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ELLIPTIC EQUATIONS IN CONVEX DOMAINS

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To Yury Burago, a friend of my youth, with love and admiration

Abstract. A short survey of a series of results by the author, partly obtained in collaboration with Yu. Burago.

Introduction

It is an elementary consequence of the theory of conformal mappings that harmonic functions of two variables with zero either Dirichlet or Neumann data are differentiable at the vertex of any angle less than π. In contrast, reentrant angles produce singularities of the gradient. This is the simplest example of the beneficial influence of a domain's convexity on the regularity of solutions to classical boundary value problems.

In the present article I review several results illustrating the same fenomena obtained with my participation.

I start with a half century old work, joint with Yu. Burago, on the double and single layer harmonic potential theory.

§2 is devoted to L^p-properties of the gradient of solutions to the Neumann problem for the Poisson equation in a convex domain (see [1]).

The next section addresses a class of quasilinear elliptic equations and systems in a convex domain. Here some sharp results obtained in 2014 together with Cianchi are described.

The last section concerns the Dirichlet problem for the biharmonic equation. It reflects the joint paper [6].

§1. Neumann problem for harmonic functions inside and outside a convex domain

I start with description of solvability results for classical boundary value problems for the Laplace equation obtained by Yu. Burago and myself.

Our starting point was a remark made in the famous course of functional analysis by F. Riesz and B. Sz.-Nagy, namely that "in the case of the spatial problem an analog of curves with bounded rotation has not yet been found". These curves form the largest class of contours known by that time for which the classical harmonic potential theory was developed. According to the note by Burago, Maz′ya, Sapozhnikova [2], it turned out that a "proper" generalization of Radon's result to higher dimensions can be achieved in terms of a certain function $\omega(\xi, B)$ replacing in a sense the solid angle at which the set B is seen from the point ξ. Within a few years Yu. Burago and myself developed a comprehensive

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theory of double and single layer potentials for a large class of n-dimensional domains. An exposition of that theory was given in the book [3].

The basic results, i.e., theorems on the solvability of the integral equations of the fundamental boundary value problems, were obtained for domains subject to two conditions

$$\text{(A)} \qquad \sup\left\{ \operatorname{var}\omega(\xi, S\setminus\xi) \,:\, \xi\in S\right\} < \infty,$$

$$\text{(B)} \qquad \limsup_{r\to 0}\left\{ \operatorname{var}\omega(\xi, S\cap B_r(\xi)) \,:\, \xi\in S\right\} < \sigma_n/2.$$

Here S is the boundary of the domain Ω with compact closure, var denotes the variation of the charge, $B_r(\xi)$ is the ball with center ξ and radius r, and σ_n is the area of the unit sphere.

We wish to point out that *condition* (A) is a corner-stone of the entire theory, whereas *condition* (B) is solely needed to prove the Fredholm alternative.

An arbitrary convex domain Ω satisfies condition (B). In fact, the closed ball $B_r(p)$ lies in Ω. For every point $q\in\partial\Omega$, we have

1) $\operatorname{var}\omega_{\partial\Omega} = \omega_{\partial\Omega}$ (because the projection from p is bijective);

2) $\omega_{\partial\Omega} \le \frac{1}{2}\omega_n$ $\{$the angle at which $B_r(p)$ is seen from the point $q\}$.

It is trivial to obtain a positive minorant for the last angle depending on ρ and r, which together with 2) guarantees (B).

Therefore, in what follows, I shall always assume Ω to be a convex domain.

I give a short exposition of our results.

Definition 1.1. The harmonic double layer potential with continuous density χ is the function defined for $x\notin S$ by

$$(W_\chi)(x) = \frac{1}{\sigma_n}\int_S \chi(\xi)\omega(x,\, d\xi).$$

Theorem 1.1. *For any continuous function χ, the limit values $W_{\pm\chi}$ of the potential W_χ, which exist from inside and outside of Ω, satisfy*

$$W_{+\chi} = \frac{1}{2}(\chi + T\chi), \tag{1.1}$$

$$W_{-\chi} = \frac{1}{2}(-\chi + T\chi), \tag{1.2}$$

and the doubled direct value of the potential,

$$(T\chi)(x) = \frac{2}{\sigma_\pi}\int_S \chi(\xi)\omega(x, d\xi),$$

generates a continuous operator in the space of continuous functions $C(S)$.

Owing to formulas (1.1) and (1.2), continuous solutions of the internal and external Dirichlet problems with prescribed boundary functions φ^+ and φ^- in $C(S)$ may be determined by means of the integral equations

$$\chi + T\chi = 2\varphi_+, \tag{1.3}$$

$$-\chi + T\chi = 2\varphi_-. \tag{1.4}$$

To pose, in some sense, the Neumann problem for surfaces satisfying condition (A), we need the notion of boundary flow.

Definition 1.2. A function $u\in C^1(\operatorname{int}\Omega)$ is said to possess an inner boundary flow if

(i) for each infinitely differentiable function φ on $\mathbb{R}^n$ with compact support and for each sequence of sets $\Omega_m\subset\operatorname{int}\Omega$ with smooth boundaries that converge to Ω, the following limit exists:

$$l_u(\varphi) = \lim_{m\to\infty} \int_{\partial\Omega_m} \varphi(x) \frac{\partial u}{\partial \nu_x} H_{n-1}\,(dx);$$

(ii) the functional l_u is bounded in the norm of $C(S)$.

Definition 1.3. Let the requirements (i), (ii) of the previous definition be satisfied. The finite charge Σ^+ on S generating the extension of the functional l_u to all of $C(S)$,

$$l_u(\varphi) = \int_S \varphi(x)\Sigma^+\,(dx),$$

is called the inner boundary flow of the function u.

The outer boundary flow $\Sigma^-$of a function $u \in C^1(\mathbb{R}^n \setminus \overline{\Omega})$ is defined similarly.

In our understanding of the internal and external Neumann problems the charges Σ^+ and Σ^- play part of the normal derivatives.

Definition 1.4. Given a finite charge ϱ on $S \subset \mathbb{R}^n$ $(n \geq 3)$, the single layer potential with the charge ϱ is defined by

$$(V\varrho)(x) = \frac{1}{\sigma_n} \int_S r^{2-n} \varrho\,(d\xi), \quad x \notin S.$$

The function $V\rho$ turns out to be harmonic in $\mathbb{R}^n \setminus S$.

Theorem 1.2. *Suppose* $\operatorname{meas}_n(S) = 0$, $S = \partial(\mathbb{R}^n \setminus \overline{\Omega})$, *and condition* (A) *is fulfilled. Then the potential $V\rho$ possesses inner and outer boundary flows, which equal*

$$-\frac{1}{2}\varrho(B) + \frac{1}{\sigma_n}\int_S \omega(x,B)\varrho\,(dx),$$
$$\frac{1}{2}\varrho(B) + \frac{1}{\sigma_n}\int_S \omega(x,B)\varrho\,(dx),$$

where B is an arbitrary Borel subset of S.

The condition (A) is not only sufficient but also necessary for the boundary flows of an arbitrary $V\varrho$ to exist.

Thus, looking for the solution of the Neumann problem in the form of a single layer potential leads to the equations

$$-\varrho + T^*\varrho = 2\Sigma^+, \tag{1.5}$$

$$\varrho + T^*\varrho = 2\Sigma^-. \tag{1.6}$$

Here T^* is the operator adjoint of T, acting on the dual space $C^*(S)$ of $C(S)$.

The following theorem is our basic result on the solvability of the above integral equations.

Theorem 1.3. 1) *The integral equation* (1.5) *of the internal Dirichlet problem has a unique solution in $C(S)$ for every continuous right-hand side Σ^+.*

2) *The integral equation* (1.6) *of the external Neumann problem is uniquely solvable for every finite charge Σ^-.*

Definition 1.5. A finite charge ϱ is said to belong to the class C_V if the simple layer potential $V\rho$ generated by ϱ possesses finite and equal limits on S from inside and outside of S.

I conclude with our theorem on the solvability of the Neumann problem in the convex domain Ω and its complement. In its formulation BV stands for the space of functions whose distributional gradients are vector-valued charges.

Theorem 1.4. 1) *For every finite charge $\Sigma^+ \in C_V$ with vanishing total mass, there exists, up to an additive constant, exactly one solution of the internal Neumann problem belonging to the class $C(\overline{\Omega}) \cap BV(\operatorname{int}\Omega)$.*

2) *For every finite charge $\Sigma^- \in C_V$ there exists precisely one solution of the external Neumann problem belonging to the class $C(\mathbb{R}^n \setminus \operatorname{int}\Omega) \cap BV^{\mathrm{loc}}(\mathbb{R}^n \setminus \overline{\Omega})$ and tending to zero at infinity.*

Note that ten years later several publications on the solvability of the boundary integral equations (1.3)–(1.6) in the spaces L^p, $1 < p < \infty$, on Lipschitz surfaces appeared (see [7, Section 3]).

§2. Neumann problem for the Poisson equation inside and outside of a convex domain

We denote by F a linear functional on the space $L^{1,p'}(\Omega)$, $p+p' = pp'$, $p \in (1,\infty)$. By a distributional solution to the Neumann problem for the Poisson equation $-\Delta u = F$ we mean a function $u \in L^{1,p}(\Omega)$ that is orthogonal to 1 in Ω and satisfies the equality

$$\int_\Omega \nabla u \cdot \nabla\psi \, dx = F(\psi) \quad \forall \psi \in L^{1,p'}(\Omega). \tag{2.1}$$

As is known (cf., for example, [8, Theorem 1.1.15/1]), any linear functional F on $L^{1,p'}(\Omega)$ can be represented as

$$F(\psi) = \int_\Omega \vec{f} \cdot \nabla\psi \, dx, \tag{2.2}$$

where $\vec{f} \in L^p(\Omega)$; moreover,

$$\|F\| = \inf \|\vec{f}\|_{L^p(\Omega)},$$

where the infimum is taken over all vector-valued functions $\vec{f}$ satisfying (2.2) for any $\psi \in L^{1,p'}(\Omega)$. The space of functionals F is denoted by $(L^{1,p'}(\Omega))^*$.

Suppose that $\psi \in L^{1,p}(\Omega)$ and $\operatorname{tr}\psi$ denotes the trace of ψ on $\partial\Omega$. As is known, the set $\{\operatorname{tr}\psi\}$ is the space $B^{1/p,p'}(\partial\Omega)$. We introduce the space of distributions $B^{-1/p,p'}(\partial\Omega)$ dual to $B^{1/p,p'}(\partial\Omega)$. It is clear that the mapping

$$L^{1,p'}(\Omega) \ni \psi \longrightarrow \int_{\partial\Omega} h(x) \operatorname{tr}\psi(x)\, ds_x,$$

where $h \in B^{-1/p,p}(\partial\Omega)$ and $h \perp 1$ on $\partial\Omega$, is a linear functional. If $h \in B^{-1/p,p}(\partial\Omega)$, then a particular case of the problem (2.1) is the problem

$$\Delta u = 0 \text{ in } \Omega, \quad \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = h, \quad u \in L^{1,p}(\Omega),$$

where ν is the unit vector of the outward normal to the boundary $\partial\Omega$.

By the Sobolev embedding theorems (cf., for example, [8, Theorem 1.4.5]), the mapping

$$L^{1,p'}(\Omega) \ni \psi \longrightarrow \int_\Omega f_0(x)\psi(x)\, dx + \int_{\partial\Omega} h(x) \operatorname{tr}\psi(x)\, ds_x$$

is a linear functional if $f_0 \in L^{np/(n+p)}(\Omega)$, $h \in L^{p(n-1)/n}(\partial\Omega)$ for $p' < n$, $f_0 \in L^q(\Omega)$ for any $q > 1$ with $p' = n$ and if

$$\int_\Omega f_0(x)\, dx + \int_{\partial\Omega} h(x)\, ds_x = 0.$$

Using this functional, we can define a weak $L^{1,p}(\Omega)$-solution to the problem

$$-\Delta u = f_0 \ \text{ in } \ \Omega, \quad \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = h. \tag{2.3}$$

For $p' > n$ the same is true if $h = 0$ and the right-hand side f_0 of equation (2.3) belongs to the dual $(C(\overline{\Omega}))^*$ of the space of functions that are continuous in the closure of Ω, have compact support in Ω and are orthogonal to 1 in Ω.

This fact is used in the following definition of the Green function G of the Neumann problem in the domain Ω. Suppose that $y \in \Omega$ and $p \in (1, n/(n-1))$. The function $\Omega \ni x \to G(x, y)$, orthogonal to 1 in Ω, is an $L^{1,p}(\Omega)$-solution to the Neumann problem

$$-\Delta_x G(x,y) = \delta(x-y) - |\Omega|^{-1} \text{ in } \Omega, \quad \frac{\partial G(x,y)}{\partial \nu_x}\Big|_{\partial\Omega} = 0, \tag{2.4}$$

where $|\Omega|$ is the n-dimensional measure of the domain Ω. It is known that this function admits a uniform estimate (cf., for example, [7, Theorem 3])

$$\|\nabla_x G(x,y)\|_{L^p(\Omega)} \le c(n,p,\Omega), \tag{2.5}$$

where $p \in (1, n'(n-1)$.

The proof of this result is based on the following estimates for the Green function $G(x, y)$ of the Neumann problem for the Laplace operator in a domain Ω:

$$|\nabla_y G(x,y)| \le c(n,\Omega)|x-y|^{1-n} \quad \forall x,y \in \Omega, x \ne y, \tag{2.6}$$

and

$$|\nabla_x \nabla_y G(xy)| \le c(n,\Omega)|x-y|^{-n} \quad \forall x,y \in \Omega, x \ne y. \tag{2.7}$$

Note that they are of independent interest.

From (2.6) it follows that a pointwise estimate for the gradient of the solution u to the Neumann problem for the Poisson equation

$$-\Delta u = f \in L^1(\Omega), \quad f \perp 1 \text{ in } \Omega,$$

has the form

$$|\nabla u(x)| \le c(n,\Omega) \int_\Omega |x-y|^{1-n} |f(y)|\, dy \quad \forall x \in \Omega. \tag{2.8}$$

We state main result of [1].

1.1 Theorem. *The Neumann problem* (2.1) *is uniquely solvable in the space* $L^{1,p}(\Omega)$ *for any exponent* $p \in (1,\infty)$*, and the solution* u *satisfies the inequality*

$$\|\nabla u\|_{L^p(\Omega)} \le C\|F\|_{(L^{1,p'}(\Omega))^*}, \tag{2.9}$$

where the constant C *is independent of* u *and* F*.*

A simple counterexample shows that this result is not valid for an arbitrary p if the boundary of $\Omega \subset \mathbb{R}^2$ has a reentrant angle.

The above theorem is a consequence of estimates (2.6), (2.7) and the following assertion.

The integral operator acting in $\mathbb{R}^n_+ = \{y : y_n > 0, y = (y', y_n)\}$, with kernel $J(x,y) = (|x'-y'| + x_n + y_n)^{-n}$ is bounded in $L^p(\mathbb{R}^n_+)$ for any exponent $p \in (1,\infty)$.

A result similar to the main theorem can be obtained also for the Dirichlet problem in an arbitrary convex domain

$$-\Delta u = \operatorname{div} f, \quad f \in [L^p(\Omega)]^n, \quad u \in \mathring{L}^{1,p}(\Omega),$$

where $\mathring{L}^{1,p}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ in $L^{1,p}(\Omega)$. In other words, the Laplace operator performs an isomorphism $\mathring{L}^{1,p}(\Omega) \to (\mathring{L}^{1,p}(\Omega))^*$ for any $p \in (1,\infty)$.

§3. Sharp estimates for the gradient of solutions for a class of nonlinear elliptic systems and equations

In [4], gradient boundedness up to the boundary for solutions to the Dirichlet and Neumann problems for elliptic systems with Uhlenbeck type structure was established. Nonlinearities of possibly nonpolynomial type are allowed, and minimal regularity of the data and of the boundary of the domain is assumed. The case of arbitrary bounded convex domains, considered in this section, is also included.

We are concerned with second-order nonlinear elliptic systems of the form

$$-\operatorname{div}(a(|\nabla \mathbf{u}|)\nabla \mathbf{u}) = \mathbf{f}(x) \text{ in } \Omega. \tag{3.1}$$

Our assumptions on the system (3.1) amount to what follows. The function $a\colon (0,\infty)\to(0,\infty)$ is required to be monotone (either nondecreasing or nonincreasing), of class $C^1(0,\infty)$, and to satisfy

$$-1 < i_a \le s_a < \infty, \tag{3.2}$$

where

$$i_a = \inf_{t>0} \frac{ta'(t)}{a(t)} \quad \text{and} \quad s_a = \sup_{t>0} \frac{ta'(t)}{a(t)}. \tag{3.3}$$

In particular, the standard p-Laplace operator for vector-valued functions, corresponding to the choice $a(t)=t^{p-2}$, with $p>1$, falls within this framework, because $i_a = s_a = p-2$ in this case. Thanks to the first inequality in (3.3), the function $b\colon [0,\infty)\to[0,\infty)$ defined by

$$b(t) = a(t)t \text{ if } t>0 \quad \text{and} \quad b(0)=0, \tag{3.4}$$

turns out to be strictly monotone increasing, and hence the function $B\colon [0,\infty)\to[0,\infty)$, given by

$$B(t) = \int_0^t b(\tau)\,d\tau \quad \text{for} \quad t\ge 0, \tag{3.5}$$

is strictly convex. The Orlicz–Sobolev space $W^{1,B}(\Omega,\mathbb{R}^N)$ associated with the function B, or its subspace $W_0^{1,B}(\Omega,\mathbb{R}^N)$ of the functions vanishing in a suitable sense on $\partial\Omega$, are appropriate functional settings for defining weak solutions to the boundary value problems associated with the system (3.1).

The right-hand side $\mathbf{f}$ is assumed to belong to the Lorentz space $L^{n,1}(\Omega,\mathbb{R}^N)$. This space is borderline, in a sense, for the family of Lebesgue space $L^q(\Omega,\mathbb{R}^N)$ with $q>n$, because $L^q(\Omega,\mathbb{R}^N) \subsetneqq L^{n,1}(\Omega,\mathbb{R}^N) \subsetneqq L^n(\Omega,\mathbb{R}^N)$ for every $q>n$.

Our result for the Dirichlet problem

$$\begin{cases} -\operatorname{div}(a(|\nabla \mathbf{u}|)\nabla \mathbf{u}) = \mathbf{f}(x) & \text{in } \Omega, \\ \mathbf{u} = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.6}$$

reads as follows.

Theorem 3.1. *Let Ω be a convex domain with compact closure in $\mathbb{R}^n$, $n\ge 3$. Assume that $\mathbf{f}\in L^{n,1}(\Omega,\mathbb{R}^N)$. Let $\mathbf{u}$ be the (unique) weak solution to the Dirichlet problem (3.6). Then there exists a constant C, $C=C(i_a,s_a,\Omega)$ such that*

$$\|\nabla \mathbf{u}\|_{L^\infty(\Omega,\mathbb{R}^{Nn}} \le Cb^{-1}(\|\mathbf{f}\|_{L^{n,1}(\Omega,\mathbb{R}^N)}). \tag{3.7}$$

In particular, $\mathbf{u}$ is Lipschitz continuous in Ω.

Problem (3.6) is the Euler equation of the strictly convex functional

$$J(u) = \int_\Omega (B(|\nabla \mathbf{u}|) - \mathbf{f} \cdot \mathbf{u})\, dx, \tag{3.8}$$

which is well defined for $\mathbf{u} \in W_0^{1,B}(\Omega, \mathbb{R}^N)$ under our assumption of $\mathbf{f}$. The solvability of the minimization problem for (3.8) is the content of the following corollary.

Corollary 3.2. *Let Ω be any convex domain in $\mathbb{R}^n$, $n \geq 3$, and let B be defined as in* (3.5)*. Assume that $\mathbf{f} \in L^{n,1}(\Omega, \mathbb{R}^N)$. Then the functional J admits a* (*unique*) *minimizer in the space* $\mathrm{Lip}_0(\Omega, \mathbb{R}^N)$.

Results parallel to Theorem 2.1 and Corollary 3.2 hold for the solutions to the Neumann problem

$$\begin{cases} -\operatorname{div}(a(|\nabla \mathbf{u}|)\nabla \mathbf{u}) = \mathbf{f}(x) & \text{in } \Omega, \\ \frac{\partial \mathbf{u}}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.9}$$

Clearly, here, $\mathbf{f}$ must fulfill the compatibility condition

$$\int_\Omega \mathbf{f}(x)\, dx = 0. \tag{3.10}$$

Theorem 3.2. *Let Ω and $\mathbf{f}$ be as in Theorem* 3.1*. Assume, in addition, that* (3.10) *is true. Let $\mathbf{u}$ be the* (*unique up to additive constant vectors*) *weak solution to problem* (3.9)*. Then there exists a constant $C = C(i_a, s_a, \Omega)$ such that*

$$\|\nabla \mathbf{u}\|_{L^\infty(\Omega, \mathbb{R}^{Nn})} \leq C b^{-1}(\|\mathbf{f}\|_{L^{n,1}(\Omega, \mathbb{R}^N)}). \tag{3.11}$$

In particular, u is Lipschitz continuous in Ω.

The minimization problem for the functional J, whose Euler equation is (2.9), is properly set in the subspace $W_\perp^{1,B}(\Omega, \mathbb{R}^N)$ of functions in $W^{1,B}(\Omega, \mathbb{R}^N)$ with vanishing mean-value. An analog of Corollary 3.2 can thus be formulated in terms of the space $\mathrm{Lip}_\perp(\Omega, \mathbb{R}^N)$ of $\mathbb{R}^N$-valued Lipschitz continuous functions on Ω with vanishing mean-value.

Corollary 3.3. *Let Ω be any convex domain in $\mathbb{R}^n$, $n \geq 3$, and let B be defined as in* (3.5)*. Assume that $\mathbf{f} \in L^{n,\perp}(\Omega, \mathbb{R}^N)$. Then the functional J admits a* (*unique*) *minimizer in the class* $\mathrm{Lip}_\perp(\Omega, \mathbb{R}^N)$.

Now I turn to the article [5], where a sharp estimate for the distribution function of the gradient of solutions to a class of nonlinear Dirichlet and Neumann elliptic boundary value problems was established under weak regularity assumptions on the domain Ω. In particular, Ω can be an arbitrary convex domain with compact closure.

As a consequence, the problem of gradient bounds in norms depending on global integrability properties is reduced to one-dimensional Hardy-type inequalities. Applications to gradient estimates in Lebesgue, Lorentz, Zygmund, and Orlicz spaces are presented.

We consider the scalar equation

$$-\operatorname{div}\big(a(|\nabla u|)\nabla u\big) = f(x) \quad \text{in } \Omega \tag{3.12}$$

coupled with either the Dirichlet condition

$$u = 0 \quad \text{on } \partial\Omega, \tag{3.13}$$

or the Neumann condition

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \tag{3.14}$$

We assume that $a\colon [0,\infty) \to [0,\infty)$ is of class $C^1(0,\infty)$, and there exists $p \in [2,n)$ and $C > 0$ such that

$$\frac{ta'(t)}{a(t)} \geq p-2 \quad \text{for} \quad t > 0, \tag{3.15}$$

and

$$ta(t) \leq C(t^{p-1}+1) \quad \text{for} \quad t > 0. \tag{3.16}$$

Equation (3.12) is patterned on the model

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f(x) \quad \text{in} \quad \Omega, \tag{3.17}$$

the so called p-Laplace equation, corresponding to the choice $a(t) = t^{p-2}$ for $t > 0$.

The main result is the following theorem.

Theorem 3.3. *Let Ω be a convex domain in with compact closure $\mathbb{R}^n$, $n \geq 3$. Assume that $f \in L^1(\Omega)$, and let u be a solution either to the Dirichlet problem* (3.12), (3.13) *or to the Neumann problem* (3.12), (3.14). *Then there exists a constant $C = C(p,\Omega)$ such that*

$$|\nabla u|^*(s)^{p-1} \leq C \int_s^{|\Omega|} f^{**}(r) r^{-\frac{1}{n'}}\, dr \quad \textit{for} \quad s \in (0, |\Omega|). \tag{3.18}$$

Here, $n' = \frac{n}{n-1}$, the Hölder conjugate to n, f^* is the decreasing rearrangement of f and $f^{**}(r) = \frac{1}{r}\int_0^r f^*(p)\, dp$ for $r \in (0, |\Omega|)$.

§4. Boundedness of second derivatives of a solution to $\Delta^2 u = f$

Given a bounded domain $\Omega \subset \mathbb{R}^n$, denote by $\mathring{W}^2_2(\Omega)$ the completion of $C_0^\infty(\Omega)$ in the norm of the Sobolev space of functions with second distributional derivatives in L^2. Consider the variational solution of the boundary value problem

$$\Delta^2 u = f \quad \text{in} \quad \Omega, \quad f \in C_0^\infty(\Omega), \quad u \in \mathring{W}^2_2(\Omega), \tag{4.1}$$

that is, a function $u \in \mathring{W}^2_2(\Omega)$ such that

$$\int_\Omega \Delta u \Delta v\, dx = \int_\Omega f v\, dx \quad \text{for every} \quad v \in \mathring{W}^2_2(\Omega). \tag{4.2}$$

The main result of this section is the following theorem proved in [6].

Theorem 4.1. *Let Ω be a convex domain in $\mathbb{R}^n$, $O \in \partial\Omega$, and fix some $R \in (0, \operatorname{diam}(\Omega) \setminus 10)$. Suppose u is a solution of the Dirichlet problem* (4.1) *with $f \in C_0^\infty(\Omega \setminus B_{10R})$. Then*

$$|\nabla^2 u(x)| \leq \frac{C}{R^2}\left(\fint_{C_{R/2,5R}\cap\Omega} |u(x)|^2\, dx\right)^{1/2} \quad \textit{for every} \quad x \in B_{R/5}\cap\Omega, \tag{4.3}$$

where $\nabla^2 u$ is the Hessian matrix of u,

$$C_{R/2,5R} = \{x \in \mathbb{R}^n : R/2 \leq |x| \leq 5R\}, \quad B_{R/5} = \{x \in \mathbb{R}^n : |x| < R/5\}, \tag{4.4}$$

and the constant C depends on the dimension only.

In particular,

$$|\nabla^2 u| \in L^\infty(\Omega). \tag{4.5}$$

The proof is based on some new global integral inequalities which will be formulated in what follows.

Let Ω be an arbitrary domain in $\mathbb{R}^2$, $n \geq 2$. We assume that the origin belongs to the complement of Ω and $r = |x|$, $\omega = x/|x|$ are the spherical coordinates centered at the origin. In fact, we will mostly use the coordinate system (t, ω), where $t = \log r^{-1}$, and the mapping $\varkappa$ defined by

$$\mathbb{R}^n \ni x \xrightarrow{\varkappa} (t, \omega) \in \mathbb{R} \times S^{n-1}. \tag{4.6}$$

Here S^{n-1} denotes the unit sphere in $\mathbb{R}^n$. I state the first of them.

Lemma 4.1. *Let Ω be a bounded convex domain in $\mathbb{R}^n$ and let $O \in \mathbb{R}^n \setminus \Omega$. Suppose that*

$$u \in C^2(\overline{\Omega}), \quad u\big|_{\partial\Omega} = 0, \quad \nabla u\big|_{\partial\Omega} = 0, \quad v = e^{2t}(u \circ \varkappa^{-1}). \tag{4.7}$$

Then

$$\begin{aligned} &\int_{\mathbb{R}^n} \Delta u(x) \Delta\left(\frac{u(x) g(\log(|\xi|/|x|))}{|x|^n}\right) dx \\ &\quad \geq -\int_{\mathbb{R}} \int_{S^{n-1}} \big(2\partial_t^2 g(t-\tau) + 3n\partial_t g(t-\tau) \\ &\qquad + (n^2-2) g(t-\tau)\big)(\partial_t v(t,\omega))^2 \, d\omega \, dt + \frac{1}{2}\int_{S^{n-1}} v^2(\tau,\omega)\, d\omega, \end{aligned} \tag{4.8}$$

for every $\xi \in \Omega$, $\tau = \log|\xi|^{-1}$. Here g is a bounded solution of the equation

$$\frac{d^4 g}{dt^4} + 2n\frac{d^3 g}{dt^3} + (n^2-2)\frac{d^2 g}{dt^2} - 2n\frac{dg}{dt} = \delta \tag{4.9}$$

subject to the restriction

$$g(t) \to 0 \quad as \quad t \to +\infty. \tag{4.10}$$

It is given explicity by

$$g(t) = -\frac{1}{2n\sqrt{n^2+8}} \begin{cases} n e^{-1/2(n-\sqrt{n^2+8})t} - \sqrt{n^2+8}, & t < 0, \\ n e^{-1/2(n+\sqrt{n^2+8})t} - \sqrt{n^2+8}\, e^{-nt}, & t > 0. \end{cases} \tag{4.11}$$

The second auxiliary global estimate is contained in the following lemma.

Lemma 4.2. *Suppose Ω is a bounded convex domain in $\mathbb{R}^n$, let $O \in \partial\Omega$,*

$$u \in C^4(\overline{\Omega}), \quad u\big|_{\partial\Omega} = 0, \quad \nabla u\big|_{\partial\Omega} = 0, \quad u = e^{2t}(u \circ \varkappa^{-1}), \tag{4.12}$$

and g is defined by (4.11)*. Then*

$$\begin{aligned} &2\int_{\mathbb{R}^n} \Delta u(x) \Delta\left(\frac{u(x) g(\log|\xi|/|x|}{|x|^n}\right) dx - \int_{\mathbb{R}^n} \Delta^2 u(x)\left(\frac{(x\cdot\nabla u(x)) g(\log|\xi|/|x|)}{|x|^n}\right) dx \\ &\quad \geq -\frac{1}{2}\int_{\mathbb{R}} \int_{S^{n-1}} \Big(\partial_t^3 g(t-\tau) + 2n\partial_t^2 g(t-\tau) \\ &\qquad\qquad + (n^2-2)\partial_t g(t-\tau) - 4n g(t-\tau)\Big)(\partial_t v(t,\omega))^2 \, d\omega\, dt, \end{aligned} \tag{4.13}$$

for every $\xi \in \Omega$, $\tau = \log|\xi|^{-1}$.

The above global estimates imply the local ones.

Theorem 4.2. *Let Ω be a bounded smooth convex domain in $\mathbb{R}^n$, let $Q \in \partial\Omega$, and let $R \in (0, \operatorname{diam}(\Omega)/5)$. Suppose*

$$\Delta^2 u = f \ \ in\ \Omega, \quad f \in C_0^\infty(\Omega \setminus B_{5R}(\Omega)), \quad u \in \mathring{W}_2^2(\Omega). \tag{4.14}$$

Then

$$\frac{1}{p^4}\fint_{S_p(Q)\cap\Omega}|u(x)|^2 d\sigma_x \le \frac{C}{R^r}\fint_{C_{R,4R}(Q)\cap\Omega}|u(x)|^2\,dx \quad \text{for every} \quad p<R, \tag{4.15}$$

where the constant C *depends on the dimension only.*

Finally (4.15) leads to Theorem 4.1.

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