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PROPERTIES OF THE INTRINSIC FLAT DISTANCE

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Dedicated to Yu. D. Burago on his 80th birthday

ABSTRACT. In this paper written in honor of Yuri Burago, we explore a variety of properties of intrinsic flat convergence. We introduce the sliced filling volume and interval sliced filling volume and explore the relationship between these notions, the tetrahedral property and the disappearance of points under intrinsic flat convergence. We prove two new Gromov–Hausdorff and intrinsic flat compactness theorems including the Tetrahedral Compactness Theorem. Much of the work in this paper builds upon Ambrosio–Kirchheim's Slicing Theorem combined with an adapted version of Gromov's Filling Volume. We are grateful to have been invited to submit a paper in honor of Yuri Burago, in thanks not only for his beautiful book written jointly with Dimitri Burago and Sergei Ivanov but also for his many thoughtful communications with us and other young mathematicians over the years.

§1. INTRODUCTION

The intrinsic flat convergence of Riemannian manifolds has been applied to study the stability of the Positive Mass Theorem, the rectifiability of Gromov–Hausdorff limits of Riemannian manifolds, and smooth convergence away from singular sets. Applications of the intrinsic flat convergence to Riemannian General Relativity appear in joint work of the second author, Lan-Hsuan Huang, Jeffrey Jauregui, Dan Lee, and Philippe LeFloch [19, 20, 13, 14]. Sajjad Lakzian applied intrinsic flat convergence to study smooth convergence away from singular sets [17, 16] with applications to Ricci flow through singularities in [15]. Raquel Perales applied it to study the limits of manifolds with boundary in [26, 27]. Some of the properties of intrinsic flat convergence applied in those papers are proved for the first time in this paper. In fact Matveev and the first author applied techniques developed here to prove that the Gromov–Hausdorff and Intrinsic Flat limits of noncollapsing sequences of manifolds with uniform lower bounds on Ricci curvature agree [23]. Other potential applications of intrinsic flat convergence and the properties proved within this article were suggested by Misha Gromov in [12] and the second author in [32].

The initial notions of the intrinsic flat distance and integral current spaces appeared in joint work of the second author with Wenger [34] building upon Ambrosio–Kirchheim's important work on currents in metric spaces [1]. Here we explore new properties and their relationship with intrinsic flat convergence building upon Ambrosio–Kirchheim's Slicing Theorem [1] (see Theorem 2.23) combined with a slightly adapted version of Gromov's

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Filling Volume [11] (see Definition 2.46). These ideas were intuitively applied in prior work of the second author with Wenger to prove the continuity of the filling volumes of spheres under intrinsic flat convergence and prevent the disappearance of points under intrinsic flat convergence [33]. Recall that under intrinsic flat convergence, points may disappear in the limit. In fact the limit space could simply be the **0** space and one must try to avoid this in most applications.

In this paper, we use the full iterative strength of Ambrosio–Kirchheim's Slicing Theorem to introduce and study the *Sliced Filling Volume* (Definitions 3.20 and 3.21), the *Interval Filling Volume* (Definition 3.43), and the *Sliced Interval Filling Volume* (Definition 3.45). We prove that the sliced filling volume is bounded below by constants in the *Tetrahedral Property* and the *Integral Tetrahedral Property* (see Definitions 3.30 and 3.36 and Theorem 3.38). The three dimensional version of the tetrahedral property appears in (1)–(2) and is depicted in Figure 1. Note that some of these notions were first announced by the second author in [30].

We prove the continuity of the Sliced Filling Volumes with respect to intrinsic flat convergence in Theorem 4.20. We prove the continuity of the Interval Filling Volumes and Sliced Interval Filling Volumes in Theorems 4.23 and 4.24. The first author proved the semicontinuity of eigenvalues under volume preserving intrinsic flat convergence in [28]. Here we do not make any assumptions on the preservation of volume in the limit.

We then use the notion of the sliced filling volume to explore when a point does not disappear under intrinsic flat convergence (Theorems 4.27, 4.30 and 4.31). Note that the disappearance of points was also studied in prior work of the second author [31]. However, in that paper, one could not determine if a sequence of points converged to a limit point that was only in the metric completion of the limit space. Here we are able to determine if the limit of the points lies in the intrinsic flat limit itself. Theorems 4.30 and 4.31 are Bolzano–Weierstrass type theorems, producing converging subsequences of points.

This paper culminates with two compactness theorems: the Sliced Filling Compactness Theorem (Theorem 5.1) and the Tetrahedral Compactness Theorem (Theorem 5.2). We state the three dimensional version of the Tetrahedral Compactness Theorem here (including the three dimensional Tetrahedral Property within the statement).

Theorem 1.1. *Given $r_0 > 0, \beta \in (0,1), C > 0, V_0 > 0$, suppose a sequence of Riemannian manifolds, M_i^3, satisfies the C, β* ***tetrahedral property*** *for all balls, $B_p(r) \subset M_i^3$, of radius $r \le r_0$ as in Figure* 1*. That is,*

(1) *there exist $p_1, p_2 \in \partial B_p(r)$ such that for all $t_1, t_2 \in [(1-\beta)r, (1+\beta)r]$ we have*
(2) $\inf\{d(x,y) : x \neq y,\ x,y \in \partial B_p(r) \cap \partial B_{p_1}(t_1) \cap \partial B_{p_2}(t_2)\} \in [Cr, \infty)$.

Assume in addition that each M_i has

$$\mathrm{Vol}(M_i^3) \le V_0 \quad \textit{and} \quad \mathrm{Diam}(M_i^3) \le D_0.$$

Then a subsequence of the M_i converges in the Gromov–Hausdorff and the Intrinsic Flat sense to the same space. In particular, the limit is countably $\mathcal{H}^3$ rectifiable.

One might view this compactness theorem as a higher dimensional analog of the compactness of Alexandrov spaces. The Sliced Filling Compactness Theorem is applied to prove this Tetrahedral Compactness Theorem. It assumes a uniform lower bound on the sliced filling volumes of balls and draws the same conclusion. To prove this theorem, we first prove the Gromov–Hausdorff convergence of a subsequence (Theorem 3.23). We only obtain the fact that the intrinsic flat limit agrees with the Gromov–Hausdorff limit in the final section of the paper by applying our theorems which avoid the disappearance

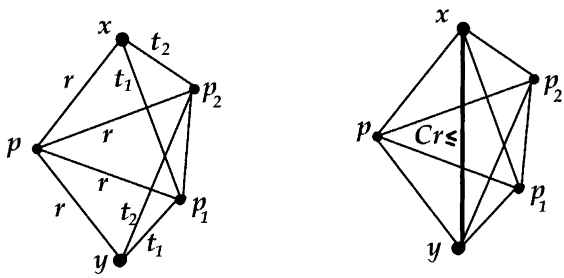

FIGURE 1. Tetrahedral Property as depicted in [30].

of points. Once the two notions of convergence agree, then we can conclude that the limits are noncollapsed countably $\mathcal{H}^m$ rectifiable metric spaces.

These theorems were announced by the second author in [30] but the rigorous proof has required the development of the full theory of sliced filling volumes elaborated herein. Prior results relating intrinsic flat limits to Gromov–Hausdorff limits appear in the joint work of the second author with Wenger concerning sequences of spaces with contractibility functions and noncollapsing manifolds with nonnegative Ricci curvature [33], in the work of Li-Perales concerning Alexandrov spaces [21], in the work of Munn concerning noncollapsing manifolds with pinched Ricci curvature [25], and in the work of Perales concerning noncollapsing Riemannian manifolds with boundary [26]. These prior results apply powerful theorems from the Cheeger–Colding Theory and Alexandrov Geometry. The results contained herein are built only upon the theorems in Ambrosio–Kirchheim's "*Currents on Metric Spaces*" [1] and the ideas in Gromov's "*Filling Riemannian Manifolds*" [11].

Recommended reading. This paper attempts to be completely self-contained, providing all necessary background material within the paper. However, students reading this paper are encouraged to read Burago–Burago–Ivanov's award winning textbook [3] which provides a thorough background in Gromov–Hausdorff convergence and also to read the second author's joint paper with Wenger [34] and the second author's recent paper [31]. Those who would like to understand the Geometric Measure Theory more deeply should read Morgan's textbook [24] or Fanghua Lin's textbook [22] and then the work of Ambrosio–Kirchheim [1].

§2. Background

In this section we review the Gromov–Hausdorff distance introduced by Gromov in [10], then various topics from Ambrosio–Kirchheim's work in [1], then intrinsic flat convergence and integral current spaces from prior work of the second author with Wenger in [34] and end with a review of filling volumes which are related to Gromov's notion from [11] but defined by using the work of Ambrosio–Kirchheim.

2.1. Review of the Gromov–Hausdorff distance. First recall that $\varphi\colon X \to Y$ is an isometric embedding if and only if

$$d_Y(\varphi(x_1), \varphi(x_2)) = d_X(x_1, x_2), \quad x_1, x_2 \in X.$$

This is referred to as a metric isometric embedding in [19] and it should be distinguished from a Riemannian isometric embedding.

Definition 2.1 (Gromov)**.** The Gromov–Hausdorff distance between two compact metric spaces (X, d_X) and (Y, d_Y) is defined as

$$d_{GH}(X,Y) := \inf d_H^Z\big(\varphi(X), \psi(Y)\big) \tag{3}$$

where Z is a complete metric space, and $\varphi\colon X \to Z$ and $\psi\colon Y \to Z$ are isometric embeddings and where the Hausdorff distance in Z is defined as

$$d_H^Z(A,B) = \inf\big\{\epsilon > 0 \,:\, A \subset T_\epsilon(B) \;\text{ and }\; B \subset T_\epsilon(A)\big\}.$$

Gromov proved that this is indeed a distance on compact metric spaces: $d_{GH}(X,Y)=0$ if and only if there is an isometry between X and Y. When studying metric spaces which are only precompact, one may take their metric completions before studying the Gromov–Hausdorff distance between them.

Definition 2.2. A collection of metric spaces is said to be equibounded or uniformly bounded if there is a uniform upper bound on the diameter of the spaces.

Remark 2.3. We will write $N(X,r)$ to denote the maximal number of disjoint balls of radius r in a space X. Note that X can always be covered by $N(X,r)$ balls of radius $2r$.

Note that Ilmanen's Example of [34] of a sequence of spheres with increasingly many splines is not equicompact, as the number of balls centered on the tips approaches infinity.

Definition 2.4. A collection of spaces is said to be equicompact or uniformly compact if they have a common upper bound $N(r)$ such that $N(X,r) \le N(r)$ for all spaces X in the collection.

Gromov's Compactness Theorem states that sequences of equibounded and equicompact metric spaces have a Gromov–Hausdorff converging subsequence [10]. In fact, Gromov proved a stronger version of this statement in [9]:

Theorem 2.5 (Gromov's Compactness Theorem)**.** *If a sequence of compact metric spaces, X_j, is equibounded and equicompact, then a subsequence of the X_j converges to a compact metric space X_∞.*

Gromov also proved the following useful theorem:

Theorem 2.6. *If a sequence of compact metric spaces X_j converges to a compact metric space X_∞, then the X_j are equibounded and equicompact. Furthermore, there is a compact metric space, Z, and isometric embeddings $\varphi_j\colon X_j \to Z$ such that*

$$d_H^Z\big(\varphi_j(X_j), \varphi_\infty(X_\infty)\big) \le 2 d_{GH}(X_j, X_\infty) \to 0.$$

This theorem allows one to define converging sequences of points:

Definition 2.7. We say that $x_j \in X_j$ converges to $x_\infty \in X_\infty$ if there is a common space Z as in Theorem 2.6 such that $\varphi_j(x_j) \to \varphi_\infty(x)$ as points in Z. If one discusses the limits of multiple sequences of points, then one uses a common Z to determine the convergence to avoid difficulties arising from isometries in the limit space. Then one immediately has

$$\lim_{j\to\infty} d_{X_j}(x_j, x'_j) = d_{X_\infty}(x_\infty, x'_\infty)$$

whenever $x_j \to x_\infty$ and $x'_j \to x'_\infty$ via a common Z.

Theorem 2.6 also allows one to extend the Arzela–Ascoli Theorem.

Definition 2.8. A collection of functions, $f_j\colon X_j \to X'_j$ is said to be equicontinuous if for all $\epsilon > 0$ there exists $\delta_\epsilon > 0$ independent of j such that

$$f_j\left(B_x(\delta_\epsilon)\right) \subset B_{f_j(x)}(\epsilon), \quad x \in X_j.$$

Theorem 2.9. *Suppose X_j and X'_j are compact metric spaces converging in the Gromov–Hausdorff sense to compact metric spaces X_∞ and X'_∞, and suppose $f_j\colon X_j \to X'_j$ are equicontinuous, then a subsequence f_{j_i} converges to a continuous function $f_\infty\colon X_\infty \to X'_\infty$ such that for any sequence $x_j \to x_\infty$ via a common Z we have $f_{j_i}(x_{j_i}) \to f_\infty(x_\infty)$.*

In particular, one can define limits of curves $C_i\colon [0,1] \to X_i$ (parametrized proportional to arclength with a uniform upper bound on length) to obtain curves $C_\infty\colon [0,1] \to X_\infty$. So that when X_i are compact length spaces whose distances are achieved by minimizing geodesics, so are the limit spaces X_∞.

One only needs uniform lower bounds on Ricci curvature and upper bounds on diameter to prove equicompactness on a sequence of Riemannian manifolds. This is a consequence of the Bishop–Gromov Volume Comparison Theorem [10]. Colding and Cheeger–Colding studied the limits of such sequences of spaces proving volume convergence and eigenvalue convergence and many other interesting properties [6, 4, 5]. One property of particular interest here, is that when the sequence of manifolds is noncollapsing (i.e., is assumed to have a uniform lower bound on volume), Cheeger–Colding proved that the limit space is countably $\mathcal{H}^m$ rectifiable with the same dimension as the sequence [4].

Greene–Petersen showed that conditions on contractibility and uniform upper bounds on diameter also suffice to achieve compactness without any assumption on Ricci curvature or volume [8]. Sormani–Wenger shown that if one assumes a uniform linear contractibility function on the sequence of manifolds then the limit spaces achieved in their setting are also countably $\mathcal{H}^m$ rectifiable with the same dimension as the sequence. Without the assumption of linearity, Schul–Wenger provided an example where the Gromov–Hausdorff limit is not countably $\mathcal{H}^m$ rectifiable [33]. The proofs here involve the Intrinsic Flat Convergence.

2.2. Review of Ambrosio–Kirchheim currents on metric spaces. In [1], Ambrosio–Kirchheim extended Federer–Fleming's notion of integral currents using De Giorgi's notion of k-tuples of functions. Here we review their ideas. Here Z denotes a complete metric space.

Federer–Fleming currents were defined as linear functionals on differential forms [7]. This approach extends naturally to smooth manifolds but not to complete metric spaces which do not have differential forms. In place of differential forms, Ambrosio–Kirchheim use De Giorgi's $(m+1)$-tuples, $\omega \in \mathcal{D}^m(Z)$,

$$\omega = (f, \pi_1 \dots \pi_m) \in \mathcal{D}^m(Z)$$

where $f\colon X \to \mathbb{R}$ is a bounded Lipschitz function and the $\pi_i\colon X \to \mathbb{R}$ are Lipschitz.

In [1, Definitions 2.1, 2.2, 2.6 and 3.1], an m dimensional current $T \in \mathbf{M}_m(Z)$ was defined. Here we combine them into a single definition.

Definition 2.10. On a complete metric space, Z, an m dimensional ***current***, denoted $T \in \mathbf{M}_m(Z)$, is a real valued *multilinear functional* on $\mathcal{D}^m(Z)$, with the following three required properties.

i) Locality.

$T(f, \pi_1, \dots, \pi_m) = 0$ if there exists $i \in \{1, \dots, m\}$ such that π_i is constant on a neighborhood of $\{f \neq 0\}$.

ii) Continuity.

The continuity of T with respect to the pointwise convergence of π_i with $\mathrm{Lip}(\pi_i) \leq 1$.

iii) Finite mass.

There exists a finite Borel μ such that

$$|T(f,\pi_1,\dots,\pi_m)| \le \prod_{i=1}^m \mathrm{Lip}(\pi_i) \int_Z |f|\,d\mu \quad \text{for all} \quad (f,\pi_1,\dots,\pi_m) \in \mathcal{D}^m(Z).$$

In [1, Definition 2.6] Ambrosio–Kirchheim introduced their mass measure which is distinct from the masses used in work of Gromov [11] and Burago–Ivanov [2]. This definition was later employed to define the notion of filling volume used in this paper.

Definition 2.11. The mass measure, $\|T\|$, of a current, $T \in \mathbf{M}_m(Z)$, is the smallest Borel measure μ such that

$$\big|T(f,\pi)\big| \le \int_X |f|\,d\mu \quad \text{for all} \quad (f,\pi) \quad \text{where} \quad \mathrm{Lip}(\pi_i) \le 1. \tag{4}$$

The mass of T is defined by the formula

$$\mathbf{M}(T) = \|T\|(Z) = \int_Z d\|T\|. \tag{5}$$

In particular

$$\big|T(f,\pi_1,\dots,\pi_m)\big| \le \mathbf{M}(T)\,|f|_\infty\,\mathrm{Lip}(\pi_1)\dots\mathrm{Lip}(\pi_m). \tag{6}$$

Stronger versions of locality and continuity, as well as product and chain rules were proved in [1, Theorem 3.5]. In particular, $T(f,\pi_1,\dots,\pi_m)$ was defined there for f that are only Borel functions as limits of $T(f_j,\pi_1,\dots,\pi_m)$, where the f_j are bounded Lipschitz functions converging to f in $L^1(E,\|T\|)$. Also, it was proved in [1] that

$$T(f,\pi_\sigma(1),\dots,\pi_\sigma(m)) = \mathrm{sgn}(\sigma)\,T(f,\pi_1,\dots,\pi_m)$$

for any permutation, σ, of $\{1,2,\dots,m\}$.

Ambrosio–Kirchheim then defined restriction [1, Definition 2.5].

Definition 2.12. The *restriction* $T\llcorner\omega \in \mathbf{M}_m(Z)$ of a current $T \in M_{m+k}(Z)$ by a $k+1$ tuple $\omega = (g,\tau_1,\dots,\tau_k) \in \mathcal{D}^k(Z)$ is given by

$$(T\llcorner\omega)(f,\pi_1,\dots,\pi_m) := T(f\cdot g,\tau_1,\dots,\tau_k,\pi_1,\dots,\pi_m).$$

Given a Borel set, A,

$$T\llcorner A := T\llcorner\omega$$

where $\omega = \mathbf{1}_A \in \mathcal{D}^0(Z)$ is the indicator function of the set. In this case,

$$\mathbf{M}(T\llcorner\omega) = \|T\|(A).$$

Ambrosio–Kirchheim then defined the push forward map:

Definition 2.13. Given a Lipschitz map $\varphi\colon Z \to Z'$, the push forward of a current $T \in \mathbf{M}_m(Z)$ to a current $\varphi_\# T \in \mathbf{M}_m(Z')$ is given in [1, Definition 2.4] by

$$\varphi_\# T(f,\pi_1,\dots,\pi_m) := T(f\circ\varphi,\pi_1\circ\varphi,\dots,\pi_m\circ\varphi) \tag{7}$$

exactly as in the smooth setting.

Remark 2.14. One immediately sees that

$$(\varphi_\# T)\llcorner(f,\pi_1,\dots,\pi_k) = \varphi_\#\big(T\llcorner(f\circ\varphi,\pi_1\circ\varphi,\dots,\pi_k\circ\varphi)\big)$$

and

$$(\varphi_\# T)\llcorner A = (\varphi_\# T)\llcorner(\mathbf{1}_A) = \varphi_\#\big(T\llcorner(\mathbf{1}_A\circ\varphi)\big) = \varphi_\#\big(T\llcorner\varphi^{-1}(A)\big).$$

In [1, (2.4)], Ambrosio–Kirchheim showed that

$$\|\varphi_\# T\| \le [\mathrm{Lip}(\varphi)]^m \varphi_\# \|T\|, \tag{8}$$

so that when φ is an isometric embedding, we have

$$\|\varphi_\# T\| = \varphi_\# \|T\| \ \text{ and } \ \mathbf{M}(T) = \mathbf{M}(\varphi_\# T). \tag{9}$$

The simplest example of a current is as follows.

Example 2.15. If one has a bi-Lipschitz map, $\varphi\colon \mathbb{R}^m \to Z$, and a Lebesgue function $h \in L^1(A, \mathbb{Z})$ where $A \in \mathbb{R}^m$ is Borel, then $\varphi_\# [\![h]\!] \in \mathbf{M}_m(Z)$ is an m dimensional current in Z. Note that

$$\varphi_\# [\![h]\!](f, \pi_1, \ldots, \pi_m) = \int_{A \subset \mathbb{R}^m} (h \circ \varphi)(f \circ \varphi)\, d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_m \circ \varphi)$$

where $d(\pi_i \circ \varphi)$ is well defined almost everywhere by Rademacher's Theorem. Here the mass measure is

$$\|[\![h]\!]\| = h\, d\mathcal{L}_m$$

and the mass is

$$\mathbf{M}([\![h]\!]) = \int_A h\, d\mathcal{L}_m.$$

In [1, Theorem 4.6] Ambrosio–Kirchheim defined a canonical set associated with any integer rectifiable current.

Definition 2.16. The (canonical) set of a current, T, is the collection of points in Z with positive lower density:

$$\mathrm{set}(T) = \big\{p \in Z \,:\, \Theta_{*m}(\|T\|, p) > 0\big\}, \tag{10}$$

where the definition of the lower density is

$$\Theta_{*m}(\mu, p) = \liminf_{r \to 0} \frac{\mu(B_p(r))}{\omega_m r^m}. \tag{11}$$

In [1, Definition 4.2 and Theorems 4.5–4.6], an integer rectifiable current was defined using the Hausdorff measure, $\mathcal{H}^m$.

Definition 2.17. Let $m \ge 1$. A current, $T \in \mathcal{D}_m(Z)$, is rectifiable if $\mathrm{set}(T)$ is countably $\mathcal{H}^m$ rectifiable and if $\|T\|(A) = 0$ for any set A such that $\mathcal{H}^m(A) = 0$. We write $T \in \mathcal{R}_m(Z)$.

We say that $T \in \mathcal{R}_m(Z)$ is integer rectifiable, denoted $T \in \mathcal{I}_m(Z)$, if for any $\varphi \in \mathrm{Lip}(Z, \mathbb{R}^m)$ and any open set $A \in Z$, we have

$$\text{there exists } \ \theta \in \mathcal{L}^1(\mathbb{R}^m, Z) \ \text{ such that } \ \varphi_\#(T \llcorner A) = [\![\theta]\!].$$

In fact, $T \in \mathcal{I}_m(Z)$ if and only if it has a parametrization. A parametrization $(\{\varphi_i\}, \{\theta_i\})$ of an integer rectifiable current $T \in \mathcal{I}_m(Z)$ is a collection of bi-Lipschitz maps $\varphi_i\colon A_i \to Z$ with $A_i \subset \mathbb{R}^m$ precompact Borel measurable and with pairwise disjoint images and weight functions $\theta_i \in L^1(A_i, \mathbb{N})$ such that

$$T = \sum_{i=1}^{\infty} \varphi_{i\#} [\![\theta_i]\!] \ \text{ and } \ \mathbf{M}(T) = \sum_{i=1}^{\infty} \mathbf{M}\big(\varphi_{i\#}[\![\theta_i]\!]\big). \tag{12}$$

A 0 dimensional rectifiable current is defined by the existence of countably many distinct points $\{x_i\} \in Z$, weights $\theta_i \in \mathbb{R}^+$, and orientation, $\sigma_i \in \{-1, +1\}$ such that

$$T(f) = \sum_i \sigma_i \theta_i f(x_i), \quad f \in \mathcal{B}^\infty(Z), \tag{13}$$

where $\mathcal{B}^\infty(Z)$ is the class of bounded Borel functions on Z and where

$$\mathbf{M}(T) = \sum_i \theta_i < \infty.$$

If T is integer rectifiable, then $\theta_i \in \mathbb{Z}^+$, so the sum must be finite.

In particular, the mass measure of $T \in \mathbf{I}_m(Z)$ satisfies

$$\|T\| = \sum_{i=1}^{\infty} \left\|\varphi_{i\#}[\![\theta_i]\!]\right\|.$$

Theorems 4.3 and 8.8 of [1] provide necessary and sufficient criteria for determining when a current is integer rectifiable.

Note that the current in Example 2.15 is an integer rectifiable current.

Example 2.18. For a Riemannian manifold, M^m, and a bi-Lipschitz map $\varphi\colon M^m \to Z$, the formula $T = \varphi_\#[\![\mathbf{1}_M]\!]$ defines an integer rectifiable current of dimension m in Z. If φ is an isometric embedding, and $Z = M$, then $\mathbf{M}(T) = \mathrm{Vol}(M^m)$. Note further that $\mathrm{set}(T) = \varphi(M)$.

If M has a conical singularity, then $\mathrm{set}(T) = \varphi(M)$. However, if M has a cusp singularity at a point $p \in M$, then $\mathrm{set}(T) = \varphi(M \setminus \{p\})$.

Definition 2.19 (see [1, Definition 2.3]). The *boundary* of $T \in \mathbf{M}_m(Z)$ is defined by

$$\partial T(f, \pi_1, \ldots, \pi_{m-1}) := T(1, f, \pi_1, \ldots, \pi_{m-1}) \in \mathbf{M}_{m-1}(Z) \tag{14}$$

When $m = 0$, we set $\partial T = 0$.

Note that

$$\varphi_\#(\partial T) = \partial(\varphi_\# T).$$

Definition 2.20 (see [1, Definition 3.4 and 4.2]). An integer rectifiable current $T \in \mathcal{I}_m(Z)$ is called an integral current, denoted $T \in \mathbf{I}_m(Z)$, if ∂T defined as in (14) has finite mass. The total mass of an integral current is

$$\mathbf{N}(T) = \mathbf{M}(T) + \mathbf{M}(\partial T).$$

Observe that $\partial\partial T = 0$. In [1, Theorem 8.6], Ambrosio–Kirchheim proved that

$$\partial\colon \mathbf{I}_m(Z) \to \mathbf{I}_{m-1}(Z)$$

whenever $m \geq 1$. By (8) one can see that if $\varphi\colon Z_1 \to Z_2$ is Lipschitz, then

$$\varphi_\#\colon \mathbf{I}_m(Z_1) \to \mathbf{I}_m(Z_2).$$

However, the restriction of an integral current need not be an integral current except in special circumstances. For example, T might be integration over $[0,1]^2$ with the Euclidean metric and $A \subset [0,1]^2$ could have an infinitely long boundary, so that $T \llcorner A \notin \mathbf{I}_2([0,1]^2)$ because $\partial(T \llcorner A)$ has infinite mass.

Remark 2.21. If T is an $\mathcal{H}^1$ integral current, then ∂T is an $\mathcal{H}^0$ integer rectifiable current so $H = \mathrm{set}\,\partial T$ must be finite and $\theta_{p_h} = \|\partial T\|(p_h) \in \mathbb{Z}^+$ for all $p \in H$ and

$$\partial T(f) = \sum_{h \in H} \sigma_h \theta_h f(p_h), \quad f \in \mathcal{B}^\infty(Z), \tag{15}$$

as described above. In addition, we have

$$0 = T(1,1) = \partial T(1) = \sum_{h \in H} \sigma_h \theta_h.$$

Example 2.22. If T is an $\mathcal{H}^1$ rectifiable current, then

$$T=\sum_{i=1}^{\infty}\sigma_i\theta_i\varphi_{i\#}[\![\chi_{A_i}]\!]$$

where $\theta_i\in\mathbb{Z}^+$, $\sigma_i\in\{+1,-1\}$ and A_i is an interval with $\bar{A}_i=[a_i,b_i]$ because all Borel sets are unions of intervals and all integer valued Borel functions can be written up to Lebesgue measure 0 as a countable sum of characteristic functions of intervals. One might like to write:

$$\partial T(f)=\sum_i\sigma_i\theta_i\left(f(\varphi_i(b_i))-f(\varphi_i(a_i))\right),\quad f\in\mathcal{B}^\infty(Z).$$

This works when the sum happens to be a finite sum. Yet if T is a infinite collection of circles based at a common point, $(0,0)\in\mathbb{R}^2$, defined with $\sigma_i=1$, $\theta_i=1$, $A_i=[0,\pi]$ and

$$\varphi_i(s)=(r_i\cos(s)+r_i,r_i\sin(s))\ \text{ for }\ i\ \text{ odd and}$$
$$\varphi_i(s)=(r_i\cos(s+\pi)+r_i,r_i\sin(s+\pi))\ \text{ for }\ i\ \text{ even}$$

where $r_{2i}=r_{2i-1}=1/i^2$, then

$$\varphi_i(a_i)=(2r_i,0)\ \text{ and }\ \varphi_i(b_i)=(0,0)\ \text{ for }\ i\ \text{ odd and}$$
$$\varphi_i(a_i)=(0,0)\ \text{ and }\ \varphi_i(b_i)=(2r_i,0)\ \text{ for }\ i\ \text{ even.}$$

So when $f(0,0)=1$, we end up with an infinite sum whose terms are all $+1$ and -1.

2.3. Review of Ambrosio–Kirchheim slicing theorems. As in Federer–Fleming, Ambrosio–Kirchheim considered the slices of currents.

Theorem 2.23 (see [1, Theorems 5.6–5.7])**.** *Let Z be a complete metric space, $T\in\mathbf{I}_mZ$, and $f\colon Z\to\mathbb{R}$ a Lipschitz function. Let*

$$\langle T,f,s\rangle:=(\partial T)\llcorner f^{-1}(s,\infty)-\partial\big(T\llcorner f^{-1}(s,\infty)\big).$$

Observe that

$$\operatorname{set}\big(\langle T,f,s\rangle\big)\subset\big(\operatorname{set}(T)\cup\operatorname{set}(\partial T)\big)\cap f^{-1}(s),$$

and

$$\partial\langle T,f,s\rangle=\langle-\partial T,f,s\rangle. \tag{16}$$

Furthermore, $\langle T_1+T_2,f,s\rangle=\langle T_1,f,s\rangle+\langle T_2,f,s\rangle$. For almost every slice $s\in\mathbb{R}$, $\langle T,f,s\rangle$ is an integral current and we can integrate the masses to obtain:

$$\int_{s\in\mathbb{R}}\mathbf{M}\big(\langle T,f,s\rangle\big)\,ds=\mathbf{M}(T\llcorner df)\le\operatorname{Lip}(f)\,\mathbf{M}(T)$$

where

$$(T\llcorner df)(h,\pi_1,\dots,\pi_{m-1})=T(h,f,\pi_1,\dots,\pi_{m-1}).$$

In particular, for almost every $s>0$ one has

$$T\llcorner f^{-1}(-\infty,s]\in\mathbf{I}_{m-1}(Z).$$

Furthermore, for all Borel sets A we have

$$\langle T\llcorner A,f,s\rangle=\langle T,f,S\rangle\llcorner A$$

and

$$\int_{s\in\mathbb{R}}\|\langle T,f,s\rangle\|(A)\,ds=\|T\llcorner\, df\|(A).$$

Remark 2.24. Observe that for any $T \in \mathbf{I}_m(Z')$, and any Lipschitz functions, $\varphi\colon Z \to Z'$ and $f\colon Z' \to \mathbb{R}$ and any $s > 0$, we have

$$\begin{aligned}\langle \varphi_\# T, f, s\rangle &= \partial\big((\varphi_\# T) \llcorner f^{-1}(-\infty, s]\big) - (\partial \varphi_\# T) \llcorner f^{-1}(-\infty, s]\big) \\ &= \partial\big(\varphi_\#(T \llcorner \varphi^{-1}(f^{-1}(-\infty, s]))\big) - (\varphi_\# \partial T) \llcorner f^{-1}(-\infty, s]) \\ &= \partial\big(\varphi_\#(T \llcorner (f\circ\varphi)^{-1}(-\infty, s]\big) - \varphi_\#\big(\partial T \llcorner \varphi^{-1}(f^{-1}(-\infty, s])\big) \\ &= \big(\varphi_\# \partial(T \llcorner (f\circ\varphi)^{-1}(-\infty, s])\big) - \varphi_\#\big(\partial T \llcorner (f\circ\varphi)^{-1}(-\infty, s]\big) \\ &= \varphi_\# \langle T, (f\circ\varphi), s\rangle.\end{aligned}$$

Remark 2.25. Ambrosio–Kirchheim then iterated this definition, $f_i\colon Z \to \mathbb{R}$, $s_i \in \mathbb{R}$, to define iterated slices:

$$\langle T, f_1, \ldots, f_k, s_1, \ldots, s_k\rangle = \big\langle \langle T, f_1, \ldots, f_{k-1}, s_1, \ldots, s_{k-1}\rangle, f_k, s_k\big\rangle, \tag{17}$$

so that

$$\begin{aligned}&\langle T_1 + T_2, f_1, \ldots, f_k, s_1, \ldots, s_k\rangle \\ &\qquad = \langle T_1, f_1, \ldots, f_k, s_1, \ldots, s_k\rangle + \langle T_2, f_1, \ldots, f_k, s_1, \ldots, s_k\rangle.\end{aligned} \tag{18}$$

In [1, Lemma 5.9] they proved the formula

$$\begin{aligned}&\langle T, f_1, \ldots, f_k, s_1, \ldots, s_k\rangle \\ &\qquad = \big\langle \langle T, f_1, \ldots, f_i, s_1, \ldots, s_i\rangle, f_{i+1}, \ldots, f_k, s_{i+1}, \ldots, s_k\big\rangle.\end{aligned} \tag{19}$$

In [1, (5.9)] they proved that

$$\int_{\mathbb{R}^k} \|\langle T, f_1, \ldots, f_k, s_1, \ldots, s_k\rangle\| \, ds_1 \ldots ds_k = \|T \llcorner (1, f_1, \ldots, f_k)\|, \tag{20}$$

where

$$(T \llcorner df)(h, \pi_1, \ldots, \pi_{m-k}) = T(h, f_1, \ldots, f_k, \pi_1, \ldots, \pi_{m-k}),$$

so

$$\int_{\mathbb{R}^k} \mathbf{M}\big(\langle T, f_1, \ldots, f_k, s_1, \ldots, s_k\rangle\big) \, d\mathcal{L}^k = \mathbf{M}(T \llcorner df) \le \prod_{j=1}^k \operatorname{Lip}(f_j)\, \mathbf{M}(T). \tag{21}$$

In [1, (5.15)] they proved that

$$\langle T \llcorner A, f_1, \ldots, f_k, s_1, \ldots, s_k\rangle = \langle T, f_1, \ldots, f_k, s_1, \ldots, s_k\rangle \llcorner A \tag{22}$$

for any Borel set $A \subset Z$ and $\mathcal{L}^m$ almost every $(s_1, \ldots, s_k) \in \mathbb{R}^k$ and

$$\int_{s\in\mathbb{R}^k} \|\langle T, f_1, \ldots, f_k, s_1, \ldots, s_k\rangle\|(A) \, ds = \|T \llcorner df\|(A)$$

for any Borel set $A \subset Z$ and $\mathcal{L}^m$ almost every $(s_1, \ldots, s_k) \in \mathbb{R}^k$.

By (23) one can easily prove by induction that

$$\partial\langle T, f_1, \ldots, f_k, s_1, \ldots, s_k\rangle = (-1)^k \langle \partial T, f_1, \ldots, f_k, s_1, \ldots, s_k\rangle. \tag{23}$$

In [1, Theorem 5.7] they proved that

$$\langle T, f_1, \ldots, f_k, s_1, \ldots, s_k\rangle \in \mathbf{I}_{m-k}(Z)$$

for $\mathcal{L}^k$ almost every $(s_1, \ldots, s_k) \in \mathbb{R}^k$. By Remark 2.24 one can prove inductively that

$$\langle \varphi_\# T, f_1, \ldots, f_k, s_1, \ldots, s_k\rangle = \varphi_\# \langle T, f_1 \circ \varphi, \ldots, f_k \circ \varphi, s_1, \ldots, s_k\rangle. \tag{24}$$

2.4. Review of convergence of currents. Ambrosio–Kirchheim's Compactness Theorem, which extends Federer–Fleming's Flat Norm Compactness Theorem, is stated in terms of weak convergence of currents. See Definition 3.6 in [1], which extends Federer–Fleming's notion of weak convergence except that they do not require compact support.

Definition 2.26. A sequence of integral currents $T_j \in \mathbf{I}_m(Z)$ is said to converge weakly to a current T if and only if the pointwise limits satisfy

$$\lim_{j\to\infty} T_j\,(f, \pi_1, \ldots, \pi_m) = T\,(f, \pi_1, \ldots, \pi_m)$$

for all bounded Lipschitz $f\colon Z \to \mathbb{R}$ and Lipschitz $\pi_i\colon Z \to \mathbb{R}$. We write

$$T_j \to T.$$

One sees immediately that $T_j \to T$ implies

$$\partial T_j \to \partial T,$$
$$\varphi_\# T_j \to \varphi_\# T$$

and

$$T_j \llcorner (f, \pi_1, \ldots, \pi_k) \to T \llcorner (f, \pi_1, \ldots, \pi_k).$$

However $T_j \llcorner A$ need not converge weakly to $T_j \llcorner A$ as seen in the following example.

Example 2.27. Let $Z = \mathbb{R}^2$ with the Euclidean metric. Let $\varphi_j\colon [0,1] \to Z$ be $\varphi_j(t) = (1/j, t)$ and $\varphi_\infty(t) = (0,t)$. Let $S \in \mathbf{I}_1([0,1])$ be

$$S(f, \pi_1) = \int_0^1 f\, d\pi_1.$$

Let $T_j \in \mathbf{I}_1(Z)$ be defined by $T_j = \varphi_{j\#}(S)$. Then $T_j \to T_\infty$. Taking $A = [0,1] \times (0,1)$, we see that $T_j \llcorner A = T_j$ but $T_\infty \llcorner A = 0$.

Immediately below the definition of weak convergence [1, Definition 3.6], Ambrosio–Kirchheim proved the lower semicontinuity of mass.

Remark 2.28. If T_j converges weakly to T, then $\liminf_{j\to\infty} \mathbf{M}(T_j) \geq \mathbf{M}(T)$.

Theorem 2.29 (Ambrosio–Kirchheim Compactness)**.** *Consider any complete metric space Z, a compact set $K \subset Z$, and $A_0, V_0 > 0$. Given any sequence of integral currents $T_j \in \mathbf{I}_m(Z)$ satisfying*

$$\mathbf{M}(T_j) \leq V_0, \quad \mathbf{M}(\partial T_j) \leq A_0, \quad \textit{and} \quad \mathrm{set}(T_j) \subset K,$$

there exists a subsequence, T_{j_i}, and a limit current $T \in \mathbf{I}_m(Z)$ such that T_{j_i} converges weakly to T.

2.5. Review of integral current spaces. The notion of an integral current space was introduced by the second author and Stefan Wenger in [34].

Definition 2.30. An m dimensional metric space (X, d, T) is called an integral current space if it has an integral current structure $T \in \mathbf{I}_m(\bar{X})$, where $\bar{X}$ is the metric completion of X and $\mathrm{set}(T) = X$. Given an integral current space $M = (X, d, T)$, we will use $\mathrm{set}(M)$ or X_M to denote X, $d_M = d$ and $[\![M]\!] = T$.

Note that $\mathrm{set}(\partial T) \subset \bar{X}$. The boundary of (X, d, T) is then the integral current space:

$$\partial(X, d_X, T) := \big(\mathrm{set}(\partial T), d_{\bar{X}}, \partial T\big).$$

If $\partial T = 0$, then we say (X, d, T) is an integral current without boundary.

Remark 2.31. Note that any m dimensional integral current space is countably $\mathcal{H}^m$ rectifiable with orientated charts, φ_i and weights θ_i provided as in (12). A 0 dimensional integral current space is a finite collection of points with orientations σ_i and weights θ_i provided as in (13). If this space is the boundary of a 1 dimensional integral current space, then as in Remark 2.21, the sum of the signed weights is 0.

Example 2.32. A compact oriented Riemannian manifold with boundary, M^m, is an integral current space, where $X = M^m$, d is the standard metric on M and T is integration over M. In this case $\mathbf{M}(M) = \mathrm{Vol}(M)$ and ∂M is the boundary manifold. When M has no boundary, $\partial M = 0$.

Definition 2.33. The space of $m \geq 0$ dimensional integral current spaces, $\mathcal{M}^m$, consists of all metric spaces that are integral current spaces with currents of dimension m as in Definition 2.30 as well as the $\mathbf{0}$ spaces. Then $\partial\colon \mathcal{M}^{m+1} \to \mathcal{M}^m$.

Remark 2.34. A 0 dimensional integral current space, $M = (X, d, T)$, is a finite collection of points, $\{p_1, \dots, p_N\}$, with a metric $d_{i,j} = d(p_i, p_j)$ and a current structure defined by assigning a weight, $\theta_i \in \mathbb{Z}^+$, and an orientation, $\sigma_i \in \{+1, -1\}$ to each $p_i \in X$ and

$$\mathbf{M}(M) = \sum_{i=1}^{N} \theta_i.$$

If M is the boundary of a 1 dimensional integral current space then, as in Remark 2.21, we have

$$\sum_{i=1}^{N} \sigma_i \theta_i = 0.$$

In particular $N \geq 2$ if $M \neq \mathbf{0}$.

Any compact Riemannian manifold with boundary is an integral current space. Additional examples appear in the work of Wenger and the second author [34].

We end this subsection with an example of an integral current space that is applied in this paper to justify the hypothesis of many of our results.

Example 2.35. Consider the one dimensional integral current space (X, d, T), where

$$X = \{0\} \cup \bigcup_{j=1}^{\infty} \partial B(0, R_j) \subset \mathbb{E}^2$$

where $(\mathbb{E}^2, d_{\mathbb{E}^2})$ is the Euclidean plane, with the restricted metric, $d = d_{\mathbb{E}^2}$, where

$$T(\omega) = \sum_{j=1}^{\infty} [\![\partial B(0, R_j)]\!]$$

is the integral current in $\bar{X}$ and in $\mathbb{E}^2$ and where $R_j = 1/2^j$. Observe that for

$$N_r = \inf\{j \,:\, 1/2^j < r\} \subset [\log_2(1/r), \log_2(r) + 1]$$

we have

$$\|T\|(B(0,r)) = \sum_{j \geq N_r}^{\infty} \mathcal{H}^1(\partial B(0, R_j)) = \sum_{j \geq N_r}^{\infty} \frac{2\pi}{2^j} = \frac{4\pi}{2^{N_r}} \in \left[\frac{8\pi}{r}, \frac{4\pi}{r}\right].$$

In this way the total mass is finite and $\{0\} \in X$. Observe that $\partial T = 0$.

2.6. Review of the Intrinsic Flat distance. The Intrinsic Flat distance was defined in the work of the second author and Stefan Wenger, see [34], as a new distance between Riemannian manifolds based upon the work of Ambrosio–Kirchheim reviewed above.

Recall that the flat distance between m dimensional integral currents $S, T \in \mathbf{I}_m(Z)$ is given by

$$d_F^Z(S,T) := \inf\left\{\mathbf{M}(U) + \mathbf{M}(V) \,:\, S - T = U + \partial V\right\} \tag{25}$$

where $U \in \mathbf{I}_m(Z)$ and $V \in \mathbf{I}_{m+1}(Z)$. This notion of a flat distance was first introduced by Whitney in [37] and later adapted to rectifiable currents by Federer–Fleming [7]. The flat distance between Ambrosio–Kirchheim's integral currents was studied by Wenger in [35]. In particular, Wenger proved that if $T_j \in \mathbf{I}_m(Z)$ has $\mathbf{M}(T_j) \le V_0$ and $\mathbf{M}(\partial T_j) \le A_0$, then

$$T_j \to T \ \text{ if and only if } \ d_F^Z(T_j, T) \to 0$$

exactly as in Federer–Fleming.

The intrinsic flat distance between integral current spaces was first defined in [34, Definition 1.1].

Definition 2.36. For $M_1 = (X_1, d_1, T_1)$ and $M_2 = (X_2, d_2, T_2) \in \mathcal{M}^m$, let the intrinsic flat distance be defined as follows:

$$d_{\mathcal{F}}(M_1, M_2) := \inf d_F^Z\big(\varphi_{1\#}T_1, \varphi_{2\#}T_2\big), \tag{26}$$

where the infimum is taken over all complete metric spaces (Z, d) and isometric embeddings $\varphi_1 \colon (\bar{X}_1, d_1) \to (Z, d)$ and $\varphi_2 \colon (\bar{X}_2, d_2) \to (Z, d)$ and the flat norm d_F^Z is taken in Z. Here $\bar{X}_i$ denotes the metric completion of X_i and d_i is the extension of d_i to $\bar{X}_i$, while $\phi_\# T$ denotes the push forward of T.

In [34], it was observed that

$$d_{\mathcal{F}}(M_1, M_2) \le d_{\mathcal{F}}(M_1, 0) + d_{\mathcal{F}}(0, M_2) \le \mathbf{M}(M_1) + \mathbf{M}(M_2).$$

There it was also proved that $d_{\mathcal{F}}$ satisfies the triangle inequality [34, Theorem 3.2] and is a distance.

Theorem 2.37 (see [34, Theorem 3.27]). *Let M, N be precompact integral current spaces and suppose that $d_{\mathcal{F}}(M, N) = 0$. Then there is a current preserving isometry from M to N where an isometry $f \colon X_M \to X_N$ is called a current preserving isometry between M and N if its extension $\bar{f} \colon \bar{X}_M \to \bar{X}_N$ pushes forward the current structure on M to the current structure on N: $\bar{f}_\# T_M = T_N$.*

In [34, Theorem 3.23], the following fact was also proved.

Theorem 2.38 (see [34, Theorem 4.23]). *Given a pair of precompact integral current spaces, $M_1^m = (X_1, d_1, T_1)$ and $M_2^m = (X_2, d_2, T_2)$, there exists a compact metric space, (Z, d_Z), integral currents $U \in \mathbf{I}_m(Z)$ and $V \in \mathbf{I}_{m+1}(Z)$, and isometric embeddings $\varphi_1 \colon \bar{X}_1 \to Z$ and $\varphi_2 \colon \bar{X}_2 \to Z$ with*

$$\varphi_\# T_1 - \varphi'_\# T_2 = U + \partial V \tag{27}$$

such that

$$d_{\mathcal{F}}(M_1, M_2) = \mathbf{M}(U) + \mathbf{M}(V). \tag{28}$$

Remark 2.39. The metric space Z in Theorem 2.38 has

$$\mathrm{Diam}(Z) \le 3\,\mathrm{Diam}(X_1) + 3\,\mathrm{Diam}(X_2).$$

This is seen by consulting the proof of Theorem 3.23 in [34], where Z is constructed as the injective envelope of the Gromov–Hausdorff limit of a sequence of spaces Z_n with this same diameter bound.

The following theorem in [34] is an immediate consequence of Gromov's and Ambrosio–Kirchheim's Compactness Theorems.

Theorem 2.40. *Given a sequence of m dimensional integral current spaces $M_j = (X_j, d_j, T_j)$ such that X_j are equibounded and equicompact and with uniform upper bounds on mass and boundary mass, some subsequence of it converges in the Gromov–Hausdorff sense $(X_{j_i}, d_{j_i}) \xrightarrow{\mathrm{GH}} (Y, d_Y)$ and in the intrinsic flat sense $(X_{j_i}, d_{j_i}, T_{j_i}) \xrightarrow{\mathcal{F}} (X, d, T)$ where either (X, d, T) is an m dimensional integral current space with $X \subset Y$ or it is the $\mathbf{0}$ current space.*

Obviously, if Y has Hausdorff dimension less than m, then $(X, d, T) = \mathbf{0}$. In [34, Example A.7], there is an example where M_j are compact three dimensional Riemannian manifolds with positive scalar curvature that converge in the Gromov–Hausdorff sense to a standard three sphere but in the Intrinsic Flat sense to $\mathbf{0}$. It was proved in [33] that if the (X_j, d_j, T_j) are compact Riemannian manifolds with nonnegative Ricci curvature or a uniform linear contractibility function, then the intrinsic flat and Gromov–Hausdorff limits agree.

There are many examples of sequences of Riemannian manifolds which have no Gromov–Hausdorff limit but have an intrinsic flat limit. The first is Ilmanen's Example of an increasingly hairy three sphere with positive scalar curvature described in [34, Example A.7]. Other examples appear in the work of the second author with Dan Lee concerning the stability of the Positive Mass Theorem [19, 18] and in the work of the second author with Sajjad Lakzian concerning smooth convergence away from singular sets [17].

The following three theorems were proved in the work of the second author with Wenger [34]. Combining these theorems with the work of Ambrosio–Kirchheim reviewed earlier will lead to many of the properties of Intrinsic Flat Convergence described in this paper.

Theorem 2.41 (see [34, Theorem 4.2])**.** *If a sequence of integral current spaces $M_j = (X_j, d_j, T_j)$ converges in the intrinsic flat sense to an integral current space, $M_0 = (X_0, d_0, T_0)$, then there is a separable complete metric space, Z, and isometric embeddings $\varphi_j \colon X_j \to Z$ such that $\varphi_{j\#} T_j$ flat converges to $\varphi_{0\#} T_0$ in Z and thus converges weakly as well.*

Theorem 2.42 (see [34, Theorem 4.3])**.** *If a sequence of integral current spaces $M_j = (X_j, d_j, T_j)$ converges in the intrinsic flat sense to the zero integral current space, $\mathbf{0}$, then we may choose points $x_j \in X_j$ and a separable complete metric space, Z, and isometric embeddings $\varphi_j \colon X_j \to Z$ such that $\varphi_j(x_j) = z_0 \in Z$ and $\varphi_{j\#} T_j$ flat converges to 0 in Z and thus converges weakly as well.*

Theorem 2.43. *If a sequence of integral current spaces M_j converges in the intrinsic flat sense to an integral current space, M_∞, then*

$$\liminf_{i \to \infty} \mathbf{M}(M_i) \geq \mathbf{M}(M_\infty).$$

Proof. This follows from Theorems 2.41 and 2.42 combined with Ambrosio–Kirchheim's lower semicontinuity of mass [cf. Remark 2.28]. □

Finally there is Wenger's Compactness Theorem [36].

Theorem 2.44 (Wenger)**.** *Given $A_0, V_0, D_0 > 0$, if $M_j = (X_j, d_j, T_j)$ are integral current spaces such that*

$$\mathrm{Diam}(M_j) \leq D_0, \quad \mathbf{M}(M_j) \leq V_0, \quad \mathbf{M}(\partial(M_j)) \leq A_0,$$

then a subsequence converges in the Intrinsic Flat Sense to an integral current space of the same dimension, possibly the **0** *space.*

Recall that this theorem applies to oriented Riemannian manifolds of the same dimension with a uniform upper bound on volume and a uniform upper bound on the volumes of the boundaries. One immediately sees that the conditions required to apply Wenger's Compactness Theorem are far weaker than the conditions required for Gromov's Compactness Theorem. The only difficulty lies in determining whether the limit space is **0** or not. Wenger's proof involves a thick thin decomposition, a study of filling volumes and involves the notion of an ultralimit.

It should be noted that Theorems 2.41–2.43 and all other theorems reviewed and proved within this paper are proved without applying Wenger's Compactness Theorem. Thus one may wish to attempt alternate proofs of Wenger's Compactness Theorem using the results in this paper.

We end this subsection with an example of a converging sequence of integral current spaces that is applied in this paper to justify many of our hypotheses. Many other examples appear in work of Wenger and the second author [34].

Example 2.45. We will construct a particular sequence of one-dimensional integral current spaces M_ℓ that converges in the intrinsic flat sense to the integral current space M induced by the standard one-dimensional torus of length 1 denoted by $\mathbb{T}$.

We define a sequence $T_k \in \mathbf{I}_1(\mathbb{T})$ as follows. Let $A_{i,n}$ $(i = 0, \ldots, 2^n - 1)$ denote the dyadic interval

$$A_{i,n} = \left[\frac{i}{2^n}, \frac{i+1}{2^n}\right] \subset \mathbb{T},$$

and let $T_{i,j,n} \in \mathbf{I}_1(\mathbb{T})$ for $0 \le i < j \le 2^n - 1$ be defined by

$$T_{i,j,n} = [\![\chi_{A_{i,n}}]\!] + [\![\chi_{A_{j,n}}]\!],$$

where χ_A denotes the characteristic function of a set $A \subset \mathbb{T}$. Reindex $T_k = T_{i,j,n}$ according to $k = k(i,j,n)$ such that k is one-to-one, onto $\mathbb{N}$ and $k(i_1, j_1, n_1) \le k(i_2, j_2, n_2)$ if and only if $n_1 \le n_2$.

Let $T = [\![1]\!] \in \mathbf{I}_1(\mathbb{T})$, let for every $k \in \mathbb{N}$, M_{2k} and M_{2k+1} be the one-dimensional integral current spaces associated with the currents $T - T_k$ and $T + T_k$ respectively. Note moreover that M is the integral current space associated with T.

Then

$$d_{\mathcal{F}}(M_{2k}, M) \le d_F^Z(T - T_k, T) \le \mathbf{M}(T_k) \to 0.$$

Similarly, $M_{2k+1} \xrightarrow{\mathcal{F}} M$, so that $M_\ell \to M$ as $\ell \to \infty$.

2.7. Filling volumes. The notion of a filling volume was first introduced by Gromov in [11]. Wenger studied the filling volumes of integral currents in metric spaces in [35]. This was applied in the joint work of the second author with Wenger in [33].

First we discuss the Plateau Problem on complete metric spaces. Given an integral current $T \in \mathbf{I}_m Z$, one may define the filling volume of ∂T within Z as

$$\operatorname{FillVol}_Z(\partial T) = \inf \left\{ \mathbf{M}(S) \,:\, S \in \mathbf{I}_m(Z) \text{ such that } \partial S = \partial T \right\}.$$

This immediately provides an upper bound on the flat distance:

$$d_F^Z(\partial T, \mathbf{0}) \le \operatorname{FillVol}_Z(\partial T) \le \mathbf{M}(T).$$

Ambrosio–Kirchheim proved that this infimum is attained on Banach spaces Z, see [1, Theorem 10.2].

Wenger defined the absolute filling volume of $T \in \mathbf{I}_m Y$ to be

$$\operatorname{FillVol}_\infty(\partial T) = \inf \left\{ \mathbf{M}(S) \,:\, S \in \mathbf{I}_m(Z) \text{ s.t. } \partial S = \varphi_\# \partial T \right\}$$

where the infimum is taken over all isometric embeddings $\varphi\colon Y \to Z$, all complete metric spaces, Z, and all $S \in \mathbf{I}_m(Z)$ such that $\partial S = \varphi_\# T$. Clearly

$$\operatorname{FillVol}_\infty(\partial T) \le \operatorname{FillVol}_Y(\partial T).$$

Here we will use the following notion of a filling of an integral current space.

Definition 2.46. Given an integral current space $M = (X, d, T) \in \mathcal{M}^m$ with $m \ge 1$, we define

$$\operatorname{FillVol}(\partial M) := \inf \big\{ \mathbf{M}(N) \,:\, N \in \mathcal{M}^m \ \text{ and } \ \partial N = \partial M \big\}. \tag{29}$$

That is, we require that there exist a current preserving isometry from ∂N onto ∂M, where as usual, we have taken the metrics on the boundary spaces to be the restrictions of the metrics on the metric completions of N and M respectively.

We note that for $M = (X, d, T) \in \mathcal{M}^m$, we have

$$\operatorname{FillVol}(\partial M) = \operatorname{FillVol}_\infty(\partial T).$$

It is also easy to see that

$$\operatorname{FillVol}(\partial M) \le \mathbf{M}(M), \tag{30}$$

and

$$d_{\mathcal{F}}(\partial M, 0) \le \operatorname{FillVol}(\partial M) \le \mathbf{M}(M)$$

for any integral current space M.

Remark 2.47. The infimum in the definition of the filling volume is attained when the space is precompact. This may be seen by imitating the proof that the infimum in the definition of the intrinsic flat norm is attained in [34]. Since the N achieving the infimum has $\partial N \ne 0$, the filling volume is positive.

Any integral current space, $M = (X, d, T)$, is separable and so one can map the space into a Banach space, Z, via the Kuratowski Embedding theorem, $\iota\colon X \to Z$. By Ambrosio–Kirchheim's solution to the Plateau problem on Banach spaces [1, Proposition 10.2],

$$\operatorname{FillVol}(\partial M) \le \operatorname{FillVol}_Z(\varphi_\#(\partial T)) \le \operatorname{Diam}(X)\,\mathbf{M}(\partial T) = \operatorname{Diam}(M)\,\mathbf{M}(\partial M). \tag{31}$$

Wenger showed that the filling volume is continuous with respect to weak convergence (and thus also intrinsic flat convergence when applying Theorem 2.41). Here we provide a precise estimate which will be needed later in the paper.

Theorem 2.48. *For any pair of integral current spaces, M_i, we have*

$$\operatorname{FillVol}(\partial M_1) \le \operatorname{FillVol}(\partial M_2) + d_{\mathcal{F}}(M_1, M_2), \tag{32}$$

and if M_i have finite diameter, then

$$\operatorname{FillVol}(\partial M_1) \le \operatorname{FillVol}(\partial M_2) + \big(1 + 3\operatorname{Diam}(M_1) + 3\operatorname{Diam}(M_2)\big)\, d_{\mathcal{F}}(\partial M_1, \partial M_2).$$

Proof. Let $M_k = (X_{M_k}, d_{M_k}, T_{M_k})$ for $k = 1, 2$.

By the definition of intrinsic flat distance there exist integral currents A_i, B_i in Z_i and isometric embeddings, $\varphi_{i,k}\colon X_{M_k} \to Z_i$, such that

$$\varphi_{i,1\#} T_{M_1} - \varphi_{i,2\#} T_{M_2} = \partial B_i + A_i$$

where

$$d_{\mathcal{F}}(M_1, M_2) = \lim_{i\to\infty} \mathbf{M}(A_i) + \mathbf{M}(B_i).$$

In particular

$$\varphi_{i,1\#}\partial T_{M_1} - \varphi_{i,2\#}\partial T_{M_2} = \partial A_i.$$

Now by (29), there exists $N_i = (X_{N_i}, d_{N_i}, T_{N_i}) \in \mathcal{M}^{m+1}$ such that $\partial N_i = \partial M_2$ and

$$\mathrm{FillVol}(\partial M_2) = \lim_{i\to\infty} \mathbf{M}(N_i).$$

Applying the gluing techniques that are developed clearly in the Appendix, we may glue the integral current space $(\mathrm{set}(A_i) \subset Z_i, d_{Z_i}, A_i)$ to $N_i = (X_{N_i}, d_{N_i}, T_{N_i})$ along $\partial N_i = \partial M_2$ to create an integral current space M such that $\partial M = \partial M_1$ and $\mathbf{M}(M) \le \mathbf{M}(A_i) + \mathbf{M}(N_i)$.

Then

$$\mathrm{FillVol}(\partial M_1) = \mathrm{FillVol}(\partial M) \le \mathbf{M}(M) \le \mathbf{M}(A_i) + \mathbf{M}(N_i)$$

and taking $i \to \infty$ we have (32).

For the second half of the theorem, we observe that there exists a new pair of integral currents B_j, A_j and isometric embeddings, $\varphi_{j,k}\colon \mathrm{spt}(\partial T_k) \to Z'_j$, such that

$$\varphi_{j,1\#}\partial T_{M_1} - \varphi_{j,2\#}\partial T_{M_2} = \partial B_j + A_j$$

where

$$d_{\mathcal{F}}(\partial M_1, \partial M_2) = \lim_{j\to\infty} \mathbf{M}(B_j) + \mathbf{M}(A_j).$$

Let

$$M_{B_j} = (\mathrm{set}(B_j), d_{Z'_j}, B_j) \quad \text{and} \quad M_{A_j} = (\mathrm{set}(A_j), d_{Z'_j}, A_j).$$

As in the proof of Theorem 3.23 in [34] (see also Remark 2.39), we may assume that

$$\mathrm{Diam}(M_{B_j}), \mathrm{Diam}(M_{A_j}) \le 3\,\mathrm{Diam}(M_1) + 3\,\mathrm{Diam}(M_2).$$

Observe that

$$\partial A_j = \partial(\varphi_{j,1\#}\partial T_{M_1} - \varphi_{j,2\#}\partial T_{M_2}) = 0.$$

So we can study the filling volume of A_j. By [1, Proposition 10.2], we see that

$$\mathrm{FillVol}(M_{A_j}) \le \mathrm{Diam}(M_{A_j})\,\mathbf{M}(A_j).$$

Let N_j be integral current spaces such that $\partial N_j = \partial M_2$ and

$$\mathrm{FillVol}(\partial M_2) \ge \mathbf{M}(N_j) - 1/j.$$

Let N'_j be integral current spaces such that $\partial N'_j = A_j$ and

$$\mathrm{FillVol}(A_j) \ge \mathbf{M}(N'_j) - 1/j.$$

We glue N_j to M_{B_j} along $\partial N_j = \partial M_2$ and we also glue N'_j to M_{B_j} along $\partial N'_j = M_{A_j}$. The glued space M'_j will have $\partial M'_j = \partial M_1$ and

$$\mathbf{M}(M'_j) \le \mathbf{M}(N_j) + \mathbf{M}(M_{B_j}) + \mathbf{M}(N'_j).$$

Thus

$$\mathrm{FillVol}(\partial M_1) = \mathrm{FillVol}(\partial M'_j) \le \mathbf{M}(M'_j).$$

Combining these equations we have

$$\begin{aligned}\mathrm{FillVol}(\partial M_1) - \tfrac{2}{j} &\le \mathrm{FillVol}(\partial M_2) + \mathbf{M}(M_{B_j}) + \mathrm{FillVol}(M_{A_j})\\ &\le \mathrm{FillVol}(\partial M_2) + \mathbf{M}(M_{B_j}) + \mathrm{Diam}(M_{A_j})\mathbf{M}(M_{A_j})\\ &\le \mathrm{FillVol}(\partial M_2) + \big(\mathrm{Diam}(M_{A_j}) + 1\big)\big(\mathbf{M}(M_{B_j}) + \mathbf{M}(M_{A_j})\big)\end{aligned}$$

and letting $j \to \infty$ we have our second claim. □

Remark 2.49. Gromov's Filling Volume in [11] is defined as in (29) where the infimum is taken over N^{n+1} that are Riemannian manifolds. Thus it is conceivable that the filling volume in Definition 2.46 might have a smaller value both because integral current spaces have integer weight and because we have a wider class of metrics to choose from, including metrics that are not length metrics.

Remark 2.50. Note also that the mass used in Definition 2.46 is Ambrosio–Kirchheim's mass ([1, Definition 2.6]) stated as Definition 2.11 here. Even when the weight is 1 and one has a Finsler manifold, the Ambrosio–Kirchheim mass has a different value than any of Gromov's mass [11] and the masses used by Burago–Ivanov [2]. We need Ambrosio–Kirchheim's mass to have continuity of the filling volumes under intrinsic flat convergence (Theorem 2.48) which is an essential tool in this paper.

§3. Metric properties of integral current spaces

In this section we prove a number of properties of integral current spaces as well as a new Gromov–Hausdorff Compactness Theorem. After describing the natural notions of balls, isometric products, slices, spheres and filling volumes in the first three subsections, we move on to key new notions.

We introduce the Sliced Filling Volume (see Definition 3.20) and $\mathbf{SF}_k(p,r)$ (see Definition 3.21). Then we prove a new Gromov–Hausdorff Compactness Theorem (Theorem 3.23).

We explore the filling volumes of 0 dimensional spaces, apply them to bound the volumes of balls, and then introduce the Tetrahedral Property (Definition 3.30) and the Integral Tetrahedral Property (Definition 3.36).

We close this section with the notion of interval filling volumes in Definition 3.43 and Sliced Interval Filling Volumes in Definition 3.45.

Those studying the proof of the Tetrahedral Compactness Theorem need to read all Sections except 3.2 and 3.12 before continuing to Section 4. Those studying the Bolzano–Weierstrass and Arzela–Ascoli Theorems need only read Sections 3.1 and 3.3–3.6 before continuing to Section 4.

3.1. Balls.

Many theorems in Riemannian geometry involve balls,

$$B(p,r) = \{x \in X \,:\, d_X(x,p) < r\}, \quad \bar{B}(p,r) = \{x \in X \,:\, d_X(x,p) \le r\}.$$

In this subsection we quickly review key lemmas about balls proved in the background of the second author's recent paper [31].

Lemma 3.1. *A ball in an integral current space, $M = (X, d, T)$, with the current restricted from the current structure of the Riemannian manifold is an integral current space itself,*

$$S(p,r) = \big(\mathrm{set}(T \llcorner B(p,r)), d, T \llcorner B(p,r)\big)$$

for almost every $r > 0$. Furthermore,

$$B(p,r) \subset \mathrm{set}\,\big(S(p,r)\big) \subset \bar{B}(p,r) \subset X. \tag{33}$$

One may imagine that it is possible that a ball is cusp shaped when we are not in a length space and that some points in the closure of the ball that lie in X do not lie in the set of $S(p,r)$. In a manifold, the set of $S(p,r)$ is a closed ball.

Lemma 3.2. *When M is a Riemannian manifold with boundary,*

$$S(p,r) = \big(\bar{B}(p,r), d, T \llcorner B(p,r)\big)$$

is an integral current space for all $r > 0$.

Example 3.3. See [31] for an example of an integral current space with a ball that is not an integral current space because its boundary has infinite mass.

Remark 3.4. Note that the outside of the ball, $(M \setminus B(p,r), d, T - S(p,r))$, is also an integral current space for almost every $r > 0$.

3.2. Isometric products. One of the most useful notions in Riemannian geometry is that of an isometric product $M \times I$ of a Riemannian manifold M with an interval I, endowed with the metric

$$d_{M\times I}((p_1,t_1),(p_2,t_2)) = \sqrt{d_M(p_1,p_2)^2 + |t_1-t_2|^2}. \tag{34}$$

We need to define the isometric product of an integral current space with an interval.

Definition 3.5. The product of an integral current space, $M^m = (X, d_X, T)$, with an interval $I_\epsilon = [0,\epsilon]$, denoted

$$M \times I_\epsilon = (X \times I_\epsilon, d_{X\times I_\epsilon}, T \times I_\epsilon)$$

where $d_{X\times I_\epsilon}$ is defined by as in (34) and

$$\begin{aligned}(T \times I_\epsilon)(f, \pi_1, \dots, \pi_{m+1}) \\ = \sum_{i=1}^{m+1} (-1)^{i+1} \int_0^\epsilon T\Big(f_t \frac{\partial \pi_i}{\partial t}, \pi_{1t}, \dots, \widehat{\pi}_{it}, \dots, \pi_{(m+1)t}\Big)\, dt,\end{aligned} \tag{35}$$

where $h_t\colon \bar{X} \to \mathbb{R}$ is defined by $h_t(x) = h(x,t)$ for any $h\colon \bar{X} \times I_\epsilon \to \mathbb{R}$ and where

$$\big(\pi_{1t}, \dots, \widehat{\pi}_{it}, \dots, \pi_{(m+1)t}\big) = \big(\pi_{1t}, \pi_{2t}, \dots, \pi_{(i-1)t}, \pi_{(i+1)t}, \dots, \pi_{(m+1)t}\big).$$

We prove that this defines an integral current space in Proposition 3.7 below.

Remark 3.6. This is closely related to the cone construction in Definition 10.1 of [1], however our ambient metric space changes after taking the product and we do not contract to a point. Ambrosio–Kirchheim observe that (35) is well defined because for $\mathcal{L}^1$ almost every $t \in I_\epsilon$ the partial derivatives are defined for $\|T\|$ almost every $x \in X$. This is also true in our setting. The proof that their cone construction defines a current [1, Theorem 10.2], however, does not extend to our setting because our construction does not close up at a point as theirs does and our construction depends on ϵ but not on the size of a bounding ball.

Proposition 3.7. *Given an integral current space $M = (X, d, T)$, the isometric product $M \times I_\epsilon$ is an integral current space such that*

$$\mathbf{M}(M \times I_\epsilon) = \epsilon \mathbf{M}(M) \tag{36}$$

and such that

$$\partial(T \times I_\epsilon) = -(\partial T) \times I_\epsilon + T \times \partial I_\epsilon, \tag{37}$$

where

$$T \times \partial I_\epsilon := \psi_{\epsilon\#}T - \psi_{0\#}T$$

where $\psi_t\colon \bar{X} \to \bar{X} \times I_\epsilon$ is the isometric embedding $\psi_t(x) = (x,t)$.

Proof. First we must show that $T \times I_\epsilon$ satisfies the three conditions of a current.

Multilinearity follows from the multilinearity of T and the use of the alternating sum in the definition of $T \times I$.

To see locality we suppose there is a π_i that is constant on a neighborhood of $\{f \neq 0\}$. Then $\partial\pi_i/\partial t = 0$ on a neighborhood of $\{f \neq 0\}$ so the ith term in the sum is 0. Since for all $t \in I_\epsilon$, π_{it} is constant on a neighborhood of $\{f_t \neq 0\}$, the rest of the terms are 0 as well by the locality of T.

To prove continuity and finite mass, we will use the fact that T is integer rectifiable. In particular there exists a parametrization as $\varphi_i\colon A_i \subset \mathbb{R}^m \to \bar{X}$ and weight functions $\theta_i \in L^1(A_i, \mathbb{N})$ such that

$$T = \sum_{k=1}^{\infty} \varphi_{k\#}[\![\theta_k]\!].$$

So

$$
\begin{aligned}
&(T\times I_\epsilon)(f,\pi_1,\dots,\pi_{m+1})\\
&\quad=\sum_{i=1}^{m+1}(-1)^{i+1}\int_0^\epsilon\sum_{k=1}^\infty \varphi_{k\#}[\![\theta_k]\!]\left(f_t\frac{\partial\pi_{it}}{\partial t},\pi_{1t},\dots,\widehat{\pi}_{it},\dots,\pi_{(m+1)t}\right)dt\\
&\quad=\sum_{i=1}^{m+1}(-1)^{i+1}\sum_{k=1}^\infty\int_{t=0}^\epsilon\int_{A_k}\theta_k\, f_t\circ\varphi_k\,\frac{\partial\pi_{it}}{\partial t}\circ\varphi_k\,d(\pi_{1t}\circ\varphi_k)\wedge\dots\\
&\qquad\qquad\wedge d\widehat{\pi}_{it}\wedge\dots\wedge d(\pi_{(m+1)t}\circ\varphi_k)\,dt\\
&\quad=\sum_{k=1}^\infty\int_{A_k}\int_{t=0}^\epsilon\theta_k(x)f(\varphi_k(x),t)\\
&\qquad\times\left(\sum_{i=1}^{m+1}(-1)^{i+1}\frac{\partial\pi_{it}}{\partial t}\circ\varphi_k\,d(\pi_{1t}\circ\varphi_k)\wedge\cdots\wedge d\widehat{\pi}_{it}\wedge\cdots\wedge d(\pi_{(m+1)t}\circ\varphi_k)\right)dt\\
&\quad=\sum_{k=1}^\infty\int_{A_k\times I_\epsilon}\theta_k(x)f(\varphi_k(x),t)\,d(\pi_1\circ\varphi)\wedge\cdots\wedge d(\pi_{m+1}\circ\varphi).
\end{aligned}
$$

Thus

$$T\times I_\epsilon=\sum_{k=1}^\infty\varphi'_{k\#}[\![\theta'_k]\!],$$

where

$$\varphi'_k\colon A_k\times I_\epsilon\to\bar X\times I_\epsilon \ \text{ satisfies }\ \varphi'_k(x,t)=(\varphi_k(x),t)$$

and $\theta'_k\in L^1(A_k\times I_\epsilon,\mathbb{N})$ satisfies $\theta'_k(x,t)=\theta_k(x)$. Observe that the images of these charts are disjoint and that

$$
\begin{aligned}
\mathbf{M}(T\times I_\epsilon)&=\sum_{k=1}^\infty\mathbf{M}(\varphi'_{k\#}[\![\theta'_k]\!])\\
&=\sum_{k=1}^\infty\int_{A_k\times I_\epsilon}|\theta'_k|\mathcal{L}^{m+1}=\sum_{k=1}^\infty\epsilon\int_{A_k}|\theta_k|\mathcal{L}^m\\
&=\sum_{k=1}^\infty\epsilon\mathbf{M}(\varphi_{k\#}[\![\theta_k]\!])=\epsilon\mathbf{M}(T).
\end{aligned}
$$

The continuity of $T\times I_\epsilon$ now follows because all integer rectifiable currents defined by parametrizations are currents.

Observe also that if $A\subset\bar X$ and $(a_1,a_2)\subset I$, then

$$
\begin{aligned}
\|T\times I_\epsilon\|(A\times(a_1,a_2))&=\mathbf{M}\big((T\times I_\epsilon)\llcorner(A\times(a_1,a_2))\big)\\
&=\sum_{k=1}^\infty\mathbf{M}\big((\varphi'_{k\#}[\![\theta'_k]\!])\llcorner(A\times(a_1,a_2))\big)\\
&=\sum_{k=1}^\infty\int_{(A\cap A_k)\times(a_1,a_2)}|\theta'_k|\mathcal{L}^{m+1}\\
&=\sum_{k=1}^\infty(a_2-a_1)\int_{A\cap A_k}|\theta_k|\mathcal{L}^m\\
&=\sum_{k=1}^\infty(a_2-a_1)\mathbf{M}(\varphi_{k\#}[\![\theta_k]\!]\llcorner A)\\
&=(a_2-a_1)\mathbf{M}(T\llcorner A)=(a_2-a_1)\|T\|(A).
\end{aligned}
$$

Thus $\|T\times I_\epsilon\|=\|T\|\times\mathcal{L}^1$.

To prove that $T \times I_\epsilon$ is an integral current, we need only verify that the current $\partial(T \times I_\epsilon)$ has finite mass.

Let $\tau_1, \ldots, \tau_m \in \mathrm{Lip}(X \times I_\epsilon)$ be such that $\partial\tau_i/\partial t$ is Lipschitz as well for $i = 1, \ldots, m$. Applying the Chain Rule [1, Theorem 3.5] and Lemma 3.8 (proved below), we have

$$\begin{aligned}
&\partial(T \times I_\epsilon)(f, \tau_1, \ldots, \tau_m) + ((\partial T) \times I_\epsilon)(f, \tau_1, \ldots, \tau_m)\\
&\quad = (T \times I_\epsilon)(1, f, \tau_1, \ldots, \tau_m) + \sum_{i=1}^m (-1)^{i+1} \int_0^\epsilon \partial T\Big(f_t \frac{\partial \tau_i}{\partial t}, \tau_{1t}, \ldots, \widehat{\tau}_{it}, \ldots, \tau_{mt}\Big)\, dt\\
&\quad = \int_0^\epsilon T\Big(\frac{\partial f}{\partial t}, \tau_{1t}, \ldots, \tau_{mt}\Big)\, dt - \sum_{i=1}^m (-1)^{i+1} \int_0^\epsilon T\Big(\frac{\partial \tau_i}{\partial t}, f_t, \tau_{1t}, \ldots, \widehat{\tau}_{it}, \ldots, \tau_{mt}\Big)\, dt\\
&\qquad\qquad + \sum_{i=1}^m (-1)^{i+1} \int_0^\epsilon T\Big(1, f_t \frac{\partial \tau_i}{\partial t}, \tau_{1t}, \ldots, \widehat{\tau}_{it}, \ldots, \tau_{mt}\Big)\, dt\\
&\quad = \int_0^\epsilon T\Big(\frac{\partial f}{\partial t}, \tau_{1t}, \ldots, \tau_{mt}\Big)\, dt + \sum_{i=1}^m (-1)^{i+1} \int_0^\epsilon T\Big(f_t, \frac{\partial \tau_i}{\partial t}, \tau_{1t}, \ldots, \widehat{\tau}_{it}, \ldots, \tau_{mt}\Big)\, dt\\
&\quad = \int_0^\epsilon T\Big(\frac{\partial f}{\partial t}, \tau_{1t}, \ldots, \tau_{mt}\Big)\, dt + \sum_{i=1}^m \int_0^\epsilon T\Big(f_t, \tau_{1t}, \ldots, \tau_{(i-1)t}, \frac{\partial \tau_i}{\partial t}, \tau_{(i+1)t}, \ldots, \tau_{mt}\Big)\, dt\\
&\quad = \int_0^\epsilon \frac{\partial}{\partial t} T\left(f_t, \tau_{1t}, \ldots, \tau_{mt}\right)\, dt\\
&\quad = T(f_\epsilon, \tau_{1\epsilon}, \ldots, \tau_{m\epsilon}) - T(f_0, \tau_{10}, \ldots, \tau_{m0})\\
&\quad = \psi_{\epsilon\#} T(f, \tau_1, \ldots, \tau_m) - \psi_{0\#} T(f, \tau_1, \ldots, \tau_m)\\
&\quad = T \times \partial I_\epsilon(f, \tau_1, \ldots, \tau_m).
\end{aligned}$$

By mollification in the t-variable and by using the continuity properties of currents, we conclude that for arbitrary

$$\tau_1, \ldots, \tau_m \in \mathrm{Lip}(X \times I_\epsilon),$$

it is still true that

$$\partial(T \times I_\epsilon)(f, \tau_1, \ldots, \tau_m) + ((\partial T) \times I_\epsilon)(f, \tau_1, \ldots, \tau_m) = T \times \partial I_\epsilon(f, \tau_1, \ldots, \tau_m).$$

Thus we have (37).

Observe that $T \times \partial I_\epsilon$ is an integral current because it is the sum of push forward of integral currents and that

$$\mathbf{M}(T \times \partial I_\epsilon) = 2\mathbf{M}(T). \tag{38}$$

Since we know products are rectifiable, $(\partial T) \times I_\epsilon$ is rectifiable and has finite mass of at most $\epsilon\mathbf{M}(\partial T)$. Thus applying (37) we see that

$$\mathbf{M}(\partial(T \times I_\epsilon)) \le \mathbf{M}((\partial T) \times I_\epsilon) + \mathbf{M}(T \times \partial I_\epsilon) \le \epsilon\mathbf{M}(\partial T) + 2\mathbf{M}(T).$$

Thus the current structure of $M \times I_\epsilon$ is an integral current.

Lastly we verify that

$$\mathrm{set}(T \times I_\epsilon) = \mathrm{set}(T) \times I_\epsilon.$$

Given $(p,t) \in \bar{X} \times I_\epsilon$, the following statements are equivalent:

$$\begin{aligned}
&(p,t) \in \operatorname{set}(T \times I_\epsilon),\\
&\quad 0 < \liminf_{r\to 0} \frac{\|T \times I_\epsilon\|(B_{(p,t)}(r))}{r^{m+1}},\\
&\quad 0 < \liminf_{r\to 0} \frac{\|T \times I_\epsilon\|(B_p(r) \times (t-r,t+r))}{r^{m+1}},\\
&\quad 0 < \liminf_{r\to 0} \frac{2r\|T\|(B_p(r))}{r^{m+1}},\\
&\quad 0 < \liminf_{r\to 0} \frac{\|T\|(B_p(r))}{r^{m}},\\
&\quad p \in \operatorname{set}(T).
\end{aligned}$$

The proposition follows. □

Lemma 3.8. *If π_{it} and $\partial_t \pi_{it}$ are Lipschitz in $Z \times I_\epsilon$, and $T \in \mathbf{I}_m(Z)$, then for almost every $t \in I_\epsilon$,*

$$\frac{\partial}{\partial t} T(\pi_{0t}, \ldots, \pi_{mt}) = \sum_{i=0}^{m} T\left(\pi_{0t}, \ldots, \pi_{(i-1)t}, \frac{\partial \pi_i}{\partial t}, \pi_{(i+1)t}, \ldots, \pi_{mt}\right)$$

Proof. This follows from the multilinearity of T, the usual expansion of the difference quotient as a sum of difference quotients in which one term changes at a time, the fact that T is continuous with respect to pointwise convergence and that the difference quotients have pointwise limits for almost every $t \in I_\epsilon$ because the π_i are Lipschitz in t. □

The following proposition will be applied later when studying limits under intrinsic flat and Gromov–Hausdorff convergence.

Proposition 3.9. *Suppose $M_i^m = (X_i, d_i, T_i)$ are integral current spaces and $\epsilon > 0$, then*

$$d_{\mathcal{F}}(M_1^m \times I_\epsilon, M_2^m \times I_\epsilon) \le (2+\epsilon) d_{\mathcal{F}}(M_1^m, M_2^m)$$

and, when the M_i are precompact,

$$d_{GH}(X_1^m \times I_\epsilon, X_2^m \times I_\epsilon) \le d_{GH}(X_1^m, X_2^m).$$

Proof. Let $\delta > 0$. There exists a metric space Z and isometric embeddings $\varphi_i \colon X_i \to Z$, and integral currents A, B on Z such that

$$\varphi_{1\#} T_1 - \varphi_{2\#} T_2 = A + \partial B$$

and

$$d_{\mathcal{F}}(M_1^m, M_2^m) \le \mathbf{M}(A) + \mathbf{M}(B) + \delta.$$

Setting $Z' = Z \times I_\epsilon$ endowed with the product metric, we have isometric embeddings $\varphi_i' \colon X_i \times I_\epsilon \to Z'$ and we have integral currents $A' = A \times I_\epsilon$ and $B' = B \times I_\epsilon$ such that

$$\begin{aligned}
\varphi'_{1\#}(T_1 \times I_\epsilon) - \varphi'_{2\#}(T_2 \times I_\epsilon) &= (\varphi'_{1\#} T_1) \times I_\epsilon - (\varphi_{2\#} T_2) \times I_\epsilon\\
&= (\varphi'_{1\#} T_1 - \varphi_{2\#} T_2) \times I_\epsilon\\
&= (A + \partial B) \times I_\epsilon\\
&= A \times I_\epsilon - \partial(B \times I_\epsilon) - B \times (\partial I_\epsilon).
\end{aligned}$$

Thus by Proposition 3.7 and (38) we have

$$\begin{aligned} d_{\mathcal{F}}(M_1^m \times I_\epsilon, M_2^m \times I_\epsilon) &\le \mathbf{M}(A \times I_\epsilon) + \mathbf{M}(B \times I_\epsilon) + \mathbf{M}(B \times (\partial I_\epsilon)) \\ &\le \epsilon \mathbf{M}(A) + \epsilon \mathbf{M}(B) + 2\mathbf{M}(B) \\ &\le (2+\epsilon)\mathbf{M}(A) + (2+\epsilon)\mathbf{M}(B) \\ &= (2+\epsilon)(d_{\mathcal{F}}(M_1^m, M_2^m) + \delta). \end{aligned}$$

Finally, we let $\delta \downarrow 0$.

To see the Gromov–Hausdorff estimate, one needs only observe that whenever $Y_1 \subset T_r(Y_2) \subset Z$, we have

$$Y_1 \times I_\epsilon \subset T_r(Y_2 \times I_\epsilon) \subset Z \times I_\epsilon. \qquad \square$$

3.3. Slices and spheres. While balls are a very natural object in metric spaces, a more important notion in integral current spaces is that of a slice. The following proposition follows immediately from the Ambrosio–Kirchheim Slicing Theorem (cf. Theorem 2.23 and Remark 2.25).

Proposition 3.10. *Given an m dimensional integral current space $M = (X, d, T)$ and Lipschitz functions $F\colon X \to \mathbb{R}^k$ where $k < m$, for almost every $t \in \mathbb{R}^k$ we can define an $m-k$ dimensional integral current space called the slice of (X, d, T):*

$$\text{Slice}(M, F, t) = \text{Slice}(F, t) = \big(\operatorname{set}\langle T, F, t\rangle, d, \langle T, F, t\rangle\big) \tag{39}$$

where $\langle T, F, t\rangle = \langle T, F_1, \ldots, F_k, t_1, \ldots, t_k\rangle$ is an integral current on $\bar{X}$ defined using the Ambrosio–Kirchheim Slicing Theorem and $\operatorname{set}\langle T, F, t\rangle \subset F^{-1}(t)$. We can integrate the masses of slices to obtain lower bounds of the mass of the original space:

$$\int_{t \in \mathbb{R}^k} \mathbf{M}(\text{Slice}(M, F, t))\, \mathcal{L}^k \le \prod_{j=1}^{k} \text{Lip}(F_j)\, \mathbf{M}(T)$$

and $\partial\,\text{Slice}(M, F, t) = (-1)^k\,\text{Slice}(\partial M, F, t)$.

Proof. This proposition follows immediately from the Ambrosio–Kirchheim Slicing Theorem 5.6 in [1] by using the fact that F has a unique extension to $\bar{X}$ and Definition 2.5. The last part follows from Lemma 5.8. $\square$

Remark 3.11. Observe that in Example 2.35 where M is a 1 dimensional current space formed by concentric circles and a center point $p_0 = 0$, if $F(x) = d(x, p_0)$ then almost every slice is the $\mathbf{0}$ integral current space.

Lemma 3.12. *Given an m dimensional integral current space (X, d, T) and a point p, for almost every $r \in \mathbb{R}$ we can define an $m-1$ dimensional integral current space called the sphere about p of radius r:*

$$\text{Sphere}(p, r) = \text{Slice}(\rho_p, r)$$

where $\rho_p(x) = d(p, x)$. On a Riemannian manifold with boundary,

$$\text{Sphere}(p, r)(f, \pi_1, \ldots, \pi_{m-1}) = \int_{\rho_p^{-1}(r)} f\, d\pi_1 \wedge \cdots \wedge d\pi_{m-1}$$

is an integral current space for all $r \in \mathbb{R}$.

Proof. This follows from Proposition 3.10 and the Ambrosio–Kirchheim Slicing Theorem (cf. Theorem 2.23) and the fact that $\text{Lip}(\rho_p) = 1$. The Riemannian part follows from Stokes' Theorem and the fact that spheres of all radii in Riemannian manifolds have finite volume as can be seen either by applying the Ricatti equation or Jacobi fields. $\square$

Observe the distinction between the sphere and the boundary of a ball in Lemma 3.12 when M has boundary.

Next we examine the setting when we do not hit the boundary.

Lemma 3.13. *If* $\mathrm{set}(\partial T) \cap \bar{B}(p,R) \subset \bar{X}$ *is empty, then for almost every* $r \le R$

$$\mathrm{Sphere}(p,r) = \partial\, S(p,r).$$

Furthermore,

$$\int_0^R \mathbf{M}\big(\partial S(p,r)\big)\, d\mathcal{L}(r) \le \mathbf{M}\big(S(p,R)\big). \tag{40}$$

In particular, on an open Riemannian manifold, for any $p \in M$*, there is a sufficiently small* $R > 0$ *such that this lemma holds true. On a Riemannian manifold without boundary, these hold true for all* $R > 0$.

Proof. This follows from Proposition 3.10 and Theorem 2.23. □

Lemma 3.14. *Given an* m *dimensional integral current space* (X, d, T) *and* $\rho\colon X \to \mathbb{R}$ *a Lipschitz function with* $\mathrm{Lip}(\rho) \le 1$*, for almost every* $r \in \mathbb{R}$ *we can define an* $m-1$ *dimensional integral current space* $\mathrm{Slice}(\rho, r)$*, where*

$$\int_{-\infty}^{\infty} \mathbf{M}\big(\,\mathrm{Slice}(\rho,r)\big)\, d\mathcal{L}(r) \le \mathbf{M}(T).$$

On a Riemannian manifold with boundary,

$$\mathrm{Slice}(\rho,r)(f,\pi_1,\ldots,\pi_{m-1}) = \int_{\rho^{-1}(r)} f\, d\pi_1 \wedge \cdots \wedge d\pi_{m-1}$$

is defined for all $r \in \mathbb{R}$.

Proof. This follows from Proposition 3.10 and Theorem 2.23. □

Lemma 3.15. *Given an* m *dimensional integral current space* (X, d, T) *and a* $\rho\colon X \to \mathbb{R}^k$ *with* $\mathrm{Lip}(\rho_i) \le 1$*, for almost every* $r \in \mathbb{R}^k$ *we can define an* $m-k$ *dimensional integral current space* $\mathrm{Slice}(\rho, r)$*, where*

$$\int_{\mathbb{R}^k} \mathbf{M}\big(\,\mathrm{Slice}(\rho,r)\big)\, d\mathcal{L}(r) \le \mathbf{M}(T).$$

Proof. This follows from Proposition 3.10 and Theorem 2.23. □

Remark 3.16. On a Riemannian manifold with boundary

$$\mathrm{Slice}(\rho,r)(f,\pi_1,\ldots,\pi_{m-1}) = \int_{\rho^{-1}(r)} f\, d\pi_1 \wedge \cdots \wedge d\pi_{m-1}$$

is defined for all $r \in \mathbb{R}$ such that $\rho_p^{-1}(r)$ is $m-1$ dimensional. By the above lemma this will be true for almost every r. Note, however, that if ρ_i are distance functions from poorly chosen points, the slice may be the 0 space for almost every r because $\rho_p^{-1}(r) = \varnothing$. This occurs for example on the standard three dimensional sphere if we take ρ_1, ρ_2 to be distance functions from opposite poles.

3.4. Filling volumes of spheres and slices. The following lemmas were applied without proof in [33]. We may now easily prove them. First recall Definition 2.46 for the notion of filling volume used in this paper.

Lemma 3.17. *Given an integral current space, $M^m = (X, d, T)$, for all $p \in \bar{X}$ and almost every $r > 0$ we have*

$$\mathbf{M}(S(p,r)) \geq \mathrm{FillVol}(\partial S(p,r)).$$

Thus $p \in \bar{X}$ lies in $X = \mathrm{set}(T)$ if

$$\operatorname{ess}\liminf_{r\to 0} \mathrm{FillVol}(\partial S(p,r))/r^m > 0. \tag{41}$$

Here we have the essential lim inf which is a lim inf as $r \to 0$ where the r are selected from a set of full measure.

Proof. This follows immediately from the definition of filling volume in Definition 2.46 and (30), the definition of $S(p,r)$ which is only defined for almost every $r > 0$, and the definition of $\mathrm{set}(T)$. □

Note that the converse of Lemma 3.17 is not true as can be seen by observing that in Example 2.35 we have the point $0 \in X = \mathrm{set}(T)$, but $\partial S(0,r) = \mathbf{0}$ for almost every $r > 0$. So (41) fails for $p = 0$ in this example. The same is true for (42) in the next lemma.

Lemma 3.18. *Given an integral current space $M = (X, d, T)$, if $B_p(R) \cap \partial M = \varnothing$, then for almost every $r \in (0, R)$ we have*

$$\mathbf{M}(S(p,r)) \geq \mathrm{FillVol}(\mathrm{Sphere}(p,R)).$$

Thus if $\partial M = \varnothing$, we know that $p \in \bar{X}$ lies in $X = \mathrm{set}(T)$ if

$$\operatorname{ess}\liminf_{r\to 0} \mathrm{FillVol}(\mathrm{Sphere}(p,r))/r^m > 0. \tag{42}$$

Proof. This follows immediately from Lemma 3.13 and Lemma 3.17. □

Theorem 4.1 of [33] can be stated as follows:

Theorem 3.19 (Sormani–Wenger). *Suppose $M^m = (X, d, T)$ is a compact Riemannian manifold such that there exist $r_0 > 0, k > 0$ such that $\bar{B}(p, kr_0) \cap \partial M = \varnothing$ and every $B(x,r) \subset \bar{B}(p, r_0)$ is contractible within $B(x, kr) \subset \bar{B}(p, r_0)$; then there exists C_k such that*

$$\mathrm{Vol}(\bar{B}(x,r)) = \|T\|(\bar{B}(x,r)\| \geq \mathrm{FillVol}(\partial S(p,r)) \geq C_k r^m.$$

This theorem essentially follows from a result of Greene–Petersen [8] combined with Lemma 3.18. The statement in [34] applies to a more general class of spaces and requires a much subtler proof involving Lipschitz extensions.

3.5. Sliced filling volumes of balls. Spheres are not the only slices whose filling volumes may be used to estimate the volumes of balls. We define the following new notions.

Definition 3.20. Suppose we are given an integral current space, $M^m = (X, d, T)$, and let $F_1, F_2, \ldots, F_k \colon X \to \mathbb{R}$ with $k \leq m-1$ be Lipschitz functions with Lipschitz constants $\mathrm{Lip}(F_j) = \lambda_j$; then we define the sliced filling volume of $\partial S(p,r) \in \mathbf{I}_{m-1}(\bar{X})$ to be

$$\mathbf{SF}(p, r, F_1, \ldots, F_k) = \int_{t \in A_r} \mathrm{FillVol}(\partial\, \mathrm{Slice}(S(p,r), F, t))\, \mathcal{L}^k$$

where

$$A_r = [\min F_1, \max F_1] \times [\min F_2, \max F_2] \times \cdots \times [\min F_k, \max F_k]$$

where $\min F_j = \min\{F_j(x) : x \in \bar{B}_p(r)\}$ and $\max F_j = \max\{F_j(x) : x \in \bar{B}_p(r)\}$. Given $q_1, \ldots, q_k \in X$, we set

$$\mathbf{SF}(p, r, q_1, \ldots, q_k) = \mathbf{SF}(p, r, \rho_1, \ldots, \rho_k) \quad \text{where} \quad \rho_i(x) = d_X(q_i, x).$$

Definition 3.21. Given an integral current space M^m and $p \in M^m$, for almost every r we can define the kth sliced filling,

$$\mathbf{SF}_k(p, r) = \sup\left\{\mathbf{SF}(p, r, q_1, \ldots, q_k) \, : \, q_i \in \partial B_p(r)\right\}$$

where $\partial B_p(r)$ is the boundary of the metric ball about p. In particular,

$$\mathbf{SF}_0(p, r) = \mathbf{SF}(p, r) = \mathrm{FillVol}(\partial S(p, r)).$$

Lemma 3.22. *Suppose we are given an integral current space, $M^m = (X, d, T)$, and Lipschitz functions $F_1, F_2, \ldots, F_k \colon X \to \mathbb{R}$, $k \leq m-1$, with Lipschitz constants* $\mathrm{Lip}(F_j) = \lambda_j > 0$; *then*

$$\mathbf{M}(S(p, r)) \geq \prod_{j=1}^{k} \lambda_j^{-1} \mathbf{SF}(p, r, F_1, \ldots, F_k). \tag{43}$$

Thus $p \in \bar{X}$ lies in $X = \mathrm{set}(T)$ if there exist $F_i \colon M \to \mathbb{R}$ as above such that

$$\liminf_{r \to 0} \frac{1}{r^m} \mathbf{SF}(p, r, F_1, \ldots, F_k) > 0.$$

Applying (43) *to $F_j = \rho_{q_{j,r}}$ where the $q_{i,r}$ provide the supremum in Definition* 3.21*, we see that*

$$\mathbf{M}(S(p, r)) \geq \mathbf{SF}(p, r, q_{1,r}, \ldots, q_{k,r}) = \mathbf{SF}_k(p, r). \tag{44}$$

Thus $p \in \bar{X}$ lies in $X = \mathrm{set}(T)$ if

$$\liminf_{r \to 0} \frac{1}{r^m} \mathbf{SF}_k(p, r) > 0.$$

Conversely, if $\partial S(p, r) \neq 0$, then for $k = 0$ we have

$$\mathbf{SF}_k(p, r) \neq 0.$$

Proof. By Proposition 3.10 we know that

$$\begin{aligned}
\mathbf{M}(S(p, r)) &\geq \prod_{j=1}^{k} \lambda_j^{-1} \mathbf{M}(S(p, r) \llcorner dF) \\
&\geq \prod_{j=1}^{k} \lambda_j^{-1} \int_{t \in \mathbb{R}^k} \mathbf{M}(\mathrm{Slice}(S(p, r), F, t)) \, \mathcal{L}^k \\
&= \prod_{j=1}^{k} \lambda_j^{-1} \int_{t \in A} \mathbf{M}(\mathrm{Slice}(S(p, r), F, t)) \, \mathcal{L}^k.
\end{aligned}$$

Then (43) follows because $k \leq m-1$ implies each slice is at least 1 dimensional, combined with (30) and the fact that $\partial\langle S, F, t\rangle = -\langle \partial S, F, t\rangle$.

The converse follows because $\mathbf{SF}_0(p, r) = \mathrm{FillVol}(\partial S(p, r)) > 0$ when $S(p, r) \neq 0$. $\square$

3.6. Uniform $\mathbf{SF}_k$ and Gromov–Hausdorff compactness. Now we prove a new Gromov–Hausdorff compactness theorem.

Theorem 3.23. *If $M_i^m = (X_i, d_i, T_i)$ are integral current spaces with a uniform upper bound on* $\mathrm{Vol}(M_i) \le V_0$ *and diameter* $\mathrm{Diam}(M_i) \le D_0$, *and a uniform $r_0 > 0$, $C\colon (0, r_0] \to \mathbb{R}^+$, such that we have a uniform lower bound on the kth sliced filling*

$$\mathbf{SF}_k(p, r) \ge C(r) > 0 \quad \text{for almost every} \quad r \in (0, r_0]$$

for all $p \in M_i$, for all i, then a subsequence (X_i, d_i) converges in the Gromov–Hausdorff sense to a limit space (Y, d_Y).

Later we will prove that the subsequence converges in the intrinsic flat sense to the same limit space when $C(r) \ge C_{SF} r^m > 0$ (Theorem 5.1).

Proof. For any p in any M_i, there exist $q_1, \ldots, q_k$ such that

$$\mathbf{SF}(p, r, q_1, \ldots, q_k) \ge C(r)/2 > 0.$$

So by Lemma 3.22, $\mathbf{M}(S(p, r)) \ge C(r)/2$. Thus the number of disjoint balls of radius r in M_i is at most $2V_0/C(r)$. So we may apply Gromov's Compactness Theorem. □

3.7. Filling volumes of 0 dimensional spaces. Before proceeding we need the following lemma.

Lemma 3.24. *Let M be an integral current space. Suppose $S \in \mathbf{I}_1(M)$ is such that $\partial S \neq \mathbf{0}$. Then* $\mathrm{set}(\partial S) = \{p_1, \ldots, p_N\}$ *with $N \ge 2$ and*

$$\partial S(f) = \sum_{i=1}^{N} \sigma_i \theta_i f(p_i) \tag{45}$$

where $\theta_i \in \mathbb{Z}^+$ and $|\sigma_i| = 1$ and

$$\mathrm{FillVol}(\partial S) \ge \max_{j=1,\ldots,N} \Big(|\theta_j| \min_{i \ne j} d_X(p_i, p_j) \Big) > 0. \tag{46}$$

In particular,

$$\begin{aligned} \mathrm{FillVol}(\partial S) &\ge \inf \big\{ d_X(p_i, p_j) \,:\, i, j \in \{1, 2, \ldots, N\} \big\} \\ &\ge \inf \big\{ d(x, y) \,:\, x \ne y,\ x, y \in \mathrm{set}(\partial S) \big\} > 0. \end{aligned} \tag{47}$$

Proof. Recall that by Remark 2.34, ∂S satisfies (45) where $\sum_{i=1}^N \sigma_i \theta_i = 0$. So $N \ge 2$ when $\partial S \neq \mathbf{0}$.

Suppose $M' = (Y, d_Y, T)$ is any one dimensional integral current space with a current preserving isometry $\varphi\colon\ \mathrm{set}(\partial M') \to \mathrm{set}(\partial S) \subset \bar{X}$ so that

$$\varphi_\# \partial T = \partial S \in \mathbf{I}_0(M)$$

and $d_X(\varphi(y_1), \varphi(y_2)) = d_Y(y_1, y_2)$ for all $y_1, y_2 \in \mathrm{set}(T) \subset Y$. In particular there exist distinct points

$$p'_j = \varphi^{-1}(p_j) \in \bar{Y}$$

such that for any Lipschitz $f\colon \bar{Y} \to \mathbb{R}$ we have

$$T(1, f) = \partial(T)(f) = \sum_{i=1}^{N} \sigma_i \theta_i f(p'_i).$$

By (6) we have

$$|T(1, f)| \le \mathrm{Lip}(f) \mathbf{M}(T).$$

Let $f_j(y) = \min_{i \neq j} d_Y(y, p'_i)$. Then $\mathrm{Lip}(f_j) = 1$, so

$$\begin{aligned}\mathbf{M}(T) &\geq \left| \sum_{i=1}^{N} \sigma_i \theta_i f_j(p'_i) \right| \\ &\geq \theta_j f_j(p'_j) = \theta_j \min_{i \neq j} d_Y(p'_i, p'_j) \\ &= \theta_j \min_{i \neq j} d_X(p_i, p_j).\end{aligned}$$

Taking an infimum over all T, we have

$$\mathrm{FillVol}(\partial S) \geq \theta_j \min_{i \neq j} d_X(p_i, p_j).$$

As this is true for all $j = 1, \ldots, N$, we have (46). Since $\theta_j \in \mathbb{Z}^+$, we have the simpler lower bound given in (47). □

3.8. Masses of balls from distances. Here we provide a lower bound on the mass of a ball using a sliced filling volume and estimates on the filling volumes of 0 dimensional currents. First we introduce the notation:

$$P(p, r, t_1, \ldots, t_{m-1}) = \rho_p^{-1}(r) \cap \rho_{p_1}^{-1}(t_1) \cap \cdots \cap \rho_{p_{m-1}}^{-1}(t_{m-1}). \tag{48}$$

Theorem 3.25. *Suppose we are given an integral current space, $M^m = (X, d, T)$, and points $p_1, \ldots, p_{m-1} \in X$; then, if $\bar{B}_p(R) \cap \mathrm{set}(\partial T) = \varnothing$, we have*

$$\begin{aligned}\mathbf{M}(S(p, r)) &\geq \mathbf{SF}(p, r, p_1, \ldots, p_{m-1}) \\ &\geq \int_{s_1 - r}^{s_1 + r} \cdots \int_{s_{m-1} - r}^{s_{m-1} + r} h(p, r, t_1, \ldots, t_{m-1})\, dt_1 dt_2 \ldots dt_{m-1}\end{aligned}$$

for almost every $r \in (0, R)$, where $t_i = d(p_i, p_0)$ and

$$\begin{aligned}h(p, r, t_1, \ldots, t_{m-1}) &= \inf\left\{ d(x, y) \,:\, x \neq y,\;\; x, y \in P(p, r, t_1, \ldots, t_{m-1}) \right\} \quad \textit{when} \\ &\qquad P(p, r, t_1, \ldots, t_{m-1}) \;\; \textit{of (48) is a nonempty discrete} \\ &\qquad \textit{set of points and} \\ h(p, r, t_1, \ldots, t_{m-1}) &= 0 \quad \textit{otherwise.}\end{aligned}$$

Thus $p \in \bar{X} \setminus Cl(\mathrm{set}(\partial T))$ lies in $X = \mathrm{set}(T)$ if

$$\liminf_{r \to 0} \,(1/r^m) \int_{t_1 = s_1 - r}^{s_1 + r} \cdots \int_{t_{m-1} = s_{m-1} - r}^{s_{m-1} + r} h(p, r, t_1, \ldots, t_{m-1})\, dt_1\, dt_2 \ldots dt_{m-1} > 0.$$

Theorem 3.25 is in fact a special case of the following theorem.

Theorem 3.26. *Consider an integral current space $M^m = (X, d, T)$ and Lipschitz functions $F_1, F_2, \ldots, F_{m-1} \colon X \to \mathbb{R}$ with Lipschitz constants $\mathrm{Lip}(F_j) = \lambda_j$. Then for almost every $r > 0$ we have*

$$\mathbf{M}(S(p, r)) \geq \mathbf{SF}(p, r, F_1, \ldots, F_k) \geq \prod_{j=1}^{k} \lambda_j^{-1} \int_{t \in A_r} h(p, r, F, t)\, d\mathcal{L}^k$$

where

$$\begin{aligned}h(p, r, F, t) &= \inf\left\{ d(x, y) \,:\, x \neq y,\;\; x, y \in \mathrm{set}(\partial\, \mathrm{Slice}(S(p, r), F, t)) \right\} > 0 \\ &\qquad \textit{when } \partial\, \mathrm{Slice}(S(p, r), F, t) \in \mathbf{I}_0(\bar{X}) \setminus \{\mathbf{0}\} \textit{ and} \\ h(p, r, F, t) &= 0 \textit{ otherwise,}\end{aligned}$$

and where

$$A_r = [\min F_1, \max F_1] \times [\min F_2, \max F_2] \times \cdots \times [\min F_k, \max F_k]$$

with $\min F_j = \min\{F_j(x) : x \in \bar{B}_p(r)\}$ *and* $\max F_j = \max\{F_j(x) : x \in \bar{B}_p(r)\}$.

Before presenting the proof, we give two important examples.

Example 3.27. On Euclidean space $\mathbb{E}^m$, taking $F_i \colon \mathbb{E}^m \to \mathbb{R}$ to be a collection of perpendicular coordinate functions for $i = 1, \ldots, m$, $F_i(x_1, \ldots, x_m) = x_i$, we have $\lambda_i = 1$ and

$$h(p, r, F_1, \ldots, F_{m-1}, t_1, \ldots, t_{m-1}) = 2\sqrt{r^2 - (t_1^2 + \cdots + t_{m-1}^2)}.$$

So

$$\omega_m r^m = \mathbf{M}(S(p, r)) \geq \mathbf{SF}(p, r, F_1, \ldots, F_k) = \omega_m r^m.$$

Example 3.28. On the standard sphere S^2, taking $p_1 \in \partial B_p(\pi/2)$, $r = \pi/2$, and $F_1(x) = d(p_1, x)$, we have

$$h(p, \pi/2, F_1, t) = \min\{2t, 2(\pi - t)\}$$

because the distances are shortest if one travels within the great circle, $\partial B_p(\pi/2)$. So

$$2\pi = \mathrm{Vol}(S^2_+) = \mathbf{M}(S(p, \pi/2)) \geq \mathbf{SF}(p, \pi/2, F_1)$$

with

$$\begin{aligned}\mathbf{SF}(p, \pi/2, F_1) &= \int_0^\pi h(p, \pi/2, F_1, t)\, dt \\ &= 2\int_0^{\pi/2} 2t\, dt = 2(\pi/2)^2 = \pi^2/2.\end{aligned}$$

Proof. Theorem 3.25 follows from Theorem 3.26 by taking $F(x) = (F_1(x), \ldots, F_{m-1}(x))$, where $F_i(x) = \rho_{p_i}(x)$. If $\bar{B}(p, r) \cap \operatorname{set} \partial T = \varnothing$, then for almost every $r \in \mathbb{R}$ and $t_1 \in \mathbb{R}, \ldots, t_{m-1} \in \mathbb{R}$ we have $\partial\,\mathrm{Slice}(S(p, r), F, t)) \in \mathbf{I}_0(\bar{X})$ and

$$\operatorname{set}\big(\partial\,\mathrm{Slice}(S(p, r), F, t)\big) = \rho_p^{-1}(r) \cap F_1^{-1}(t_1) \cap \cdots \cap F_{m-1}^{-1}(t_{m-1}),$$

so this set either has 0 points or at least two points. ☐

Proof. Theorem 3.26 is proved by applying Lemma 3.22 to F and then computing the filling volume of the 0 dimensional current, $\partial(\mathrm{Slice}(S(p, r), F, t))$, using Lemma 3.24. Note that $t_i < s_i - r$ or $t_i > s_i + r$ implies $h(p, r, t_1, \ldots, t_{m-1}) = 0$ because $\rho_p^{-1}(r) \cap \rho_{p_i}^{-1}(t_i) = \varnothing$. ☐

Remark 3.29. Naturally we could combine Theorem 3.26 with any other lower bound on the filling volumes of 0 dimensional sets, like, for example, (46).

3.9. Tetrahedral property. Theorem 3.25 allows us to estimate the masses of balls using a tetrahedral property (see Figure 1).

Definition 3.30. Given $C > 0$ and $\beta \in (0, 1)$, a metric space X is said to have the m dimensional C, β tetrahedral property at a point p for radius r if one can find points $p_1, \ldots, p_{m-1} \subset \partial B_p(r) \subset \bar{X}$ such that

$$h(p, r, t_1, \ldots, t_{m-1}) \geq Cr, \quad (t_1, \ldots, t_{m-1}) \in [(1-\beta)r, (1+\beta)r]^m$$

where

$$h(p, r, t_1, \ldots, t_{m-1}) = \inf\big\{d(x, y) : x \neq y,\ \ x, y \in P(p, r, t_1, \ldots, t_{m-1})\big\}$$

when

$$P(p, r, t_1, \ldots, t_{m-1}) = \rho_p^{-1}(r) \cap \rho_{p_1}^{-1}(t_1) \cap \cdots \cap \rho_{p_{m-1}}^{-1}(t_{m-1})$$

is nonempty and 0 otherwise. Observe that $P(p, r, t_1, \ldots, t_{m-1})$ is the set of a 0 current for almost every $(t_1, \ldots, t_{m-1})$ and is thus a discrete set of points.

Example 3.31. On Euclidean space $\mathbb{E}^3$, taking $p_1, p_2 \in \partial B(p,r)$ such that $d(p_1,p_2) = r$, there exists exactly two points $x, y \in P(p,r,r,r)$ each forming a tetrahedron with p, p_1, p_2. See Figure 1. As we vary $t_1, t_2 \in (r/2, 3r/2)$, we still have exactly two points in $P(p,r,t_1,t_2)$. By scaling we see that

$$h(p,r,t_1,t_2) = rh(p,1,t_1/r,t_2/r) \geq C_{\mathbb{E}^3} r,$$

where

$$C_{\mathbb{E}^3} = \inf\{h(p,1,s_1,s_2) \,:\, s_i \in (1/2,3/2)\} > 0$$

could be computed explicitly. So $\mathbb{E}^3$ satisfies the $C_{\mathbb{E}^3}, (1/2)$ tetrahedral property at p for all r.

Example 3.32. On a torus, $M^3_\epsilon = S^1 \times S^1 \times S^1_\epsilon$ where S^1_ϵ has been scaled to have diameter ϵ instead of π, we see that M^3 satisfies the $C_{\mathbb{E}^3}, (1/2)$ tetrahedral property at p for all $r < \epsilon/4$. By taking $r < \epsilon/4$, we guarantee that the shortest paths between x and y stay within the ball $B(p,r)$ allowing us to use the Euclidean estimates. If r is too large, $P(p,r,t_1,t_2) = \varnothing$.

Remark 3.33. On a Riemannian manifold or an integral current space, we know that $P(p,r,t_1,\ldots,t_{m-1})$ is the set of a 0 current which is a boundary. So if it is not empty, it has at least two points, one with positive weight and one with negative weight.

Remark 3.34. It is not merely a simple application of the triangle inequality to proceed from knowing the relation $h(p,r,r,\ldots,r) \geq Cr$ to having $h(p,r,t_1,\ldots,t_{m-1}) \geq C_2 r$. There is the possibility that $P(p,r,t_1,\ldots,t_{m-1})$ is empty or has a closest pair of points both near a single point of $C(p,r,r,\ldots,r)$ even in a Riemannian manifold. However one expects the same type of curvature conditions that would lead to control of $h(p,r,\ldots,r)$ could be used to study $h(p,t_1,\ldots,t_{m-1})$.

Remark 3.35. On a manifold with sectional curvature bounded below, one should have the $C, 1/2$ tetrahedral property at any point p as long as $r < \mathrm{injrad}(p)/4$ where C depends on the lower sectional curvature bound. This should be provable using the Toponogov Comparison Theorem. One would like to replace the condition on injectivity radius with radius depending upon a lower bound on volume. Work in this direction is under preparation by the second author's doctoral students. Note that there is no uniform tetrahedral property on manifolds with positive scalar curvature even when the volumes of the balls are uniformly bounded below by that of Euclidean balls [Remark 5.3]. With lower bounds on Ricci curvature one might expect to have the $C, 1/2$ tetrahedral property or an integral version of this property. Again a uniform lower bound on volume will be necessary as seen in the torus example above.

3.10. Integral tetrahedral property. For our applications we need only the following property which clearly holds at any point with the tetrahedral property.

Definition 3.36. Given $C > 0$ and $\beta \in (0,1)$, a metric space X is said to have the m dimensional integral C, β tetrahedral property at a point p for radius r if one can find points $p_1, \ldots, p_{m-1} \subset \partial B_p(r) \subset \bar{X}$ such that

$$\int_{t_1=(1-\beta)r}^{(1+\beta)r} \cdots \int_{t_{m-1}=(1-\beta)r}^{(1+\beta)r} h(p,r,t_1,\ldots,t_{m-1})\, dt_1\, dt_2 \ldots dt_{m-1} \geq C(2\beta)^{m-1} r^m.$$

Proposition 3.37. *If X is a metric space that satisfies the C, β tetrahedral property at p for radius r, then it has the C, β integral tetrahedral property.*

Proof.

$$\begin{aligned}
&\int_{t_1=(1-\beta)r}^{(1+\beta)r} \cdots \int_{t_{m-1}=(1-\beta)r}^{(1+\beta)r} h(p,r,t_1,\ldots,t_{m-1})\, dt_1\, dt_2 \ldots dt_{m-1} \\
&\qquad \geq \int_{t_1=(1-\beta)r}^{(1+\beta)r} \cdots \int_{t_{m-1}=(1-\beta)r}^{(1+\beta)r} CR\, dt_1\, dt_2 \ldots dt_{m-1} \\
&\geq Cr\left((1+\beta)r-(1-\beta)r\right)^{m-1}.
\end{aligned}$$

□

3.11. Tetrahedral property and masses of balls.

Theorem 3.38. *Suppose (X,d,T) is an integral current space and suppose that for $p \in X$ we have $\bar{B}_p(R) \cap \operatorname{set}(\partial T) = \varnothing$. Then for almost every $r \in (0,R)$, if the m dimensional (integral) C,β tetrahedral property at a point p for radius r holds on $\bar{X}$, then*

$$\mathbf{M}(S(p,r)) \geq \mathbf{SF}_{m-1}(p,r) \geq C(2\beta)^{m-1}r^m.$$

Proof. By Theorem 3.25 with $s_i = r$ we have

$$\begin{aligned}
\mathbf{M}(S(p,r)) &\geq \mathbf{SF}(p,r,p_1,\ldots,p_{m-1}) \\
&\geq \int_{t_1=s_1-r}^{s_1+r} \cdots \int_{t_{m-1}=s_{m-1}-r}^{s_{m-1}+r} h(P_{(r,t_1,\ldots,t_{m-1})})\, dt_1\, dt_2 \ldots dt_{m-1} \\
&\geq \mathbf{SF}(p,r,p_1,\ldots,p_{m-1}) \\
&\geq \int_{t_1=0}^{2r} \cdots \int_{t_{m-1}=0}^{2r} h(P_{(r,t_1,\ldots,t_{m-1})})\, dt_1\, dt_2 \ldots dt_{m-1} \\
&> \int_{t_1=(1-\beta)r}^{(1+\beta)r} \cdots \int_{t_{m-1}=(1-\beta)r}^{(1+\beta)r} h(p,r,t_1,\ldots,t_{m-1})\, dt_1\, dt_2 \ldots dt_{m-1} \\
&> C(2\beta)^{m-1}r^m.
\end{aligned}$$

□

Theorem 3.39. *Suppose p_0 lies in a Riemannian manifold M with boundary, and $B_{p_0}(R) \cap \partial M = \varnothing$. For almost every $r \in (0,R)$, if the m dimensional (integral) C,β tetrahedral property holds at a point p for radius r, then*

$$\operatorname{Vol}(B(p,r)) \geq C(2\beta)^{m-1}r^m.$$

Proof. This is an immediate consequence of Theorem 3.38. □

Remark 3.40. In Example 3.32, as $\epsilon \to 0$, we have $\operatorname{Vol}(B(p,r)) \leq \operatorname{Vol}(M^3_\epsilon) \to 0$, so we could not have a uniform tetrahedral property on these spaces.

Theorem 3.41. *Given $r_0 > 0, \beta \in (0,1), C > 0, V_0 > 0$, if for a sequence of compact Riemannian manifolds M^m we have $\operatorname{Vol}(M^m) \leq V_0$, $\operatorname{Diam}(M^m) \leq D_0$, and the C,β (integral) tetrahedral property for all balls of radius $\leq r_0$, then a subsequence converges in the Gromov–Hausdorff sense. In particular, they have a uniform upper bound on diameter depending only on these constants.*

The proof of this theorem strongly requires that the manifold have no boundary.

Proof. This follows immediately from Theorem 3.39 and Gromov's Compactness Theorem, by using the fact that we can bound the number of disjoint balls of radius $\epsilon > 0$ in M^m. In a manifold, this provides an upper bound on the diameter of M^m. □

Later we will apply the following theorem to prove that the Gromov–Hausdorff limit is in fact an Intrinsic Flat limit and thus is countably $\mathcal{H}^m$ rectifiable (Theorem 5.2).

Theorem 3.42. *Suppose we are given an integral current space (X,d,T) and a point $p_0 \in \bar{X} \setminus Cl(\mathrm{set}(\partial T))$, then $p_0 \in X = \mathrm{set}(T)$ if there exists a pair of constants $\beta \in (0,1)$ and $C > 0$ such that $\bar{X}$ has the tetrahedral property at p_0 for all sufficiently small $r > 0$.*

Proof. By Theorem 3.38 we have

$$\|T\|(B_p(r)) \geq C(2\beta)^{m-1}r^m \tag{49}$$

for almost every r sufficiently small. For any R sufficiently small, there exists $r = r_j < R$ satisfying (49) with $r_j \to R$, so

$$\|T\|(B(p,R)) \geq \limsup_{j\to\infty} \|T\|(B(p,r_j)) \geq \limsup_{j\to\infty} C(2\beta)^{m-1}r_j^m = C(2\beta)^{m-1}R^m.$$

Thus $p_0 \in X = \mathrm{set}(T)$ by the definition of $\mathrm{set}(T)$. □

3.12. Fillings, slices and intervals. In the above sections, a key step consisted of estimating $\mathbf{M}(M) \geq \mathrm{FillVol}(\partial M)$. This is only a worthwhile estimate when ∂M is not zero or has a filling volume close to the mass.

A better estimate can be obtained by using the following trick. Given a Riemannian manifold M,

$$\mathrm{Vol}(M) = \mathrm{Vol}(M \times I) \geq \mathrm{FillVol}(\partial(M \times I))$$

where the metric on $M \times I$ is defined in (34). This has the advantage that $M \times I$ is always a manifold with boundary. It may also be worthwhile to use an interval, I_ϵ, of length ϵ, then

$$\mathrm{Vol}(M) = \frac{\mathrm{Vol}(M \times I_\epsilon)}{\epsilon} \geq \frac{\mathrm{FillVol}(\partial(M \times I_\epsilon))}{\epsilon}.$$

Intuitively it would seem reasonable to conjecture that

$$\mathrm{Vol}(M) = \lim_{\epsilon\to 0} \frac{\mathrm{Vol}(M \times I_\epsilon)}{\epsilon} = \lim_{\epsilon\to 0} \frac{\mathrm{FillVol}(\partial(M \times I_\epsilon))}{\epsilon}.$$

We introduce the following notion made rigorous on arbitrary integral current spaces $M = (X, d_X, T)$ by applying Definition 3.5 and Proposition 3.7.

Definition 3.43. Given any $\epsilon > 0$, we define the ϵ interval filling volume,

$$\mathbf{IFV}_\epsilon(M) = \mathrm{FillVol}(\partial(M \times I_\epsilon)).$$

Lemma 3.44. *Given an integral current space $M = (X,d,T)$,*

$$\mathbf{M}(M) = \epsilon^{-1}\mathbf{M}(M \times I_\epsilon) \geq \epsilon^{-1}\mathbf{IFV}_\epsilon(M).$$

Proof. This follows immediately from Proposition 3.7. □

Definition 3.45. For an integral current space $M^m = (X,d,T)$ and Lipschitz functions $F_1, F_2, \ldots, F_k \colon X \to \mathbb{R}$, $k \leq m$, with Lipschitz constant $\mathrm{Lip}(F_j) = \lambda_j$, whenever $\epsilon > 0$, for almost every $r > 0$ we can define the ϵ sliced interval filling volume of $\partial S(p,r) \in \mathbf{I}_{m-1}(\bar{X})$ to be

$$\mathbf{SIF}_\epsilon(p,r,F_1,\ldots,F_k) = \int_{t\in A_r} \mathrm{FillVol}(\partial(\mathrm{Slice}(S(p,r),F,t) \times I_\epsilon))\,\mathcal{L}^k$$

where

$$A_r = [\min F_1, \max F_1] \times [\min F_2, \max F_2] \times \cdots \times [\min F_k, \max F_k]$$

where $\min F_j = \min\{F_j(x) : x \in \bar{B}_p(r)\}$ and $\max F_j = \max\{F_j(x) : x \in \bar{B}_p(r)\}$. When the F_j are distance functions ρ_{p_j} we write,

$$\mathbf{SIF}_\epsilon(p,r,p_1,\ldots,p_k) := \mathbf{SIF}_\epsilon(p,r,\rho_{p_1},\ldots,\rho_{p_k}).$$

Proposition 3.46. *Given an integral current space $M^m = (X, d, T)$ and Lipschitz functions $F_1, F_2, \ldots, F_k \colon X \to \mathbb{R}$, $k \le m$, with Lipschitz constant $\mathrm{Lip}(F_j) = \lambda_j$, for all $\epsilon > 0$ and almost every $r > 0$ we can bound the mass of a ball in M as follows*:

$$\mathbf{M}(S(p,r)) \ge \prod_{j=1}^{k} \lambda_j^{-1}\epsilon^{-1}\mathbf{SIF}_\epsilon(p, r, F_1, \ldots, F_k). \tag{50}$$

Thus for any $p_1, \ldots, p_k \in X$, and almost every $r > 0$ we have

$$\mathbf{M}(S(p,r)) \ge \epsilon^{-1}\mathbf{SIF}_\epsilon(p, r, p_1 \ldots, p_k). \tag{51}$$

Proof. By Proposition 3.10 and Lemma 3.44 we have

$$\begin{aligned}
\mathbf{M}(S(p,r)) &\ge \prod_{j=1}^{k} \lambda_j^{-1}\mathbf{M}(S(p,r) \llcorner dF)\\
&\ge \prod_{j=1}^{k} \lambda_j^{-1} \int_{t\in\mathbb{R}^k} \mathbf{M}(\mathrm{Slice}(S(p,r), F, t))\, \mathcal{L}^k\\
&= \prod_{j=1}^{k} \lambda_j^{-1} \int_{t\in A} \mathbf{M}(\mathrm{Slice}(S(p,r), F, t))\, \mathcal{L}^k\\
&\ge \prod_{j=1}^{k} \lambda_j^{-1}\epsilon^{-1} \int_{t\in A} \mathrm{FillVol}(\partial(\mathrm{Slice}(S(p,r), F, t) \times I_\epsilon))\, \mathcal{L}^k.
\end{aligned}$$
□

Corollary 3.47. *A point $p \in \bar{X}$ lies in $X = \mathrm{set}(T)$ if there exists $\epsilon > 0$ and points $p_1, \ldots, p_k$ such that*

$$\operatorname{ess}\liminf_{r\to 0} \frac{1}{\epsilon r^m}\mathbf{SIF}_\epsilon(p, r, p_1, \ldots, p_k) > 0.$$

Corollary 3.48. *A point $p \in \bar{X}$ lies in $X = \mathrm{set}(T)$ if there exists $C > 0$, and points $p_1, \ldots, p_k$ such that*

$$\operatorname{ess}\liminf_{r\to 0} \frac{1}{Cr^{m+1}}\mathbf{SIF}_{Cr}(p, r, p_1, \ldots, p_k) > 0.$$

Corollary 3.49. *Given an integral current space M, we have*

$$\mathbf{M}(M) \ge \prod_{j=1}^{k} \lambda_j^{-1}\epsilon^{-1} \int_{t\in\mathbb{R}^k} \mathrm{FillVol}(\partial(\mathrm{Slice}(M, F, t) \times I_\epsilon))\, \mathcal{L}^k$$

§4. Convergence and continuity

In this section we examine the limits of points in sequences of integral current spaces that converge in the intrinsic flat sense and prove various continuity theorems and close with a pair of Bolzano–Weierstrass Theorems.

Recall that Theorem 2.41 (which was proved by the second author jointly with Wenger in [34]) states that a sequence of integral current spaces that converges in the intrinsic flat sense, $M_i \xrightarrow{\mathcal{F}} M_\infty$, can be embedded into a common complete metric space, Z, via distance preserving maps, $\varphi_i \colon M_i \to Z$, such that $\varphi_{i\#}T_i \xrightarrow{\mathcal{F}} \varphi_{\infty\#}T_\infty$. This allowed the second author to define converging, Cauchy, and disappearing sequences of points in [31]:

Definition 4.1. If $M_i = (X_i, d_i, T_i) \xrightarrow{\mathcal{F}} M_\infty = (X_\infty, d_\infty, T_\infty)$, then we say $x_i \in X_i$ constitute a converging sequence that converges to $x_\infty \in X_\infty$ if there exists a complete metric space Z and isometric embeddings $\varphi_i \colon M_i \to Z$ such that $\varphi_{i\#}T_i \xrightarrow{\mathcal{F}} \varphi_{\infty\#}T_\infty$

and $\varphi_i(x_i) \to \varphi_\infty(x_\infty)$. We say a collection of points, $\{p_{1,i}, p_{2,i}, \ldots, p_{k,i}\}$, converges to a corresponding collection of points, $\{p_{1,\infty}, p_{2,\infty}, \ldots, p_{k,\infty}\}$, if $\varphi_i(p_{j,i}) \to \varphi_\infty(p_{j,\infty})$ for $j = 1, 2 \ldots k$.

Unlike in Gromov–Hausdorff convergence, we have the possibility of disappearing sequences of points.

Definition 4.2. If $M_i = (X_i, d_i, T_i) \xrightarrow{\mathcal{F}} M_\infty = (X_\infty, d_\infty, T_\infty)$, then we say $x_i \in X_i$ are Cauchy if there exists a complete metric space Z and isometric embeddings $\varphi_i \colon M_i \to Z$ such that $\varphi_{i\#} T_i \xrightarrow{\mathcal{F}} \varphi_{\infty\#} T_\infty$ and $\varphi_i(x_i) \to z_\infty \in Z$.

We say the sequence is disappearing if $z_\infty \notin \varphi_\infty(X_\infty)$.

Examples with disappearing splines from [34] demonstrate that there exist Cauchy sequences of points which disappear. In [31] the second author proved theorems demonstrating when z_∞ lies in the metric completion of the limit space, $\varphi_\infty(\bar{X}_\infty)$. This material is reviewed in the first two subsections of this paper: including Theorems 4.3 and 4.6 and some related open questions.

Here we study when z_∞ lies in the limit space itself:

$$z_\infty \in X_\infty = \operatorname{set}(\varphi_{\infty\#}(T_\infty)),$$

which happens if and only if

$$\liminf_{r \to 0} \mathbf{M}\left(\varphi_{\infty\#}(T_\infty) \llcorner B(z_\infty, r)\right) / r^m > 0.$$

In [33] the second author and Wenger intuitively applied the idea that the filling volumes are continuous to prove sequences of points in certain sequences of spaces do not disappear. Here we will use sliced filling volumes and also provide the complete details not provided in [33] to justify the convergence of filling volumes. We first prove that slices of converging spaces converge in Proposition 4.13. We apply this proposition to prove that the sliced filling volumes are continuous (Theorem 4.20). These results are technically difficult and require a sequence of propositions and lemmas. Using similar methods, we prove the continuity of the sliced interval filling volumes (Theorem 4.24) and the interval filling volumes (Theorem 4.23).

In the penultimate subsection, we apply these continuity theorems to prove Theorem 4.27 which describes when a Cauchy sequence of points converges. This theorem is a crucial step in the proof of the Tetrahedral Compactness Theorem.

We close the section with two Bolzano–Weiestrass Theorems. Theorem 4.30 concerns sequences of points $p_i \in M_i$ with lower bounds on the filling volumes of spheres around them and produces a subsequence which converges to a point in the intrinsic flat limit of the M_i. Theorem 4.31 assumes that the points have a lower bound on the sliced filling volumes of balls about them and obtains a converging subsequence as well.

4.1. Review of limit points and diameter lower semicontinuity. Recall Definition 4.1. Recall the following theorems proved by the second author in [31]:

Theorem 4.3. *If a sequence of integral current spaces, $M_i = (X_i, d_i, T_i) \in \mathcal{M}^m$, converges to an integral current space, $M = (X, d, T) \in \mathcal{M}^m$, in the intrinsic flat sense, then every point z in the limit space M is the limit of points $x_i \in M_i$. In fact there exists a sequence of maps $F_i \colon X \to X_i$ such that $x_i = F_i(x)$ converges to x and*

$$\lim_{i \to \infty} d_i(F_i(x), F_i(y)) = d(x, y).$$

This sequence of maps F_i is not uniquely determined and is not even unique up to isometry.

Definition 4.4. Like in any metric space, one can define the diameter of an integral current space, $M = (X, d, T)$, to be

$$\mathrm{Diam}(M) = \sup\left\{d_X(x, y) \,:\, x, y \in X\right\} \in [0, \infty].$$

However, we explicitly define the diameter of the 0 integral current space to be 0. A space is bounded if the diameter is finite.

Theorem 4.5. *Suppose $M_i \xrightarrow{\mathcal{F}} M$ are integral current spaces, then*

$$\mathrm{Diam}(M) \le \liminf_{i\to\infty} \mathrm{Diam}(M_i) \in [0, \infty].$$

4.2. Review of flat convergence to Gromov–Hausdorff convergence. Recall the following theorem proved by the second author in [31].

Theorem 4.6. *If a sequence of precompact integral current spaces, $M_i = (X_i, d_i, T_i) \in \mathcal{M}_0^m$, converges to a precompact integral current space, $M = (X, d, T) \in \mathcal{M}_0^m$, in the intrinsic flat sense, then there exists $S_i \in \mathbf{I}_m(\bar{X}_i)$ such that the sequence $N_i = \big(\mathrm{set}(S_i), d_i\big)$ converges to $(\bar{X}, d)$ in the Gromov–Hausdorff sense and*

$$\liminf_{i\to\infty} \mathbf{M}(S_i) \ge \mathbf{M}(M). \tag{52}$$

When the M_i are Riemannian manifolds, the N_i can be taken to be settled completions of open submanifolds of M_i.

Remark 4.7. If in addition it is assumed that $\lim_{i\to\infty} \mathbf{M}(M_i) = \mathbf{M}(M)$, then by (52) we have $\mathbf{M}(\mathrm{set}(T_i - S_i), d_i, T_i - S_i) = 0$. In the Riemannian setting, we have $\mathrm{Vol}(M_i \setminus N_i) \to 0$.

Remark 4.8. In Ilmanen's example [34] of a sphere with increasingly many spikes, the $\mathrm{set}(S_i)$ are spheres with the spikes removed.

Remark 4.9. The precompactness of the limit integral current spaces is necessary in this theorem because a noncompact limit space can never be the Gromov-Hausdorff limit of precompact spaces.

Remark 4.10. Gromov's Compactness Theorem combined with Theorem 4.6 implies that any sequence of $x_i \in N_i \subset M_i$ has a subsequence converging to a point x in the metric completion of M. Other sequences of points may not have converging subsequences, as can be seen when the tips of thin spikes disappear. Below we will use filling volumes to determine which sequences have converging subsequences using filling volumes (Theorem 4.30). Another such Bolzano–Weierstrass Theorem with different hypothesis was proved in [31].

Remark 4.11. It is not immediately clear whether the integral current spaces, N_i, constructed in the proof of Theorem 4.6 actually converge in the intrinsic flat sense to M. One expects an extra assumption on total mass would be needed to interchange between flat and weak convergence, but even so it is not completely clear. One would need to uniformly control the masses of ∂N_i using a common upper bound on $\mathbf{M}(N)$ which can be done using theorems in Section 5 of [1], but is highly technical. It is worth investigating.

4.3. Limits of slices, spheres, and balls. In this section we prove the following two theorems via a sequence of lemmas and propositions, which will be applied elsewhere in this paper.

Recall that for any point p, for almost every radius r, the ball about p of radius r may be viewed as an integral current space, $S(p, r)$, as in Lemma 3.1. In prior work of the second author [31] it was shown that if $M_i \xrightarrow{\mathcal{F}} M_\infty$ and $p_i \to p_\infty$, then for almost every $r \in \mathbb{R}$ there is a subsequence such that

$$d_{\mathcal{F}}\big(S(p_i, r), S(p_\infty, r)\big) \to 0 \quad \text{for almost every } r \in \mathbb{R}.$$

Here we prove the following more precise estimate.

Theorem 4.12. *Suppose we have a sequence of m dimensional integral current spaces, $M_i = (X_i, d_i, T_i)$ and $M_\infty = (X_\infty, d_\infty, T_\infty)$ and isometric embeddings $\varphi_i \colon X_i \to Z$ such that*

$$d_F^Z\big(\phi_{i\#}T_i, \phi_{\infty\#}T_\infty\big) < d_{\mathcal{F}}(M_i, M_\infty) + \epsilon_i$$

and points $p_i \in M_i$ such that

$$d_Z\big(\varphi_i(p_i), \varphi_\infty(p_\infty)\big) \le \delta_i.$$

Then, for almost every $r \in \mathbb{R}$, the integral current spaces $S(p_i, r)$ satisfy

$$d_{\mathcal{F}}\big(S(p_i, r), S(p_\infty, r)\big) \le \varepsilon_i(r) + d_{\mathcal{F}}(M_i, M_\infty) + \epsilon_i + \|T_\infty\|\big(\rho_{x_\infty}^{-1}(r - \delta_i, r + \delta_i)\big) \tag{53}$$

and

$$\int_{-\infty}^{\infty} \varepsilon_i(r)\, dr \le d_{\mathcal{F}}(M_i, M_\infty) + \epsilon_i.$$

If $d_{\mathcal{F}}(M_i, M_\infty) \to 0$ and $p_i \to p_\infty$, then there is a subsequence (*that we do not relabel*) *such that for almost every $r \in \mathbb{R}$ we have*

$$\lim_{i\to\infty} d_{\mathcal{F}}\big(S(p_i, r), S(p_\infty, r)\big) = 0.$$

In fact we will prove a more general statement in Proposition 4.16. Note that a subsequence is required to obtain the final limit as can be seen in Example 2.45.

Recall that in Proposition 3.10 we defined the slices of an integral current space. In this section we will also prove the following theorem concerning limits of slices.

Theorem 4.13. *Suppose we have a sequence of m dimensional integral current spaces, $M_i = (X_i, d_i, T_i)$ and $M_\infty = (X_\infty, d_\infty, T_\infty)$ and isometric embeddings $\varphi_i \colon X_i \to Z$ such that*

$$d_F^Z\big(\phi_{i\#}T_i, \phi_{\infty\#}T_\infty\big) < d_{\mathcal{F}}(M_i, M_\infty) + \epsilon_i$$

and points $p_{j,i} \in M_i$ such that

$$d_Z\big(\varphi_i(p_{j,i}), \varphi_\infty(p_{j,\infty})\big) \le \delta_i \quad \text{for} \quad j = 1, \ldots, k \le m.$$

Then

$$\begin{aligned}\int_{R^k} d_{\mathcal{F}}\big(\, &\mathrm{Slice}(M_i, \rho_{p_{i,1}}, \ldots, \rho_{p_{i,k}}, r_1, \ldots, r_k), \\ &\mathrm{Slice}(M_\infty, \rho_{p_{\infty,1}}, \ldots, \rho_{p_{\infty,k}}, r_1, \ldots, r_k)\big)\, dr_1, \ldots, dr_k \\ &\le d_{\mathcal{F}}(M_i, M_\infty) + \epsilon_i + 2\delta_i\big(\mathbf{M}(T_\infty) + \mathbf{M}(\partial T_\infty)\big).\end{aligned} \tag{54}$$

If in addition $M_i \xrightarrow{\mathcal{F}} M_\infty$ and $p_{i,j} \to p_{\infty,j}$, then there is a subsequence (*which we do not relabel*) *such that for almost every $r \in \mathbb{R}^k$ we have*

$$\begin{aligned}\lim_{i\to\infty} d_{\mathcal{F}}\big(\, &\mathrm{Slice}(M_i, \rho_{p_{i,1}}, \ldots, \rho_{p_{i,k}}, r_1, \ldots, r_k), \\ &\mathrm{Slice}(M_\infty, \rho_{p_{\infty,1}}, \ldots, \rho_{p_{\infty,k}}, r_1, \ldots, r_k)\big) = 0.\end{aligned}$$

Before proving either of the key propositions leading to these theorems, we will prove a proposition (Proposition 4.14) which captures the main idea leading to these results, followed by a technical lemma (Lemma 4.15). Later we will prove Proposition 4.17 by iterating the idea in this proposition.

Proposition 4.14. *If an integral current space $M = (X, d, T)$ and Lipschitz functions $\rho \colon X \to \mathbb{R}$ and $f \colon X \to \mathbb{R}$ are such that*

$$|f(x) - \rho(x)| < \delta \quad \text{for all} \quad x \in X, \tag{55}$$

then for almost every $r \in \mathbb{R}$ we have

$$d_{\mathcal{F}}(\mathrm{Slice}(M, \rho, r), \mathrm{Slice}(M, f, r)) \le \|T\|(\rho^{-1}(r-\delta, r+\delta)) + \|\partial T\|(\rho^{-1}(r-\delta, r+\delta)). \tag{56}$$

Proof. First observe that by the definition of intrinsic flat distance,

$$d_{\mathcal{F}}(\mathrm{Slice}(M, p, r), \mathrm{Slice}(M, f, r)) \le d_F^{\bar{X}}\big(\langle T, \rho, r\rangle, \langle T, f, r\rangle\big) \le \mathbf{M}(B) + \mathbf{M}(A)$$

where

$$B = T \llcorner \rho^{-1}(-\infty, r] - T \llcorner f^{-1}(-\infty, r],$$
$$A = (\partial T) \llcorner f^{-1}(-\infty, r] - (\partial T) \llcorner \rho^{-1}(-\infty, r].$$

Next note that for any pair of sets $U, V \subset X$,

$$\mathbf{M}(T \llcorner U - T \llcorner V) = \mathbf{M}(T \llcorner (\chi_U - \chi_V)) = \mathbf{M}(T \llcorner (U \setminus V)) + \mathbf{M}(T \llcorner (V \setminus U))$$

and the same is true for ∂T. Since

$$\rho^{-1}(-\infty, r] \setminus f^{-1}(-\infty, r] \subset \rho^{-1}(r-\delta, r+\delta)$$

and

$$f^{-1}(-\infty, r] \setminus \rho^{-1}(-\infty, r] \subset \rho^{-1}(r-\delta, r+\delta),$$

we have

$$\mathbf{M}(B) \le \mathbf{M}(T \llcorner (\rho^{-1}(r-\delta, r+\delta))),$$
$$\mathbf{M}(A) \le \mathbf{M}(\partial T \llcorner (\rho^{-1}(r-\delta, r+\delta))).$$ □

The following technical lemma is used in the proof of Proposition 4.16 and again in the proof of Proposition 4.17 below.

Lemma 4.15. *Let μ be a finite Borel measure on $\mathbb{R}$. Then for every $\delta > 0$,*

$$\frac{1}{2\delta}\int_{-\infty}^{\infty} \mu(t-\delta, t+\delta)dt = \mu(\mathbb{R}).$$

Moreover, the set of $a \in \mathbb{R}$ such that $\mu(\{a\}) > 0$ is at most countable.

In particular, given an integral current space, (X, d, T), and any Borel function, $f\colon X \to \mathbb{R}$, we have for all $r \in \mathbb{R}$ outside an at most countable set,

$$\lim_{\delta \to 0} \|T\|\big(f^{-1}(r-\delta, r+\delta)\big) = 0$$

and

$$\int_{r \in \mathbb{R}} \|T\|\big(f^{-1}(r-\delta, r+\delta)\big)\, dr = 2\delta \mathbf{M}(T).$$

Proof. Identity (4.15) follows by changing the order of integration (or rather Tonelli's Theorem, which is an analog of Fubini's Theorem for nonnegative functions, cf. [29, Chapter 12, Theorem 20]) as follows

$$\begin{aligned}\int_{-\infty}^{\infty} \mu(t-\delta, t+\delta)dt &= \int_{-\infty}^{\infty}\left(\int_{\mathbb{R}} \chi_{(t-\delta, t+\delta)}\, d\mu\right) dt \\ &= \int_{R^2} \chi_{\{(x,y)\in\mathbb{R}^2 \,|\, -\delta < y-x < \delta\}}\, d\mathcal{L}^1\, d\mu \\ &= \int_{\mathbb{R}} 2\delta\, d\mu = 2\delta\mu(\mathbb{R}).\end{aligned}$$

That the set $A_\mu := \{a \in \mathbb{R} \,|\, \mu(\{a\}) > 0\}$ for any finite Borel measure μ is at most countable is a well-known fact. It follows because the number of $b \in \mathbb{R}$ such that $\mu(\{b\}) >$

$1/n$ is bounded by $n\mu(\mathbb{R})$, and therefore the set A_μ is at most countable, as it is a countable union of at most finite sets.

To conclude the second half of the lemma, we simply apply the first part to $\mu = f_\# \|T\|$. Indeed, by the σ-additivity of the measure,

$$\lim_{\delta\to 0} \|T\|(f^{-1}(r-\delta, r+\delta)) = \lim_{\delta\to 0} f_\#\|T\|(r-\delta, r+\delta) = f_\#\|T\|(\{r\}),$$

which is strictly positive for at most countably many $r \in \mathbb{R}$. □

Observe that Theorem 4.12 is an immediate consequence of the next proposition by taking $z_i = \varphi_i(p_i)$:

Proposition 4.16. *Let a sequence of integral current spaces $M_i = (X_i, d_i, T_i)$ and isometric embeddings $\varphi_i \colon X_i \to Z$ satisfy*

$$\lim_{i\to\infty} d_F^Z\big(\varphi_{i\#}T_i, \varphi_{\infty\#}T_\infty\big) = 0,$$

and let $z_i \in Z$ be such that $\delta_i = d_Z(z_i, z_\infty)$. Then, for almost every $r \in \mathbb{R}$, the balls $S_i(r) = \varphi_{i\#}T_i \llcorner B(z_i, r)$ satisfy

$$d_F^Z\big(S_i(r), S_\infty(r)\big) \le \varepsilon_i(r) + d_F^Z\big(\varphi_{i\#}T_i, \varphi_{\infty\#}T_\infty\big) + \|\varphi_{\infty\#}T_\infty\|\big(f^{-1}(r-\delta_i, r+\delta_i)\big) \tag{57}$$

where $f(x) = \rho_{z_\infty}(\varphi_\infty(x))$ and

$$\int_{-\infty}^{\infty} \varepsilon_i(r)\, dr \le d_F^Z\big(\varphi_{i\#}T_i, \varphi_{\infty\#}T_\infty\big).$$

If $\delta_i \to 0$, then there is a subsequence (that we do not relabel) such that for almost every $r \in \mathbb{R}$ we have

$$\lim_{i\to\infty} d_F^Z\big(S_i(r), S_\infty(r)\big) = 0.$$

Proof. There exist integral currents A_i, B_i in Z such that

$$\varphi_{i\#}T_i - \varphi_{\infty\#}T_\infty = A_i + \partial B_i$$

and

$$d_F^Z\big(\varphi_{i\#}T_i, \varphi_{\infty\#}T_\infty\big) = \mathbf{M}(A_i) + \mathbf{M}(B_i).$$

For almost every r, the restrictions of these spaces to balls, $B(z_i, r)$, are integral current spaces such that

$$S_i(r) - S_\infty(r) = \big(\varphi_{i\#}T_i - \varphi_{\infty\#}T_\infty\big) \llcorner \bar{B}(z_i, r) + S_i'(r)$$

where

$$S_i'(r) = (\varphi_{\infty\#}T_\infty) \llcorner \bar{B}(z_i, r) - (\varphi_{\infty\#}T_\infty) \llcorner \bar{B}(z_\infty, r).$$

Thus

$$\begin{aligned} S_i(r) - S_\infty(r) &= A_i \llcorner \bar{B}(z_i, r) + (\partial B_i) \llcorner \bar{B}(z_i, r) + S_i'(r) \\ &= A_i \llcorner \bar{B}(z_i, r) + \langle B_i, \rho_i, r\rangle + \partial(B_i \llcorner \bar{B}(z_i, r)) + S_i'(r). \end{aligned}$$

Since these are integral currents for almost every r, we have

$$d_F^Z(S_i(r), S_\infty(r)) \le \mathbf{M}(A_i \llcorner \bar{B}(z_i, r) + \langle B_i, \rho_i, r\rangle) + \mathbf{M}(B_i \llcorner \bar{B}(z_i, r)) + \mathbf{M}(S_i'(r)).$$

By the Ambrosio–Kirchheim Slicing Theorem and $\mathrm{Lip}(\rho_i) \le 1$ we have a Lebesgue measurable function $\epsilon_i \colon \mathbb{R} \to [0, \infty)$ such that

$$\mathbf{M}(A_i \llcorner \bar{B}(z_i, r) + \langle B_i, \rho_i, r\rangle) \le \mathbf{M}(A_i) + \varepsilon_i(r),$$

where

$$\int_{-\infty}^{\infty} \varepsilon_i(r)\, dr \le \mathbf{M}(B_i).$$

Naturally

$$\mathbf{M}(B_i \llcorner \bar{B}(z_i, r)) \le \mathbf{M}(B_i)$$

and

$$\begin{aligned}\mathbf{M}(S_i'(r)) &= \mathbf{M}\big((\varphi_{\infty\#}T_\infty) \llcorner \bar{B}(z_i, r) - (\varphi_{\infty\#}T_\infty) \llcorner \bar{B}(z_\infty, r)\big) \\ &\le \|\varphi_{\infty\#}T_\infty\| \rho_{z_\infty}^{-1}(r - \delta_i, r + \delta_i).\end{aligned}$$

Thus we have (57). The rest follows from Lemma 4.15 and the fact that for a subsequence and almost every r we have $\lim_{i\to\infty} \varepsilon_i(r) = 0$. ☐

In the next proposition, we will iterate the proof of Proposition 4.14 to bound the flat distance between lower-dimensional slices of two different currents with two different Lipschitz functions.

Proposition 4.17. *Let T_1 and T_2 be two m dimensional integral currents on a complete metric space Z. Let $k \in \{1, \dots, m\}$ and let $\pi\colon Z \to \mathbb{R}^k$ and $\widetilde{\pi}\colon Z \to \mathbb{R}^k$ be two Lipschitz functions such that*

$$|\pi_j(z) - \widetilde{\pi}_j(z)| < \delta, \quad z \in Z, \; j \in \{1, \dots, k\},$$

and such that

$$\mathrm{Lip}(\pi) \le L \quad \textit{and} \quad \mathrm{Lip}(\widetilde{\pi}) \le L.$$

Then

$$\int_{\mathbb{R}^k} d_F^Z\big(\langle T_1, \pi, t\rangle, \langle T_2, \widetilde{\pi}, t\rangle\big)\, dt \le L^k d_F^Z(T_1, T_2) + 2k\delta L^{k-1}\big(\mathbf{M}(T_2) + \mathbf{M}(\partial T_2)\big).$$

Note that this proposition implies Theorem 4.13 by taking $T_1 = \varphi_{i\#}T_i$ and $T_2 = \varphi_{\infty\#}T_\infty$, $\pi = (\rho_{z_{1,i}}, \dots, \rho_{z_{k,i}})$ where $z_{j,i} = \varphi_i(p_{j,i})$ and $\widetilde{\pi} = (\rho_{z_{1,\infty}}, \dots, \rho_{z_{k,\infty}})$ where $z_{j,\infty} = \varphi_\infty(p_{j,\infty})$. Then $\delta = \delta_i$ and $L = 1$. It may also be applied to study Cauchy sequences of points in a similar way. This proposition is applied later in the paper to study sliced filling volumes.

Proof. First, we write

$$\langle T_1, \pi, t\rangle - \langle T_2, \widetilde{\pi}, t\rangle = \langle T_1 - T_2, \pi, t\rangle + \langle T_2, \pi, t\rangle - \langle T_2, \widetilde{\pi}, t\rangle.$$

Let $\epsilon > 0$ be arbitrary. There exist integral currents $A \in \mathbf{I}_m(Z)$ and $B \in \mathbf{I}_{m+1}(Z)$ such that

$$T_1 - T_2 = A + \partial B$$

and

$$\mathbf{M}(A) + \mathbf{M}(B) \le d_F^Z(T_1, T_2) + \epsilon.$$

Then

$$\langle T_1 - T_2, \pi, t\rangle = \langle A, \pi, t\rangle + \langle \partial B, \pi, t\rangle = \langle A, \pi, t\rangle + (-1)^k \partial\langle B, \pi, t\rangle.$$

Note that by the Ambrosio–Kirchheim Slicing Theorem

$$\int_{\mathbb{R}^k} \mathbf{M}(\langle A, \pi, t\rangle)\, dt \le L^k \mathbf{M}(A),$$
$$\int_{\mathbb{R}^k} \mathbf{M}(\langle B, \pi, t\rangle)\, dt \le L^k \mathbf{M}(B).$$

We define the projections $P_j\colon \mathbb{R}^k \to \mathbb{R}^j$ and $Q_j\colon \mathbb{R}^k \to \mathbb{R}^{k-j}$ by

$$P_j(x_1, \dots, x_j, x_{j+1}, \dots, x_k) := (x_1, \dots, x_j),$$
$$Q_j(x_1, \dots, x_j, x_{j+1}, \dots, x_k) := (x_{j+1}, \dots, x_k)$$

so that P_k and Q_0 are identity maps. Using the following slight abuse of notation:

$$T = \langle T, P_0 \circ \widetilde{\pi}, P_0(t) \rangle,$$
$$T = \langle T, Q_k \circ \pi, Q_k(t) \rangle$$

we have for $\mathcal{L}^k$-a.e. $t \in \mathbb{R}^k$,

$$\begin{aligned}
&\langle T_2, \pi, t\rangle - \langle T_2, \widetilde{\pi}, t\rangle \\
&\quad = \big\langle \langle T_2, P_0 \circ \widetilde{\pi}, P_0(t)\rangle, Q_0 \circ \pi, Q_0(t)\big\rangle - \big\langle \langle T_2, P_k \circ \widetilde{\pi}, P_k(t)\rangle, Q_k \circ \pi, Q_k(t)\big\rangle \\
&\quad = \sum_{j=0}^{k-1} \big[\big\langle \langle T_2, P_j \circ \widetilde{\pi}, P_j(t)\rangle, Q_j \circ \pi, Q_j(t)\big\rangle - \big\langle \langle T_2, P_{j+1} \circ \widetilde{\pi}, P_{j+1}(t)\rangle, Q_{j+1} \circ \pi, Q_{j+1}(t)\big\rangle\big].
\end{aligned}$$

We calculate each term in the sum using the iterated definition of a slice:

$$\begin{aligned}
&\langle T_2, (\widetilde{\pi}_1, \ldots, \widetilde{\pi}_j, \pi_{j+1}, \ldots, \pi_k), (t_1, \ldots, t_k)\rangle \\
&\qquad - \langle T_2, (\widetilde{\pi}_1, \ldots, \widetilde{\pi}_{j+1}, \pi_{j+2}, \ldots, \pi_k), (t_1, \ldots, t_k)\rangle \\
&\quad = \big\langle \big\langle \langle T_2, P_j \circ \widetilde{\pi}, P_j(t)\rangle, \pi_{j+1}, t_{j+1}\big\rangle, Q_{j+1} \circ \pi, Q_{j+1}(t)\big\rangle \\
&\qquad - \big\langle \big\langle \langle T_2, P_j \circ \widetilde{\pi}, P_j(t)\rangle, \widetilde{\pi}_{j+1}, t_{j+1}\big\rangle, Q_{j+1} \circ \pi, Q_{j+1}(t)\big\rangle \\
&\quad = \big\langle \partial \langle T_2, P_j \circ \widetilde{\pi}, P_j(t)\rangle \llcorner \pi_{j+1}^{-1}(t_{j+1}, \infty), Q_{j+1} \circ \pi, Q_{j+1}(t)\big\rangle \\
&\qquad - \big\langle \partial\big(\langle T_2, P_j \circ \widetilde{\pi}, P_j(t)\rangle \llcorner \pi_{j+1}^{-1}(t_{j+1}, \infty)\big), Q_{j+1} \circ \pi, Q_{j+1}(t)\big\rangle \\
&\qquad - \big\langle \partial \langle T_2, P_j \circ \widetilde{\pi}, P_j(t)\rangle \llcorner \widetilde{\pi}_{j+1}^{-1}(t_{j+1}, \infty), Q_{j+1} \circ \pi, Q_{j+1}(t)\big\rangle \\
&\qquad + \big\langle \partial\big(\langle T_2, P_j \circ \widetilde{\pi}, P_j(t)\rangle \llcorner \widetilde{\pi}_{j+1}^{-1}(t_{j+1}, \infty)\big), Q_{j+1} \circ \pi, Q_{j+1}(t)\big\rangle \\
&\quad = \big\langle \partial \langle T_2, P_j \circ \widetilde{\pi}, P_j(t)\rangle \llcorner \pi_{j+1}^{-1}(t_{j+1}, \infty), Q_{j+1} \circ \pi, Q_{j+1}(t)\big\rangle \\
&\qquad - (-1)^{k-j} \partial \big\langle \langle T_2, P_j \circ \widetilde{\pi}, P_j(t)\rangle \llcorner \pi_{j+1}^{-1}(t_{j+1}, \infty), Q_{j+1} \circ \pi, Q_{j+1}(t)\big\rangle \\
&\qquad - \big\langle \partial \langle T_2, P_j \circ \widetilde{\pi}, P_j(t)\rangle \llcorner \widetilde{\pi}_{j+1}^{-1}(t_{j+1}, \infty), Q_{j+1} \circ \pi, Q_{j+1}(t)\big\rangle \\
&\qquad + (-1)^{k-j} \partial \big\langle \langle T_2, P_j \circ \widetilde{\pi}, P_j(t)\rangle \llcorner \widetilde{\pi}_{j+1}^{-1}(t_{j+1}, \infty), Q_{j+1} \circ \pi, Q_{j+1}(t)\big\rangle \\
&\quad = A_j(t) + \partial B_j(t),
\end{aligned}$$

where

$$A_j(t) := \big\langle \partial \langle T_2, P_j \circ \widetilde{\pi}, P_j(t)\rangle \llcorner \kappa_j, Q_{j+1} \circ \pi, Q_{j+1}(t)\big\rangle,$$
$$B_j(t) := (-1)^{k-j} \big\langle \langle T_2, P_j \circ \widetilde{\pi}, P_j(t)\rangle \llcorner \kappa_j, Q_{j+1} \circ \pi, Q_{j+1}(t)\big\rangle.$$

Here, $\kappa_j = \chi_{U_j} - \chi_{\widetilde{U}_j}$ is the difference of characteristic functions of the following sets,

$$U_j = \pi_{j+1}^{-1}(t_{j+1}, \infty) \text{ and } \widetilde{U}_j = \widetilde{\pi}_{j+1}^{-1}(t_{j+1}, \infty).$$

It follows that

$$\langle T_2, \pi, t\rangle - \langle T_2, \widetilde{\pi}, t\rangle = \sum_{j=0}^{k-1} (A_j(t) + \partial B_j(t)).$$

Since κ_j is supported on

$$\pi_{j+1}^{-1}\big[t_{j+1} - \delta, t_{j+1} + \delta\big],$$

Lemma 4.15 implies that

$$\begin{aligned}\int_{\mathbb{R}^m} \mathbf{M}(A_j(t))\, dt &\le L^{k-j-1} \int_{\mathbb{R}^m} \mathbf{M}(\partial\langle T_1, P_j \circ \widetilde{\pi}, P_j(t)\rangle \llcorner \kappa_j)\, dt_1 \dots dt_{j+1} \\ &\le 2\delta L^{k-j-1} \int_{\mathbb{R}^m} \mathbf{M}\big(\partial\langle T_1, P_j \circ \widetilde{\pi}, P_j(t)\rangle\big)\, dt_1 \dots dt_j \\ &\le 2\delta L^{k-1}\mathbf{M}(\partial T_1).\end{aligned}$$

In the same way,

$$\begin{aligned}&\int_{\mathbb{R}^m} \mathbf{M}\big(\big\langle \langle T_1, P_j \circ \widetilde{\pi}, P_j(t)\rangle \llcorner \kappa_j, Q_{j+1} \circ \pi, Q_{j+1}(t)\big\rangle\big)\, dt_1 \dots dt_k \\ &\quad\le L^{k-j-1} \int_{\mathbb{R}^m} \mathbf{M}\big(\langle T_1, P_j \circ \widetilde{\pi}, P_j(t)\rangle \llcorner \kappa_j\big)\, dt_1 \dots dt_{j+1} \\ &\quad\le 2\delta L^{k-j-1} \int_{\mathbb{R}^m} \mathbf{M}\big(\langle T_1, P_j \circ \widetilde{\pi}, P_j(t)\rangle\big)\, dt_1 \dots dt_j \\ &\quad\le 2\delta L^{k-1}\mathbf{M}(T_1).\end{aligned}$$

We conclude by applying the triangle inequality and by taking the limit as $\epsilon \downarrow 0$. □

Remark 4.18. One might be able to strengthen the results in this section if one assumes that the sequence of integral current spaces is a sequence of Riemannian manifolds with boundary. Or one may wish to try to produce an example of a sequence of manifolds which have the same sorts of disturbing properties as Examples 2.35 and 2.45 at least in a limiting sense.

4.4. Continuity of sliced filling volumes. In this section we prove continuity and semicontinuity of the various Sliced Filling Volumes [Theorem 4.20]. Recall that Theorem 2.48 implies the continuity of filling volume in the following sense:

$$M_i \xrightarrow{\mathcal{F}} M_\infty \implies \mathrm{FillVol}(\partial M_i) \to \mathrm{FillVol}(\partial M_\infty)$$

where the filling volume is defined as in Definition 2.46. In this section we combine Theorem 2.48 with the convergence of slices proved in Proposition 4.13. An immediate consequence of these results is that the filling volumes of slices converge. In particular the filling volumes of spheres converge to the filling volumes of spheres, as stated in the prior work of the second author with Wenger [34].

The situation is more complicated when one considers sliced filling volumes.

Example 4.19. Recall the integral current spaces M and M_ℓ defined in Example 2.45. One may observe that there exists a sequence $p_\ell \in M_\ell$ converging to p such that for $\mathcal{L}$-a.e. $r \in (0, 1/4)$,

$$\liminf_{\ell\to\infty} \mathbf{SF}_0(p_\ell, r) = 0 < 2r = \mathbf{SF}_0(p, r) < 4r = \limsup_{\ell\to\infty} \mathbf{SF}_0(p_\ell, r).$$

In fact these inequalities are true for all sequences p_ℓ that converge to p at a sufficiently high rate.

Nevertheless we are able to prove the following continuity theorem.

Theorem 4.20. *Let $M_i = (X_i, d_i, T_i)$ be a sequence of m dimensional integral current spaces such that $M_i \xrightarrow{\mathcal{F}} M_\infty = (X_\infty, d_\infty, T_\infty)$ and let the collections of points $p_i, p_{i,1}, \dots, p_{i,k}$ converge to $p_\infty, p_{\infty,1}, \dots, p_{\infty,k}$ as $i \to \infty$ for $k \in \{1, \dots, m-1\}$. Then there exists a subsequence, which we do not relabel, such that for $\mathcal{L}^1$-a.e. $r > 0$,*

$$\lim_{i\to\infty} \mathbf{SF}(p_i, r, p_{i,1}, \dots, p_{i,k}) = \mathbf{SF}(p_\infty, r, p_{\infty,1}, \dots, p_{\infty,k}). \tag{58}$$

In the $k = 0$ case we have for every $r > 0$,

$$\lim_{i\to\infty} \frac{1}{r}\int_0^r |\mathbf{SF}_0(p_i,\tau) - \mathbf{SF}_0(p_\infty,\tau)|\, d\tau = 0. \tag{59}$$

Consequently, there is a subsequence i_j, such that for $\mathcal{L}^1$-a.e. $r > 0$,

$$\lim_{j\to\infty} \mathbf{SF}_0(p_{i_j}, r) = \mathbf{SF}_0(p_\infty, r) \le \mathbf{M}(S(p_\infty, r)).$$

For $k \in \{0, \dots, m-1\}$, we also obtain the following inequality for $\mathcal{L}^1$-a.e. $r > 0$,

$$\liminf_{i\to\infty} \mathbf{SF}_k(p_i, r) \le \mathbf{M}(S(p_\infty, r)).$$

Finally, for every $r > 0$, we have

$$\limsup_{i\to\infty} \frac{1}{r}\int_0^r \mathbf{SF}_k(p_i,\tau)\, d\tau \le \frac{1}{r}\int_0^r \mathbf{M}(S(p_\infty,\tau))\, d\tau.$$

Before we prove Theorem 4.20, we state and prove two key ingredients toward the proof (Proposition 4.21 and Lemma 4.22).

Proposition 4.21. *Suppose we have a sequence of m dimensional integral current spaces $M_i = (X_i, d_i, T_i)$, isometric embeddings $\phi_i\colon X_i \to Z$, points $z_{j,i} \in Z$, and $\delta_i > 0$ such that $d_Z(z_{j,i}, z_{j,\infty}) < \delta_i$ for $j = 1, \dots, k$ for some $k \in \{0, \dots, m-1\}$, and $p_i \in X_i$ such that $d_Z(\varphi_i(p_i), \varphi(p)) < \delta_i$; then for almost every $t \in \mathbb{R}^k$ we have*

$$\begin{aligned} &\big|\operatorname{FillVol}\big(\partial \operatorname{Slice}(M_i,\rho_i,t)\big) - \operatorname{FillVol}\big(\partial \operatorname{Slice}(M_\infty,\rho_\infty,t)\big)\big| \\ &\quad\le d_{\mathcal{F}}\big(\partial \operatorname{Slice}(M_i,\rho_i,t), \partial \operatorname{Slice}(M_\infty,\rho_\infty,t)\big) \end{aligned} \tag{60}$$

and thus

$$\begin{aligned} &\int_{\mathbb{R}^k} \big|\operatorname{FillVol}\big(\partial \operatorname{Slice}(M_i,\rho_i,t)\big) - \operatorname{FillVol}\big(\partial \operatorname{Slice}(M_\infty,\rho,t)\big)\big|\, dt \\ &\quad\le \int_{R^k} d_{\mathcal{F}}\big(\partial \operatorname{Slice}(M_i,f,t), \partial \operatorname{Slice}(M_\infty,\rho,t)\big)\, dt \\ &\quad\le d_F^Z(\phi_{i\#}M_i, \phi_{\infty\#}M_\infty) + 2\delta(\mathbf{M}(T_\infty) + \mathbf{M}(\partial T_\infty)). \end{aligned} \tag{61}$$

If $\lim_{i\to\infty} \delta_i = 0$ and

$$\lim_{i\to\infty} d_F^Z(\varphi_{i\#}T_i, \varphi_{\infty\#}T_\infty) = 0,$$

then for almost every $t \in \mathbb{R}^k$ the masses satisfy

$$\liminf_{i\to\infty} \mathbf{M}(\operatorname{Slice}(M_i,\rho_i,t)) \ge \mathbf{M}(\operatorname{Slice}(M_\infty,\rho_\infty,t)).$$

Finally, there is a subsequence such that (without relabeling) for almost every $t \in \mathbb{R}^k$,

$$\lim_{i\to\infty} d_{\mathcal{F}}\big(\partial \operatorname{Slice}(M_i,\rho_i,t), \partial \operatorname{Slice}(M_\infty,\rho_\infty,t)\big) = 0. \tag{62}$$

Proof. That one can estimate the difference in Filling Volume of the boundaries of two currents in terms of the flat distance between them as in (60) was explained in Theorem 2.48. Inequality (61) is then a direct consequence of inequality (54) in Lemma 4.13. We select a subsequence of M_{i_j} of M_i such that

$$\lim_{j\to\infty} \mathbf{M}(\operatorname{Slice}(M_{i_j},\rho_{i_j},t)) = \liminf_{i\to\infty} \mathbf{M}(\operatorname{Slice}(M_i,\rho_i,t)).$$

The integral bound (54) implies that for a subsequence of the M_{i_j} (that we do not relabel), for almost every $t \in \mathbb{R}^k$ equation (62) holds true. Since flat convergence implies weak convergence and the mass is lower-semicontinuous under weak convergence,

$$\liminf_{i\to\infty} \mathbf{M}(\operatorname{Slice}(M_i,\rho_i,t)) = \liminf_{j\to\infty} \mathbf{M}(\operatorname{Slice}(M_{i_j},\rho_{i_j},t)) \ge \mathbf{M}(\operatorname{Slice}(M_\infty,\rho_\infty,t)). \qquad \square$$

Lemma 4.22. *Let $M = (X, d, T)$ be an m dimensional integral current space, and let $\pi\colon X \to \mathbb{R}^k$ be a Lipschitz function with* $\mathrm{Lip}(\pi_j) \le 1$. *Then*

$$\int_{\mathbb{R}^k} \mathrm{FillVol}(\partial\,\mathrm{Slice}(M, \pi, t))\, dt \le \mathrm{FillVol}(\partial M). \tag{63}$$

In particular,

$$\mathbf{SF}_k(p, r) \le \mathbf{SF}_0(p, r) \le \mathbf{M}(S(p, r)). \tag{64}$$

Proof. Let $\epsilon > 0$. There is a m dimensional integral current space A such that $\partial A = \partial M$ and

$$\mathrm{FillVol}(\partial M) + \epsilon \ge \mathbf{M}(A).$$

By the Ambrosio–Kirchheim slicing theorem,

$$\begin{aligned}
\mathrm{FillVol}(\partial M) + \epsilon \ge \mathbf{M}(A) &\ge \int_{\mathbb{R}^k} \mathbf{M}(\mathrm{Slice}(A, \pi, t))\, dt \\
&\ge \int_{\mathbb{R}^k} \mathrm{FillVol}(\partial\,\mathrm{Slice}(A, \pi, t))\, dt \\
&= \int_{\mathbb{R}^k} \mathrm{FillVol}(\mathrm{Slice}(\partial A, \pi, t))\, dt \\
&= \int_{\mathbb{R}^k} \mathrm{FillVol}(\mathrm{Slice}(\partial M, \pi, t))\, dt \\
&= \int_{\mathbb{R}^k} \mathrm{FillVol}(\partial\,\mathrm{Slice}(M, \pi, t))\, dt.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, estimate (63) holds true.

Taking $M = S(p, r)$, we have by Definition 3.20,

$$\begin{aligned}
\mathbf{SF}(p, r, F_1, \dots, F_k) &= \int_{t \in A_r} \mathrm{FillVol}(\partial\,\mathrm{Slice}(S(p, r), F, t))\, \mathcal{L}^k \\
&\le \int_{t \in A_r} \mathrm{FillVol}(\partial S(p, r))\, \mathcal{L}^k = \mathbf{SF}_0(p, r).
\end{aligned}$$

Taking $F_i = \rho_{q_i}$, we have

$$\mathbf{SF}(p, r, q_1, \dots, q_k) \le \mathbf{SF}_0(p, r)$$

and taking the supremum over $q_i \in \partial B_p(r)$ we obtain (64). □

We may now prove Theorem 4.20.

Proof. By the definition of convergence of points, there exists a complete separable metric space Z and isometric embeddings $\phi_i\colon \overline{X_i} \to Z$ such that

$$d_F^Z(\phi_{i\#} T_i, \phi_{\infty\#} T_\infty) = 0,$$

and $\delta_i := d_Z(\phi_i(p_i), \phi_\infty(p_\infty)) \to 0$, $\delta_{i,j} := d_Z(\phi_i(p_{i,j}, \phi_\infty(p_{\infty,j})) \to 0$ as $i \to \infty$, $j = 1, \dots, k$.

By Proposition 4.16 there exists a subsequence such that for $\mathcal{L}^1$-a.e. $r > 0$,

$$d_F^Z\big(\phi_{i\#} T_i \llcorner B_r(\phi_i(p_i)), \phi_{\infty\#} T_\infty \llcorner B_r(\phi_\infty(p_\infty))\big) \to 0$$

as $i \to \infty$.

Hence, for such a value of $r > 0$ we can apply Proposition (4.21) to the integral current spaces associated with $\phi_{i\#} T_i \llcorner B_r(p_i)$. In particular, inequality (61) yields the continuity property expressed by (58).

Similarly, if $M_i \xrightarrow{\mathcal{F}} M$ and the points $p_i \in M_i$ converge to p_∞, there is a complete separable metric space Z and isometric embeddings $\phi_i\colon \overline{X_i} \to Z$ such that

$$\lim_{i\to\infty} d_F^Z(\phi_{i\#}T_i, \phi_{\infty\#}T_\infty) = 0,$$

and $\delta_i := d_Z(\phi_i(p_i), \phi_\infty(p_\infty)) \to 0$ in Z as $i \to \infty$.

By using respectively Theorem 2.48, Proposition 4.16, and Lemma 4.15, we find

$$\begin{aligned}
\frac{1}{r}\int_0^r \left|\mathbf{SF}_0(p_i,\tau) - \mathbf{SF}_0(p_\infty,\tau)\right| d\tau &\le \frac{1}{r}\int_0^r d_{\mathcal{F}}(\partial S(p_i,r), \partial S(p_\infty,r))\, d\tau\\
&\le \left(\frac{1}{r}+1\right) d_F^Z(\phi_{i\#}T_i, \phi_{\infty\#}T_\infty) + \frac{1}{r}\int_0^r \|T_\infty\|\big(\rho_{p_\infty}^{-1}(\tau-\delta_i, \tau+\delta_i)\big)\, d\tau\\
&\le \left(\frac{1}{r}+1\right) d_F^Z(\phi_{i\#}T_i, \phi_{\infty\#}T_\infty) + \frac{2}{r}\,\delta_i\,\mathbf{M}(T_\infty).
\end{aligned}$$

When we take the limit as $i \to \infty$, we obtain (59).

By Lemma 4.22, for $\mathcal{L}^1$-a.e. $r > 0$, we have

$$\liminf_{i\to\infty} \mathbf{SF}_k(p_i, r) \le \lim_{j\to\infty} \mathbf{SF}_0(p_{i_j}, r) = \mathbf{SF}_0(p_\infty, r) \le \mathbf{M}(S(p_\infty, r)).$$

Thus for every $r > 0$, we have

$$\begin{aligned}
\limsup_{i\to\infty} \frac{1}{r}\int_0^r \mathbf{SF}_k(p_i,\tau)\, d\tau &\le \limsup_{i\to\infty} \frac{1}{r}\int_0^r \mathbf{SF}_0(p_i,\tau)\, d\tau\\
&= \frac{1}{r}\int_0^r \mathbf{SF}_0(p_\infty,\tau)\, d\tau\\
&\le \frac{1}{r}\int_0^r \mathbf{M}\big(S(p_\infty,\tau)\big)\, d\tau. \qquad \square
\end{aligned}$$

4.5. Continuity of interval filling volumes. Recall the definition of the interval filling volume of a manifold or integral current space in Definition 3.43,

$$\mathbf{IFV}_\epsilon(M) = \mathrm{FillVol}(\partial(M \times I_\epsilon)) \le \epsilon\mathbf{M}(M).$$

This notion was particularly useful for M without boundary. In this section we prove that the interval filling volume is continuous with respect to intrinsic flat convergence (Theorem 4.23). Taking more precise estimates, we prove that the sliced interval filling volumes are continuous as well (Theorem 4.24).

Theorem 4.23. *Suppose we have m dimensional integral current spaces $M_i = (X_i, d_i, T_i)$ such that $M_i \xrightarrow{\mathcal{F}} M_\infty$, then for any fixed $\epsilon > 0$ their interval filling volumes converge,*

$$\lim_{i\to\infty} \mathbf{IFV}_\epsilon(M_i) = \mathbf{IFV}_\epsilon(M_\infty).$$

Proof. By Proposition 3.9, we see that $M_i \times I_\epsilon \xrightarrow{\mathcal{F}} M_\infty \times I_\epsilon$. Thus we have continuity applying Theorem 2.48. $\square$

We now prove the continuity of the sliced interval filling volume defined in Definition 3.45.

Theorem 4.24. *Suppose $M_i \xrightarrow{\mathcal{F}} M$. If $p_i \in M_i$ converge to $p_\infty \in M_\infty$, and $q_{j,i} \in M_\infty$ converge to $q_{j,\infty} \in M_\infty$ for $j = 1, \ldots, k$, then for any fixed $\epsilon > 0$ there is a subsequence such that for almost every $r \in \mathbb{R}$ we have*

$$\lim_{i\to\infty} \mathbf{SIF}_\epsilon(p_i, r, q_{1,i}, \ldots, q_{k,i}) = \mathbf{SIF}_\epsilon(p_\infty, r, q_{1,\infty}, \ldots, q_{k,\infty}). \tag{65}$$

Proof. This theorem is a consequence of Proposition 4.25 stated and proved immediately below, combined with Proposition 4.16 and Definition 3.45. $\square$

Proposition 4.25. *Suppose we have a sequence of m dimensional integral current spaces $M_i = (X_i, d_i, T_i)$ and isometric embeddings $\varphi_i\colon X_i \to Z$, constants $\delta_i > 0$, and points $z_{j,i} \in Z$ such that $d_Z(z_{j,i}, z_{j,\infty}) < \delta_i$ for $j = 1, \ldots, k$ for some $k \in \{0, \ldots, m-1\}$, and $p_i \in X_i$ such that $d_Z(\varphi_i(p_i), \varphi(p)) < \delta$; then for any fixed $\epsilon > 0$ and almost every $t \in \mathbb{R}^k$ we have*

$$\begin{aligned} &\left|\mathbf{IFV}_\epsilon(\mathrm{Slice}(M_i, \rho_i, t)) - \mathbf{IFV}_\epsilon(\mathrm{Slice}(M_\infty, \rho_\infty, t))\right| \\ &\qquad \le (2+\epsilon)\, d_{\mathcal{F}}\big(\mathrm{Slice}(M_i, \rho_i, t), \mathrm{Slice}(M_\infty, \rho_\infty, t)\big). \end{aligned} \tag{66}$$

In particular,

$$\begin{aligned} &\int \left|\mathbf{IFV}_\epsilon(\mathrm{Slice}(M_i, \rho_i, t)) - \mathbf{IFV}_\epsilon(\mathrm{Slice}(M_\infty, \rho_\infty, t))\right| dt \\ &\qquad \le (2+\epsilon)\, d_{\mathcal{F}}(M_i, M_\infty) + 2\delta_i(\mathbf{M}(T_\infty) + \mathbf{M}(\partial T_\infty)). \end{aligned} \tag{67}$$

If $\lim_{i\to\infty} \delta_i = 0$ and

$$\lim_{i\to\infty} d_F^Z(\varphi_{i\#}T_i, \varphi_{\infty\#}T_\infty) = 0,$$

there is a subsequence such that for almost every $t \in \mathbb{R}^k$, the interval filling volumes of slices converge,

$$\lim_{i\to\infty} \mathbf{IFV}_\epsilon(\mathrm{Slice}(M_i, \rho_i, t)) = \mathbf{IFV}_\epsilon(\mathrm{Slice}(M_\infty, \rho_\infty, t)). \tag{68}$$

Proof. Since

$$\mathbf{IFV}_\epsilon(\mathrm{Slice}(M_i, \rho_i, t)) = \mathrm{FillVol}(\partial(\mathrm{Slice}(M_i, \rho_i, t) \times I_\epsilon)),$$

Theorem 2.48 and Proposition 3.9 imply that inequality (66) holds true for almost every $t \in \mathbb{R}^k$. Estimate (67) on the integrated quantity then follows from Proposition 4.13. □

4.6. Limits of points. In this section we prove two statements about Cauchy and converging sequences of points. Recall also Definition 4.1 and Definition 4.2. In [33], the second author and Stefan Wenger proved that certain points in Gromov–Hausdorff limits of sequences are also in the intrinsic flat limit by bounding Gromov's filling volumes of spheres about points converging to these points. In [31] the second author removed the assumption of a Gromov–Hausdorff limit, and studied whether or not Cauchy sequences of points converge to points in the metric completion of the intrinsic flat limit. The technique there involved uniformly bounding the intrinsic flat distance of spheres away from 0 and is not sufficiently precise to distinguish between points in the intrinsic flat limit and its metric completion.

Here we use sliced filling volumes and filling volumes to determine when there is a limit point in the intrinsic flat limit space and not merely in its metric completion. We prove Theorem 4.27 which assumes one has a Cauchy sequence and determines when the Cauchy sequence has a limit point in the limit space, not merely in the metric completion of the limit space. Before we state and prove this theorem, we discuss the difficulties arising in identifying points in the limit space.

Recall that, given an integral current space (X, d, T), any isometric embedding $\varphi\colon X \to Z$ with Z complete maps X isometrically onto $\mathrm{set}(\varphi_\# T)$ and extends to $\varphi\colon \bar{X} \to Z$, which maps X isometrically onto

$$\mathrm{set}(\varphi_\# T) = \left\{ z \in Z \,:\, \liminf_{r\to 0} \frac{\|\varphi_\# T\|(B(z,r)))}{r^m} > 0 \right\}.$$

In Lemma 4.22 we proved that

$$\mathbf{SF}_k(p,r) \le \mathbf{SF}_0(p,r) = \mathrm{FillVol}(\partial S(p,r)) \le \mathbf{M}(S(p,r)).$$

In the work of the second author with Wenger, continuity of the filling volume was applied to prove that for certain sequences of spaces a point in the Gromov–Hausdorff limit lies in the intrinsic flat limit. In general it may be tricky to use the filling volume in this way.

Remark 4.26. Let $M = (X, d, T)$ be the m dimensional integral current space of Example 2.35. It has a point $p \in X$ which is the center of the concentric spheres. This $p \in X = \mathrm{set}(T)$ because $\mathbf{M}(B(p,r)) \le Cr^m$. However $\mathbf{SF}_0(p,r) = 0$ for $\mathcal{L}$-a.e. small $r > 0$. This shows that although $\mathbf{SF}_k(p,r)$ for $k = 0, \ldots, m-1$ provide lower bounds for $\mathbf{M}(S(p,r))$, in general these lower bounds could be far from sharp and thus not be able to identify when a point lies in X.

The next theorem concerns Cauchy sequences of points in the sense of Definition 4.2.

Theorem 4.27. *Suppose M_i are integral current spaces of dimension m with M_i converging to M_∞ in the intrinsic flat sense. Let $k \in \{0, 1, 2, \ldots, m-1\}$. Suppose that $p_i \in M_i$ are Cauchy. If there is a function $c\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\frac{1}{r}\int_0^r \mathbf{SF}_k(p_i, r)\, d\tau \ge c(r), \tag{69}$$

then the p_i converge to a point $p_\infty \in \bar{M}_\infty$. If in addition

$$\liminf_{r\downarrow 0} \frac{c(r)}{r^m} > 0, \tag{70}$$

then the p_i converge to a point $p_\infty \in M_\infty$.

Proof. By Lemma 4.22, it suffices to prove the case of $k = 0$. If $M_i \xrightarrow{\mathcal{F}} M$ and the points $p_i \in M_i$ are Cauchy, there is a complete separable metric space Z and isometric embeddings $\phi_i\colon X_i \to Z$ such that

$$\lim_{i\to\infty} d_F^Z(\phi_{i\#}T_i, \phi_{\infty\#}T_\infty) = 0,$$

and there exists a $z_\infty \in Z$ such that $\delta_i := d_Z(\phi_i(p_i), z_\infty) \to 0$ in Z as $i \to \infty$.

Again, by using Theorem 2.48, Proposition 4.16, and Lemma 4.15, we find

$$\begin{aligned}
\mathbf{M}\big(\phi_{\infty\#}T_\infty \llcorner B_r(z_\infty)\big) &\ge \frac{1}{r}\int_0^r \mathbf{M}\big(\phi_{\infty\#}T_\infty \llcorner B_\tau(z_\infty)\big)\, d\tau \\
&\ge \frac{1}{r}\int_0^r \mathrm{FillVol}_\infty\left(\partial(\phi_{\infty\#}T_\infty \llcorner B_\tau(z_\infty))\right) d\tau \\
&\ge c(r) - \left(\frac{1}{r}+1\right) d_F^Z(\phi_{i\#}T_i, \phi_{\infty\#}T_\infty) - \frac{1}{r}\int_0^r \|T\|_\infty\left(\rho_{p_\infty}^{-1}(\tau - \delta_i, \tau + \delta_i)\right) d\tau \\
&\ge c(r) - \left(\frac{1}{r}+1\right) d_F^Z(\phi_{i\#}T_i, \phi_{\infty\#}T_\infty) - \frac{2\delta_i}{r}\mathbf{M}(T_\infty).
\end{aligned}$$

We take the limit as $i \to \infty$ and conclude that

$$\mathbf{M}(\phi_{\infty\#}T_\infty \llcorner B_r(z_\infty)) \ge c(r).$$

Therefore, M_∞ is not the $\mathbf{0}$ space, and $z_\infty \in \overline{\mathrm{set}(\phi_{\infty\#}T_\infty)}$. Moreover, if inequality (365) is satisfied, $z_\infty \in \mathrm{set}(\phi_{\infty\#}T_\infty)$. □

Example 4.28. It is quite possible for a Cauchy sequence of points to have more than one limit as can be seen simply by taking the constant sequence of integral current spaces, S^1, and noting that due to the isometries, any point may be set up as the limit of a Cauchy sequence of points. One may also use isometries of S^1 to relocate a Cauchy sequence so that the images are no longer Cauchy in Z. This is also true of converging sequences in the theory of Gromov–Hausdorff convergence.

Remark 4.29. Note that in order to apply the first part of Theorem 4.27, it is sufficient to find a function $c(r)\colon \mathbb{R}_+ \to \mathbb{R}_+$ and a constant $r_0 > 0$ such that (69) is satisfied for $0 < r < r_0$. In particular, (69) holds true if $\mathbf{SF}_k(p_i, r) > \widetilde{c}(r)$ for $\mathcal{L}$-a.e. $0 < r < r_0$. Finally, (65) holds true if there exists a constant C such that $\mathbf{SF}_k(p_i, r) \geq Cr^m$ for $0 < r < r_0$.

4.7. Bolzano–Weierstrass theorems. When one has a sequence of compact metric spaces converging in the Gromov–Hausdorff sense to a compact metric space, and one has a sequence of points in those metric spaces, then a subsequence converges to a point in the Gromov–Hausdorff limit. This is the Gromov–Hausdorff Bolzano–Weierstrass Theorem and is an immediate consequence of Gromov's Embedding Theorem which provides a common metric space which is compact. The immediate restatement of the Gromov–Hausdorff Bolzano–Weierstrass Theorem is not true when the spaces converge in the intrinsic flat sense instead of the Gromov–Hausdorff sense. This can be seen in Ilmanen's Example with disappearing tips. The key difficulty lies in the fact that, unlike Gromov's Embedding Theorem, Theorem 2.41 does not provide a compact common metric space.

Nevertheless we are able to prove the following two Bolzano–Weierstrass Theorems by assuming the limit space is compact and preventing the points in the sequence from disappearing. These theorems require bounding the Gromov Filling Volumes and Sliced Filling Volumes of spheres. A simpler Bolzano–Weierstrass Theorem was proved by the second author in [31]. It requires uniformly bounding the intrinsic flat distance of spheres away from 0 but only produces a subsequence that converges in the metric completion of the intrinsic flat limit space.

Theorem 4.30. *Suppose $M_i^m = (X_i, d_i, T_i)$ are m-dimensional integral current spaces converging in the intrinsic flat sense to a limit integral current space $M_\infty^m = (X_\infty, d_\infty, T_\infty)$. Suppose there exists a sequence $p_i \in M_i$ and a function $c\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\frac{1}{r}\int_0^r \operatorname{FillVol}(\partial S(p_i, \tau))\, d\tau \geq c(r) > 0.$$

Then there exists a subsequence p_{i_j} that converges to $p_\infty \in \bar{M}_\infty^m$. In particular, M_∞^m is nonzero.

If in addition

$$\liminf_{r\downarrow 0} \frac{c(r)}{r^m} > 0, \tag{71}$$

then the subsequence converges to a point $p_\infty \in M_\infty^m$.

Proof. By Theorem 2.41 there is a complete metric space Z and isometric embeddings $\phi_i\colon X_i \to Z$ such that $d_F^Z(\phi_{i\#}T_i, \phi_{\infty\#}T_\infty) \to 0$. Set $z_i := \phi_i(p_i)$.

Note that by Proposition 4.16

$$\int_0^{r_0} d_F^Z\big(\phi_{i\#}T_i \llcorner B_r(z_i), \phi_{\infty\#}T_\infty \llcorner B_r(z_i)\big)\, dr \leq 2d_{\mathcal{F}}(M_i, M_\infty).$$

Hence, by Theorem 2.48 and our hypothesis,

$$\begin{aligned}\mathbf{M}\big(\phi_{\infty\#}T_\infty \llcorner B_r(z_i)\big) &\geq \liminf_{i\to\infty} \frac{1}{r}\int_0^r \mathbf{M}\big(\phi_{\infty\#}T_\infty \llcorner B_\tau(z_i)\big)\, d\tau \\ &\geq \liminf_{i\to\infty} \frac{1}{r}\int_0^r \operatorname{FillVol}_\infty\big(\partial(\phi_{\infty\#}T_\infty \llcorner B_\tau(z_i))\big)\, d\tau \\ &\geq c(r).\end{aligned} \tag{72}$$

In particular, M_∞ is not the $\mathbf{0}$ space.

We claim that the z_i have a Cauchy subsequence.

We argue by contradiction. If not, the metric space $(\{z_i\}, d_Z)$ is complete, and therefore not totally bounded, so that there is a $\delta > 0$ such that for a subsequence (without relabeling) $d_Z(z_i, z_j) > 4\delta$ for $i \neq j$. As the balls $B_\delta(z_i)$ are mutually disjoint, this would mean that

$$\lim_{i\to\infty} \mathbf{M}\big(\phi_{\infty\#}T_\infty \llcorner B_\delta(z_i)\big) = 0,$$

which contradicts (72).

Consequently, the z_i have a Cauchy subsequence. We could conclude now by applying Theorem 4.27, or by mimicking its proof. With the established notation, however, we can easily finish the proof in an alternative fashion.

Since Z is complete, this subsequence, also denoted z_i, converges to a limit $z_\infty \in Z$. Since for every $\tau < r$, for i large we have $B_\tau(z_i) \subset B_r(z_\infty)$ by (72), it follows that

$$\mathbf{M}\big(\phi_{\infty\#}T_\infty \llcorner B_r(z_\infty)\big) \geq \limsup_{i\to\infty} \mathbf{M}\big(\phi_{\infty\#}T_\infty \llcorner B_\tau(z_i)\big) \geq c(\tau)$$

for every $\tau < r$. Consequently, $z_\infty \in \overline{\mathrm{set}(\phi_{\infty\#}T_\infty)}$, and if inequality (71) holds true, even $z_\infty \in \mathrm{set}(\phi_{\infty\#}T_\infty)$. □

This theorem is a special case of the following more general Bolzano–Weierstrass Theorem. The generalization follows from the special case and Lemma 4.22.

Theorem 4.31. *Suppose $M_i^m = (X_i, d_i, T_i)$ are m dimensional integral current spaces converging in the intrinsic flat sense to a limit integral current space $M_\infty^m = (X_\infty, d_\infty, T_\infty)$. Suppose there exists $k \in \{0, 1, \ldots, (m-1)\}$, $r_0 > 0$, a sequence $p_i \in M_i$, and a function $c\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\frac{1}{r}\int_0^r \mathbf{SF}_k(p_i, \tau)\, d\tau \geq c(r) > 0. \tag{73}$$

Then there exists a subsequence p_{i_j} that converges to $p_\infty \in \bar{M}_\infty^m$. In particular, M_∞^m is nonzero.

If in addition

$$\liminf_{r\downarrow 0} \frac{c(r)}{r^m} > 0, \tag{74}$$

then in fact $p_\infty \in M_\infty^m$.

§5. Compactness theorems

In this section we complete the proofs of our main two compactness theorems: Theorem 5.2 and Theorem 5.1. These theorems were announced by the second author in [30]. Both of these theorems prove that certain sequences of spaces have subsequences that converge in both the intrinsic flat and the Gromov–Hausdorff sense to the same space. Theorem 5.2 is the Tetrahedral Compactness Theorem concerning sequences of Riemannian manifolds satisfying the tetrahedral property. It was partially stated in the introduction. It is a consequence of Theorem 5.1, which applies to integral current spaces that have uniform lower bounds on the sliced filling volumes of the form $\mathbf{SF}_k(p, r) \geq C_{SF} r^m$.

In prior work of the second author with Wenger [33], another pair of compactness theorems was proved providing subsequences of manifolds that converge both in the intrinsic flat and Gromov Hausdorff sense to the same limit. One theorem concerned noncollapsing sequences of Riemannian manifolds with nonnegative Ricci curvature (extending Gromov's Ricci Compactness Theorem [10]). The other concerned sequences of Riemannian manifolds with a uniform linear contractibility function and a uniform upper bound on volume (extending Greene–Petersen's Compactness Theorem [8]). The techniques used in the proof of the Contractibility Function Compactness Theorem in

[33] involve the continuity of the filling volumes of balls. Here we use the continuity of sliced filling volumes in a similar way.

The proofs of the theorems in this section are very short because they build upon the prior theorems proved in the previous sections of this paper. Those theorems have applications in other situations and so it was important to prove them separately rather than hiding those results within the proofs of these theorems.

5.1. Sliced filling compactness theorem.

Theorem 5.1. *Suppose we have a sequence of m dimensional integral current spaces $M_i = (X_i, d_i, T_i)$ with $\mathbf{M}(M_i) \le V_0$, $\mathbf{M}(\partial M_i) \le A_0$, $\mathrm{Diam}(M_i) \le D_0$ and a uniform constant $C_{SF} > 0$ such that*

$$\mathbf{SF}_k(p, r) \ge C_{SF} r^m$$

for some $k \in 0, \ldots, (m-1)$. Then a subsequence of the M_i converges in the Gromov–Hausdorff sense and the Intrinsic Flat sense to a nonzero integral current space M_∞.

Proof. By Theorem 3.23, we know a subsequence (X_i, d_i) has a Gromov–Hausdorff limit (Y, d_Y). Thus by Gromov, there exists a common compact metric space Z and isometric embeddings $\varphi_i \colon X_i \to Z$, $\varphi \colon Y \to Z$ such that $d_H^Z(\varphi(X_i), \varphi(Y)) \to 0$. By the Ambrosio–Kirchheim Compactness Theorem, a subsequence of $\varphi_{1\#} T_i$ converges to $T_\infty \in \mathbf{I}_m(Z)$. Let $M_\infty = (\mathrm{set}(T_\infty), d_Z, T_\infty)$.

We need only show that $\varphi(Y) = \mathrm{set}(T_\infty)$. Let $z_\infty \in Y$, and let $p_i \in X_i$ be such that $z_i = \varphi_i(p_i) \to z$. By Theorem 4.27, we see that $z_\infty = \varphi(p_\infty)$. □

5.2. Tetrahedral compactness theorem.

Theorem 5.2. *Given $r_0 > 0$, $\beta \in (0,1)$, $C > 0$, $V_0 > 0$, $A_0 > 0$, if a sequence of compact Riemannian manifolds, M^m, has $\mathrm{Vol}(M^m) \le V_0$, $\mathrm{Diam}(M^m) \le D_0$ and the C, β (integral) tetrahedral property for all balls of radius $\le r_0$, then a subsequence converges in the Gromov–Hausdorff and Intrinsic Flat sense to a nonzero integral current space.*

Here our manifolds do not have boundary.

Proof. The C, β (integral) tetrahedral property implies that there exists $C_{SF} > 0$ such that

$$\mathbf{SF}_{m-1}(p, r) \ge C_{SF} r^m.$$

Theorem 3.41 implies there exists a uniform upper bound on diameter. So we apply Theorem 5.1. □

Remark 5.3. As a consequence of this theorem, we see that there is no uniform tetrahedral property on manifolds with positive scalar curvature even when the volumes of the balls are uniformly bounded below by those of Euclidean balls. In fact there exists a sequence of such manifolds M_j^3 whose intrinsic flat limit is 0, described in [33].

§6. Appendix.
Gluing integral current spaces

Here we define how to glue two integral current spaces along an isometric boundary to produce a new integral current space. This is applied to prove Theorem 2.48. It is straightforward but included because it has some technical difficulties that must be handled.

Theorem 6.1. *Given two integral current spaces, $M_i = (X_i, d_i, T_i)$, with $S_i, S_i' \in \mathbf{I}_{m-1}(\bar{X}_i)$ such that*

$$\partial T_i = S_i + S_i' \quad \textit{with} \quad \operatorname{spt}(S_i) \subset \operatorname{spt}(\partial T_i)$$

and a current reversing distance preserving bijection

$$F\colon \operatorname{spt}(S_1) \to \operatorname{spt}(S_2) \quad \textit{such that} \quad F_\# S_1 = -S_2,$$

we define the glued integral current space

$$M = M_1 \sqcup_F M_2 = (X, d, T)$$

such that there are distance preserving maps

$$f_i \colon \bar{X}_i \to Y \tag{75}$$

where Y is the glued metric space:

$$Y = \bar{X}_1 \sqcup (\bar{X}_2 \setminus \operatorname{spt}(S_2))$$

such that

$$f_2 \circ F = f_1 \quad \textit{when restricted to} \quad \operatorname{spt}(S_1),$$

and such that

$$T = f_{1\#}T_1 + f_{2\#}T_2 \quad \textit{and} \quad X = \operatorname{set} T \subset Y. \tag{76}$$

Since f_i are distance preserving,

$$\mathbf{M}(M) \le \mathbf{M}(M_1) + \mathbf{M}(M_2).$$

Note that it is possible that the glued integral current space, X, is a proper subset of the glued metric space, Y. In fact the glued integral current space could be the 0 space (see Example 6.3).

Before we prove this theorem, we apply it to prove the following useful corollary which glues integral current spaces together in order to provide an estimate on the filling volume. This corollary is applied to prove Theorem 2.48. See Definition 2.46 for the definition of filling volume being applied here.

Corollary 6.2. *Suppose we are given two integral current spaces $M_i = (X_i, d_i, T_i)$ with $S_i, S_i' \in \mathbf{I}_{m-1}(\bar{X}_i)$ such that*

$$\partial T_i = S_i + S_i',$$

and a current reversing distance preserving map

$$F\colon \operatorname{spt}(S_1) \to \operatorname{spt}(S_2) \quad \textit{such that} \quad F_\# S_1 = -S_2.$$

If $S_2' = 0$, then

$$\operatorname{FillVol}(N) \le \mathbf{M}(M_1) + \mathbf{M}(M_2),$$

where $N = (\operatorname{set}(S_1'), d_1, S_1')$.

Proof. By Theorem 6.1 we have

$$\begin{aligned}
\partial T &= f_{1\#}\partial T_1 + f_{2\#}\partial T_2 \\
&= f_{1\#}S_1 + f_{1\#}S_1' + f_{2\#}S_2 + f_{2\#}S_2' \\
&= f_{2\#}F_\# S_1 + f_{2\#}S_2 + f_{1\#}S_1' + f_{2\#}S_2' \\
&= f_{1\#}S_1' + f_{2\#}S_2'.
\end{aligned}$$

Thus if $S_2' = 0$, then $\partial T = f_{1\#}S_1'$ and so by (75) we have a current preserving isometry

$$f_1 \colon N = (\operatorname{set}(S_1'), d_1, S_1') \to \partial M = (\operatorname{set}(\partial T), d, \partial T).$$

Thus

$$\operatorname{FillVol}(N) = \operatorname{FillVol}(\partial M) \le \mathbf{M}(T) \le \mathbf{M}(M_1) + \mathbf{M}(M_2). \qquad \square$$

We now prove Theorem 6.1:

Proof. First we prove the glued metric space, Y, is well defined. This is included for completeness of exposition and because there are different methods used for gluing metric spaces.

Let $d\colon Y \times Y \to [0,\infty)$ be symmetric and such that

$$d(x,y) = \begin{cases} d_1(x,y), & x,y \in \bar{X}_1, \\ d_2(x,y), & x,y \in \bar{X}_2 \setminus \operatorname{spt}(S_2), \\ \inf\{d_1(x,w) + d_2(F(w),y) \,:\, w \in \operatorname{spt}(S_1)\}, & x\in\bar{X}_1,\ y\in\bar{X}_2 \setminus \operatorname{spt}(S_2). \end{cases}$$

Observe that $d(x,y) \geq 0$.

Observe that $d(x,y) = 0$ implies that $x = y$ if both $x,y \in \bar{X}_1$ or both $x,y \in \bar{X}_2\setminus\operatorname{spt}(S_2)$ since d_1 and d_2 are metrics. The third case, where $x \in \bar{X}_1$ and $y \in \bar{X}_2 \setminus \operatorname{spt}(S_2)$ cannot produce $d(x,y) = 0$. If it did occur then we would have $w_j \in \operatorname{spt}(S_1)$ such that

$$0 = d(x,y) = \lim_{j\to\infty} d_1(x,w_j) + d_2(F(w_j),y)$$

so

$$\lim_{j\to\infty} d_2(F(w_j),y) = 0$$

and y is in the closure of the image of F, which means $y \in \operatorname{spt}(S_2)$ which is a contradiction.

To see the triangle inequality observe that if $x_i \in \bar{X}_1$ and $y_i \in \bar{X}_2 \setminus \operatorname{spt}(S_2)$, then

$$\begin{aligned}
&d(x_1,x_3) + d(x_3,x_2) = d_1(x_1,x_3) + d_1(x_3,x_2) \geq d_1(x_1,x_2) = d(x_1,x_2),\\
&d(x_1,y_3) + d(y_3,x_2)\\
&\qquad = \inf\{d_1(x_1,w) + d_2(F(w),y_3) + d_1(x_2,w') + d_2(F(w'),y_3) \,:\, w,w' \in \operatorname{spt}(S_1)\}\\
&\qquad \geq \inf\{d_1(x_1,w) + d_2(F(w),F(w')) + d_1(x_2,w') \,:\, w,w' \in \operatorname{spt}(S_1)\}\\
&\qquad = \inf\{d_1(x_1,w) + d_1(w,w') + d_1(x_2,w') \,:\, w,w' \in \operatorname{spt}(S_1)\}\\
&\qquad \geq d_1(x_1,x_2) = d(x_1,x_2),\\
&d(y_1,y_3) + d(y_3,y_2) = d_2(y_1,y_3) + d_2(y_3,y_2) \geq d_2(y_1,y_2) = d(y_1,y_2),\\
&d(y_1,x_3) + d(x_3,y_2)\\
&\qquad = \inf\{d_1(x_3,w) + d_2(F(w),y_1) + d_1(x_3,w') + d_2(F(w'),y_2) \,:\, w,w' \in \operatorname{spt}(S_1)\}\\
&\qquad \geq \inf\{d_1(w',w) + d_2(F(w),y_1) + d_2(F(w'),y_2) \,:\, w,w' \in \operatorname{spt}(S_1)\}\\
&\qquad = \inf\{d_2(F(w'),F(w)) + d_2(F(w),y_1) + d_2(F(w'),y_2) \,:\, w,w' \in \operatorname{spt}(S_1)\}\\
&\qquad \geq d_2(y_1,y_2) = d(y_1,y_2),\\
&d(x_1,x_3) + d(x_3,y_2) = \inf\{d_1(x_1,x_3) + d_1(x_3,w) + d_2(F(w),y_2) \,:\, w \in \operatorname{spt}(S_1)\}\\
&\qquad \geq \inf\{d_1(x_1,w) + d_2(F(w),y_2) \,:\, w \in \operatorname{spt}(S_1)\} = d(x_1,y_2),\\
&d(x_1,y_3) + d(y_3,y_2) = \inf\{d_1(x_1,w) + d_2(F(w),y_3) + d_2(y_3,y_2) \,:\, w \in \operatorname{spt}(S_1)\}\\
&\qquad \geq \inf\{d_1(x_1,w) + d_2(F(w),y_2) \,:\, w \in \operatorname{spt}(S_1)\} = d(x_1,y_2).
\end{aligned}$$

Thus d is a metric on $\bar{X}$.

By the definition of d, immediately we have natural identification maps

$$f_1\colon \bar{X}_1 \to \bar{X} \text{ such that } f_1(x) = x,$$
$$f_2\colon \bar{X}_2 \setminus \operatorname{spt}(S_2) \to X \text{ such that } f_2(x) = x,$$

which are immediately distance preserving. We define

$$f_2\colon \operatorname{spt}(S_2) \to Y \text{ such that } f_2(y) = f_1(F^{-1}(y)).$$

Then for $y_1, y_2 \in \operatorname{spt}(S_2)$ and $y_3 \in \bar{X}_2 \setminus \operatorname{spt}(S_2)$ we have

$$\begin{aligned}
d(f_2(y_1), f_2(y_2)) &= d\left(f_1(F^{-1}(y_1)), f_1(F^{-1}(y_2))\right) \\
&= d_1\left(F^{-1}(y_1), F^{-1}(y_2)\right) = d_2(y_1, y_2), \\
d(f_2(y_1), f_2(y_3)) &= d\left(f_1(F^{-1}(y_1)), f_2(y_3)\right) \\
&= \inf\left\{d_1\left(F^{-1}(y_1), w\right) + d_2(F(w), f_2(y_3))\ :\ w \in \operatorname{spt}(S_1)\right\} \\
&= \inf\left\{d_2(y_1, F(w)) + d_2(F(w), f_2(y_3))\ :\ w \in \operatorname{spt}(S_1)\right\} \\
&= d_2(y_1, y_3).
\end{aligned}$$

Thus $f_2\colon \bar{X}_2 \to Y$ is also distance preserving.

Since the f_i are distance preserving, they are Lipschitz, and so $f_{i\#}T_i$ is well defined and $\mathbf{M}(f_{i\#}T_i) = \mathbf{M}(T_i)$. Defining $M = (X, d, T)$ as in (76), we have

$$\mathbf{M}(M) = \mathbf{M}(T) \leq \mathbf{M}(f_{1\#}T_1) + \mathbf{M}(f_{2\#}T_2) = \mathbf{M}(T_1) + \mathbf{M}(T_2). \qquad \square$$

Example 6.3. Let $M_1 = (X_1, d_1, T_1)$ where T_1 is a two dimensional integral current in $X_1 \subset [0,1]^2$, endowed with the standard Euclidean metric $d_1(x,y) = |x - y|$. Let $M_2 = (X_2, d_2, T_2)$ where $T_2 = -T_1$ and $d_2 = d_1$, so $X_1 = X_2$. Then one can glue M_1 to M_2 along their boundary.

Suppose that ∂T_1 is dense in X_1 so that $\operatorname{set}(T) \subset \operatorname{spt}(\partial T)$. If we glue M_1 to M_2 along their boundary, they are completely glued together with opposite orientation and we obtain the $\mathbf{0}$ space. More precisely, we have

$$Y = \operatorname{spt}(T) = \operatorname{spt}(\partial T) = \bar{X}_1$$

and $f_i\colon [0,1] \to [0,1]$ are identity maps. So

$$f_{1\#}T_1 = f_{2\#}T_1 = f_{2\#}(-T_2) = -f_{2\#}T_2.$$

Thus $T = f_{1\#}T_1 + f_{2\#}T_2 = 0$.

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