Vvar}(v)(x) := \int_0^1 |\nabla v|(x_y)\, dy. \tag{2.6}$$

This quantity bounds from above the variation of v on the "vertical" line $(x, x_1] \subset O$. If v is harmonic on O, then the gradient $|\nabla v|$ (along with $|\nabla v_y|$ for any positive y) is subharmonic on O (see [SW, Chapter 6, §4]). Assume that $\lim_{|p|\to+\infty} |\nabla v(p)| = 0$ (this condition is fulfilled for $v = u$, because u is harmonic outside $\mathbb{B}^d(0, r)$ and vanishes at infinity). Put

$$h_v^{[y]}(q) := P^O(|\nabla v_y||S)(q), \quad q \in O,\ y > 0. \tag{2.7}$$

We see that $h_v^{[y]}$ is the smallest harmonic majorant of $|\nabla v_y|$ in O.
Let $x \in S$. The *mean vertical variation of* v *at* $x \in S$ is

$$V(x)\, (= V^v(x)) := \int_0^1 (h_v^{[2y]})_y(x)\, dy. \tag{2.8}$$

Here (unlike (2.7), where the gradient $|\nabla v_y(x)|$ is treated) we have the additional averaging of $\zeta \mapsto |\nabla v_{2y}(\zeta)|$ with respect to the probability measure $p^O(x_y, \zeta)\, ds(\zeta)$ on S. Assume that v is harmonic on O and outside $\mathbb{B}^d(0, r)$, and that $\lim_{|p|\to\infty} v(p) = 0$. We claim that

$$V^v(x) \geq \frac{1}{3} \mathrm{Vvar}(v)(x), \quad x \in S. \tag{2.9}$$

Indeed,

$$V(x) \geq \int_0^1 |\nabla v_{2y}(x_y)|\, dy = \int_0^1 |\nabla v(x_{3y})|\, dy \geq \frac{1}{3} \int_0^1 |\nabla v_y(x)|\, dy.$$

In §9 it will be shown that $V(x) \geq A(\vec{\mathrm{N}}\,\mathrm{var}\, u)(x)$, $x \in S$, where A is some positive absolute constant (this is trivial for $\Phi \equiv 0$, i.e., for $O = \mathbb{R}^d_+$, because then we have $\vec{\mathrm{N}}\,\mathrm{var}(u) \leq \mathrm{Vvar}(u)$).

A point $x \in S$ is called *a Bourgain point* (B-point) of u if $V^u(x) < +\infty$. The set of such points is denoted by $\mathcal{B}(u)$. By what has been mentioned above, in our special case (i.e., when $O = O_\Phi$) we have $\mathcal{B}(u) \subset \mathcal{V}_{\mathrm{grad}}(u)$.

Theorem 3. *If conditions* (2.1) *and* (2.2) *are satisfied, then* $\mathcal{B}(u)$ *is ultradense in* S.

§3. Preparation for the proof of Theorem 3. The operators B_y

3.1. Our immediate goal is to produce a sort of "partial linearization" for the function $V = V^u$. We shall obtain the representation

$$V^u = \mathcal{L}^u(u), \tag{3.1}$$

where $\mathcal{L}^u$ is a linear operator on a certain set of functions defined on O. In the next subsection we do some preliminary work to obtain (3.1); the derivation itself is done in Subsection 3.3.3.

3.2. Kernels of some integral operators. In what follows by a "kernel" we mean a function defined on $S \times S$ (in some disagreement with the term "Poisson kernel for O", because the latter is actually defined on $O \times S$). The kernels are denoted by lowercase Latin or Greek letters, and the corresponding integral operators by the respective capital letters. For example, a kernel k gives rise to the operator K given by

$$K(\varphi)(x) := \int_S k(x,\xi)\varphi(\xi)\, ds(\xi), \quad x \in S. \tag{3.2}$$

In each individual case we pay close attention to the conditions under which for any $x \in S$ the function $\xi \mapsto k(x,\xi)\varphi(\xi)$ in (3.2) is integrable on S with respect to s.

The composition $k_1 \circ k_2$ of two kernels k_1 and k_2 is defined as follows:

$$(k_1 \circ k_2)(x,\xi) := \int_S k_1(x,\eta) k_2(\eta,\xi)\, ds(\eta), \quad x,\xi \in S. \tag{3.3}$$

Here we assume that the integrand in (3.3) is integrable with respect to s for any $x, \xi \in S$. We take special care for ensuring the identity $K_1(K_2(\varphi)) = K(\varphi)$, where K is the integral operator with the kernel $k := k_1 \circ k_2$. We shall also need the compositions $k_1 \circ \cdots \circ k_n$ defined in the same way for any natural n.

By k^* we denote the kernel adjoint to k,

$$k^*(x,\xi) := k(\xi,x), \quad x,\xi \in S, \tag{3.4}$$

and K^* is the respective integral operator. For the situations under study the following formula holds true:

$$\int_S K^*(\varphi) \cdot \psi\, ds = \int_S \varphi \cdot K(\psi)\, ds.$$

3.3. The kernels p_y, c_y, b_y. The first of these kernels depends only on y and S, while the other two also depend on u.

Put

$$p_y(x,\xi) := p^O(x_y,\xi), \quad x,\xi \in S, \ \ y > 0. \tag{3.5}$$

Given $x \in S$, $y > 0$, the measure $p_y(x,\,\cdot\,)\, ds$ on S is the harmonic measure in O with the pole at $x_y \in O$. In particular, $\int_S p_y(x,\xi)\, ds(\xi) = 1$.

The properties of the mapping $\varphi \mapsto P^O(\varphi)$ (see (2.5)) imply the *semigroup property of the family* $(p_y)_{y>0}$:

$$p_{y_1+y_2} = p_{y_1} \circ p_{y_2}, \quad y_1, y_2 > 0. \tag{3.6}$$

3.3.1. We also need the following property of p_y: *everywhere* on $S \times S$, we have

$$\frac{p_{y_2}}{p_{y_1}} \le c(S) \left(\frac{y_2}{y_1}\right) \tag{3.7}$$

whenever $0 < y_1 \le y_2 \le 1$.

Proof. We recall that S is the graph of a compactly supported function $\Phi \in C^2(\mathbb{R}^{d-1})$ (see Subsection 2.1). Hence, there exists $R = R(S) > 0$ such that for any $x \in S$ the ball $\mathbb{B}\,(= \mathbb{B}(x + R\vec{N}(x), R))$ is tangent to S at x and lies in $O = O_\Phi$. Let $x, \xi \in S$, and let $z_j := x + y_j \cdot \vec{e}_d$, $j = 1, 2$. Now we apply Corollary 1 to Lemma 10.1 (see Subsection 10.1) to the ball $\mathbb{B}$, the vector $\nu := \vec{e}_d$, the points z_1, z_2, and the function

$$v \colon z \mapsto p^O(z,\xi), \quad z \in \mathbb{B}.$$

It is possible for $y_2 \le \frac{R}{4}$, because ν and the (inward) normal to S at x are "almost parallel" (see (2.2)), and $z_1, z_2 \in \mathbb{B}$. We arrive at

$$\frac{v(z_2)}{v(z_1)} \le C(S)\left(\frac{y_2}{y_1}\right).$$

If $y_1 > \frac{R}{4}$, then, in accordance with Subsection 10.3.2, we have

$$\begin{aligned}\frac{v(z_2)}{v(z_1)} &\le C_1(S)\left(\frac{y_2}{y_1}\right)^{C_1(S)} \le C_1(S)y_1^{-C_1(S)}\\ &\le C_1(S)\left(\frac{4}{R}\right)^{C_1(S)} =: C_2(S) \le C_2(S)\frac{y_2}{y_1}.\end{aligned}$$

Finally, if $y_1 \le \frac{R}{4} < y_2$, then the desired inequality follows from the identity

$$\frac{v(z_2)}{v(z_1)} = \frac{v(z_2)}{v(\bar z)} \cdot \frac{v(\bar z)}{v(z_1)},$$

where $\bar z := x_{\frac{R}{4}}$. □

Remark. We also need a slightly more general version of (3.7), where we assume that $0 < y_1 \le y_2 \le Y$. Then, clearly,

$$\frac{p_{y_2}}{p_{y_1}} \le c(S)(1+Y)^{c(S)}\frac{y_2}{y_1}. \tag{3.8}$$

3.3.2. Now we proceed to the definition of the kernels c_y. Given $\vec a \in \mathbb{R}^d$, we put

$$\operatorname{sgn}\vec a := \begin{cases}\vec a/|\vec a| & \text{if } \vec a \ne 0,\\ 0 & \text{if } \vec a = 0.\end{cases}$$

The scalar product of $\vec a$ and $\vec b$ is denoted by $\langle \vec a, \vec b\rangle$. Let $\vec\sigma(q) := \operatorname{sgn}(\nabla u)(q)$, $q \in O$. The vector field $\vec\sigma$ vanishes on the closed (in O) set $\mathcal{Z} = \mathcal{Z}(u) := \{\nabla u = 0\}$. This field is C^∞-smooth on $O \setminus \mathcal{Z}$, and $|\vec\sigma||_{O\setminus\mathcal{Z}} \equiv 1$. In Subsection 10.4 it will be shown that (unless $u \equiv$ const) $s(\mathcal{Z} \cap S_y) = 0$ for any $y > 0$ (in the case of $d = 2$ the set $\mathcal{Z}$ is discrete). Moreover, $\vec\sigma$ is discontinuous at any point of $\mathcal{Z}$.

Put

$$c_y(x,\xi) := \langle (\nabla^1 p^O)(x_y,\xi), \vec\sigma(x_{2y})\rangle = \frac{\partial^1 p^O}{\partial\vec\sigma}(x_{2y})(x_y,\xi), \quad x,\xi \in S,\ \ y > 0, \tag{3.9}$$

(the superscript 1 means that we differentiate with respect to the first variable; we also assume that $\frac{\partial\varphi}{\partial\vec u} = 0$ if $\vec u = 0$). Both $\vec\sigma$ and c_y depend on u, but, for simplicity, we do not reflect this in the notation.

From the properties of σ it follows that, for any $y > 0$, the kernel c_y vanishes on the set $(\mathcal{Z}_{-2y} \cap S) \times S$, and its trace on $(S \setminus \mathcal{Z}_{2y}) \times S$ is continuous. Using (10.8) and (10.10), we obtain

$$|c_y(x,\xi)| \le \left|(\nabla^1 p^O)(x_y,\xi)\right| \le \frac{c(S)p^O(x_y,\xi)}{y}, \quad x,\xi \in S,\ \ y > 0. \tag{3.10}$$

If we differentiate $\int_S p^O(z,\xi)ds(\xi) \equiv 1$, $z \in O$, along $\vec\sigma(x_{2y})$, we get

$$C_y(1) = 0 \ \text{ in } \ S, \quad y > 0. \tag{3.11}$$

3.3.3. Now we define the kernel as follows: $b_y := p_y \circ c_y, y > 0$. The properties of the kernels p_y and c_y combined with (3.10) imply that this definition is consistent on $S \times S$, and the resulting kernel depends on u (more precisely, on $\vec{\sigma}$). To start the "linearization" of $u \mapsto V^u$ (see Subsection 3.1), we observe that, for $x \in S$, $y > 0$,

$$\begin{aligned} |(\nabla u)(x_{2y})| &= \big\langle (\nabla u)(x_{2y}), \vec{\sigma}(x_{2y}) \big\rangle = \big\langle (\nabla u_y)(x_y), \vec{\sigma}(x_{2y}) \big\rangle \\ &= \Big\langle \int_S \nabla^1 p^O(x_y, \xi) u_y(\xi)\, ds(\xi), \vec{\sigma}(x_{2y}) \Big\rangle = C_y(u_y)(x). \end{aligned} \tag{3.12}$$

By (3.12)), the least harmonic majorant $h^{[2y]}$ for $|\nabla u_{2y}|$ in O satisfies

$$h^{[2y]}(q) = P^O(|\nabla u_{2y}||_S)(q) = P^O(C_y(u_y|_S))(q), \quad q \in O. \tag{3.13}$$

Putting $q = x_y$, we obtain

$$h^{[2y]}(x_y) = (h^{[2y]})_y(x) = ((P_y C_y)(u_y|_S))(x) = (B_y(u_y|_S))(x), \quad x \in S, \tag{3.14}$$

and, finally,

$$V(x) = \int_0^1 (B_y(u_y|_S))(x)\, dy, \quad x \in S. \tag{3.15}$$

Note that $B_y(u_y|_S) > 0$.

Now, the operator $\mathcal{L}^u$ that "linearizes" the mapping $u \mapsto V^u$ (see (3.1)) is defined as follows:

$$\mathcal{L}^u = \int_0^1 B_y\, dy,$$

and the Bourgain points of u are the points $x \in S$ where the integral $\int_0^1 (B_y(u_y))(x)\, dy$ is finite (from now on we write $B_y(u_y)$ instead of $B_y(u_y|_S)$).

We shall need the following properties of b_y:

for any $y \in (0, 1)$,

$$|b_y| \le \frac{c(S)}{y} p_y, \tag{3.16a}$$

$$B_y(1) \equiv 0 \quad \text{on} \quad S. \tag{3.16b}$$

Indeed,

$$|b_y| \le p_y \circ |c_y| \le c(S) \frac{(p_y \circ p_y)}{y} = c(S) \frac{p_{2y}}{y} \le c'(S) \frac{p_y}{y},$$

(we have used (3.10) and the results of Subsection 10.3.2). We see that (3.16b) follows from (3.11).

In Subsection 10.5 we shall show that the function $(x, \xi, y) \mapsto b_y(x, \xi)$ is continuous on $S \times S \times (0, +\infty)$.

3.4. Outline of the proof of Theorem 3. The measures $\nu_{\varepsilon,u}$ and properties (*a*)**,** (*b*)**,** (*c*)**.** Assume that there exists a Borel measure ν supported on S such that $\int_S V\, d\nu < +\infty$ and $\nu(\mathbb{B}) > 0$ for any ball $\mathbb{B}$ with center on S. Then the set $\mathcal{B}(u)$ of Bourgain points of u is *dense* in S. However, we cannot yet say that $\mathcal{B}(u)$ is *ultradense* in S – such a measure ν can still be supported on some countable subset of S.

To prove the *ultradensity* of $\mathcal{B}(u)$ in S, we construct the family $(\nu_\varepsilon)_{\varepsilon \in (0, \varepsilon(S))}$ of measures supported on S such that

(a) $\int_S V\, d\nu_\varepsilon < +\infty$ *for any* $\varepsilon \in (0, \varepsilon(S))$;

(b) *there exist positive constants* $c_1(S)$, $c_2(S)$ *such that for any* $\rho > 0$ *we have*

$$\nu_\varepsilon(\mathbb{B}) \le c_1(S) \rho^{d-1-c_2(S)\varepsilon} \tag{3.17}$$

for any number $\varepsilon \in (0, \varepsilon(S))$ *and any ball* $\mathbb{B}$ *with radius* ρ *and center on* S;

(c) *for any ball* $\mathbb{B}$ *with center on* S *there exists* $\varepsilon(\mathbb{B}) > 0$ *such that* $\nu_\varepsilon(\mathbb{B}) > 0$ *whenever* $0 < \varepsilon < \varepsilon(\mathbb{B})$.

Now we verify that *the existence of such a family of measures* (ν_ε) *guarantees the ultradensity of* $\mathcal{B}(u)$ *in* S.

Let $\mathbb{B}$ be a ball with center on S. Put $\alpha := \mathcal{B}(u) \cap \mathbb{B}$, so that $\nu_\varepsilon(\alpha) = \nu_\varepsilon(\mathbb{B})$, due to (a). For any covering $(b_j)_{j=1}^{\infty}$ of the set α with balls of radii r_j with centers on S, we have

$$0 < \nu_\varepsilon(\mathbb{B}) = \nu_\varepsilon(\alpha) \le \sum_j \nu_\varepsilon(b_j) \le c_1 \sum_j r_j^{d-1-c_2\varepsilon}$$

if $0 < \varepsilon < \min(\varepsilon(S), \varepsilon(\mathbb{B}), \frac{d-1}{c_2})$. For such ε we see that $\mathcal{H}_{d-1-c_2\varepsilon}(\alpha) > 0$, hence $\dim \alpha = d-1$.

The plan of the construction of ν_ε is presented in §4. The construction itself (and the proof of (a)–(c)) is done in §§5–8. We note that the ν_ε are probability measures (i.e., $\nu_\varepsilon(S) = 1$).

We call the measures ν_ε $(= \nu_{\varepsilon,\,u})$ the Bourgain measures (B-measures) of the function u. The idea to use these measures to prove the ultradensity of B-points is borrowed from [B1, B2]. The main difference here lies in the construction of B-measures. Our argument works not only for the case when O is the ball or the upper half-space, but also where O is the "almost upper half-space" (see §§4–8, where we construct the measures (ν_ε) that satisfy (a), (b), (c)).

§4. Construction of the measures ν_ε: an outline

4.1. It remains to construct a family of B-measures that satisfy conditions (a), (b), and (c) (see Subsection 3.4). It is done in §§5–8 in accordance with the plan we introduce below.

By $M_+(S)$ we denote the set of finite Borel measures on $\mathbb{R}^d$ supported on S. In Subsection 4.2 we construct a family of mappings $(\mathcal{W}_{\varepsilon,u})_{0<\varepsilon<\varepsilon(S)}$ of the set $M_+(S)$ into itself such that $\nu_\varepsilon := \mathcal{W}_{\varepsilon,u}(\kappa)$ satisfies (a), (b), (c) for any nonzero $\kappa \in M_+(S)$. This proves Theorem 3.

Remark. The mappings $\mathcal{W}_{\varepsilon,u}$ are actually restrictions to $M_+(S)$ of the linear operators that map the set $M(S)$ of finite Borel charges on $\mathbb{R}^d$ supported on S into itself, and moreover, $\mathcal{W}_{\varepsilon,u}(\kappa)(S) = \kappa(S)$ for any $\kappa \in M(S)$.

4.2. The kernels $\omega_{y,u,\varepsilon}$. Let $\varepsilon \in (0, \varepsilon(S))$ be sufficiently small. The measure $\mathcal{W}_{\varepsilon,u}(\kappa)$ is obtained from the measure $\kappa \in M_+(S)$ via a continuous transformation depending on the parameter y. With $y \in (0,1)$, in Subsection 4.5 we associate a positive kernel ω_y $(= \omega_{y,\varepsilon,u}) \in C(\bar{S} \times \bar{S})$ (and consequently, an integral operator Ω_y) such that

$$\int_S \omega_y(x,\xi)\, ds(\xi) = \Omega_y(1)(x) = 1 \tag{4.1}$$

for any $x \in S$, $y \in (0,1)$. Given a probability measure $\kappa \in M_+(S)$, put

$$\gamma_y(x) := \int_S \omega_y(\xi, x)\, d\kappa(\xi) = \int_S \omega^*(x,\xi)\, d\kappa(\xi) =: \Omega_y^*(\kappa)(x), \quad x \in S,\ y \in (0,1). \tag{4.2}$$

Clearly, $\gamma_y\, ds$ is a probability measure supported on S. The measure $\nu_\varepsilon := \mathcal{W}(\kappa)$ $(= \mathcal{W}_{\varepsilon,u}(\kappa))$ is defined as the weak limit $\lim_{y\downarrow 0} \gamma_y\, ds$ (its existence is proved in Subsection 5.4). The operators Ω_y tend to the identity as $y \uparrow 1$ (see Subsection 5.3). More precisely, $\lim_{y\uparrow 1} \Omega_y(\varphi) = \varphi$ uniformly on S for any $\varphi \in C(\bar{S})$. From now on, Ω_1 is the identity operator on $C(\bar{S})$, and Ω_1^* is the identity mapping of $M(\bar{S})$ into itself.

Thus, for $y = 1$ we start with a unit mass κ on S, and then we gradually redistribute it as y tends to zero. Every $y \in (0,1)$ corresponds to the distribution $\Omega_y^*(\kappa) = \gamma_y\, ds$.

Passing to the limit, we obtain the desired distribution $\nu_\varepsilon = \mathcal{W}(\kappa)$ "adjusted" to u in the sense that $\int_S V\,d\nu_\varepsilon$ is finite. Hence, the *existence* of Bourgain points of u is ensured. Their *ultradensity* is implied by the fact that the family $(\nu_\varepsilon)_{0<\varepsilon<\varepsilon(S)}$ satisfies (b) and (c). These conditions are proved in §7. Condition (a) will be deduced at once from some properties of the kernels ω_y.

We finally observe that the functions γ_y are continuous on S, and that $\lim_{x\to\infty}\gamma_y(x) = 0$. Letting $\gamma_y(\infty)=0$, we may assume that $\gamma_y \in C(\bar{S})$. This follows from (4.2) and from the inequality $\omega_y \le c_1 p_{1-y} + c_2 p_y$ (see Subsection 5.1)

4.3. Two key facts about ω_y. These facts will be proved in §§5–6.

Let φ be a positive harmonic function on O that has a finite limit $\lim_{z\to\infty}\varphi(z)$ (i.e., $\varphi \in \mathcal{H}_+(\bar{O}) = \mathcal{H}_+(O\cup\{\infty\})$). Put $\varphi_y(x) := \varphi(x + y\vec{e}_d)$, $x \in S$, $y > 0$, so that $\varphi_y \in C(S) = C(S\cup\{\infty\})$. Let $0<\eta<y\le 1$ (whence $\Omega_\eta(\varphi_y)(x) < +\infty$ for any $x\in S$). *Then*

(i) $\Omega_\eta(\varphi_y) \le c(S)\Omega_y(\varphi_y)$;
(ii) if $\lim_\infty \varphi = 0$, then for any $x \in S$ *the function* $f^x : y \mapsto \Omega_y(\varphi_y)(x)$ *is continuously differentiable on* $(0,1]$, and

$$\left(\frac{\partial}{\partial y} f^x\right)(y) = \varepsilon\Omega_y(B_y(\varphi_y))(x), \quad x\in S,\ y\in(0,1] \tag{4.3}$$

(statement (i) is proved in Subsection 5.3.2, statement (ii) and the fact that $B_y(\varphi_y) \in C(\bar{S})$ for $y\in(0,1]$ is deduced in §6).

4.4. Now we show that (i) and (ii) imply that $\int_S V\,d\nu_\varepsilon$ is finite, which coincides, essentially, with property (a) of the measure ν_ε.

Given $y \in (0,1]$, we put

$$g_y := (h^{[2y]})_y = B_y(u_y) = P_y(|\nabla u_{2y}|) \tag{4.4}$$

(see (3.12)–(3.14)). We note that for any $y\in(0,1]$ the function g_y coincides on S with some positive harmonic function on O_{-y} that vanishes at infinity (because $|\nabla u_{2y}|$ vanishes at infinity, see Subsection 2.4). To prove (a), it suffices to show that $J_\delta := \int_S\int_\delta^1 g_y\,dy\,d\nu_\varepsilon$ is uniformly bounded for $\delta\in(0,1)$, because Levy's theorem implies that $\lim_{\delta\downarrow 0} J_\delta = \int_S V\,d\nu_\varepsilon$. For such δ, the function $x\mapsto \int_\delta^1 g_y(x)dy$, $x\in S$, coincides on S with some function in $\mathcal{H}_+(\bar{O}_{-\delta})$ that vanishes at infinity (see Subsection 10.5). Therefore, due to (i) we have

$$\begin{aligned} J_\delta &= \lim_{\eta\to 0}\int_S\left(\int_\delta^1 g_y\,dy\right)\gamma_\eta\,ds = \lim_{\eta\to0}\int_S\left(\Omega_\eta\left(\int_\delta^1 g_y\,dy\right)\right)d\kappa \\ &= \lim_{\eta\to0}\int_S\left(\int_\delta^1\Omega_\eta(g_y)\,dy\right)d\kappa \le c\int_S\left(\int_\delta^1\Omega_y(g_y)\,dy\right)d\kappa. \end{aligned}$$

But from (ii), (4.3), and (4.4) it follows that the last integral $c\int_S\left(\int_\delta^1\Omega_y(B_y(u_y))\,dy\right)d\kappa$ is equal to

$$\begin{aligned} \frac{c}{\varepsilon}\int_S\left(\int_\delta^1\frac{\partial}{\partial y}\Omega_y(u_y)\,dy\right)d\kappa &= \frac{c}{\varepsilon}\int_S(\Omega_1(u_1)-\Omega_\delta(u_\delta))\,d\kappa \\ &\le \frac{c}{\varepsilon}\int_S\Omega_1(u_1)\,d\kappa \le \frac{c}{\varepsilon}\sup_S u_1 \end{aligned}$$

(it is precisely in the penultimate inequality where we have used the positivity of u). Recall that u vanishes at infinity, and therefore, u_1 is bounded on S. We are done.

4.5. We need to solve the differential equations (4.3) (where $y \mapsto \Omega_y$, $y \in (0,1]$, is the unknown operator-valued function). For this, we use (a version of) the well-known method of solving linear differential equations in a vector space (see [DK]). We construct an operator-valued function $\Delta \mapsto \Omega_\Delta$ that maps a compact interval to an integral operator Ω_Δ with positive kernel $\omega_\Delta \in C(S \times S)$. This function ("multiplicative integral") satisfies the following condition:

$$0 < a < b < c \Rightarrow \omega_{[a,c]} = \omega_{[b,c]} \circ \omega_{[a,b]} \tag{4.5}$$

(the actual construction is done in §8). The kernels ω_y (see Subsection 4.2) are defined as follows: $\omega_y := \omega_{[y,1]}$, $0 < y < 1$. In §§5–8 we shall make sure that this choice of kernels provides all the necessary properties of ν_ε.

§5. The kernels ω_Δ. Weak convergence of $\gamma_y\,ds$ as $y \downarrow 0$ and condition (i)

5.1. The kernels b_Δ, $\widetilde{\omega}_\Delta$, ω_Δ. By segm_+ we denote the set of all nondegenerate compact intervals (segments) in $(0,+\infty)$. Given $\Delta \in \mathrm{segm}_+$ let

$$\begin{gathered} m(\Delta) := \min \Delta,\ M(\Delta) := \max \Delta,\ |\Delta| := M(\Delta) - m(\Delta), \\ b_\Delta(x,\xi) := \int_\Delta b_\theta(x,\xi)\,d\theta, \quad x,\xi \in S \end{gathered} \tag{5.1}$$

(these kernels were defined in Subsection 3.3).

From (3.16a) and (3.7) it follows that

$$|b_\Delta| \le c(S)\frac{P_{m(\Delta)}}{m(\Delta)}|\Delta| \tag{5.2}$$

for $M(\Delta) \le 1$. We say that a segment $\Delta \in \mathrm{segm}_+$ is *short* if $|\Delta| \le m(\Delta)$. Given a short segment $\Delta \subset (0,1]$, we have

$$|b_\Delta| \le c'(S)p_{|\Delta|} \tag{5.3}$$

by (3.7). Note that for any $\Delta \in \mathrm{segm}_+$ we have

$$B_\Delta(1) = \int_\Delta B_\theta(1)\,d\theta = 0. \tag{5.4}$$

Let

$$\widetilde{\omega}_\Delta := p_{|\Delta|} - \varepsilon b_\Delta, \quad \varepsilon > 0,\ \Delta \in \mathrm{segm}_+\,. \tag{5.5}$$

Clearly, $\widetilde{\Omega}_\Delta(1) = 1$. The kernel $\widetilde{\omega}_\Delta$ *is positive* if $\varepsilon \in (0, \varepsilon(S))$ and Δ is short (see (5.3)). Moreover, under these conditions we have

$$c_1(S)p_{|\Delta|} \le \widetilde{\omega}_\Delta \le c_2(S)p_{|\Delta|}; \tag{5.6}$$

here $c_1(S) > (1 - \varepsilon(S))c'(S)$.

5.1.1. We proceed with describing the kernels ω_Δ that play a crucial role in what follows (we already mentioned them in Subsection 4.5, see (4.5)). They are constructed in §8. Identity (4.5) will also be proved there, as well as the properties to be listed below.

For sufficiently small $|\Delta|$, the kernel ω_Δ can be viewed as a small "correction" of $\widetilde{\omega}_\Delta$. Namely, if we put $r_\Delta := \omega_\Delta - \widetilde{\omega}_\Delta$, we obtain

$$|r_\Delta| \le c(S)\varepsilon^2 \frac{|\Delta|^2}{m(\Delta)^2} p_{m(\Delta)}. \tag{5.7}$$

From (5.7) and (3.7) it follows that

$$|r_\Delta| \le c'(S)\varepsilon^2 \frac{|\Delta|}{m(\Delta)} p_{|\Delta|} \tag{5.8}$$

provided $\Delta \subset (0,1]$ is short. *The kernel* ω_Δ *is positive if* $\varepsilon \in (0, \varepsilon(S))$ (here $\varepsilon(S)$ is a small positive constant that depends only on S; in what follows it can change from line to line). Indeed, (5.8) implies that

$$\omega_\Delta = \widetilde{\omega}_\Delta + r_\Delta \geq (c_1(S) - c''(S)\varepsilon)\, p_{|\Delta|}$$

for short Δ. Generally, $\Delta = \bigsqcup_{s=1}^N \Delta_s$, where the Δ_s are pairwise disjoint short segments, $m(\Delta_s) < m(\Delta_{s+1})$, so that $\omega_\Delta = \omega_{\Delta_N} \circ \cdots \circ \omega_{\Delta_1} > 0$ (see (4.4)).

We also note that $R_\Delta(1) = 0$, whence

$$\Omega_\Delta(1) = 1. \tag{5.9}$$

5.2. Here we estimate the kernels $\omega_{[\rho,1]}$ for small positive ρ. First, we observe that $[y, 2y]$ is short for $y > 0$, so that for any $\varepsilon \in (0, \varepsilon(S))$ we have

$$\omega_{[y,2y]} \leq \widetilde{\omega}_{[y,2y]} + c\varepsilon p_y \leq p_y + c'\varepsilon \int_y^{2y} \frac{p_\theta}{\theta}\, d\theta + c\varepsilon p_y,$$

where c and c' are positive constants depending only on S (see (3.16), (5.3)). Next,

$$\int_y^{2y} \frac{p_\theta}{\theta}\, d\theta \leq \int_y^{2y} c' \frac{p_y}{y} \cdot \frac{\theta}{y}\, d\theta = c' p_y, \quad c' = c'(S) > 0$$

(we have used (3.7)). This means that

$$\omega_{[y,2y]} \leq C(S) p_y, \quad C(S) = 1 + \varepsilon c'(S) > 0, \; 0 < y \leq 1, \tag{5.10}$$

assuming that $\varepsilon < 1$. An estimate for $\omega_{[\rho,1]}$ follows from (5.10): *if* $0 < \rho < \frac{1}{2}$, $\varepsilon \in (0, \varepsilon(S))$, *then*

$$\omega_{[\rho,1]} \leq c \cdot \frac{1}{\rho^{c\varepsilon}} \cdot p_{1-\rho}, \quad c = c(S) > 0. \tag{5.11}$$

Proof. Let $K = K(\rho)$ be a natural number such that $2^K \rho \leq 1 \leq 2^{K+1}\rho$. Letting $\Delta_j := [2^j \rho, 2^{j+1}\rho]$, we obtain $[\rho, 1] = \bigcup_{j=0}^{K-1} \Delta_j \cup [2^K \rho, 1]$. Now (4.5) implies

$$\omega_{[\rho,1]} = \omega_{[2^k \rho,1]} \circ \omega_{\Delta_{K-1}} \cdots \circ \omega_{\Delta_0},$$

and combining this with (5.10), we arrive at

$$\begin{aligned}
\omega_{[\rho,1]} &\leq (1+c\varepsilon)^{K+1} p_{1-2^k\rho} \circ p_{2^{K-1}} \circ \cdots \circ p_\rho = (1+c\varepsilon)^{K+1} p_{(1-2^k\rho)+(2^{K-1}+\cdots+1)\rho} \\
&= (1+c\varepsilon)^{K+1} p_{1-\rho} \leq 2(1+c\varepsilon)^K p_{1-\rho} \leq 2e^{a \log \frac{1}{\rho} \log(1+c\varepsilon)} \leq 2e^{ac\varepsilon \log \frac{1}{\rho}} p_{1-\rho} \\
&= 2c \cdot \frac{1}{\rho^{ac\varepsilon}} p_{1-\rho}
\end{aligned}$$

for $\varepsilon \in (0, \varepsilon(S))$; here $c = c(S) > 0$, $a = \frac{1}{\log 2}$. □

Similarly, under the same conditions, from (5.10) we deduce the following estimate:

$$\omega_{[\rho,1]} \geq c_- \rho^{c-\varepsilon} \cdot p_{1-\rho}, \quad c_- = c_-(S) > 0. \tag{5.12}$$

5.3. Φ-property of the operator Ω_Δ. For a given short segment Δ, the operator Ω_Δ behaves like the Poisson operator $P_{|\Delta|}$: for many functions ψ defined on S the function $\Omega_\Delta(\psi)$ converges to ψ itself as $|\Delta| \to 0$, and the function $\xi \mapsto \omega_\Delta(x, \xi)$, $\xi \in S$ "is focusing to $x \in S$", becoming similar to δ_x. In what follows we use this "focusing property" repeatedly. Now we proceed to more rigorous formulation.

Lemma 5.1. *Let $\varepsilon, y \in (0,1)$, and let a function ψ defined on S coincide with some positive harmonic function v on O_{-y} such that $v|_O = P^O(\psi)$. Then for any $\Delta \in \mathrm{segm}_+$, $\Delta \subset (0, y]$, we have*

$$|\Omega_{\Delta,\varepsilon}(\psi) - \psi| \le c\frac{|\Delta|}{y}\psi \tag{5.13}$$

everywhere on S (from now on c, c_1, c_2, ... denote positive constants that depend only on S).

The condition $v|_O = P^O(v|_S) = P^O(\psi)$ is satisfied for $\psi = P_y(\varphi)$, where $y > 0$ φ is a nonnegative function defined on S.

Proof. Let $J \in \mathrm{segm}_+$, $J \subset \Delta$. Then

$$|\Omega_J(\psi) - \psi| \le |P_{|J|}(\psi) - \psi| + |B_J(\psi)| + |R_J(\psi)| =: I + II + III. \tag{5.14}$$

For any $x \in S$ we have

$$I(x) = |v(x_{|J|}) - v(x)| \le |\nabla v(x_\eta)| \cdot |J|, \tag{5.15}$$

where $\eta = \eta(x) \in (0, |J|)$, so that $x_\eta := x + \eta\vec{e}_d \in O$, and $d(x_\eta, S_{-y}) \ge c_1 y$ (see Subsection 10.3.1). Therefore (see (10.10) and (3.7)),

$$|\nabla v(x_\eta)| \le c_2\frac{v(x_\eta)}{y} \le c_2\frac{v(x)}{y} = c_3\frac{\psi(x)}{y}. \tag{5.16}$$

First, we deal with II. For this, we note that for $\theta \in J$ and $x \in S$ we have

$$|C_\theta(\psi)(x)| \le |\nabla P^O(\psi)(x_\theta)| = |\nabla v(x_\theta)| \le c_3\frac{v_\theta(x)}{y} = c_3\frac{P_\theta(\psi)(x)}{y}.$$

This means that

$$II \le \int_J P_\theta(|C_\theta(\psi)|)\,d\theta \le \frac{c_3}{y}\int_J P_{2\theta}(\psi)\,d\theta = \frac{c_3}{y}\int_J v_{2\theta}\,d\theta \le \frac{c_4}{y}v|J|. \tag{5.17}$$

Finally, (5.7) implies that for $x \in S$ we have

$$\begin{aligned} III(x) &\le c_5\int_S \frac{|J|^2}{m(J)^2}p_{m(J)}(x,\xi)\psi(\xi)\,ds(\xi) \\ &\le c_5\frac{|\Delta|\,|J|}{(m(\Delta))^2}v(x_{m(J)}) \le c_6\psi(x)\frac{|\Delta|\,|J|^2}{y(m(\Delta))^2} \end{aligned} \tag{5.18}$$

(we recall that $y \in (0,1)$). Relations (5.14)–(5.18) imply that

$$(1-\rho_J)\psi \le \Omega_J(\psi) \le (1+\rho_j)\psi, \tag{5.19}$$

where $0 < \rho_j \le c_7\frac{|J|}{y}\big(1 + \frac{|\Delta|\,|J|}{(m(\Delta))^2}\big)$.

Now we decompose Δ into K nonoverlapping segments $J_1, J_2, \dots, J_k$ in such a way that $\Delta = \bigcup_{k=1}^K J_k$, $|J_k| = \frac{|\Delta|}{K}$, $m(J_k) < m(J_{k+1})$, $k = 1, \dots, K-1$. Let $K = K(\Delta, y)$ be so large that

$$\rho_{J_k}(:= \rho_k) \le 2c_7\frac{|\Delta|}{Ky} =: \sigma_K < \frac{1}{2}, \quad k = 1, \dots, K.$$

Then, by (5.19) and (4.4),

$$\begin{aligned} \Omega_\Delta &\le \Omega_{J_K}\Omega_{J_{K-1}}\dots\Omega_{J_2}((1+\sigma_K)\psi) \le \dots \le (1+\sigma_K)^K\psi \\ &= \Big(1 + 2c_7\frac{|\Delta|}{yK}\Big)^K\psi < e^{2c_7\frac{|\Delta|}{y}}\psi < \Big(1 + c_8\frac{|\Delta|}{y}\Big)\psi \end{aligned} \tag{5.20}$$

(we recall that $\frac{|\Delta|}{y} \le 1$, and the kernel of Ω_{J_k} is positive). Next, we have

$$\Omega_\Delta(\psi) \ge (1-\sigma_K)^K \psi \ge e^{-c_9 \frac{|\Delta|}{y}} \psi \ge \Big(1 - c_{10}\frac{|\Delta|}{y}\Big)\psi. \tag{5.21}$$

Now (5.20) and (5.21) imply (5.13). □

5.3.1. Here we mention another version of the Φ-property of the operators Ω_Δ, which can be applied to $\psi \in C(\bar{S})$ (not necessarily positive).

Let $0 < \varepsilon < \varepsilon(S)$, *and let* $\Delta \in \mathrm{segm}_+$ *be a short segment. Then*

$$\|\psi - \Omega_\Delta(\psi)\|_\infty \le \|\psi - P_{|\Delta|}(\psi)\|_\infty + c(S)\frac{|\Delta|}{m(\Delta)}\|\psi\|_\infty. \tag{5.22}$$

The proof is deduced from (5.19) (with Δ in place of J). We only need to estimate II in a different way:

$$|II| \le \int_\Delta |B_\theta|(\psi)\,d\theta \le c(S)\int_\Delta \frac{P_\theta(|\psi|)}{\theta}\,d\theta \le c(S)\frac{|\Delta|}{m(\Delta)}\|\psi\|_\infty.$$

In (5.18) we only use the first inequality, replacing ψ by $|\psi|$ and taking into account that $\|P_{m(J)}(|\psi|)\|_\infty \le \|\psi\|_\infty$, $\frac{|J|}{m(J)} \le 1$.

5.3.2. As an immediate corollary to Lemma 5.1 we get statement (i) from Subsection 4.3. To prove this, first we put $\Delta := [\eta, y]$.

Due to Lemma 5.1, letting $\psi = \varphi_y$, we obtain

$$\Omega_\Delta(\varphi_y) \le (1 + c(S))\varphi_y.$$

By assumption, we have $[\eta, 1] = \Delta \cup [y, 1]$, so that

$$\Omega_\eta(\varphi_y) = \Omega_y(\Omega_\Delta(\varphi_y)) \le (1 + c(S))\Omega_y(\varphi_y).$$

5.4. Weak convergence of the measures $\gamma_y\,ds$**.** Here we show that as $y \downarrow 0$ *the measures* $\gamma_y s$ *converge weakly on* $\bar{S}$ *to some measure* $\nu\ (= \nu_\varepsilon = \nu_{\varepsilon,u})$ *supported on* S *and such that* $\nu(S) = 1$ (we recall that $\gamma_y(\infty) = 0$). The functions γ_y are defined by (4.2) in Subsection 4.5 ($\omega_y := \omega_{[y,1]}$, $0 < y < 1$, Ω_y is the integral operator with the kernel ω_y, and Ω_1 is the identity operator).

Proof. First we find a monotone decreasing sequence $(y_k)_{k\in\mathbb{N}}$ in $(0,1)$ such that

$$\lim_{k\to\infty} y_k = 0$$

and $\gamma_{y_k}\,ds$ converges weakly on $\bar{S}$ to some measure:

$$\lim_{k\to\infty}\int_S \alpha\gamma_{y_K}\,ds = \int_{\bar{S}} \alpha\,d\nu, \tag{5.23}$$

for any $\alpha \in C(\bar{S})$ (we recall that $\gamma_y \in C(\bar{S})$, $\gamma_y(\infty) = 0$). Let us verify that $\nu(\{\infty\}) = 0$, so that $\nu(S) = 1$, and we can replace $\bar{S}$ by S in the last integral. Consider a ball $\mathbb{B}_L = \mathbb{B}^d(0, L)$ (with L large) such that S is flat outside $\mathbb{B}_L$: $S \setminus \mathbb{B}_L \subset \mathbb{R}^{d-1}$ (see Subsection 4.1). By β^L and δ^L we denote the harmonic measures of $\mathbb{B}_L \cap S$ and $S \setminus \mathbb{B}_L$ in O, so that $\beta^L + \delta^L = 1$ in O. The function β^L is harmonic on O and vanishes on $S \setminus \bar{\mathbb{B}}_L$, therefore it admits a harmonic extension to the larger domain $O \cup (\mathbb{R}^d \setminus \bar{\mathbb{B}}_L)$. This function is also bounded, so it vanishes at infinity. We choose a sufficiently large $\rho > 0$ so that $\{x_d = \rho\} \subset O$, and let $L' > L$ be such that $(\delta^L)_\rho(x)\ (= \delta^L(x + \rho\vec{e}_d)) > \frac{1}{2}$ for any $x \in S \setminus \bar{\mathbb{B}}_{L'} = \mathbb{R}^{d-1} \setminus \bar{\mathbb{B}}_{L'}$. Putting $\delta^L(\infty) = 1$, we may assume that $(\delta^L)_\rho|_{\bar{S}} \in C(\bar{S})$. Next, we have

$$\begin{aligned}\nu(\{\infty\}) \le \nu(\bar{S} \setminus \mathbb{B}_{L'}) \le 2 \int_{\bar{S}} (\delta^L)_\rho \, d\nu = \lim_{k\to\infty} 2 \int_S (\delta^L)_\rho \gamma_{y_k} \, ds \\ = \lim_{k\to\infty} \int_S \Omega_{y_k}((\delta^L)_\rho) \, d\kappa \le \frac{c(S)}{\rho} \int_S \delta^L(x + \rho \vec{e}_d) \, d\kappa(x)\end{aligned}$$

(we have used the Φ-property of Ω_{y_k} and the fact that $(\delta^L)_\rho$ is harmonic on $O_{-\rho}$ for large k). Harmonic measures of $S \setminus \mathbb{B}_L$ tend to zero in O as $L \to +\infty$ (in particular, on S_ρ). At the same time, $0 \le \delta^l \le 1$ in O and $\kappa(S) = 1$, so that the last integral vanishes as $L \to +\infty$. We have proved that $\nu(\{\infty\}) = 0$.

It remains to show that

$$\lim_{y \downarrow 0} \int_S \alpha \gamma_y \, ds = \int_S \alpha \, d\nu$$

for any $\alpha \in C(\bar{S})$. First, assume that α coincides on S with $P_\sigma(\beta)$, for some $\beta \in C(\bar{S})$, $\beta \ge 0$ and $\sigma > 0$. Let $y \in (y_{k+1}, y_k)$. Using Lemma 5.1, the multiplicative property of ω_Δ, and the fact that $\Omega_y(1) = 1$, $\kappa(S) = 1$, we obtain

$$\begin{aligned}\left| \int_S \alpha \gamma_y \, ds - \int_S \alpha \gamma_{y_k} \, ds \right| = \left| \int_S \Omega_{y_k} \left(\Omega_{[y, y_k]}(\alpha) - \alpha \right) \, d\kappa \right| \\ \le \|\Omega_{[y_k, y]}(\alpha) - \alpha\|_{\infty, S} \le c(S) \frac{y_k}{\sigma} \|\alpha\|_{\infty, S},\end{aligned}$$

arriving at the desired conclusion because $y_k \to 0$ as $y \to 0$. It remains to note that for any $\alpha \in C(\bar{S})$ we have $\|P_\sigma(\alpha) - \alpha\|_{\infty, S} \to 0$ as $\sigma \downarrow 0$. ☐

§6. Deduction of the differential equation (ii) for the operator-valued functions $y \mapsto \Omega_y$

In this section, still taking for granted the existence of ω_Δ and the properties outlined in Subsection 5.1, we deduce equations (ii), which have already been used in Subsection 4.4.

6.1. Let $\varphi \in \mathcal{H}_+(\bar{O})$ (see Subsection 4.3) be such that $\lim_\infty \varphi = 0$. We observe that φ is bounded in O_y, $y > 0$. Let

$$f^x(y) := \Omega_y(\varphi_y)(x), \quad y \in (0, 1], \; x \in S,$$

where, as before, $\Omega_y := \Omega_{[y,1]}$ for $y \in (0, 1)$, and Ω_1 is the identity operator. To compute the derivative of f^x, we start (in Subsection 6.2) with proving that it is Lipschitz on any segment $[y_0, 1]$, $0 < y_0 < 1$. After that (Subsection 6.3) we compute the *left* derivative $(f^x)'_-$. We shall see that it exists everywhere on $(0, 1]$ and is continuous (the right derivative $(f^x)'_+$ is much harder to deal with, because the kernels (4.3) may fail to commute). The Lipschitz property of $f|_{[y_0,1]}$, $y_0 \in (0, 1]$, implies that

$$f^x(y) = f^x(1) - \int_y^1 (f^x)'_-(\eta) \, d\eta, \quad y \in (0, 1].$$

Therefore, we have $(f^x)'_-(y) = (f^x)'_+$, $y \in (0, 1]$, and consequently, $f^x \in C^1((0, 1])$.

6.2. Now we show that $f^x|_{[y_0,1]}$ is Lipschitz for $x \in S, y_0 \in (0,1]$. Suppose $y \in (0,1]$, $h > 0, y - h \geq y_0$, $\Delta := [y-h, 1]$. Using (4.4), we get

$$f^x(y) - f^x(y-h) = (\Omega_y\,(I + II))\,(x) + III(x), \tag{6.1}$$

where $I := \varphi_y - \Omega_\Delta(\varphi_y)$, $II := \varphi_y - \varphi_{y-h}$, and $III := (\Omega_{y-h} - \Omega_y)(II)$. Setting $\|\varphi_y\| := \sup_S \varphi_y$, $K := \sup_{y_0 \leq y \leq 1} \|\varphi_y\|$, we see that the Φ-property of ω_Δ (Lemma 5.1) implies

$$\|I\| \leq c_1(S) K \frac{h}{y_0}$$

(we recall that the norm of the operator Ω_y from $C([y,1])$ to $C([y,1])$ does not exceed one). For any $x \in S$ there exists $\theta = \theta(x) \in (y-h, y)$ such that

$$|II|(x) \leq |\nabla\varphi(x_\theta)| h \leq c_2(S) K \frac{h}{y_0}$$

(see (10.10)). Hence,

$$\|\Omega_y(I + II)\| \leq \|I\| + \|II\| \leq c_3(S) K \frac{h}{y_0}.$$

Finally,

$$\|III\| \leq 2\|II\| \leq 2c_3(S) K \frac{h}{y_0},$$

and (see (6.1))

$$|f^x(y) - f^x(y-h)| \leq c_4(S) K \frac{h}{y_0}, \quad x \in S,\ O < y_0 \leq y - h < y \leq 1,$$

which proves that the functions in question are Lipschitz.

6.3. Computing the derivative $(f^x)'$**.** From (6.1) it follows that

$$\frac{f^x(y-h) - f^x(y)}{-h} = \Omega_y(A_1)(x) + A_2(x), \tag{6.2}$$

where $x \in S$, $y \in (0,1)$, $h \in (0, \frac{y}{2})$, $A_1 := \frac{I+II}{h}$, $A_2 := \frac{III}{h}$ (I, II, III are the same as in (6.1)). We verify that $\lim_{h\downarrow 0} A_2 = 0$ on S. Put $\alpha_h := \Omega_{y-h} - \Omega_y = \Omega_y - \Omega_y\Omega_\Delta$, where $\varphi^{(d)} := \frac{\partial\varphi}{\partial\vec{e}_d}$. We have

$$\frac{III}{h} = \alpha_h(IV + V),$$

where $IV := (\varphi^{(d)})_y$, and $V := \frac{\varphi_y - \varphi_{y-h}}{h} - IV$. Next, $\alpha_h(IV) = \Omega_y(IV - \Omega_\Delta(IV))$. Clearly, $IV \in C(\bar{S})$ (see (10.8) and (10.10), so that $|\varphi^{(d)}(S_y)| \leq \frac{c(\varphi|_{S_y})}{y}$, $\lim_\infty \varphi^{(d)}|_{S_y} = 0$), and, by Subsection 5.3, $IV - \Omega_\Delta(IV) \to 0$ as $h \downarrow 0$ uniformly on S, whence $\alpha_h(IV) \to 0$ as $h \downarrow 0$ on S (recall that the kernels ω_y, ω_{y-h} are positive, while $\Omega_y(1) = \Omega_{y-h}(1) = 1$); the norm of $\|\alpha_h\|$ as an operator from $C(\bar{S})$ to $L^\infty(S)$ does not exceed 2. Now we claim that $V \to 0$ uniformly on S as $h \downarrow 0$. Indeed, since $\varphi \in \mathcal{H}_+(\bar{O})$, φ is bounded on $O_{\frac{y}{2}}$; therefore, its second derivatives are bounded on any ball of radius $\frac{y}{4}$ with the center on $S + y\vec{e}_d$ by some constant depending only on y, so that $|V| \leq C(y)h$ everywhere on S, and $|\alpha_h(V)| \leq 2C(y)h$. It follows that $\lim_{h\downarrow 0} A_2(x) = 0$ for any $x \in S$. It remains to prove that on S we have

$$\lim_{h\downarrow 0} \frac{I + II}{h} = \varepsilon B_y(\varphi_y). \tag{6.3}$$

Indeed, formula (6.2) implies that $\lim_{h\downarrow 0} \Omega_y(A_1) = \varepsilon\Omega_y(B_y(\varphi_y))$ pointwise on S (due to the Lebesgue dominated convergence theorem: we have $\omega_y > 0$, $\Omega_y(1) = 1$, and $\frac{I+II}{h}$ is uniformly bounded on S for $h \in (\frac{y}{2}, y)$, which stems from the estimates of Subsection 6.2).

Now we check (6.3). We start with the relation $P_h\varphi_y = \varphi_{y+h}$, which is immediate because $\varphi \in \mathcal{H}_+(O))$. It follows that $\lim_{h\downarrow 0} \frac{P_h(\varphi_y)-\varphi_y}{h} = \lim_{h\downarrow 0} \frac{\varphi_y-\varphi_{y-h}}{h}$, whence

$$\lim_{h\downarrow 0} \frac{I+II}{h} = \lim_{h\to 0} \left(\frac{\varepsilon}{h}\int_{y-h}^{y} B_\theta(\varphi_y)\,d\theta + \frac{1}{h}R_\Delta(\varphi_y)\right) = \varepsilon B_y(\varphi_y) + \lim_{h\downarrow 0}\frac{1}{h}R_\Delta(\varphi_y) \tag{6.4}$$

(we have used the fact that $\theta \mapsto B_\theta(\varphi_y)(x)$ is continuous on Δ for any $x \in S$, see Subsection 10.4.3). To obtain (6.3) it suffices to prove that the last limit in (6.4) is zero. This follows directly from (5.7), because

$$|R_\Delta(\varphi_y)| \le c(S)\frac{\varepsilon^2 h^2}{y^2}\sup_S |\varphi_y|$$

everywhere on S.

§7. Properties (b) and (c) of the measures ν_ε

In Subsection 4.5 we checked statement (a) from Subsection 3.4, proving, thereby, the *existence* of B-points of u. Here we prove properties (b) and (c) of the measures ν_ε (see Subsection 3.4). The results of the present section will imply the *ultradensity* of the set of B-points, and to finish the proof of Theorem 3 it will remain to construct the kernels ω_Δ described in Subsection 5.1.1. This is done in §8.

From now on we assume that $\varepsilon \in (0, \varepsilon(S))$, so that the kernel ω_Δ is *positive* (see 5.1).

So far, we have not paid close attention to the parameter ε involved in the definition of $\omega_\Delta = \omega_{\Delta,\varepsilon,u}$. It is now when it comes to the fore.

7.1. Positivity of $\nu_\varepsilon(\mathbb{B}(\xi, r))$. Let $\mathbb{B} = \mathbb{B}(\xi, r)$ be a ball of radius $r < 1$ with center $\xi \in S$.

Lemma 7.1. *There exists a positive number $\varepsilon(\mathbb{B})$ such that for any $\varepsilon \in (0, \varepsilon(\mathbb{B}))$ we have $\nu_\varepsilon(\mathbb{B}) > c$, where $c = c(S, \mathbb{B}, \kappa) > 0$, κ is a probability measure on S (see Subsection 4.1).*

Proof. Let $\varphi\,(= \varphi_{\mathbb{B}})$ be a function on S that coincides on S with a function $\psi\,(= \psi_{\mathbb{B}})$ such that $0 \le \psi \le 1$, $\psi \equiv 1$ on $\frac{1}{2}\mathbb{B}$, and $|\nabla\psi| \le \frac{2}{r}$. As usual, we let

$$\varphi_\theta(x) := P^O(\varphi)(x_\theta) = P^O(\varphi)(x + \theta\vec{e}_d), \quad 0 < \theta,\ \ x \in S.$$

It suffices to show that for some $\varepsilon(S, \mathbb{B}) > 0$ we have

$$\int_S \varphi_y\,d\nu_\varepsilon \ge c(S, \mathbb{B}) > 0 \tag{7.1}$$

whenever $0 < \varepsilon < \varepsilon(S, \mathbb{B})$. Recalling the definition of ν_ε (see Subsection 3.4), we notice that (7.1) follows from

$$\int_S \varphi_y \Omega^*_\delta(\kappa)\,ds \ge c(S, \mathbb{B}, \kappa) > 0, \quad \text{where} \quad 0 < \delta < y < r \tag{7.2}$$

(here κ is some probability measure on S; note that $\varphi_y \in C(\bar{S})$, because $\lim_\infty P^O(\varphi) = 0$ and, therefore, $\lim_{x\to\infty}\varphi_y(x) = 0$). Estimate (7.2) is equivalent to

$$\int_S \Omega_{[\delta,y]}(\varphi_y)\Omega^*_y(\kappa)\,ds \ge c(S, \mathbb{B}) > 0,$$

which, by the Φ-property of Ω_Δ (see Lemma 5.1), follows from

$$\int_S \varphi_y \Omega^*_y(\kappa)\,ds \left(= \int_S \Omega_y(\varphi_y)\,d\kappa\right) \ge c_1(S, \mathbb{B}) > 0 \tag{7.3}$$

(observe that on S, φ_y coincides with the function v_y, where $v := P^O(\psi)$ lies in $\mathcal{H}_+(\bar{O})$; this allows us to use Lemma 5.1). If $y \in (0, r)$, then

$$\Omega_y(\varphi_y) = \Omega_r(\varphi_r) - \int_y^r \frac{d}{d\theta}(\Omega_\theta(\varphi_\theta))\, d\theta = \Omega_r(\varphi_r) - \varepsilon \int_y^r \Omega_\theta B_\theta(\varphi_\theta)\, d\theta = I - II \tag{7.4}$$

(we have used the differential equation (ii), see §6).

Now we estimate I from below. By (5.12), we have

$$I \geq c_2(S) r^\varepsilon P_{1-r}(\varphi_r) = c_2(S) r^\varepsilon \varphi_1, \tag{7.5}$$

so that

$$\int_S I\, d\kappa \geq c_3 r^\varepsilon,$$

where $c_3 = (c_3(S, \mathbb{B}, \kappa)) := c_2(S) \int_S \varphi_1\, d\kappa > 0$.

Estimating II from above, we have

$$|II| \leq \varepsilon \int_y^r \Omega_\theta(|B_\theta(\varphi_\theta)|)\, d\theta \leq \varepsilon \sup_{S, \theta \in (0,r)} |B_{\theta(\varphi_\theta)}| r. \tag{7.6}$$

On the other hand,

$$|B_\theta(\varphi_\theta)| \leq c_4(S) |P_\theta(|\nabla v_\theta|)| \leq c_4(S) \sup_O |\nabla v| =: \frac{1}{r} c_5(S, \mathbb{B}) \varepsilon \tag{7.7}$$

(see the definition of b_y in Subsection 3.3.3 and the first inequality in (3.10)). Now (7.4)–(7.7) imply (7.3):

$$\int_S \Omega_y(\varphi_y)\, d\kappa \geq c_3 r^\varepsilon - c_5 \varepsilon \geq \frac{c_3}{2}$$

for $0 < \varepsilon < \varepsilon(S, \mathbb{B}, \kappa)$. □

7.2. Here we prove property (b) of the measure ν_ε, see Subsection 3.4. For this, we need the set $\mathfrak{B}$ $(= \mathfrak{B}(S, r))$ of all open balls of radius r with center on S and the family $(\psi)_{\mathbb{B} \in \mathfrak{B}}$ of functions continuous on $\mathbb{R}^d$, harmonic on O, and such that $0 \leq \psi_\mathbb{B} \leq 1$, $\psi_\mathbb{B} \equiv 0$ on $S \setminus 2\mathbb{B}$ (here by $\lambda\mathbb{B}$ we denote the open ball of radius λr with the same center as $\mathbb{B}$), and $\psi_\mathbb{B} > q$ on $\frac{1}{2}\mathbb{B}$, where $\mathbb{B} \in \mathfrak{B}$, $r < r(S)$, and $q \in (0, 1)$ depends only on S but not on $\mathbb{B}$. The existence of such functions will be proved in Subsection 7.3.

Note that $p_{\frac{r}{8}} (:= p + \frac{1}{2}\frac{r}{4}\vec{e}_d) \in \frac{1}{2}\mathbb{B}$, $p \in \frac{1}{8}\mathbb{B}$. Hence,

$$\nu_\varepsilon\Big(\frac{1}{4}\mathbb{B}\Big) \leq q^{-1} \int_{\frac{1}{4}\mathbb{B} \cap S} \psi_{\frac{r}{8}}\, d\nu_\varepsilon \leq q^{-1} \int_S \psi_{\frac{r}{8}}\, d\nu_\varepsilon$$

(here $\psi := \psi_\mathbb{B}$; we recall that $\psi_y(p) := \psi(p_y)$, $y > 0$). The last integral is equal to $\lim_{y \downarrow 0} j_y$, where

$$j_y := \int_S \psi_{\frac{r}{8}} \Omega_y^*(\kappa)\, ds = \int_S \Omega_{[y, \frac{r}{8}]}(\psi_{\frac{r}{8}}) \Omega^*_{[\frac{r}{8}, 1]}(\kappa)\, ds \leq \int_S c_1(S) \psi_{\frac{r}{8}} \Omega^*_{[\frac{r}{8}, 1]}(\kappa)\, ds$$

(we assume that $r < r(S) \leq 1$, and $0 < y < \frac{r}{8}$; in the last estimate we have used the identity $\omega^*_{[y,1]} = \omega^*_{[y, \frac{r}{8}]} \circ \omega_{[\frac{r}{8}, 1]}$ and Lemma 5.1). Applying (5.11) to $\omega_{[\frac{r}{8}, 1]}$, we obtain

$$\begin{aligned} j_y &\leq c_2(S) \int_S P^O_{1 - \frac{r}{8}}(\psi_{\frac{r}{8}}) \cdot \Big(\frac{8}{r}\Big)^{c_r \varepsilon} d\kappa = c_s(S, \varepsilon) r^{-c_4 \varepsilon} \int_S P^O_1(\psi)\, d\kappa \\ &= c_3 r^{-c_4 \varepsilon} \int_S \psi(\xi) \Big(\int_S p^O_1(x, \xi)\, d\kappa(x)\Big)\, ds(\xi) \leq c_3 r^{-c_4 \varepsilon} \sup_{S \times S} p_1 \cdot \int_{S \cap 2\mathbb{B}} \psi(\xi)\, ds(\xi) \\ &\leq c_5 r^{d - 1 - c_4 \varepsilon} \end{aligned}$$

(we recall that κ is a probability measure, $0 \le \psi \le 1$, and $\sup_{S\times S} p_1^O < +\infty$, see Subsection 10.5.1). It follows that

$$\nu_\varepsilon(\mathbb{B}) \le c_5\,(4r)^{d-1-c_4\varepsilon} \le c_6 r^{d-1-c_4\varepsilon},$$

where c_j depends only on S, and $c_4\varepsilon < d-1$, $r < r(S) < 1$. Finally, if $r \ge r(S)$, then

$$\frac{\nu_\varepsilon(\mathbb{B})}{r^{d-1-c_4\varepsilon}} \le \frac{1}{(r(S))^{d-1-c_4\varepsilon}}.$$

7.3. The functions $\psi_{\mathbb{B}}$. We start with some additional notation.

Let $\mathbb{B}$ be an open ball with center c and radius $r > 0$, and g a function on $\mathbb{R}^d$. Let

$$(g)_{\mathbb{B}}(p) := g\left(\frac{p-c}{r}\right), \quad p \in \mathbb{R}^d.$$

By $\lambda\mathbb{B}$ (where $\lambda > 0$) we denote the open ball with center c and radius λr.

Given a set $A \subset \mathbb{R}^d$, by $A_+ := A \cap O$ we denote its "upper part".

Let U be a domain in $\mathbb{R}^d$, bU its boundary, $\bar{U} := U \cup bU$ its closure, and f a function continuous on bU. By $\mathcal{P}^U(f)$ we denote a function in $C(\bar{U})$ that is harmonic on U and coincides with f on bU.

By $\mathbb{B}_1$ we denote the unit ball in $\mathbb{R}^d$.

Now we define the functions $\psi_{\mathbb{B}}$ mentioned in Subsection 7.2. For this, we choose a function $\varphi \in C^\infty(\mathbb{R}^d)$ such that $0 \le \varphi \le 1$, $\varphi(\mathbb{R}^d \setminus 2\mathbb{B}_1) = 0$, $\varphi|_{(\frac{3}{2}\mathbb{B}_1)} = 1$, and $|\nabla\varphi| \le 3$, and let

$$\psi_{\mathbb{B}} = \begin{cases} \mathcal{P}^O((\varphi)_{\mathbb{B}}) & \text{on } \ O \\ \varphi_{\mathbb{B}} & \text{on } \ \mathbb{R}^d \setminus O. \end{cases} \tag{7.8}$$

Clearly, $\psi_{\mathbb{B}} \in C(\mathbb{R}^d)$, $0 \le \psi \le 1$, ψ is harmonic on O, $\psi_{\mathbb{B}}|_{(S\cap\frac{3}{2}\mathbb{B})} = 1$, and $\psi_{\mathbb{B}}|_{(S\setminus 2\mathbb{B})} = 0$. It remains to show that

$$\psi_{\mathbb{B}} \ge q \ \text{ in } \ \frac{1}{2}\mathbb{B}_+ \tag{7.9}$$

for any ball $\mathbb{B}$ in $\mathfrak{B}(S,r)$, and $q \in (0,1)$ *may depend on S but not on* $\mathbb{B}$.

7.3.1. Let $\mathbb{B}\,(=\mathbb{B}(x,r)) \in \mathfrak{B}(S,r)$, $x \in S$, $r > 0$. Put $(b\mathbb{B})_- = b\mathbb{B}\setminus O$. From the maximum principle it follows that

$$\psi_{\mathbb{B}}(x') \ge \delta_{x'}^{\mathbb{B}_+}(\mathbb{B}\cap S) \ge \delta_{x'}^{\mathbb{B}}((b\mathbb{B})_-),$$

where $x' \in \left(\frac{1}{2}\mathbb{B}\right)_+$, whereas $\delta_{x'}^{\mathbb{B}_+}$ and $\delta_{x'}^{\mathbb{B}}$ are harmonic measures in $\mathbb{B}_+$ and $\mathbb{B}$ with the pole at x'. The inequality $|\nabla\Phi| < \frac{1}{100}$ implies (see Subsection 2.1, we recall that S is the graph of Φ) that there exists a positive constant $c_1 = c_1(S)$ such that

$$b\mathbb{B} \cap (c_1\mathbb{B} - r\vec{e}_d) \subset (b\mathbb{B})_-,$$

and therefore, the surface area of the part $(b\mathbb{B})_-$ of the sphere is bounded from below by $c_2(S)r^{d-1}$. It remains to note that

$$p^{\mathbb{B}}(x',\zeta) \ge c_3(d), \quad x' \in \frac{1}{2}\mathbb{B}, \ \zeta \in b\mathbb{B}$$

(see Subsection 10.1), whence we see that $\delta_{x'}^{\mathbb{B}}((b\mathbb{B})_-) \ge q > 0$ with q depending only on S.

§8. The kernels ω_Δ: existence, properties

8.1. Let $\mu \subset \mathrm{segm}_+$ be a finite set of nonoverlapping intervals such that $\Delta = \bigcup_{j\in\mu} j$. We call such a set a *partition* of $\Delta \in \mathrm{segm}_+$. Sometimes we understand μ as a *family* $(j_k)_{k=1}^K$ of segments with positive (and nondecreasing) left endpoints: $0 < m(\Delta) = m(j_1) < m(j_2) < \cdots < m(j_K) < M(j_K) = M(\Delta)$. The number $\lambda(\mu) := \max_{j\in\mu} |j|$ is called the *mesh* of the partition μ; we write $\mu_2 \succ \mu_1$ if any element of μ_2 lies in some element of μ_1. Given a partition μ of Δ, we define the kernel Π^μ by

$$\Pi^\mu := \widetilde{\omega}_{j_K} \circ \widetilde{\omega}_{j_{K-1}} \circ \dots \widetilde{\omega}_{j_1}$$

(see the related definitions in Subsection 5.1). The kernel ω_Δ is defined (in Subsection 8.5) as the limit of the sequence of kernels $(\Pi^{\mu_n})_{n=1}^\infty$, where the μ_n form some sequence of partitions of Δ, with $\lambda(\mu_n) \to 0$. Subsections 8.2–8.4 are devoted to some preparatory work.

8.2. By N_q we denote the set of subsets of $\{1, 2, \dots, K\}$ of cardinality q. We start with the following identity:

$$\Pi^\mu = (p_{|j_1|} - \varepsilon b_{j_1}) \circ (p_{|j_2|} - \varepsilon b_{j_2}) \circ \dots \circ (p_{|j_K|} - \varepsilon b_{j_K}) = p_{|\Delta|} + \sum_{q=1}^K \sum_{l\in N_q} \pi_l, \tag{8.1}$$

where $\pi_l := r_K^l \circ r_{K-1}^l \circ \dots \circ r_1^l$,

$$r_q^l = \begin{cases} -\varepsilon b_{j_q} & \text{if } q \in l, \\ p_{|j_q|} & \text{if } q \notin l. \end{cases}$$

Consider the sum in (8.1) that corresponds to $q = 1$, taking into account that $b_\Delta = \sum_{q=1}^K b_{j_q}$, while

$$\sum_{l\in N_1} \pi_l = -\varepsilon \sum_{q=1}^K p_{|j_q^+|} \circ b_{j_q} \circ p_{|j_q^-|},$$

where the $j_q^\pm$ are the segments $[M(j_q), M(\Delta)]$ or $[m(\Delta), m(j_q)]$, respectively (if one of them degenerates to a point, then $p_{|j_q^\pm|}$ is understood as the composition identity). Formula (8.1) shows that

$$\Pi^\mu = \widetilde{\omega}_\Delta + \varepsilon \sum_{k=1}^K w_k + \rho_\mu, \tag{8.2}$$

where $w_k := b_{j_k} - p_{|j_k^+|} \circ b_{j_k} \circ p_{|j_k^-|}$, $\rho_\mu := \sum_{q=2}^K \sum_{l\in N_q} \pi_l$.

8.3. Estimating $\Pi^\mu - \widetilde{\omega}_\Delta$. First we estimate the kernel w_k (Subsection 8.3.1) and ρ_μ (Subsection 8.3.2).

8.3.1. *The kernels w_j.* Note that for any $\theta, \lambda > 0$ we have

$$|b_\theta| + |c_\theta| \le c(S) \frac{p_\theta}{\theta} \tag{8.3a}$$

$$|p_{\theta+\lambda} - p_\theta| \le c(S, \Lambda) \frac{\lambda}{\theta} p_\theta, \tag{8.3b}$$

whenever $\theta + \lambda < \Lambda$; the function $x \mapsto c(S, x)$ is monotone increasing on $(0, +\infty)$. Now (8.3a) follows from (3.10) and (3.16). Next we prove (8.3b):

$$|p_{\theta+\lambda} - p_\theta| \le \int_0^\lambda \left| \frac{d}{dt}(p_{\theta+t}) \right| dt \le c(S) \int_0^\lambda \frac{p_{\theta+t}}{\theta + t} dt \le c^1 \int_0^\lambda \Big(\frac{p_\theta}{\theta + t} \Big) \Big(1 + \frac{t}{\theta} \Big) dt = c^1 \frac{\lambda}{\theta} p_\theta,$$

$c^1 = c^1(S,\lambda)$, and the function $x \mapsto c^1(S,x)$ is monotone increasing on $(0,+\infty)$ (we have used (3.8)).

Suppose $j \in \mathrm{segm}_+$, $j \subset \Delta(\in \mathrm{segm}_+)$. By $L(j)$ we denote $\frac{|j|}{m(j)}$ (note that $L(j)$ is a monotone nondecreasing function of the segment j). Put $w_j := b_j - p_{|j^+|} \circ b_j \circ p_{|j^-|}$. We have

$$|w_j| \le I + II, \quad I := |b_j - p_{|j^+|} \circ b_j|, \quad II := |p_{|j^+|} \circ b_j - p_{|j^+|} \circ b_j \circ p_{|j^-|}|. \tag{8.4}$$

Using (8.3b) and (3.10), we obtain

$$\begin{aligned} I &\le \int_j |p_\theta c_\theta - p_{|j^+|+\theta} \circ c_\theta|\, d\theta \le c(S,\Delta)|j^+| \int_j \frac{p_\theta}{\theta} \circ \frac{p_\theta}{\theta}\, d\theta \\ &= c(S,\Delta)|j^+| \int_j \frac{p_{2\theta}}{\theta^2}\, d\theta \le c'(S,\Delta)|j^+| \int_j \frac{p_{m(\Delta)}}{\theta^2} \cdot \frac{2\theta}{m(\Delta)}\, d\theta \\ &\le L(\Delta) \frac{|j|}{m(\Delta)} p_{m(\Delta)}. \end{aligned} \tag{8.5}$$

We pass to estimating the kernel II. Observe that (see Subsection 3.3)

$$c_\theta = \frac{\partial^1 p_\theta}{\partial \vec{\sigma}_{2\theta}} = \frac{\partial^1 p_{\frac{\theta}{2}}}{\partial \sigma_{2\theta}} \circ p_{\frac{\theta}{2}}, \quad \theta > 0.$$

Hence,

$$\begin{aligned} II &\le \int_j |p_{|j^+|+\theta} \circ (c_\theta - c_\theta \circ p_{|j^-|})|\, d\theta = \int_j \left| p_{|j^+|+\theta} \circ \frac{\partial^1 p_{\frac{\theta}{2}}}{\partial \vec{\sigma}_{2\theta}} \circ (p_{\frac{\theta}{2}} - p_{\frac{\theta}{2}+|j^-|}) \right| d\theta \\ &\le c(S) \int_j \frac{1}{\theta} \frac{|j^-|}{\theta} p_{|j^+|+\frac{3}{2}\theta} \circ p_{\frac{\theta}{2}}\, d\theta \le c''(S) \int_j \frac{|j^-|}{\theta^2} \frac{|j^+|+2\theta}{m(\Delta)} p_{m(\Delta)}\, d\theta \\ &\le c''(S,\Delta) L(\Delta)(3L(\Delta)+2) \frac{|j|}{m(\Delta)} p_{m(\Delta)} \le 3c''(S,\Delta) \frac{M(\Delta)}{m(\Delta)} L(\Delta) \frac{|j|}{m(\Delta)} p_{m(\Delta)} \end{aligned} \tag{8.6}$$

(we recall that $L(\Delta) = \frac{M(\Delta)}{m(\Delta)} - 1$). From (8.5), (8.6), and (8.4) it follows that

$$|w_j| \le c(S,\Delta) \frac{M(\Delta)}{m(\Delta)} L(\Delta) \frac{|j|}{m(\Delta)} p_{m(\Delta)}, \tag{8.7}$$

and $c(S,\Delta)$ increases with Δ. Returning to the partition μ of the segment Δ (see (8.2)), we get

$$\sum_{k=1}^K |w_k| \le c(S,\Delta) \frac{M(\Delta)}{m(\Delta)} L(\Delta) \sum_{k=1}^K \frac{|j|_k}{m(\Delta)} p_m(\Delta) = c(S,\Delta) \frac{M(\Delta)}{m(\Delta)} (L(\Delta))^2 p_{m(\Delta)}. \tag{8.8}$$

8.3.2. *Estimating the kernel ρ_μ.* By (8.1), we see that $\rho_\mu = \sum_{q=2}^K \sum_{l \in N_q} \pi_l$, where $\pi_l = r_K^l \circ r_{K-1}^l \circ \cdots \circ r_1^l$. If $k \in l$, then $|r_k^l| = \varepsilon |b_{j_k}| \le c(S) \varepsilon L(j_k) p_{m(j_k)}$ (see (5.2)). Assume that μ is a *regular* partition, i.e., $\lambda(\mu) \le \frac{2|\Delta|}{K}$. Then, putting $\nu := 2c(S)\varepsilon L(\Delta)$, we obtain

$$|r_k^l| \le \frac{\nu}{K} p_{m(j_k)}, \quad k \in l.$$

On the other hand, if $k \notin l$, then $r_k^l = p_{|j_k|}$. This means that for $l \in N_q$ we have

$$|\pi_l| \le \nu^q K^{-q} p_{a(l)},$$

where $a(l) := \sum_{k\in l} m(j_k) + \sum_{k\in l} |j_k| \le sM(\Delta) + |\Delta|$. Since $\sharp N_q = C_K^q \le \frac{K^q}{q!}$, we arrive at

$$
\begin{aligned}
|\rho_\mu| &\le \sum_{q=2}^{K} \frac{K^q}{q!} \nu^q K^{-q} c(S) \frac{qM(\Delta)+|\Delta|}{m(\Delta)} p_{m(\Delta)} \\
&\le \nu^2 \sum_{q=2}^{\infty} \frac{\nu^{q-2}}{(q-1)!} c(S) \frac{M(\Delta)+|\Delta|}{m(\Delta)} p_{m(\Delta)} \qquad (8.9)\\
&\le (2L(\Delta)+1)\nu^2 c(S) e^{\nu} p_{m(\Delta)}.
\end{aligned}
$$

Now we are ready to estimate $\Pi^\mu - \widetilde{\omega}_\Delta$. From (8.2), (8.8), and (8.9) we deduce the inequality

$$
|\Pi^\mu - \widetilde{\omega}_\Delta| \le c(S, L(\Delta))(L(\Delta))^2 p_{m(\Delta)}, \qquad (8.10)
$$

where the function $x \mapsto c(S,x)$ is monotone increasing on $[0,+\infty)$ (we recall that $L(\Delta)$ increases with $\Delta \in \mathrm{segm}_+$). In (8.10) it is assumed that the partition μ of Δ is regular and that $\varepsilon \in (0,1)$.

An estimate for the kernel Π^μ also follows from (8.10):

$$
\begin{aligned}
|\Pi^\mu| &\le |\widetilde{\omega}(\Delta)| + c(S, L(\Delta))(L(\Delta))^2 p_{m(\Delta)} \\
&\le p_{|\Delta|} + \big(c(S)\varepsilon L(\Delta) + c(S, L(\Delta))(L(\Delta))^2 p_{m(\Delta)}\big) \qquad (8.11)\\
&= p_{|\Delta|} + A \cdot (L(\Delta))^2 p_{m(\Delta)},
\end{aligned}
$$

where $A = A(L(\Delta), S)$, and the function $x \mapsto A(x,S)$ is monotone increasing on $[0,+\infty)$.

8.4. Estimating the kernel $\Pi^\tau - \Pi^\sigma$, $\sigma \succ \tau$. This is the main estimate of this section, after which we shall complete the construction of the kernel ω_Δ easily.

Lemma 8.1. *Let τ be a partition of $\Delta \in \mathrm{segm}_+$, and let $\sigma \succ \tau$. Then*

$$
|\Pi^\tau - \Pi^\sigma| \le C(S,\Delta)\lambda(\tau) p_{m(\Delta)} \qquad (8.12)
$$

($\lambda(\tau)$ is the mesh of a partition, see Subsection 8.1*).*

Proof. Suppose $\tau = \{\Delta_1, \Delta_2, \dots, \Delta_K\}$, $m(\Delta_1) < m(\Delta_2) \cdots < m(\Delta_K)$. Put $\sigma_k := \{j \in \sigma : j \subset \Delta_k\}$; then σ_k is a partition of Δ_k, $\sigma = \bigcup_{k=1}^K \sigma_k$. For $i = 2, 3, \dots, K$, we denote by σ_i^- the part of σ that lies in Δ_i^-: $\sigma_i^- = \bigcup_{1\le q<i} \sigma_q$; $\sigma_1^- := \varnothing$. For $i = 1, \dots, K$, we denote by τ_i^+ the part of τ that lies in Δ_i^+ : $\tau_i^+ = \bigcup_{i<q\le K} \Delta_q$; $\tau_{K+1}^+ := \varnothing$. Finally, let $\tau(i) := \sigma_i^- \cup \{\Delta_i\} \cup \tau_i^+$, $1 \le i \le K$; $\tau(K+1) := \sigma$. Here the kernel $\Pi^{\tau(i)}$ is written as Π_i, $i = 1, \dots, K+1$. In particular, $\Pi_1 = \Pi^\tau$, $\Pi_{K+1} = \Pi^\sigma$, and

$$
\Pi^\tau - \Pi^\sigma = \sum_{i=1}^{K} (\Pi_i - \Pi_{i+1}). \qquad (8.13)
$$

If $i \ne 1$, K, then

$$
\Pi_i - \Pi_{i+1} = \Pi^{\tau_i^+} \circ (\widetilde{\omega}_i - \Pi^{\sigma_i}) \circ \Pi^{\sigma_i^-} \qquad (8.14)
$$

(here we write $\widetilde{\omega}_i$ instead of $\widetilde{\omega}_{\Delta_i}$). This is also true for $i = 1, K$ if $\Pi^\emptyset$ is understood as the convolution identity operator. Relations (8.14), (8.10), (8.11) and the inequality $|\Delta_i|^2 \le \lambda(\tau)|\Delta_i|$ imply that

$$
|\Pi_i - \Pi_{i+1}| \le (p_{|\Delta_i^+|} + Cp_{m_i^+}) \circ \lambda(\tau)|\Delta_i| p_{m(\Delta)} \circ (p_{|\Delta_i^-|} + Cp_{m_i^-}) \qquad (8.15)
$$

(here $C = C(\Delta, S)$; $m_i^\pm = m(\Delta_i^\pm)$). The right-hand side in (8.15) does not exceed

$$
\frac{A(M(\Delta)+|\Delta|)}{m(\Delta)} \lambda(\tau)|\Delta_i| p_{m(\Delta)} = A(1+2L(\Delta))\lambda(\tau)|\Delta_i| p_{m(\Delta)},
$$

where $a = A(\Delta, S)$. Therefore,

$$|\Pi^\tau - \Pi^\sigma| \le \sum_{i=1}^{K} |\Pi_i - \Pi_{i+1}| \le A(1 + 2L(\Delta))|\Delta|\lambda(\tau)p_{m(\tau)}. \qquad \square$$

8.5. Dyadic partitions of ω_Δ. Let $n \in \mathbb{Z}_+$, $\Delta \in \mathrm{segm}_+$. We denote by $\tau_n(\Delta)$ the partition of Δ that consists of all intersections of the form $\Delta \cap \left[\frac{s}{2^n}, \frac{s+1}{2^n}\right]$, $s \in \mathbb{Z}_+$. We say that such a partition is *dyadic of rank n*. Clearly this is a regular partition, $\tau_{n+1}(\Delta) \succ \tau_n(\Delta)$, and $\lambda(\tau_n(\Delta)) \le \frac{|\Delta|}{2^n}$. Put $\Pi_n := \Pi^{\tau_n(\Delta)}$. Lemma 8.1 implies that $|\Pi_n - \Pi_{n+1}| \le C(\Delta, S)\frac{\Delta}{2^n}p_{m(\Delta)}$, $n = 1, \dots$. Therefore, the series

$$\lim_{n\to\infty} \Pi_n \,(=: \omega_\Delta) = \Pi_1 + (\Pi_2 - \Pi_1) + (\Pi_3 - \Pi_2) + \dots \tag{8.16}$$

converges uniformly on $S \times S$ and gives rise to a kernel ω_Δ that satisfies

(1) $\omega_\Delta \in C(S \times S)$;

(2) $\omega_\Delta > 0$ if $\varepsilon \in (0, \varepsilon(S))$;

(3) $\int_S \omega_\Delta(x, \xi)\, ds(\xi) = 1$ for every $x \in S$;

(4) if $0 < a < b < c$, then $\omega_{[a,c]} = \omega_{[b,c]} \circ \omega_{[a,b]}$;

(5) $|\omega_\Delta - \widetilde{\omega}_\Delta| \le C(S)\varepsilon^2(L(\Delta))^2 p_{m(\Delta)}$ if Δ is short (i.e., if $L(\Delta) \le 1$).

Proof. (1) follows from the continuity of b_Δ (see Subsection 10.5) and from the fact that the series in (8.16) converges uniformly on $S \times S$. Statement (2) follows from the positivity of $\widetilde{\omega}_\Delta$ for $\varepsilon \in (0, \varepsilon(S))$ (see (5.3)). To obtain (3), we observe that $\widetilde{\omega}_\Delta(x, \xi)\, ds(\xi)$ is a probability measure on S (see Subsection 5.1), so that we have $\int_S \Pi_n(x, \xi)\, ds(\xi) = 1$, $x \in S$, $n = 1, 2, \dots$; the possibility of passing to the limit in the integral is due to the estimate

$$|\Pi_n - \omega_\Delta| \le C|\Delta|2^{-n}p_{m(\Delta)},$$

which, in its turn, is implied by (8.12). Now we show (4). Put $\Delta := [a, c]$, $\Delta^- := [a, b]$, $\Delta^+ := [b, c]$, $\tau_n'(\Delta) := \tau_n(\Delta^-) \cup \tau_n(\Delta^+)$. Since, clearly, $\tau_n'(\Delta) \succ \tau_n(\Delta)$, we see that

$$|\Pi^{\tau_n'} - \Pi^{\tau_n}| \le c(S, \Delta)\frac{|\Delta|}{2^n}p_{m(\Delta)}$$

by Lemma 8.1. Therefore, $\lim_{n\to\infty} \Pi^{\tau_n'(\Delta)} = \omega_\Delta$ everywhere on $S \times S$. On the other hand, we see that $\lim_{n\to\infty} \Pi^{\tau_n(\Delta^+)} \circ \Pi^{\tau_n(\Delta^-)} = \omega_{\Delta^+} \circ \omega_{\Delta^-}$. The limit passage is justified by the estimates $|\Pi^{\tau_n(\Delta^\pm)}| \le \widetilde{\omega}_{\Delta^\pm} + cp_{m(\Delta)}$, which follow from (8.10) and provide the existence of s-integrable majorants for the functions $\xi \mapsto \Pi^{\tau_n(\Delta^+)}(x, \xi) \cdot \Pi^{\tau_n(\Delta^-)}(\xi, x')$ for $x, x' \in S$, $n = 1, 2, \dots$. It remains to note that $\Pi^{\tau_n'(\Delta)} \equiv \Pi^{\tau_n(\Delta^+)} \circ \Pi^{\tau_n(\Delta^-)}$. It remains to prove (5). Estimate (8.10) shows that for $\mu := \tau_n(\Delta)$ we have

$$|\Pi^{\tau_n(\Delta)} - \widetilde{\omega}(\Delta)| \le c(S, L(\Delta))\varepsilon^2(L(\Delta))^2 p_{m(\Delta)}, \quad n = 1, 2, \dots,$$

where the function $x \mapsto c(S, x)$ is monotone increasing on $[0, +\infty)$. If Δ is a short segment (i.e., $L(\Delta) \le 1$), then, letting n go to infinity, we obtain

$$|\omega_\Delta - \widetilde{\omega}_\Delta| \le c(S, 1)\varepsilon^2 \left(\frac{|\Delta|}{m(\Delta)}\right)^2 p_{m(\Delta)}. \qquad \square$$

Theorem 3 is proved (up to some minor details collected in §10). In the next section we use it to deduce Theorem 2.

§9. Proof of Theorem 2

9.1. First, we specify the smoothness conditions for $S\,(=bO)$ in Theorems 1 and 2 (here bE denotes the boundary of $E\subset\mathcal{E}^d$). We introduce a system of Cartesian coordinates in $\mathcal{E}^d$ by putting the origin at $p\in\mathcal{E}^d$ and aligning the axes along the elements of some orthonormal basis $\vec{e}_1,\vec{e}_3,\dots,\vec{e}_d$ in $\mathcal{E}^d$. We thus transform $\mathcal{E}^d$ into the space $\mathbb{R}^d\cong\mathbb{R}^{d-1}\times\mathbb{R}$, writing its points as (x,y) with $x=(x_1,\dots,x_{d-1})\in\mathbb{R}^{d-1}$ and $y\in\mathbb{R}$. By a *cylinder* in $\mathcal{E}^d$ we call a set $\mathfrak{C}\subset\mathcal{E}^d$ that is defined in the above coordinate system as follows: $\mathfrak{C}=\{(x,y)\in\mathbb{R}^d\,:\,|x|<r,\ |y|<h\}$, where r,h are some positive numbers.

9.2. In Theorems 1 and 2 we assume that the following condition is satisfied: for any $p\in S$ there exists a cylinder $\mathfrak{C}=\mathfrak{C}(p,r(p),h(p))$ with center p and a function $\varphi\in C^2(\{x\in\mathbb{R}^{d-1}\,:\,|x|<2r(p)\})$ such that

(1) $O\cap\mathfrak{C}\,(=:\mathfrak{C}^+)=\{(x,y)\in\mathbb{R}^d\,:\,|x|<r(p),\,\varphi(x)<y<h(p)\}$, where $|\varphi|\le\frac{h(p)}{2}$ in $\{x\,:\,|x|<r(p)\}$, $\varphi(0)=0$, $\nabla\varphi(0)=0$, and $\vec{e}_d$ coincides with $\vec{N}(p)$;

(2) $\mathfrak{C}_-:=\{(x,y)\in\mathbb{R}^d\,:\,|x|<r(p),\,-h(p)<y<\varphi(x)\}\subset\mathcal{E}^d\setminus\bar{O}$

In this case $S\cap\mathfrak{C}(p)$ is the graph of the function $\varphi|_{\mathbb{B}^{d-1}(0,r(p))}$, and the hyperplane $\{y=0\}$ is tangent to S at the point p. We call such a domain O a C^2-*domain* (see Subsection 1.1).

9.3. In Subsections 9.5, 9.6 we shall show that *for any $p\in S$ and some cylinder $\mathfrak{C}(p)$* we have

$$\dim\left(\mathcal{V}_{\mathrm{grad}}(u)\cap\mathfrak{C}(p,\rho,h(p))\right)=d-1 \tag{9.1}$$

whenever $\rho\in(0,r(p))$. This proves Theorem 2.

In what follows, we fix a point $p\in S$, some coordinate system with origin at p, and a cylinder $\mathfrak{C}(p)$ as in Subsection 9.2.

9.4. Construction of a "near half-space" W. We extend the function $\varphi|_{\mathbb{B}^{d-1}(0,r(p))}$ up to a function $\Phi\in C^2(\mathbb{R}^{d-1})$. Taking $r(p)$ to be sufficiently small, we may assume that $|\nabla\varphi|<\kappa$ on $\mathbb{B}^{d-1}(0,r(p))$, and $|\nabla\Phi|<\kappa$ on $\mathbb{R}^d$, where κ is some small positive number to be chosen later. We also assume that $\Phi\equiv0$ outside $\mathbb{B}^{d-1}(0,2r(p))$. Let

$$W:=\{(x,y)\,:\,x\in\mathbb{R}^{d-1},\,y>\Phi(x)\}$$

be the open epigraph of the "almost constant" Φ. Clearly, $\mathfrak{C}_+\subset W$.

9.5. With the function u (see Theorems 1 and 2) we shall associate a function w that is positive and harmonic on W and is such that for sufficiently small $r=r(p)$, we have

$$\mathcal{V}_{\mathrm{grad}}(w)\cap S\cap\mathfrak{C}=\mathcal{V}_{\mathrm{grad}}(u)\cap S\cap\mathfrak{C}\,. \tag{9.2}$$

This function w will be defined in Subsection 9.5.2.

9.5.1. Consider a subdomain U of $W\cap O$,

$$U:=\{(x,y)\in\mathfrak{C}\,:\,|x|<r,\,h>y>\varphi(x)+\beta(x)\}, \tag{9.3}$$

where $\beta\in C^\infty(\mathbb{R}^{d-1})$, $0\le\beta\le\frac{h}{4}$, $\beta\equiv0$ in $\mathbb{B}^{d-1}(0,\frac{r}{2})$, $\beta(x)>0$ for $|x|>\frac{r}{2}$. Since the "bottom" of U (i.e., $bU\cap S=:\Sigma$) coincides with $S\cap\mathbb{B}^{d-1}(0,\frac{r}{2})$, we have

$$\overline{U+y\vec{e}_d}\subset O\quad\text{for}\quad y\in(0,y_O], \tag{9.4}$$

where $y_O>0$ is sufficiently small. For such y, put $u_y(x,\eta)=u(x,y+\eta)$, where $(x,y)\in\bar{U}$, $0\le\eta<y_O$. The function u_y admits an extension harmonic near $\bar{U}$; hence, for $p\in U$, $0<y<y_O$ we have

$$u_y(p)=\int_{bU}u_y(q)g^U(p,q)\,ds(q), \tag{9.5}$$

where ds is the $(d-1)$-dimensional surface measure on bU,

$$g^U(p,q) = \frac{\partial}{\partial \vec{N}(q)} G^U(p,q), \quad p \in U, \ q \in bU,$$

G^U is the Green function of U, and $\vec{N}(q)$ is the (inward) normal to bU at q. Fixing a point $p_0 \in U$, we use the fact that $g^U(p_0,q) \geq c(r,U) > 0$ for $q \in \Sigma$ (this follows from the C^2-smoothness of φ and β, see [Br1, Chapter I, §8]). Therefore,

$$\sup_{0 \leq y \leq y_O} \int_\Sigma u_y(q)\, ds_q < +\infty.$$

The functions u_y, $0 \leq y \leq y_O$, are uniformly bounded on the compact set $bU \setminus \Sigma \subset O$. We see that

$$\sup_{0 \leq y \leq y_O} \int_{bU} u_y\, ds < +\infty,$$

and from any sequence (y_k) in $(0, y_O]$ that tends to zero we can extract a subsequence such that the $u_{y_k}\, ds$ converge weakly on U to some measure $d\mu$ as $l \to \infty$. Next, (9.5) implies that

$$u(p) = \int_{bU} g^U(p,q)\, d\mu(q), \quad p \in U. \tag{9.6}$$

If u is given, then the measure μ that satisfies (9.6) is determined uniquely ([HW1], [Br2, Chapter XIV]), so that it does not depend on the choice of (y_k). Consequently μ is the weak limit of the family $u_y\, ds$ on bU as $y \downarrow 0$ (it coincides with $u\, ds$ outside Σ). In particular,

$$\lim_{y \downarrow 0} \int_\Sigma \psi u_y\, ds = \int_\Sigma \psi\, d\mu \tag{9.7}$$

for any $\psi \in C(bU)$ supported on Σ.

9.5.2. Here we define the function w (see (9.2)):

$$w(p) := \int_\Sigma g^W(p,q)\, d\mu(q), \quad p \in W, \tag{9.8}$$

where μ is the measure occurring in (9.6) (we recall that $\Sigma \subset bW$). Like in Subsection 9.5.1, we show that

$$\lim_{y \downarrow 0} \int_\Sigma \psi w_y\, ds = \int_\Sigma \psi\, d\mu,$$

where $w_y(x,\eta) := w(x, \eta + y)$, $(x,\eta) \in W$, $y > 0$, and ψ is the same as in (9.7) (w admits in W the integral representation $\int_{bW} g^W(p,q)\, d\xi(q)$ with a *uniquely determined* measure ξ, which is the weak limit on bW of $w_y\, ds$ as $y \downarrow 0$; hence, $\xi = \mu$ on Σ and $\xi = 0$ on $bW \setminus \Sigma$). Finally, using the same arguments as in Subsection 9.5.1, we obtain

$$w(p) = \int_{bU} g^U(p,q)\, d\lambda(q), \quad p \in U,$$

where λ is some measure on bU. As before, we see that, for the same ψ we have $\int_\Sigma \psi\, d\lambda = \lim_{y \downarrow 0} \int_\Sigma \psi w_y\, ds = \int_\Sigma \psi\, d\mu$, and $\lambda = \mu$ on Σ. Therefore, for $p \in U$ obtain

$$\begin{aligned} u(p) - w(p) &= \int_{bU} g^U(p,q)\, d\mu(q) - \int_{bU} g^U(p,q)\, d\lambda(q) \\ &= \int_{bU \setminus \Sigma} g^U(p,q)\, d\mu(q) - \int_{bU \setminus \Sigma} g^U(p,q) d\lambda(q). \end{aligned}$$

Put $\mathring{\Sigma} := bU \setminus \overline{bU \setminus \Sigma}$. The last two integrals tend to zero as $p \to p^*$, where p^* is an arbitrary point in Σ, and moreover, convergence is uniform on any compact subset of $\mathring{\Sigma}$. The smoothness condition $\varphi, \beta \in C^2$ implies that $\nabla(u - w)$ extends continuously from U to $U \cup \mathring{\Sigma}$ (see [Br1, Chapter I, §8]), and we get (9.2). Now we can apply Theorem 3 to the function w and the "near half-space" W. It follows immediately that the set $\{q \in bW \, : \, V^w(q) < +\infty\}$ is ultradense in bW. In Subsection 9.6 it will be shown that in this case

$$\mathcal{V}_{\text{grad}}(w) \text{ is ultradense in } bW. \tag{9.9}$$

Returning to the point $p \in bO$ chosen at the end of Subsection 9.3, and to the number $\rho \in (0, r(p))$, from (9.2) we deduce that $\mathfrak{C}(p, \rho, h(p)) \cap \mathcal{V}_{\text{grad}}(u)$ is ultradense in $\mathfrak{C}(p, \rho, h(p)) \cap bO$.

9.6. Proof of (9.9)**.** We start with some preliminary arguments.

9.6.1. Let $E \subset \mathbb{R}^d$, $y \in \mathbb{R}$. As before, by E_y we denote the set $E + y\vec{e}_d$.

Let $y, l > 0$, $q = (x, \eta) \in \overline{W}_{y+l}$, $q' = (x', \eta') \in \mathbb{R}^d \setminus W_y$. *Then* $|q - q'| \geq (1 - \kappa)l$. Indeed, we may assume that $|x - x'| < l$, because $|q - q'| \geq |x - x'|$. On the other hand, we see that $\eta \geq \Phi(x) + y + l$, $\eta' \leq \Phi(x) + y$, $|q - q'| \geq \eta - \eta' \geq l - |\Phi(x) - \Phi(x')| \geq l - \sup |\vec{\nabla}\Phi| \cdot l \geq l - \kappa l$. □

9.6.2. Let $\vec{k}$ be a unit vector, $\vec{k} = (k_1, \dots, k_d)$, $k_d > 0$. Let h be a function positive and harmonic on W. Given $q \in bW$, we put

$$(\mathrm{var}_{\vec{k}}\, h)(q) := \int_0^{t_q} |\nabla h(q + t\vec{k})| \, dt,$$

where $t_q > 0$, $(q, q + t_p\vec{k}) \subset W$. In Theorem 2 we are interested in the case where $\vec{k} := \vec{N}(p)$, where

$$k_j = -\frac{\partial \Phi}{\partial \vec{e}_j}(p) \frac{1}{\sqrt{1 + |\vec{\nabla}\Phi(p)|^2}};$$

$$k_d = \frac{1}{\sqrt{1 + |\vec{\nabla}\Phi(p)|^2}}.$$

Let $q = (x, \Phi(x)) \in bW$. Then, for $y > 0$, Subsection 9.6.1 implies that

$$\mathbb{B}^d\left(q_y, \frac{(1 - \kappa)y}{2}\right) \subset W_{\frac{y}{2}}. \tag{9.10}$$

Put $q_{y,\vec{k}} := (x + y\vec{k}', \Phi(x) + yk_d)$, $\vec{k}' = (k_1, \dots, k_{d-1})$. We have

$$|q_y - q_{y,\vec{k}}|^2 \leq \left(\frac{(1 - \kappa)}{4}\right)^2 y^2$$

whenever

$$|\vec{k}'|^2 \, (= (1 - k_d^2)) < \kappa^2 \leq \frac{1}{100}. \tag{9.11}$$

If (9.11) holds true, then an arbitrary function h that is positive and harmonic on $W_{\frac{y}{2}}$ (and, consequently, in $\mathbb{B}^d(q_y, \frac{(1-\kappa)y}{2})$, see (9.10)) satisfies

$$A_d \leq \frac{h(q_y)}{h(q_{y,\vec{k}})} \leq B_d, \quad y > 0 \tag{9.12}$$

(due to the Harnack inequality), where A_d, B_d are some positive constants that depend only on d.

9.6.3. The function $|\nabla w|$ is subharmonic on W, $\lim_\infty |\nabla w| = 0$, and it is continuous in $\overline{W}_y$ for any $y > 0$. Let h_y^W denote the least harmonic majorant for $|\nabla w||_{W_y}$ (i.e., the solution of the Dirichlet problem in W_y with the boundary data $|\nabla w|$; we drop the superscript in h_y^W).

Let $q = (x, \Phi(x)) \in bW$. Put

$$\mathbf{V}_{\vec{k}}(w)(q) := \int_0^1 h_{\frac{y}{2}}(q_{y,\vec{k}})\, dy. \tag{9.13}$$

We assume (9.11), so that $q_{y,\vec{k}} \in \mathbb{B}^d(q_y, \frac{99}{200}y) \subset W_{\frac{y}{2}}$. Therefore, (9.13) makes sense, and $h_{\frac{y}{2}}(q_{y,\vec{k}}) \geq |\nabla w(q_{y,\vec{k}})|$, whence

$$\mathbf{V}_{\vec{k}}(w)(q) \geq (\operatorname{var}_{\vec{k}} w)(q) \tag{9.14}$$

if (9.11) is satisfied, which is true for $\vec{k} = \vec{N}(q)$, because then we have

$$|\vec{k}'|^2 = \frac{|\vec{\nabla}\Phi(x)|^2}{1 + |\vec{\nabla}\Phi(x)|^2} \leq \kappa^2 < 10^{-4}.$$

Now we compare $\mathbf{V}_{\vec{N}(q)}(w)$ with $\mathbf{V}_{\vec{e}_d}(w) = V^w$. From (9.12) it follows that these two quantities are finite or infinite simultaneously, and we obtain the main result of this section, concluding the proof of Theorem 2.

§10. Some auxiliary results

To avoid being distracted by minor details, we omitted some of the proofs of the auxiliary results we used before. They are collected in this section.

10.1. Estimates of the ratios of Poisson kernels. We recall that by $\mathbb{B}$ $(= \mathbb{B}^d)$ and $\mathbb{S}$ we denote the d-dimensional unit ball $\{z \in \mathbb{R}^d : |z| < 1\}$ and its boundary, respectively. Let $R > 0$. Let $p^{R\mathbb{B}}$ denote the Poisson kernel for $R\mathbb{B}$ (see (2.4)):

$$p^{R\mathbb{B}}(z, \zeta) = c_d \frac{R^2 - |z|^2}{|z - \zeta|^d}, \quad z \in R\mathbb{B},\ \zeta \in R\mathbb{S}. \tag{10.1}$$

Given $z_1, z_2 \in R\mathbb{B}$ and $\zeta \in \mathbb{S}$, we put

$$\rho\, (= \rho_R(z_1, z_2, \zeta)) := \frac{p^{R\mathbb{B}}(z_2, \zeta)}{p^{R\mathbb{B}}(z_1, \zeta)}. \tag{10.2}$$

Now we estimate ρ from above for z_1, z_2 that lie on a "near radius" l.

Lemma 10.1. *Let $\zeta_0, \nu \in \mathbb{S}$, and let*

$$|\nu + \zeta_0| < \frac{1}{2}. \tag{10.3}$$

Put $z_j := \zeta_0 + Ry_j\nu$, $j = 1, 2$, where $0 < y_1 < y_2 < \frac{1}{4}$. Then for any $\zeta \in R\mathbb{S}$ we have

$$\rho \leq A^d \frac{y_2}{y_1}, \tag{10.4}$$

where A is an absolute constant.

Proof. From (10.1) and (10.2) it follows that $\rho = \rho_1\left(\frac{z_1}{R}, \frac{z_2}{R}, \frac{\zeta}{R}\right)$. This allows us to assume that $R = 1$. Condition (10.3) means essentially that $\langle \nu, \zeta_0\rangle < 0$, $|\langle \nu, \zeta_0\rangle| > \frac{7}{8}$, because $|\nu + \zeta_0|^2 = 2(1 + \langle \zeta_0, \nu\rangle)$. Next, $\rho = \rho^* \cdot (\rho^{**})^{\frac{d}{2}}$, where $\rho^* := \frac{1-|z_2|^2}{1-|z_1|^2}$, $\rho^{**} := \frac{|\zeta - z_1|^2}{|\zeta - z_2|^2}$. We estimate ρ^* and ρ^{**} separately:

$$\rho^* = \frac{1 - |y_2\nu + \zeta_0|^2}{1 - |y_1\nu + \zeta_0|^2} = \frac{2y_2\langle \zeta_0, \nu\rangle + y_2^2}{2y_1\langle \zeta_0, \nu\rangle + y_1^2} = \frac{y_2}{y_1}\frac{1 - \frac{y_2}{2|\langle \zeta_0,\nu\rangle|}}{1 - \frac{y_1}{2|\langle \zeta_0,\nu\rangle|}} < 2\frac{y_2}{y_1}, \tag{10.5}$$

because $\frac{y_1}{2|\langle \zeta_0,\nu\rangle|} < \frac{1}{7}$. We estimate ρ^{**}. Let $\zeta \in \mathbb{S}$, $|\zeta - \zeta_0| := r$, so that $\zeta = \zeta_0 + r\theta$, $\theta \in \mathbb{S}$. The identity $1 = |\zeta_0 + r\theta|^2 = 1 + 2\langle \zeta_0, \theta\rangle + r^2$ implies that $\langle \zeta_0, \theta\rangle = -\frac{r}{2}$. Next,

$$\rho^{**} = \frac{|r\theta - y_1\nu|^2}{|r\theta - y_2\nu|^2} = \frac{r^2 + y_1^2}{r^2 + y_2^2} \cdot \frac{1 - \frac{2y_1 r\langle \theta,\nu\rangle}{r^2+y_1^2}}{1 - \frac{2y_2 r\langle \theta,\nu\rangle}{r^2+y_2^2}} \le \frac{1 + |\langle \theta, \nu\rangle|}{1 - |\langle \theta, \nu\rangle|} \le \frac{2}{1 - |\langle \theta, \nu\rangle|} \tag{10.6}$$

(we have used the fact that $2|ab| \le a^2 + b^2$, $y_1^2 < y_2^2$). Assume that $r < \frac{1}{2}$. Then (see (10.3))

$$|\langle \theta, \nu\rangle| = |\langle \theta, -\zeta_0\rangle + \langle \theta, \nu + \zeta_0\rangle| \le \frac{r}{2} + \frac{1}{2} \le \frac{3}{4},$$

and (10.6) implies that $\rho^{**} \le 8$. On the other hand, if $r \ge \frac{1}{2}$, then $|\zeta - z_2| \ge r - |z_0 - z_2| = r - y_2|\nu| \ge r - \frac{1}{4} \ge \frac{1}{4}$, and the definition of ρ^{**} shows that $\rho^{**} \le 8$. □

10.1.1.

Corollary 1. *Let v be a positive and harmonic function on $R\mathbb{B}$, and z_1, z_2 are from Lemma* 10.1. *Then*

$$\frac{v(z_2)}{v(z_1)} \le A^d \left(\frac{y_2}{y_1}\right), \tag{10.7}$$

where A is the constant from (10.4).

Proof. This follows immediately from (10.4):

$$v(z_2) = \int_{R\mathbb{S}} p^{R\mathbb{B}}(z_2, \zeta)\, d\mu \le A^d \frac{y_2}{y_1} \int_{R\mathbb{S}} p^{R\mathbb{B}}(z_1, \zeta)\, d\mu(\zeta) = A^d v(z_1)\frac{y_2}{y_1},$$

where μ is some measure on $R\mathbb{S}$. □

10.2. Harnack inequalities for positive harmonic functions. We formulate these inequalities in a convenient form. Let v be a positive harmonic function on $O \subset \mathbb{R}^d$, and let $A \in O$. Then

$$|\nabla v(A)| \le d\frac{v(A)}{\mathrm{dist}(A, bO)}. \tag{10.8}$$

This inequality (see [HK, 1.5.6]) is a consequence of the following estimate of the ratio $\frac{v(A)}{v(B)}$ for $A, B \in O$ such that $|A - B| < \mathrm{dist}(A, bO)$:

$$\frac{1 - Q}{(1 + Q)^{d-1}} \le \frac{v(A)}{v(B)} \le \frac{1 + Q}{(1 - Q)^{d-1}}, \tag{10.9}$$

where $Q := \frac{|A-B|}{\mathrm{dist}(A,bO)}$.

10.3. Rough estimate of the ratios $\frac{v(q_2)}{v(q_1)}$ for positive harmonic v on O_Φ. We use this estimate (see Subsection 10.3.2 below) for the proof of (3.7). Let O_Φ denote the "near half-space" defined in Subsection 2.1, and let S denote its boundary (the graph of Φ). The point $q \in \mathbb{R}^d$ is written as (q', q_d) with $q' \in \mathbb{R}^{d-1}$, $q_d \in \mathbb{R}$.

10.3.1. We start with estimating the distance $\operatorname{dist}(q, S)$ from $q \in O_\Phi$ to S,

$$c(S)(q_d - \Phi(q')) \le \operatorname{dist}(q, S) \le q_d - \Phi(q'), \tag{10.10}$$

where $c(S) > 0$ depends only on S. Indeed, if $\tau \in \mathbb{R}^{d-1}$, then

$$\begin{aligned} q_d - \Phi(q') &\le |q_d - \Phi(q')| + |\Phi(\tau) - \Phi(q')| \le |q_d - \Phi(\tau)| + K|\tau - q'| \\ &\le \sqrt{1+K^2} \cdot \sqrt{(q_d - \Phi(\tau))^2 + |\tau - q'|^2}, \end{aligned}$$

where $K := \sup |\nabla \Phi|$. Therefore,

$$q_d - \Phi(q') \le \sqrt{1+K^2}\operatorname{dist}(q, S).$$

The right-hand side inequality in (10.10) is trivial.

10.3.2. Let v be a positive harmonic function on O_{-y}, where $O = O_\Phi$, $y > 0$. Then for $q \in \bar{O}$ and $t > 0$ we have

$$|(v(q_t))'_t| \le |(\nabla v)(q_t)| \le d\frac{v(q_t)}{\operatorname{dist}(q_t, S_{-y})} \le d\frac{v(q_t)}{c(y+t)}$$

by (10.8) and (10.10). Dividing by $v(q_t)$ and integrating over $t \in [0, h]$, we obtain

$$\frac{v(q_h)}{v(q)} \le \left(\frac{y+h}{y}\right)^{\frac{d}{c}}, \quad q \in \bar{O},\ \ h > 0.$$

In particular, putting $p = p^O$ to be the Poisson kernel for $O = O_\Phi$, we get

$$\frac{p_{y_1}}{p_{y_2}} \le \left(\frac{y_1}{y_2}\right)^{\frac{d}{c}}, \quad 0 < y_1 < y_2. \tag{10.11}$$

10.4. Zeros of the gradient of a harmonic function. *Let v be harmonic on $O = O_\Phi$ (see Subsection 2.1); suppose that $y > 0$, $E \subset S_y$ $(= S + y\vec{e}_d)$, $s(E) > 0$, $\nabla v|_E = 0$. Then v is a constant function.*

10.4.1. In order to prove this, we need the following fact. *Assume that the origin is a density point for $A \subset \mathbb{R}^m$. Then there exists a basis $\vec{e}_1, \dots, \vec{e}_m$ of $\mathbb{R}^m$ such that the set of the density points of $\{t \in \mathbb{R} : t\vec{e}_k \in A\}$ contains zero for any $k = 1, \dots, m$.*

Proof. Let $\mathbb{S}$ denote the unit sphere in $\mathbb{R}^m$ and ν the trace of the $(m-1)$-dimensional Hausdorff measure $\mathcal{H}^{m-1}$ on $\mathbb{S}$. Given $\rho > 0$ and $E \subset \mathbb{R}^m$, we put $E^\rho := E \cap \rho\mathbb{B}^m$, where $\mathbb{B}^m$ is the unit ball in $\mathbb{R}^m$, $E(\rho) := \frac{1}{\rho}(E \cap \rho\mathbb{S}) = \frac{1}{\rho}E \cap \mathbb{S}, cE := \mathbb{R}^m \setminus E$. Since $\mathcal{H}^m((cA)^\rho = o(\rho^m))$ as $\rho \downarrow 0$, i.e.,

$$\int_0^\rho \nu((cA)(r)) r^{m-1}\, dr = o(\rho^m),$$

we have $\lim_{\rho\downarrow 0} \nu((cA)(\rho)) = 0$. Therefore, there exists a positive sequence $(r_j)_{j=1}^\infty \to 0$ such that

$$\sum_{j=1}^\infty \nu((cA)(r_j)) < \nu(S),$$

and

$$\begin{aligned} \nu\Big(\bigcap_{j=1}^\infty A(r_j)\Big) = \nu(\mathbb{S}) - \nu\Big(\mathbb{S} \setminus \bigcap_{j=1}^\infty A(r_j)\Big) &= \nu(\mathbb{S}) - \Big(\bigcup_{j=1}^\infty (cA)(r_j) \cap \mathbb{S}\Big) \\ &\ge \nu(\mathbb{S}) - \sum_{j=1}^\infty \nu\big((cA)(r_j)\big) > 0. \end{aligned}$$

This means that $\bigcap_{j=1}^{\infty} A(r_j)$ does not lie in any hyperplane, and therefore, contains a basis of $\mathbb{R}^m$. □

10.4.2. Now we prove the statement formulated at the beginning of this section. First, we verify that all second derivatives of v vanish s-a.e. on E. Namely, this happens at the points $p = (p', p_d)$ such that p' is the density point for the projection PrE of E to $\mathbb{R}^{d-1}$ (this is true s-a.e. on E). If $p \in E$ is such a point, then it is a density point also for $Pr_{T(p)}(E) =: E(p)$, where $Pr_{T(p)}(E)$ is the projection of E to the tangent hyperplane $T(p)$ for S at p. By Subsection 10.4.1, there exist $d-1$ rays l in $T(p)$ that start at p and such that p is the density point of $l \cap E(p)$. The first derivatives of ∇v along these rays vanish at p; moreover, any derivative along a vector tangential to S_y also vanishes at p. We also see that $\frac{\partial^2 v}{\partial \vec{N}(p)^2}(p) = 0$, where $\vec{N}(p)$ is the normal vector to S_y at the point p, which follows from the identity $\Delta v(p) = 0$ (it suffices to write the Laplacian in some orthonormal coordinate system containing $\vec{N}(p)$). Therefore, all the second derivatives of v vanish s-a.e. on E. By iterating this argument (and using the harmonicity of the derivatives of v) we show that all the partial derivatives of v vanish s-a.e. on E. Combining this with the the fact that v is real analytic, we see that v is constant in O.

10.4.3. Here we deduce a corollary to the result of Subsection 10.4.2. *Let* $\psi \in L^\infty(S)$, $x \in S$ *(*S *is the boundary of* O_Φ*, see Subsection* 2.1*). The function* $k\colon \theta \mapsto B_\theta(\psi)(x)$, $\theta \in (0, +\infty)$*, is continuous.* Here B_θ is the integral operator with the kernel b_θ, see Subsection 3.3.3.

Proof. Let y be any positive number. We show that k is continuous at y. Put $v := P^O(\psi)$, $g_\theta(\xi) := C_\theta(\psi)(\xi) = \langle \nabla v(\xi_\theta), \vec{\sigma}(\xi_{2\theta}) \rangle$, $\xi \in S$, $\theta > 0$, so that $k(\theta) = P_\theta(g_\theta)(x)$. The function $\vec{\sigma}$ is continuous at any $p \in O$ such that $\nabla u(p) \neq 0$. In accordance with Subsection 10.4.2, this is true s-a.e. on S_y, whence $\lim_{\theta \to y} g_\theta = g_y$ s-a.e. on S_y, and also

$$\sup_{\theta \in (\frac{y}{2}, 2y)} \|g_\theta\|_{S,\infty} \leq \sup_{O_{\frac{y}{2}}} |\nabla v| < +\infty.$$

Put $w_\theta = P^O(g_\theta)$ (so that $k(\theta) = w_\theta(x_\theta)$). The above argument implies that

$$\lim_{\theta \to y} w_\theta = w_y \tag{10.12}$$

in O (by dominated convergence). The family $(w_\theta)_{\theta > \frac{y}{2}}$ of functions harmonic on O is uniformly bounded on compact subsets of O. It follows that (10.12) is true uniformly in some neighborhood of x_y, and the family $(w_\theta)_{\frac{y}{2} < \theta < y}$ is uniformly continuous there, so that we have $\lim_{\theta \to y} w_\theta(x_\theta) = w_y(x_y)$, i.e., $\lim_{\theta \to y} k(\theta) = k(y)$. □

The continuity of k follows immediately from the results in the next subsection.

10.5. The kernels b_y are continuous. Here we consider the kernels b_y defined in Subsection 3.3.3. Put $\beta(x, \xi, y) := b_y(x, \xi)$, $\gamma(x, \eta, \xi, y) := p_y(x, \eta) c_y(\eta, \xi)$, $x, \eta, \xi \in S, y > 0$. We recall that $c_y(\eta, \xi) = \langle \nabla^1 p^O(\eta_y, \xi), \vec{\sigma}(\eta_{2y}) \rangle$ (see Subsection 3.3.2). Note that

$$\beta(x, \xi, y) = \int_S \gamma(x, \eta, \xi, y)\, ds(\eta), \quad x, \xi \in S, \ \ y > 0. \tag{10.13}$$

To show that β is continuous at $(a, b, y_0) \in S \times S \times (0, +\infty)$, we observe that

$$\lim_{\substack{x \to a, \\ \xi \to b,\, y \to y_0}} \gamma(x, \eta, \xi, y) = \gamma(a, \eta, b, y_0)$$

if $\eta \in S \setminus \mathcal{Z}_{-2y_0}$, i.e., for s-a.e. $\eta \in S$ (see Subsection 10.4.2; $\mathcal{Z}$ is the zero set of ∇u). To justify the limit passage in (10.11), we show that the functions $l : \eta \mapsto |\gamma(x, \eta, \xi, y)|$ have a common majorant integrable on S with respect to s for any $\xi \in S$ and x, y close to a, y_0, respectively. Indeed,

$$|\gamma(x, \eta, \xi, y)| \le c(S) \frac{p_y(x, \eta) p_y(\eta, \xi)}{y} \tag{10.14}$$

(see (3.10)). Next, $\sup \left\{ p_y(\eta, \xi) \,:\, y \in (\frac{y_0}{2}, \frac{3y_0}{2}),\ \eta, \xi \in S \right\} < +\infty$ (see Subsection 10.5.1). Now we estimate $p_y(x, \eta)$ for any $\eta \in S$, with y close to y_0, and x close to a. For this, we choose $\eta \in S$ and consider a function v harmonic and positive in O and such that

$$v(q) := p^O(q, \eta), \quad q \in O,$$

so that $p_y(x, \eta) = v(x_y)$ for $x \in S$, $y > 0$. By Subsection 10.3.1, we have $\mathbb{B}(x_{y_0}, c_1 y_0) \subset O$, where c_1 is some positive constant depending only on S. Assume that $y > 0$, $|y - y_0| < \frac{c_1}{4} y_0$, $x \in S$, $|x - a| < \frac{c_1}{4} y_0$. Then

$$|x_y - a_{y_0}| \le |x_y - x_0| + |x_{y_0} - a_{y_0}| = |y - y_0| + |x - a| < \frac{c_1}{2} y_0,$$

and, by the Harnack inequality (we apply it to v and $\mathbb{B}(a_{y_0}, c_1 y_0)$, see Subsection 10.2), we see that

$$v(x_y) \le c_2(d) v(a_{y_0}).$$

Now (10.14) implies that $\eta \mapsto c_3 p_{y_0}(a, \eta)$, $\eta \in S$, is the desired majorant for l; here the constant $c_3 > 0$ depends only on d, S, and y_0 (we recall that $\int_S p_{y_0}(a, \eta)\, ds(\eta) = 1$). Finally, from (10.11) and dominated convergence it follows that β is continuous at (a, b, y_0).

10.5.1. Here we prove that the function $m : (\eta, \xi, y) \mapsto p_y(\eta, \xi)\ (= p^O(\eta_y, \xi))$, $\eta, \xi \in S$, $y \ge y_1$, is bounded. We recall that

$$m(\eta, \xi, y) = \frac{\partial G^O}{\partial \vec{N}(\xi)}(\eta_y, \xi),$$

where G^O is the Green function of $O = O_\Phi$ (see Subsection 2.3), and $\vec{N}(\xi)$ is the inward normal to S at ξ; we differentiate over the second variable. The surface S is the graph of a function $\Phi \in C^2(\mathbb{R}^{d-1})$ that is constant outside of some ball with sufficiently large radius. Therefore, we can choose $R = R(S) > 0$ such that for any $\xi \in S$ there exists an open ball $\mathbb{B}\ (= \mathbb{B}(\xi - R\vec{N}(\xi), R))$ of radius R that is tangent to S at ξ and lies outside $O \cup S$. Denote by $\mathbb{B}^-$ its complement, i.e., the set $\mathbb{R}^d \setminus (\mathbb{B} \cup b\mathbb{B})$.

Given $q \in O$, the function $\lambda : x \mapsto \left(G^{\mathbb{B}^-}(q, x) - G^O(q, x)\right)$ is harmonic on O, positive on S, and vanishes at infinity. By the maximum principle, this function is positive on O. Next, $\lambda(\xi) = 0$, whence $\frac{\partial \lambda}{\partial \vec{N}(\xi)}(\xi) \ge 0$ (we can differentiate because $S \in C^2$). Thus, we have $\frac{\partial G^O}{\partial \vec{N}(\xi)}(x, \xi) \le \frac{\partial G^{\mathbb{B}^-}}{\partial \vec{N}(\xi)}(q, \xi)$, i.e.,

$$p^O(q, \xi) \le p^{\mathbb{B}^-}(q, \xi), \quad q \in O,\ \ \xi \in S.$$

Putting $q = \eta_y$, where $\eta \in S$ and $y \ge y_1$, we see that the right-hand side does not exceed some constant that depends only on y_1, S, and R. This follows immediately from the explicit formula for $p^{\mathbb{B}^-}$ (see [ABR, p. 66]); we also observe that for $y \ge y_1$ the distance between η_y and the center of $\mathbb{B}$ is at least cy_1, where $c = c(S) > 0$ (see Subsection 10.3.2).

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