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REPRESENSIBILITY OF CONES OF MONOTONE FUNCTIONS IN WEIGHTED LEBESGUE SPACES AND EXTRAPOLATION OF OPERATORS ON THESE CONES

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Abstract. It is shown that a sublinear operator is bounded on the cone of monotone functions if and only if a certain new operator related to the one mentioned above is bounded on a certain ideal space defined constructively. This construction is used to provide new extrapolation theorems for operators on the cone in weighted Lebesgue spaces.

§1. Introduction

The role of sharp estimates for classical operators in harmonic analysis and related fields is well known. In the recent time, in connection with new analytic problems, it has become fashionable to estimate such operators on certain cones in a given space rather than on the entire space (see, e.g., [1, 4, 17, 20, 27, 28, 36, 38, 40, 41]). Next, for operators with positive kernels the Schur extrapolation theorem is also well known (see, e.g., [25]), saying that an integral operator $Tx(t) = \int k(t,s)x(s)\,ds$, $k(t,s) \geq 0$ is bounded on L^p if and only if there exists a positive function $u(t)$ finite a.e. and such that the operator is bounded in the couples $T\colon L^\infty_u \to L^\infty_u$ and $T\colon L^1_v \to L^1_v$, where $v = u^{1/p-1}$. Since various problems of analysis have resulted in a gradually increasing interest to extrapolation theorems, see [5, 7–9], it seems to be natural to pass from spaces to cones in the extrapolation theory for L^p.

The present work was planned as early as in the beginning of the 2000s. A short summary of the main results was given in [10]. The central result of the paper consists of the verification of the fact that, basically, the cone $K(\downarrow) \cap L^p_v$ in the Lebesgue space L^p_v is generated by the linear operator $Qx(t) = \int_t^\infty x(s)\,ds$ of integration. For the operators T in the class $\mathrm{Sub}(\alpha,\beta,\gamma,\downarrow)$ (which is described below and contains all subadditive operators), this makes it possible to prove the equivalence $T : K(\downarrow) \cap L^p_v \to X \Leftrightarrow TQ\colon L^p_{\bar{v}} \to X$. Here the weight $\bar{v}$ is defined constructively in terms of the weight function v (see Theorem 1). This approach distinguishes our paper from the well-known paper [16], which is devoted to estimates of classical operators in the couple $(K(\downarrow) \cap L^p_v, L^p_w)$ implied by certain estimates in couples of weighted Lebesgue spaces.

Our approach allows us to apply the entire technique of sharp estimates on weighted Lebesgue spaces to the derivation of sharp estimates of operators on cones. In particular, these constructions have led us to a new extrapolation theorem for operators in the class $\mathrm{Sub}(\alpha,\beta,\gamma,\downarrow)$ that are defined on cones included in $K(\downarrow) \cap L^p_v$ in a weighted Lebesgue space. This extrapolation theorem is new even for the Hardy classical operator.

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§2. Preliminaries

Let $S(\mu) = S(R_+, \Sigma, \mu)$ ($R_+ = (0, +\infty)$) denote the space of measurable functions $x\colon R_+ \to R$, let $\chi(D)$ stand for the characteristic function of a set D, and let $\|x \,|\, X\|$ be the norm of an element x in X. Recall that a Banach space $X = (X, \|\cdot\,|\,X\|)$ of measurable functions is called an ideal space (see [24, 26]) if $x \in X$ and $\|x\,|\,X\| \le \|y\,|\,X\|$ whenever x is measurable and $|x(t)| \le |y(t)|$ a.e. on R_+ for some $y \in X$. As usual, the symbol L^p ($1 \le p \le \infty$) denotes the Lebesgue space, and the exponent conjugate to $p \in [1, \infty]$ is denoted by p': $\frac{1}{p} + \frac{1}{p'} = 1$.

Let $v\colon R_+ \to R_+$ be a positive function (a weight). If X is an ideal space, we denote by X_v the new ideal space whose norm is defined as follows: $\|x\,|\,X_v\| = \|vx\,|\,X\|$. In particular, the norm in L^p_v ($1 \le p < \infty$) has the form

$$\|x\,|\,L^p_v\| = \left(\int_0^\infty |x(t)v(t)|^p \, dt \right)^{1/p},$$

which differs somewhat from the variant usually adopted (the latter presumes the incorporation of the weight in the measure).

For every ideal space X, the dual space X' is well defined. It consists of bounded linear functionals on X representable by integrals; the norm of every such functional is defined by $\|y\,|\,X'\| = \sup\left\{ \int_R y(t)x(t)\,dt \;:\; \|x\,|\,X\| \le 1 \right\}$. If v is a weight and X is an ideal space, it can easily be verified that

$$(X_v)' = (X')_{1/v}. \tag{1}$$

Let X be an ideal space in $S(\mu)$, and K a cone in $S(\mu)$. The symbol $K \cap X$ denotes the intersection of K with X_+.

Definition 1. We denote by $K(\downarrow)$ the cone in $S(\mu)$ consisting of monotone nonincreasing functions $x\colon R_+ \to R_+$, i.e., $x(t+h) \le x(t)$ for $h \ge 0$. Similarly, $K(\uparrow)$ denotes the cone of monotone nondecreasing functions in $S(\mu)$.

Now, we describe the classes of operators to be treated in the paper.

We denote by $\mathrm{Sub}(+)$ the class of sublinear operators T defined on an ideal space X or on $S(\mu)$ and taking values in $S(\mu)$. For $T \in \mathrm{Sub}(+)$, the adjoint operator may fail to exist, but its role can be played by the associated operator $T' \in \mathrm{Sub}(+)$ defined as follows.

For $T \in \mathrm{Sub}(+)$, an operator $T' \in \mathrm{Sub}(+)$ is said to be *associated* with T in the scale L^p if for all $1 \le p \le \infty$ and all weight functions u, the operator $T\colon L^p_u \to L^p_u$ is bounded if and only if $T'\colon L^{p'}_{1/u} \to L^{p'}_{1/u}$ is also bounded and, moreover,

$$C^{-1}\|T\,|\,L^p_u \to L^p_u\| \le \|T'\,|\,L^{p'}_{1/u} \to L^{p'}_{1/u}\| \le C\|T\,|\,L^p_u \to L^p_u\|$$

with some constant $C > 0$ independent of p and u.

An associated operator is not uniquely determined. If T is linear, we may take the adjoint T^* for the role of an associated operator T'. Next, for a linear operator T, the operator $x \longmapsto |Tx|$ is sublinear and possesses no adjoint, but the operator $T'x = |T^*x|$ is associated with it. If $T \in \mathrm{Sub}(+)$ and a linear operator T_1 is given, then the role of operators associated with the compositions TT_1 and T_1T can be played by $(T_1)^*T'$ and $T'(T_1)^*$. Thus, the set $T, T' \in \mathrm{Sub}(+)$ is a two-sided ideal with respect to composition with bounded linear operators.

Now, we extend the class $\mathrm{Sub}(+)$.

Definition 2. We say that an operator $T\colon X \cap K(\downarrow) \to Y$ belongs to the class $\mathrm{Sub}(\alpha, \beta, \gamma, \downarrow)$, ($\alpha \ge 1, \beta > 0, \gamma > 0$) if

a) for every $x, y \in X \cap K(\downarrow)$ we have

$$\|T(y+x)\,|\,Y\| \le \alpha(\|Ty\,|\,Y\| + \|Tx\,|\,Y\|);$$

b) for every $x \in X \cap K(\downarrow)$ and every $\lambda \in R$ we have

$$\|T(\lambda x)\,|\,Y\| = |\lambda|\|Tx\,|\,Y\|;$$

c) for every $x \in X \cap K(\downarrow)$ we have

$$\inf\big\{\|Ty\,|\,Y\| \,:\, y(t) \ge \beta x(t) \,:\, y \in X \cap K(\downarrow)\big\} \ge \gamma\|Tx\,|\,Y\|.$$

It is straightforward from the definition that every operator belonging to Sub(+) belongs also to Sub($\downarrow$). To see this, it suffices to put $\alpha = 1$, choose a positive number β arbitrarily, and define γ by $\gamma = \max\{1, \frac{1}{\beta}\}$.

For $T \in \mathrm{Sub}(\alpha, \beta, \gamma, \downarrow)$, we can define an operator T' associated with it in the scale L^p by analogy with the case of $T \in \mathrm{Sub}(+)$.

The proof of the following lemma is an easy consequences of the definitions.

Lemma 1. (a) *Suppose that $T \in \mathrm{Sub}(\alpha, \beta, \gamma, \downarrow)$ and $\delta \in (0, \infty)$. Then*
(a) *if $\delta > 1$, then $T \in \mathrm{Sub}(\alpha, \delta\beta, \gamma, \downarrow)$;*
(b) *if $\delta < 1$, then $T \in \mathrm{Sub}(\alpha, \delta\beta, \frac{1}{\delta}\gamma, \downarrow)$.*

Since precise values of the constants are irrelevant in the present paper, we introduce the notation Sub($\downarrow$) for the following class of operators:

$$\mathrm{Sub}(\downarrow) = \bigcup_{\beta,\gamma>0}\Big(\bigcup_{\alpha\ge 1}\mathrm{Sub}(\alpha,\beta,\gamma,\downarrow)\Big).$$

We present an example showing that Sub($\downarrow$) is much wider than Sub(+).

Fix a monotone increasing sequence $\{k_i\}_1^\infty$ of positive integers; let $k_1 > 4$. We introduce a function $w\colon [0,\infty) \to R_+$. On each interval $[i, i+1)$ it is given by

$$w(t) = \begin{cases} 1, & \text{for } t \in [i-1, i-\frac{1}{k_i}), \\ -\frac{k_i}{4}, & \text{for } t \in [i-\frac{1}{k_i}, i), \end{cases} \qquad i = 1, 2, \dots$$

Now, we define a functional f by the formula

$$f(x) = \int_0^\infty w(s)x(s)\,ds.$$

Then for every $x \in K(\downarrow)$ we have

$$\begin{aligned}\int_{i-1}^{i} w(s)x(s)\,ds &= \int_{i-1}^{i-\frac{1}{k_i}} w(s)x(s)\,ds + \int_{i-\frac{1}{k_i}}^{i} w(s)x(s)\,ds \\ &\ge \int_{i-1}^{i-\frac{1}{k_i}} x(s)\,ds - k_i\frac{1}{4k_i}x\Big(i-\frac{1}{k_i}\Big) \\ &\ge \int_{i-1}^{i-\frac{1}{2}} x(s)\,ds \ge \frac{1}{2}\int_{i-1}^{i} x(s)\,ds.\end{aligned}$$

This inequality shows that

$$\begin{aligned}\int_0^\infty x(s)\,ds \ge \int_0^\infty w(s)x(s)\,ds = f(x) = \sum_{i=1}^\infty \int_{i-1}^{i} w(s)x(s)\,ds \\ \ge \frac{1}{2}\sum_{i=1}^\infty \int_{i-1}^{i} x(s)\,ds = \frac{1}{2}\int_0^\infty x(s)\,ds\end{aligned}$$

for every $x \in K(\downarrow)$. But now, if $y \in K(\downarrow)$ and $y(t) \geq \beta x(t)$ for a.e. t, then, applying the last inequality, we obtain

$$f(x) \leq \int_0^\infty x(s)\,ds \leq \frac{1}{\beta}\int_0^\infty y(s)\,ds \leq \frac{2}{\beta} f(y).$$

Thus, the functional f constructed above belongs to $\mathrm{Sub}(1, \beta, \frac{2}{\beta}, \downarrow)$. On the other hand, since $\lim_{i\to\infty} k_i = \infty$, we see that there is no nonnegative function w_0 having the property that the functionals

$$f(x) = \int_0^\infty w(s)x(s)\,ds, \quad f_0(x) = \int_0^\infty w_0(s)x(s)\,ds$$

are equivalent on the cone of nonnegative functions. Thus, the functional f is not equivalent to any functional f_0 belonging to $\mathrm{Sub}(+)$.

Below we shall often use the classical integral operators given by the following formulas on their natural domains:

$$Px(t) = \int_0^t x(s)\,ds, \quad Qx(t) = \int_t^\infty x(s)\,ds.$$

The next result about the boundedness of these operators in weighted $(L^p - L^q)$-spaces is well known (see [28, 35] and [27]).

Lemma 2. (a) *Let $1 \leq p \leq q \leq \infty$. Then the operator $P\colon L^p_v \to L^q_w$ is bounded if and only if*

$$\sup_{t>0} \left\| \frac{1}{v}\chi_{[0,t]} \,|\, L^{p'} \right\| \|w\chi_{[t,\infty)} \,|\, L^q\| < \infty. \tag{2}$$

The operator $Q\colon L^p_v \to L^q_w$ is bounded if and only if

$$\sup_{t>0} \left\| \frac{1}{v}\chi_{[t,\infty)} \,|\, L^{p'} \right\| \|w\chi_{[0,t]} \,|\, L^q\| < \infty. \tag{3}$$

(b) *Let $1 < q < p < \infty$, $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$. Then the operator $P\colon L^p_v \to L^q_w$ is bounded if and only if*

$$\left(\int_0^\infty \left(\left\| \frac{1}{v}\chi_{[0,t]} \,|\, L^{p'} \right\|^{p'/q'} \|w\chi_{[t,\infty)} \,|\, L^q\| \right)^r v(t)^{-p'}\,dt \right)^{1/r} < \infty. \tag{4}$$

The operator $Q\colon L^p_v \to L^q_w$ is bounded if and only if

$$\left(\int_0^\infty \left(\left\| \frac{1}{v}\chi_{[t,\infty)} \,|\, L^{p'} \right\|^{p'/q'} \|w\chi_{[0,t]} \,|\, L^q\| \right)^r v(t)^{-p'}\,dt \right)^{1/r} < \infty. \tag{5}$$

(c) *If $1 = q < p < \infty$, we have $r = p'$. Therefore, formula* (4) *should be understood in the following way:*

$$\left(\int_0^\infty \|w\chi_{[t,\infty)} \,|\, L^1\|^{p'} v(t)^{-p'}\,dt \right)^{1/p'} < \infty. \tag{6}$$

Similarly, a limit passage in (5) *yields*

$$\left(\int_0^\infty \|w\chi_{[0,t]} \,|\, L^1\|^{p'} v(t)^{-p'}\,dt \right)^{1/p'} < \infty. \tag{7}$$

(d) *If $1 \leq q < p = \infty$, formula* (4) *becomes*

$$\left(\int_0^\infty \left(w(t) \left\| \frac{1}{v}\chi_{[0,t]} \,|\, L^1 \right\| \right)^q dt \right)^{1/q} < \infty, \tag{8}$$

and inequality (5) *transforms to*

$$\left(\int_0^\infty \left(w(t)\left\|\frac{1}{v}\chi_{[t,\infty)} \,|\, L^1\right\|\right)^q dt\right)^{1/q} < \infty. \tag{9}$$

§3. Representation of cones in weighted L^p-spaces

We begin with the statements of two main results (Theorems 1 and 2), which constitute a principal tool for the study of operators belonging to Sub(↓) on cones. We begin with the case where $1 \le p < \infty$.

Theorem 1. *Fix a number $1 \le p < \infty$ and a weight function v such that*

$$\int_0^t v(s)^p \, ds < \infty \tag{10}$$

for every $t \in R_+$ and

$$\int_0^\infty v(s)^p \, ds = \infty. \tag{11}$$

We introduce a new weight function $\overline{v}$, putting

$$\|v\chi_{[0,t]} \,|\, L^p\| \left\|\frac{1}{\overline{v}}\chi_{[t,\infty)} \,|\, L^{p'}\right\| \equiv 1. \tag{12}$$

Then

(a) $Q((L^p_{\overline{v}})_+) \subset K(\downarrow) \cap L^p_v$ *and, moreover, for every $x \in (L^p_{\overline{v}})_+$ we have*

$$\|Qx \,|\, L^p_v\| \le \|x \,|\, L^p_{\overline{v}}\|;$$

(b) *for every $x \in K(\downarrow) \cap (L^p_v)_+$ and every $\varepsilon > 0$ there exists $x_\varepsilon \in (L^p_{\overline{v}})_+$ such that*

$$\|x_\varepsilon \,|\, L^p_{\overline{v}}\| \le 16(1+\varepsilon)\|x \,|\, L^p_v\| \tag{13}$$

and

$$Q(x_\varepsilon)(t) \ge \frac{1}{8}\, x(t) \tag{14}$$

for a.e. $t > 0$.

The proof of Theorem 1 will be given in the last section of the paper. Here we comment on its assumptions and show some applications.

The assumption (10) says that for every $t > 0$ the characteristic function $\chi_{[0,t)}$ satisfies the condition $\chi_{[0,t)} \in K(\downarrow) \cap L^p_v$, i.e., the cone $K(\downarrow) \cap L^p_v$ is nondegenerate. The assumption (11) says that every $x \in K(\downarrow) \cap L^p_v$ satisfies $\lim_{t\to\infty} x(t) = 0$. It should be noted that if $\int_0^\infty v(s)^p \, ds < \infty$, then for $p \in [1,\infty)$ relation (12) must fail as $t \to +0$ or as $t \to \infty$.

Relation (12) for $\overline{v}$ can be expanded as follows:

$$\overline{v}(t) = \frac{(p-1)^{1/p'}}{v(t)^{p-1}} \int_0^t v(s)^p \, ds, \quad p > 1, \quad \overline{v}(t) = \int_0^t v(s)\, ds \ \text{ for } \ p = 1. \tag{15}$$

Now we present a series of corollaries to Theorem 1.

Corollary 1. *Let $1 \le p \le q \le \infty$ $(p \ne \infty)$, and let the weight function v satisfy conditions* (10), (11). *Then the embedding $K(\downarrow) \cap (L^p_v)_+ \subset (L^q_w)_+$ (equivalently, the inequality*

$$\|x \,|\, L^q_w\| \le C_1 \|x \,|\, L^p_v\| \tag{16}$$

for every $x \in K(\downarrow) \cap (L^p_v)_+)$ *occurs if and only if*

$$\sup_{t>0} \frac{\|w\chi_{[0,t]} \mid L^q\|}{\|v\chi_{[0,t]} \mid L^p\|} = C_2 < \infty. \tag{17}$$

Proof. The necessity of condition (17) follows because $\chi_{[0,t]} \in K(\downarrow) \cap (L^p_v)_+$. To prove sufficiency, we use Theorem 1: condition (16) is fulfilled if and only if

$$\|Qx \mid L^q_w\| \le C_3 \|x \mid L^p_{\bar{v}}\| \tag{18}$$

for every $x \in (L^p_{\bar{v}})_+$, where the weight function $\bar{v}$ is defined in (12). Applying (17) and (12), we obtain

$$\infty > C_2 = \sup_{t>0} \frac{\|w\chi_{[0,t]} \mid L^q\|}{\|v\chi_{[0,t]} \mid L^p\|} = \sup_{t>0} \|w\chi_{[0,t]} \mid L^q\| \left\| \frac{1}{\bar{v}} \chi_{[t,\infty)} \mid L^{p'} \right\|.$$

The last inequality and Lemma 2 yield (18). $\square$

Proofs of Corollary 1 based on different ideas can be found in the papers by Sawyer (see [36, Remark (i), p. 148]), Stepanov (see [41, Proposition 1(a)]), Carrro and Soria (see [14, Corollary 2.7]), and Heinig and Maligranda (see [20, Proposition 2.5(a)]). The structure of the cone $K(\downarrow) \cap L^p_v$ considered in the Lorentz quasi-Banach space Λ_{p,v^p} was also treated in [22].

Corollary 2. *Let* $1 \le q < p < \infty$, $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$, *and let a weight function* v *satisfy conditions* (10) *and* (11). *Then the embedding*

$$K(\downarrow) \cap (L^p_v)_+ \subset (L^q_w)_+ \tag{19}$$

or, equivalently, the relation

$$\|x \mid L^q_w\| \le C_1 \|x \mid L^p_v\| \text{ for every } x \in K(\downarrow) \cap (L^p_v)_+,$$

occurs if and only if

$$C_4 := \left(\int_0^\infty \left(\|w\chi_{[0,t]} \mid L^q\|\right)^r \|v\chi_{[0,t]} \mid L^p\|^{-pr/q} v^p(t)\, dt \right)^{1/r} < \infty. \tag{20}$$

Proof. Condition (19) is fulfilled if and only if the identity operator I maps boundedly the cone $K(\downarrow) \cap L^p_v$ to the cone $(L^q_w)_+$. We show that the latter is equivalent to the following relation for Q:

$$\|Q \mid L^p_{\bar{v}} \to L^q_w\| < \infty. \tag{21}$$

To see that (21) suffices, we apply (13) and (14):

$$\sup\left\{\|x \mid L^q_w\| \,:\, x \in K(\downarrow) \cap L^p_v \;\&\; \|x \mid L^p_v\| \le 1\right\}$$
$$\le 8 \sup\left\{\|x \mid L^q_w\| \,:\, x \le Qx_\varepsilon \;\&\; \|x_\varepsilon \mid L^p_{\bar{v}}\| \le 16(1+\varepsilon)\right\}.$$

The necessity of (21) follows from (12)–(14):

$$\|Q \mid L^p_{\bar{v}} \to L^q_w\| = \|I(Q) \mid L^p_{\bar{v}} \to L^q_w\| \le \|Q \mid L^p_{\bar{v}} \to L^p_v\| \, \|I \mid K(\downarrow) \cap L^p_v \to L^q_w\|.$$

By (5), condition (21) is equivalent to the inequality

$$\left(\int_0^\infty \left(\left\| \frac{1}{\bar{v}} \chi_{[t,\infty)} \mid L^{p'} \|^{p'/q'} \|w\chi_{[0,t]} \mid L^q \right\| \right)^r \bar{v}(t)^{-p'}\, dt \right)^{1/r} < \infty$$

or

$$\left(\int_0^\infty \left(\|w\chi_{[0,t]} \mid L^q\| \, \|v\chi_{[0,t]} \mid L^p\|^{-p'/q'}\right)^r \left(-\frac{d}{dt} \|v\chi_{[0,t]} \mid L^p\|^{-p'} \right) dt \right)^{1/r} < \infty. \tag{22}$$

Since

$$-\frac{d}{dt}\|v\chi_{[0,t]} \,|\, L^p\|^{-p'} = -\frac{d}{dt}\Big(\int_0^t v^p(s)\,ds\Big)^{(1-p')}$$
$$= (p'-1)\Big(\int_0^t v^p(s)\,ds\Big)^{-p'} v^p(t) = (p'-1)\|v\chi_{[0,t]} \,|\, L^p\|^{-pp'} v^p(t),$$

we see that (22) is a consequence of the relations

$$\Big(\int_0^\infty \big(\|w\chi_{[0,t]} \,|\, L^q\|\,\|v\chi_{[0,t]} \,|\, L^p\|^{-p'/q'}\big)^r\Big(-\frac{d}{dt}\|v\chi_{[0,t]} \,|\, L^p\|^{-p'}\Big)\,dt\Big)^{1/r}$$
$$= \Big((p'-1)\int_0^\infty \|w\chi_{[0,t]} \,|\, L^q\|^r \|v\chi_{[0,t]} \,|\, L^p\|^{-(p'p+\frac{p'r}{q'})} v^p(t)\,dt\Big)^{1/r}$$
$$= \Big((p'-1)\int_0^\infty \|w\chi_{[0,t]} \,|\, L^q\|^r \|v\chi_{[0,t]} \,|\, L^p\|^{-pr/q} v^p(t)\,dt\Big)^{1/r}$$
$$= (p'-1)^{1/r}\cdot C_4. \qquad \square$$

Other proofs of Corollary 2 can be found in the papers by Sawyer (see [36, Remark (i), p. 148]) and Stepanov (see [41, Proposition 1(b)].

Corollary 3. *Let* $1 \le q < p < \infty$, $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$, *and suppose that a weight function* v *satisfies conditions* (10), (11) *for* p *and a weight function* w *satisfies conditions* (10), (11) *for* q. *Then*

$$K(\downarrow) \cap (L^p_v)_+ \ne K(\downarrow) \cap (L^q_w)_+,$$

that is, for every v, w *with* (10) *and* (11) *these two cones do not coincide.*

Proof. Suppose the contrary, i.e., let

$$K(\downarrow) \cap (L^p_v)_+ = K(\downarrow) \cap (L^q_w)_+.$$

Then, by Corollary 2, the embedding $K(\downarrow)\cap(L^p_v)_+ \subset (L^q_w)_+$ is equivalent to (21) or (22):

$$\Big(\int_0^\infty \big(\|w\chi_{[0,t]} \,|\, L^q\|\,\|v\chi_{[0,t]} \,|\, L^p\|^{-p'/q'}\big)^r\Big(-\frac{d}{dt}\|v\chi_{[0,t]} \,|\, L^p\|^{-p'}\Big)\,dt\Big)^{1/r} < \infty. \tag{23}$$

Next, by Corollary 1, the embedding $K(\downarrow)\cap(L^p_w)_+ \subset (L^q_v)_+$ is equivalent to (17):

$$\sup_{t>0} \frac{\|v\chi_{[0,t]} \,|\, L^p\|}{\|w\chi_{[0,t]} \,|\, L^q\|} = C_2 < \infty. \tag{24}$$

Then, using (23) and (24), we obtain

$$\infty > \Big(\int_0^\infty \big(\|w\chi_{[0,t]} \,|\, L^q\|\,\|v\chi_{[0,t]} \,|\, L^p\|^{-p'/q'}\big)^r\Big(-\frac{d}{dt}\|v\chi_{[0,t]} \,|\, L^p\|^{-p'}\Big)\,dt\Big)^{1/r}$$
$$\ge \frac{1}{C_2}\Big(\int_0^\infty \big(\|v\chi_{[0,t]} \,|\, L^p\|\,\|v\chi_{[0,t]} \,|\, L^p\|^{-p'/q'}\big)^r\Big(-\frac{d}{dt}\|v\chi_{[0,t]} \,|\, L^p\|^{-p'}\Big)\,dt\Big)^{1/r}.$$

Recalling that $\big(1-\frac{p'}{q'}\big)r = \frac{p}{p-1} = p'$ and substituting $\tau = \|v\chi_{[0,t]} \,|\, L^p\|^{-p'}$, which is possible because

$$\lim_{t\to 0}\|v\chi_{[0,t]} \,|\, L^p\|^{p'} = 0, \quad \lim_{t\to\infty}\|v\chi_{[0,t]} \,|\, L^p\|^{p'} = \infty,$$

we arrive at

$$\infty > \frac{1}{C_2}\Big(\int_\infty^0 \frac{1}{\tau}(-d\tau)\Big)^{1/r} = \frac{1}{C_2}\Big(\int_0^\infty \frac{1}{\tau}\,d\tau\Big)^{1/r} = \infty.$$

This contradiction shows that the cones do not coincide. □

Now, we present a result about norm estimates for operators in $\mathrm{Sub}(\downarrow)$.

Theorem 2. *Fix a number $p \in [1, \infty)$, take a weight function v satisfying* (10), (11), *and construct the function $\overline{v}$ as in Theorem* 1. *Let X be an ideal space.*

An operator $T \in \mathrm{Sub}(\downarrow)$ acts boundedly from $K(\downarrow) \cap L^p_v$ to X, i.e.,

$$\|Tx \,|\, X\| \leq C_5 \|x \,|\, L^p_v\| \tag{25}$$

for every $x \in K(\downarrow) \cap (L^p_v)_+$, if and only if the composition operator TQ acts boundedly from $L^p_{\overline{v}}$ to X, i.e.,

$$\|TQx \,|\, X\| \leq C_6 \|x \,|\, L^p_{\overline{v}}\| \tag{26}$$

for every $x \in (L^p_{\overline{v}})_+$.

Proof. First, we show that (26) $\Longrightarrow$ (25). By Theorem 1, for every $x \in K(\downarrow) \cap (L^p_v)_+$ there exists $x_\varepsilon \in (L^p_{\overline{v}})_+$ such that

$$\|x_\varepsilon \,|\, L^p_{\overline{v}}\| \leq 16(1+\varepsilon)\|x \,|\, L^p_v\| \text{ and } Q(x_\varepsilon)(t) \geq \frac{1}{8}x(t) \text{ for all } t > 0.$$

The definition of the set $\mathrm{Sub}(\downarrow)$ and Lemma 1 imply the existence of constants $\alpha \geq 1$ and $\gamma > 0$ with $T \in \mathrm{Sub}(\alpha, \frac{1}{8}, \gamma; \downarrow)$. Therefore,

$$\|Tx \,|\, X\| \leq \gamma \|TQ(x_\varepsilon) \,|\, X\| \leq \gamma C_6 \|x_\varepsilon \,|\, L^p_{\overline{v}}\| \leq 16(1+\varepsilon)\gamma C_6 \|x \,|\, L^p_v\|,$$

which proves the implication (26) $\Longrightarrow$ (25).

Now, we verify the reverse implication: (25) $\Longrightarrow$ (26). The mapping Q takes any nonnegative function to a monotone nonincreasing function, i.e., $Qx \in K(\downarrow) \cap (L^p_v)_+$ for every $x \in (L^p_{\overline{v}})_+$. Next, by the definition (12) of $\overline{v}$, the operator Q is bounded when treated as an operator $Q\colon L^p_{\overline{v}} \to L^p_v$. Therefore, we have

$$\|TQx \,|\, X\| \leq C_5 \|Qx \,|\, L^p_v\| \leq C_5 \|Q \,|\, L^p_{\overline{v}} \to L^p_v\| \, \|x \,|\, L^p_{\overline{v}}\|. \qquad \square$$

Using the techniques of estimating operators $L\colon L^p_w \to X$ (see, e.g., [3, 4, 6, 27, 28]), on the basis of Theorem 2 it is possible to deduce various estimates for operators on the cone of monotone functions in Lebesgue spaces. We illustrate this by several classical examples.

First, with the help of a new approach, we shall prove the theorem of Sawyer (see [36]), which, in combination with a result by Ariño and Muckenhoupt (see [1]), resolved an important problem of harmonic analysis, namely, the boundedness problem for the Hardy operator on weighted Lorentz spaces. Furthermore, that theorem gave rise to a wide range of new problems, which remain fashionable still.

Theorem 3. *Let p, v, and $\overline{v}$ be the same as in Theorem* 1. *Consider a measurable function $g\colon R_+ \to R_+$. Then*

$$\begin{aligned} \frac{1}{C_8}\left\| \int_0^t g(s)\,ds \,\Big|\, L^{p'}_{\frac{1}{\overline{v}}} \right\| &\leq \sup\left\{ \int_0^\infty y(t)g(t)\,dt \,:\, y \in K(\downarrow) \cap L^p_v, \|y \,|\, L^p_v\| \leq 1 \right\} \\ &\leq C_8 \left\| \int_0^t g(s)\,ds \,\Big|\, L^{p'}_{\frac{1}{\overline{v}}} \right\|, \end{aligned} \tag{27}$$

where the constant $C_8 > 0$ does not depend on g.

Proof. We define a functional $F\colon K(\downarrow) \cap L^p{}_v \to R$ by $Fy(t) = \int_0^\infty y(t)g(t)\,dt$. Since this functional is quasilinear and nonnegative, we may apply Theorem 2 to deduce the existence of a constant $c > 0$ such that

$$\frac{1}{c}\|FQ \,|\, L^p{}_{\overline{v}} \to R\| \leq \|F \,|\, K(\downarrow) \cap L^p{}_v \to R\| \leq c\|FQ \,|\, L^p{}_{\overline{v}} \to R\|.$$

Integrating by parts, we arrive at

$$\int_0^\infty g(t)Qy(t)\,dt = \int_t^\infty y(s)\,ds \int_0^t g(s)\,ds\Big|_0^\infty + \int_0^\infty \Big(\int_0^t g(s)ds\Big) y(t)\,dt. \tag{28}$$

Suppose first that

$$\lim_{t\to 0} \frac{1}{\|\chi_{[0,t)}v \,|\, L^p\|} \cdot \int_0^t g(s)\,ds = 0, \qquad \lim_{t\to \infty} \frac{1}{\|\chi_{[0,t)}v \,|\, L^p\|} \cdot \int_0^t g(s)\,ds = 0. \tag{29}$$

Then, whenever $y \in K(\downarrow) \cap L^p{}_v$ satisfies $\|y \,|\, K(\downarrow) \cap L^p{}_v\| \le 1$, by (1) we can estimate the integrated term as follows:

$$\Big|\int_t^\infty y(s)\,ds \int_0^t g(s)\,ds\Big| \le \Big\|\chi_{[t,\infty)}\frac{1}{\overline{v}} \,|\, L^{p'}\Big\| \cdot \int_0^t g(s)\,ds = \frac{1}{\|\chi_{[0,t)}v \,|\, L^p\|} \cdot \int_0^t g(s)\,ds.$$

Now, (27) is a consequence of (28), the last inequality, (29), and the definition of the dual norm.

The assumption (29) can be lifted in the following way.

For a nonnegative function g and arbitrary $n \in N$, put $g_n(t) \equiv \chi_{(n^{-1},n)}(t)g(t)$. Then g_n satisfies (29), whence we obtain (27). We can easily pass to the limit here with the help of the B. Levy classical theorem (see, e.g., [24]). □

Note that, tracing the behavior of the constant in Theorem 3, it is possible to estimate the constant in (27).

Now we consider one of the most important operators in analysis, namely, the Hardy operator. On its natural domain, it is defined by the formula

$$Hx(t) = \frac{1}{t}\int_0^t x(s)\,ds.$$

Theorem 4. *Suppose that $1 \le p < \infty$, $1 \le q \le \infty$, and functions v and $\overline{v}$ satisfy the assumptions of Theorem 1. For the Hardy operator to be bounded in the sense that*

$$H\colon K(\downarrow) \cap L^p_v \to L^q_w, \tag{30}$$

it is necessary and sufficient that the following conditions be fulfilled:

a) *if $1 \le p \le q \le \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, then*

$$\sup_{t>0} \frac{\|\chi_{(0,t)}w(s) \,|\, L^q\|}{\|\chi_{(0,t)}v(s) \,|\, L^p\|} < \infty; \quad \sup_{t>0} \frac{1}{t}\Big\|\chi_{(0,t)}\frac{s}{\overline{v}(s)} \,|\, L^{p'}\Big\| \cdot \|\chi_{(t,\infty)}w(s) \,|\, L^q\| < \infty; \tag{31}$$

b) *if $1 \le q < p < \infty$, $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$, then*

$$\Big(\int_0^\infty \Big(\frac{\|\chi_{(0,t)}w(s) \,|\, L^q\|}{\|\chi_{(0,t)}v(s) \,|\, L^p\|}\Big)^r \Big(-\frac{d}{dt}\Big(\frac{1}{\|\chi_{(t,\infty)}v(s) \,|\, L^p\|}\Big)^{p'}\Big)\,dt\Big)^{1/r} < \infty; \tag{32}$$

(33)

$$\Big(\int_0^\infty \Big(\Big\|\chi_{(0,t)}\frac{s}{\overline{v}(s)} \,|\, L^{p'}\Big\| \cdot \|\chi_{(t,\infty)}w(s) \,|\, L^q\|\Big)^r \Big(-\frac{d}{dt}\Big(\frac{1}{\|\chi_{(t,\infty)}v(s) \,|\, L^p\|}\Big)^{p'}\Big)\,dt\Big)^{1/r} < \infty.$$

Proof. By Theorem 2, relation (30) is fulfilled if and only if the operator HQ acts boundedly in the couple

$$HQ\colon L^p_{\overline{v}} \to L^q_w.$$

Since HQ is positive, it suffices to verify its boundedness on nonnegative functions. Let $x(t) \ge 0$ a.e. Using the Fubini theorem (see, e.g., [24]) for nonnegative functions, we obtain

$$\begin{aligned}
HQx(t) &= \frac{1}{t}\int_0^t \left(\int_s^\infty x(\tau)\,d\tau\right) ds \\
&= \frac{1}{t}\left(\int_0^t \left[\int_s^t x(\tau)\,d\tau + \int_t^\infty x(\tau)\,d\tau\right] ds\right) \\
&= \frac{1}{t}\left(\int_0^t \left[\int_0^\tau ds\right] x(\tau)\,d\tau + t\int_t^\infty x(\tau)\,d\tau\right) \\
&= \frac{1}{t}\int_0^t \tau x(\tau)\,d\tau + \int_t^\infty x(\tau)\;d\tau.
\end{aligned}$$

The proof can be finished by application of Lemma 2 to each summand in the last identity. $\square$

Now we pass to the cone $K(\downarrow)\cap L_v^\infty$. Note that the situation will differ much from the case of $p<\infty$ considered above.

Our goal is to present certain analogs of the statements formulated above for the cone $K(\downarrow)\cap L_v^\infty$. Despite the relative ease of proofs, the central results of this subsections are Theorems 5 and 6.

Theorem 5. *Fixing $p=\infty$ and a weight function $v\colon [0,\infty)\to R_+$, we define a new function $\widetilde{v}$ by the formula*

$$\widetilde{v}(t) = \operatorname*{ess\,sup}_{0<\tau<t} v(\tau). \tag{34}$$

Then $\widetilde{v}$ is monotone nondecreasing, and the cones $K(\downarrow)\cap L_v^\infty$ and $K(\downarrow)\cap L_{\widetilde{v}}^\infty$ coincide. Moreover, for every $x\in K(\downarrow)\cap L_v^\infty$ we have

$$\|x \mid L_v^\infty\| = \|x \mid L_{\widetilde{v}}^\infty\|. \tag{35}$$

Proof. The definition (34) readily implies that the function $\widetilde{v}$ is monotonic.

We verify (35). Directly from the definition (34), it follows that for a.e. $t\in[0,\infty)$ we have $\widetilde{v}(t)\ge v(t)$. Consequently, for every $x\in K(\downarrow)\cap L_v^\infty$ we obtain the norm inequality

$$\|x \mid L_{\widetilde{v}}^\infty\| \ge \|x \mid L_v^\infty\|. \tag{36}$$

Now, let $x\in K(\downarrow)$. Then

$$\widetilde{v}(t)x(t) \le \operatorname*{ess\,sup}_{0<s\le t} v(s)x(s) \le \operatorname*{ess\,sup}_{s>0} v(s)x(s) = \|x \mid L_v^\infty\|$$

for a.e. $t\in R_+$. Thus,

$$\|x \mid L_{\widetilde{v}}^\infty\| \le \|x \mid L_v^\infty\|. \tag{37}$$

By (36)–(37), identity (35) follows. $\square$

Corollary 4. *Let $p=\infty$, $q\in[1,\infty]$. Then the embedding $K(\downarrow)\cap(L_v^\infty)_+\subset(L_w^q)_+$ or, equivalently, the inequality*

$$\|x \mid L_w^q\| \le C_1\|x \mid L_v^\infty\| \tag{38}$$

for every $x\in K(\downarrow)\cap(L_v^\infty)_+$, occur if and only if

$$\left\|\frac{1}{\widetilde{v}}w \mid L^q\right\| = C_1 < \infty. \tag{39}$$

Proof. Theorem 5 shows that the cones $K(\downarrow)\cap L_v^\infty$ and $K(\downarrow)\cap L_{\widetilde{v}}^\infty$ coincide. The function $\frac{1}{\widetilde{v}}$, which belongs to the intersection of $K(\downarrow)$ and the unit ball of $L_{\widetilde{v}}^\infty$, is a pointwise majorant for all functions in $K(\downarrow)\cap L_{\widetilde{v}}^\infty$ with unit norm. Thus, conditions (38) and (39) are equivalent. $\square$

Corollary 5. *Let $p = \infty$.*

If $q \neq \infty$, then

$$K(\downarrow) \cap (L_v^\infty)_+ \neq K(\downarrow) \cap (L_w^q)_+,$$

i.e., these two cones do not coincide for any weights v, w.

If $q = \infty$, then the identity

$$K(\downarrow) \cap (L_v^\infty)_+ = K(\downarrow) \cap (L_w^\infty)_+$$

is fulfilled if and only if

$$\sup_{t>0} \frac{\widetilde{w}(t)}{\widetilde{v}(t)} < \infty, \quad \sup_{t>0} \frac{\widetilde{v}(t)}{\widetilde{w}(t)} < \infty. \tag{40}$$

Proof. Theorem 5 shows that the cones $K(\downarrow) \cap L_v^\infty$ and $K(\downarrow) \cap L_{\widetilde{v}}^\infty$ coincide.

First, let $q < \infty$. Then for all $\tau \in R_+$ the function $\chi_{[0,\tau]} \frac{1}{\widetilde{v}(t)}$ belongs to $K(\downarrow)$, and for its norm we have

$$\Big\| \chi_{[0,\tau]} \frac{1}{\widetilde{v}} \mid L_{\widetilde{v}}^\infty \Big\| = 1, \qquad \Big\| \chi_{[0,\tau]} \frac{1}{\widetilde{v}} \mid L_w^q \Big\| = \Big(\int_0^\tau \Big(\frac{w(t)}{\widetilde{v}(t)} \Big)^q dt \Big)^{1/q}. \tag{41}$$

Letting τ tend to zero, we deduce from (41) that the norms $\| \cdot \mid L_{\widetilde{v}}^\infty \|$ and $\| \cdot \mid L_w^q \|$ cannot be equivalent on the cone $K(\downarrow)$.

Now, let $q = \infty$. Applying Theorem 5 once again, we see that the cones $K(\downarrow) \cap L_w^\infty$ and $K(\downarrow) \cap L_{\widetilde{w}}^\infty$ coincide.

The biggest function in $K(\downarrow)$ whose norm in $L_{\widetilde{v}}^\infty$ equals 1 is the function $\frac{1}{\widetilde{v}}$, and the biggest function in $K(\downarrow)$ whose norm in $L_{\widetilde{w}}^\infty$ equals 1 is the function $\frac{1}{\widetilde{w}}$. Conditions (40) precisely ensure the inclusions $\frac{1}{\widetilde{v}} \in L_{\widetilde{w}}^\infty$ and $\frac{1}{\widetilde{w}} \in L_{\widetilde{v}}^\infty$. ☐

An analog of the Sawyer theorem in the case where $p = \infty$ looks like this.

Corollary 6. *Let $p = \infty$, and suppose we are given a measurable function $g\colon R_+ \to R_+$. Then*

$$\sup \Big\{ \int_0^\infty x(t) g(t)\, dt \colon x \in K(\downarrow) \cap L_v^\infty, \|x \mid L_v^\infty\| \leq 1 \Big\} = \int_0^\infty \frac{1}{\widetilde{v}(s)} g(s)\, ds.$$

Proof. Theorem 5 shows that the cones $K(\downarrow) \cap L_v^\infty$ and $K(\downarrow) \cap L_{\widetilde{v}}^\infty$ coincide. To finish the proof, it suffices to observe that the biggest function in $K(\downarrow)$ whose norm in $L_{\widetilde{v}}^\infty$ equals 1 is the function $\frac{1}{\widetilde{v}}$. ☐

Corollary 7. *Let $p = \infty$, $1 \leq q \leq \infty$. Then the Hardy operator is bounded as an operator*

$$H\colon K(\downarrow) \cap L_v^\infty \to L^q{}_w$$

if and only if

$$\Big\| wH\Big(\frac{1}{\widetilde{v}}\Big) \mid L^q \Big\| < \infty.$$

Theorem 6. *Fix $p = \infty$, and let X be a Banach ideal space in $S(\mu)$. For $T \in \mathrm{Sub}(\downarrow)$ to act boundedly as an operator $T\colon K(\downarrow) \cap L_v^\infty \to X$, it is necessary and sufficient that*

$$\Big\| T\Big(\frac{1}{\widetilde{v}}\Big) \mid X \Big\| < \infty.$$

The two statements above are proved by much the same arguments as Corollary 6.

Now, we pass to an analog of Theorem 1 for the cone $K(\downarrow) \cap L_v^\infty$. For this, some prerequisites are needed.

To begin with, we observe that for a weight function $v\colon [0, \infty) \to R_+$ the condition

$$\limsup_{t\to\infty} v(t) = \infty \tag{42}$$

is equivalent to the relation

$$\lim_{t\to\infty} \widetilde{v}(t) = \infty. \tag{43}$$

Note that condition (42) or the equivalent condition (43) are quite natural, because any function representable in the form $y(t) = Qx(t)$ with $x \in \bigcup_{n\in N} L^1(\frac{1}{n}, \infty)$ satisfies the relation

$$\lim_{t\to\infty} y(t) = 0. \tag{44}$$

At the same time, precisely condition (43) is necessary and sufficient for an arbitrary function $x \in K(\downarrow) \cap L_v^\infty$ to satisfy $\lim_{t\to\infty} x(t) = 0$.

Theorem 7. *Fix a weight function v satisfying* (42) *and use* (34) *to define a function $\widetilde{v}$. Let there exist an absolutely continuous function $\widetilde{v}_{ac} \in K(\downarrow) \cap L_v^\infty$ such that for some constant $c > 0$ we have*

$$\frac{1}{c}\widetilde{v}_{ac}(t) \le \widetilde{v}(t) \le c\widetilde{v}_{ac}(t) \tag{45}$$

for all $t \in R_+$.

Put

$$\frac{1}{\overline{v}(t)} = -\frac{d}{dt}\frac{1}{\widetilde{v}_{ac}(t)}. \tag{46}$$

Then

(a) $Q((L_{\overline{v}}^\infty)_+) \subset K(\downarrow) \cap L_v^\infty$, *i.e., for every $x \in (L_{\overline{v}}^\infty)_+$ we have*

$$\|Qx \,|\, L_v^\infty\| \le C\|x \,|\, L_{\overline{v}}^\infty\|; \tag{47}$$

(b) *for every $x \in K(\downarrow) \cap (L_v^\infty)_+$ there exists $x_\varepsilon \in (L_{\overline{v}}^\infty)_+$ such that*

$$\|x_\varepsilon \,|\, L_{\overline{v}}^\infty\| = \|x \,|\, L_v^\infty\| \quad \textit{and} \quad Q(x_\varepsilon)(t) \ge \frac{1}{8}\, x(t) \textit{ for a.e. } t > 0.$$

The proof of Theorem 7 will be given in the last section of the paper. Also in that section, in Lemma 7, we shall indicate conditions on $\widetilde{v}$ necessary and sufficient for the existence of $\widetilde{v}_{ac} \in K(\downarrow) \cap L_v^\infty$ satisfying (45). Essentially, $\widetilde{v}_{ac} \in K(\downarrow) \cap L_v^\infty$ exists if and only if $\widetilde{v}$ satisfies a Δ_2-condition at the discontinuity points.

Theorems 1 and 7 justify the following definition.

Definition 3. Suppose we are given two ideal spaces X_0, X_1, two cones $K_0 \cap X_0$, $K_1 \cap X_1$, and a sublinear operator $T\colon K_0 \cap X_0 \to K_1 \cap X_1$. The pair $\{K_0 \cap X_0, T\}$ is said to generate the cone $K_1 \cap X_1$ if the following conditions are fulfilled:

a) there is a constant $c_0 > 0$ with the property that $\|Tx \,|\, X_1\| \le c_0\|x \,|\, X_0\|$ for every $x \in K_0 \cap X_0$;

b) there is a constant $c_1 > 0$ with the property that for every $y \in K_1 \cap X_1$ there exists $x_y \in K_0 \cap X_0$ such that the norm inequality $\|y \,|\, X_1\| \le c_1\|x_y \,|\, X_0\|$ and the a.e. inequality $y(t) \le c_1 T x_y(t)$ both hold true.

Theorem 8. *Suppose that $1 \le p < \infty$ and a weight function v satisfying* (10) *is given. Define $\overline{v}$ by* (12).

Then the pair $((L_w^p)_+, Q)$ generates the cone $K(\downarrow) \cap L_v^p$ if and only if the following conditions are fulfilled:

a) *the weight v satisfies* (11);

b) *for every $t > 0$ we have*

$$C_7^{-1} \le \|v\chi_{[0,t]} \,|\, L^p\| \cdot \left\|\frac{1}{w}\chi_{[t,\infty)} \,|\, L^{p'}\right\| \le C_7 \tag{48}$$

with a constant $C_7 > 0$ independent of t;

c) *for every $t > 0$ we have*

$$C_8^{-1}\left\|\frac{1}{v}\chi_{[t,\infty)} \,|\, L^{p'}\right\| \le \left\|\frac{1}{w}\chi_{[t,\infty)} \,|\, L^{p'}\right\| \le C_8\left\|\frac{1}{v}\chi_{[t,\infty)} \,|\, L^{p'}\right\| \tag{49}$$

with a constant $C_8 > 0$ independent of t.

Proof. First, we observe that conditions (48) and (49) are equivalent by (12).

We check the "only if" part. Suppose that the pair $((L^p_w)_+, Q)$ generates the cone $K(\downarrow) \cap L^p_v$. This implies immediately that the function $x(t) \in K(\downarrow) \cap L^p_v$ satisfies the condition

$$\lim_{t\to\infty} x(t) = 0. \tag{50}$$

For the nondegenerate cone $K(\downarrow) \cap L^p_v$, condition (12) is equivalent to the statement that the characteristic function of the entire half-line R_+ does not belong to $K(\downarrow) \cap L^p_v$. Therefore, (50) implies the necessity of (12).

We show the necessity of (48), (49). Since the pair $((L^p_w)_+, Q)$ generates the cone $K(\downarrow) \cap L^p_v$, we see that for every $t > 0$ the inequalities

$$\frac{1}{c_0}\|\chi_{(0,t)} \,|\, L^p_v\| \le \inf\left\{\|y \,|\, L^p_w\| \,:\, 1 \le \int_t^\infty y(s)\,ds\right\} \le c_0\|\chi_{(0,t)} \,|\, L^p_v\|. \tag{51}$$

must be true with a constant $c_0 > 0$ independent of t.

By duality and formula (1), we obtain

$$1 \le \int_t^\infty y(s)\,ds \le \|y\chi_{(t,\infty)} \,|\, L^p_w\|\left\|\chi_{(t,\infty)}\frac{1}{w} \,|\, L^{p'}\right\|,$$

or

$$\|y\chi_{(t,\infty)} \,|\, L^p_w\| \ge \frac{1}{\|\chi_{(t,\infty)}\frac{1}{w} \,|\, L^{p'}\|}. \tag{52}$$

The definition of the dual space shows that we can choose a sequence y_n of functions in the unit sphere of L^p_w such that

$$\begin{aligned}
&\|y_n\chi_{(t,\infty)} \,|\, L^p_w\|\left\|\chi_{(t,\infty)}\frac{1}{w} \,|\, L^{p'}\right\| \\
&\quad\ge \int_t^\infty y_n(s)\,ds \ge \|y_n\chi_{(t,\infty)} \,|\, L^p_w\|\left\|\chi_{(t,\infty)}\frac{1}{w} \,|\, L^{p'}\right\| - \frac{1}{n} \\
&\quad= \left\|\chi_{(t,\infty)}\frac{1}{w} \,|\, L^{p'}\right\| - \frac{1}{n}.
\end{aligned}$$

Now, we define a sequence of functions z_n as follows:

$$z_n(t) = \frac{y_n(t)}{\int_t^\infty y_n(s)\,ds}.$$

Then

$$\int_t^\infty z_n(s)\,ds = 1 \text{ and } \|z_n\chi_{(t,\infty)} \,|\, L^p_w\| \le \frac{1}{\|\chi_{(t,\infty)}\frac{1}{w} \,|\, L^{p'}\| - \frac{1}{n}}. \tag{53}$$

From (52)–(53) we deduce that

$$\inf\left\{\|y \,|\, L^p_w\| \,:\, 1 \le \int_t^\infty y(s)\,ds\right\} = \frac{1}{\|\chi_{(t,\infty)}\frac{1}{w} \,|\, L^{p'}\|}. \tag{54}$$

Conditions (51) and (54) proof the necessity of (48)–(49).

To prove that (a), (b), and (c) suffice, we may repeat the proof of Theorem 1 word-for-word. □

Theorem 9. *Suppose $p = \infty$ and we are given a weight function w. Define a function $\widetilde{w}$ by* (34).

Then the pair $((L_u^\infty)_+, Q)$ generates the cone $K(\downarrow) \cap L_w^\infty$ if and only if the following relations are fulfilled:

(a) *for w, we have*

$$\limsup_{t\to\infty} w(t) = \infty; \tag{55}$$

(b) *there exists an absolutely continuous function $\widetilde{w}_{ac} \in K(\downarrow) \cap L_w^\infty$ such that, with some constant C, we have*

$$\frac{1}{c}\widetilde{w}_{ac}(t) \le \widetilde{w}(t) \le c\widetilde{w}_{ac}(t) \tag{56}$$

for all $t \in R_+$;

(c) *for every $t > 0$ we have*

$$C_7^{-1}\frac{1}{\widetilde{w}(t)} \le \int_t^\infty \frac{1}{u(s)}\,ds \le C_7\frac{1}{\widetilde{w}(t)} \tag{57}$$

with a constant $C_7 > 0$ independent of t.

Proof. Theorem 5 shows that the cones $K(\downarrow) \cap L_w^\infty$ and $K(\downarrow) \cap L_{\widetilde{w}}^\infty$ coincide.

We check the "only if" part. Suppose that the pair $((L_u^\infty)_+, Q)$ generates the cone $K(\downarrow) \cap L_{\widetilde{w}}^\infty$. This readily implies that every function $x(t)$ in $K(\downarrow) \cap L_{\widetilde{w}}^\infty$ satisfies the condition $\lim_{t\to\infty} x(t) = 0$. The last is equivalent to

$$\lim_{t\to\infty} \widetilde{w}(t) = \infty, \tag{58}$$

and, as it was indicated in the proof of the equivalence of (42) and (43), condition (58) is equivalent to (55).

Next, since the pair $((L_u^\infty)_+, Q)$ generates the cone $K(\downarrow)\cap L_{\widetilde{w}}^\infty$, we see that the function $\frac{1}{\widetilde{w}} \in K(\downarrow) \cap L_{\widetilde{w}}^\infty$ satisfies the inequality

$$\frac{1}{\widetilde{w}(t)} \le C_{10}\int_t^\infty \frac{1}{u(s)}\,ds \tag{59}$$

with a constant C_{10} independent of t. However, by Lemma 2, the boundedness condition for the operator $Q\colon L_u^\infty \to L_{\widetilde{w}}^\infty$ looks like this:

$$\sup_{t\in R_+} \widetilde{w}(t)\int_t^\infty \frac{1}{u(s)}\,ds \le C_{11} < \infty. \tag{60}$$

Relations (59)–(60) prove the necessity of (b) and (c).

To prove that (a), (b), and (c) suffice, we may repeat the proof of Theorem 7 word-for-word. □

§4. Extrapolation theorems for cones

In the theory of integral operators with positive kernel, a special role is played by the so-called Schur theorem or Schur test (see [25, p. 37] and [42, p. 42]), which says that an integral operator $Kx(t) = \int k(t,s)x(s)\,ds$ with positive kernel $k(t,s) \ge 0$ is bounded in L^p for $1 < p < \infty$ if and only if there exists a positive function u such that

$$Ku^{p'}(t) \le Cu^{p'}(t) \quad\text{and}\quad K'u^p(t) \le Cu^p(t),$$

here K' is the formally adjoint operator and $1/p' + 1/p = 1$. This statement can be regarded as a factorization or extrapolation theorem: there exists a positive function u (a weight function u) such that K is bounded in the following couples of spaces:

$$K\colon L^\infty_{u^{-p'}} \to L^\infty_{u^{-p'}}, \qquad K\colon L^1_{u^p} \to L^1_{u^p}.$$

We start with a reformulation of the Schur extrapolation theorem in modern terms, see [9, Corollary 7].

Proposition 1 (Schur test)**.** *Suppose that $T, T' \in \mathrm{Sub}(+), 1 < p < \infty$, and we are given two weight functions v, w. The following conditions are equivalent*:

(a) *the operator $T\colon L^p_v \to L^p_w$ is bounded*;

(b) *there exist four weight functions v_0, v_1, w_0, w_1 such that*

$$v(t) = v_0(t)^{1/p} v_1(t)^{1-1/p}, \quad w(t) = w_0(t)^{1/p} w_1(t)^{1-1/p} \tag{61}$$

for all $t \in R_+$, and the operator T acts boundedly in the following couples:

$$T\colon L^1_{v_0} \to L^1_{w_0}, \quad T\colon L^\infty_{v_1} \to L^\infty_{w_1}. \tag{62}$$

The implication $(b) \implies (a)$ follows from interpolation theorems for positive operators for the Calderón–Lozanovskiĭ construction $X_0^\theta\ X_1^{1-\theta}$ (see [2, 26, 30, 37], [31, Theorem 15.13]) and the relations

$$(L^1_{v_0})^{1/p} (L^\infty_{v_1})^{1-1/p} = L^p_{v_0^{1/p} v_1^{1-1/p}} = L^p_v,$$
$$(L^1_{w_0})^{1/p} (L^\infty_{w_1})^{1-1/p} = L^p_{w_0^{1/p} w_1^{1-1/p}} = L^p_w.$$

The reverse implication $(a) \implies (b)$, which is the essence of theorems like the Schur test, was proved in [9, Corollary 7] (see also [5, p. 728], [8, Theorem 1], [7, p. 18]).

In this section we pass to extrapolation theorems for operators on cones. We begin with a general version of extrapolation theorems for operators on the cone $K(\downarrow)$.

Theorem 10. *Suppose that $T, T' \in \mathrm{Sub}$, $1 < p < \infty$, and we are given a weight function v satisfying* (10), (11)*. Define a new function $\bar{v}$ by* (12)*.*

Put $\theta = 1/p$. Then the following conditions are equivalent:

a) *T is bounded as an operator in the following couple*:

$$T\colon K(\downarrow) \cap L^p_v \to L^p_u;$$

b) *there exist functions v_0, v_1, u_0, u_1 satisfying*

$$v_0^\theta(t) \cdot v_1^{1-\theta}(t) \equiv \bar{v}(t), \quad u_0^\theta(t) \cdot u_1^{1-\theta}(t) \equiv u(t) \tag{63}$$

and such that TQ acts boundedly in the couples

$$TQ\colon L^1_{v_0} \to L^1_{u_0}, \quad TQ\colon L^\infty_{v_1} \to L^\infty_{u_1}. \tag{64}$$

Proof. Suppose a) is fulfilled. Then Theorem 4 shows that this statement is equivalent to the boundedness of TQ in the couple $TQ\colon L^p_{\bar{v}} \to L^p_u$. Since TQ and $(TQ)'$ belong to $\mathrm{Sub}(+)$, we may apply the Schur test to the operator $TQ\colon L^p_{\bar{v}} \to L^p_u$. This yields the conditions of item b).

Suppose b) is fulfilled. Then, by the interpolation theorem for the operator TQ, we see that TQ is bounded as indicated below:

$$TQ\colon \big(L^1_{v_0}\big)^\theta \big(L^\infty_{v_1}\big)^{1-\theta} \to \big(L^1_{u_0}\big)^\theta \big(L^\infty_{u_1}\big)^{1-\theta}.$$

The well-known identity

$$\big(L^1_{w_0}\big)^\theta \big(L^\infty_{w_1}\big)^{1-\theta} = L^p_w,$$

where $w_\theta(t) \equiv w_0^\theta(t) w_1^{1-\theta}(t)$, combined with (67) implies that TQ is bounded also in the following way:

$$TQ\colon L^p_{v_\theta} \to L^p_{u_\theta},$$

where $v_\theta(t) \equiv v_0^\theta(t) v_1^{1-\theta}(t)$ and $u_\theta(t) \equiv u_0^\theta(t) u_1^{1-\theta}(t)$. Since $v_\theta(t) \equiv \bar{v}(t)$ and $u_\theta(t) \equiv u(t)$, relation (65) is equivalent to the boundedness of TQ in the following couple:

$$TQ\colon L^p_{\bar{v}} \to L^p_u.$$

But $T \in \mathrm{Sub}(+)$, and, by Theorem 8, the pair $((L^p_{\bar v})_+, Q)$ generates the cone $K(\downarrow) \cap L^p_{\bar v}$; therefore, the last relation implies a). $\square$

Theorem 10 has an important drawback. It would be desirable to replace conditions (63) and (64) by the following more natural conditions:

there exist functions v_0, v_1, u_0, u_1 satisfying

$$v_0^\theta(t) \cdot v_1^{1-\theta}(t) \equiv v(t), \quad u_0^\theta(t) \cdot u_1^{1-\theta}(t) \equiv u(t) \tag{65}$$

and such that T acts boundedly in the following couples:

$$T\colon K(\downarrow) \cap L^1_{v_0} \to L^1_{u_0}, \quad T\colon K(\downarrow) \cap L^\infty_{v_1} \to L^\infty_{u_1}. \tag{66}$$

We are going to obtain an analog of Theorem 10 with conditions (63), (64) replaced by (65), (66) in one important particular case. Some preliminaries are required for this.

Let X_0, X_1 be two ideal spaces with $X_0, X_1 \subset S(\mu)$. Fix $0 < \theta < 1$. The new ideal space $X_0^\theta X_1^{1-\theta}$ (the Calderón–Lozanovskiĭ construction) consists of all $x \in S(\mu)$ for which the following norm is finite:

$$\begin{aligned} \|x \,|\, X_0^\theta\ X_1^{1-\theta}\| = \inf \big\{\lambda > 0 \,:\, |x(t)| \le \lambda \cdot |x_0(t)|^\theta |x_1(t)|^{1-\theta} \forall t \in [0,\infty); \\ \|x_0 \,|\, X_0\| \le 1,\ \ \|x_1 \,|\, X_1\| \le 1\big\}. \end{aligned} \tag{67}$$

The space $X_0^\theta\ X_1^{1-\theta}$ was introduced by Calderón in [13] for the study of the complex interpolation method.

Definition 4. A cone K is said to be *canonical* if for every pair x, y of functions in K and every number $\theta \in (0,1)$ the function $x^\theta \cdot y^{1-\theta}$ again belongs to K.

We observe that the cones of monotonic functions are canonical.

If K is a canonical cone in $S(\mu)$, then by analogy with the space $X_0^\theta\ X_1^{1-\theta}$ we can introduce the new cone $(K \cap X_0)^\theta (K \cap X_1)^{1-\theta}$, admitting in (67) only decompositions that involve elements of the cone.

Remark 1. It is easily seen that for a canonical cone and $\theta \in (0,1)$ we always have a continuous embedding

$$(K \cap X_0)^\theta (K \cap X_1)^{1-\theta} \subseteq K \cap X_0^\theta\ X_1^{1-\theta}.$$

On the other hand, as it usually happens in interpolation theory, for an arbitrary canonical cone K the relation

$$(K \cap X_0)^\theta (K \cap X_1)^{1-\theta} = K \cap X_0^\theta\ X_1^{1-\theta}$$

may fail. Even for the best studied cone $K(\downarrow)$, no sharp conditions are known that ensure this relation in the scale of Lebesgue spaces.

The next theorem is of interpolation nature. It is well known for the cone consisting of nonnegative functions (see, e.g., [2, 30, 31]).

Theorem 11. *Suppose that T is a positive operator and K_0, K_1 are two canonical cones in $S(\mu)_+$. Consider four Banach ideal spaces X_0, X_1, Y_0, Y_1 in $S(\mu)$ and suppose that T acts boundedly as indicated: $T\colon K_0 \cap X_i \to K_1 \cap Y_i$, $(i = 0, 1)$. Fix $\theta \in (0,1)$. Then for every $x_0 \in K_0 \cap X_0$, $x_1 \cap X_1 \in K_1$ we have the pointwise inequality*

$$T({x_0}^\theta \cdot {x_1}^{1-\theta})(t) \le (Tx_0(t))^\theta \cdot (Tx_1(t))^{1-\theta}, \tag{68}$$

and T acts boundedly when regarded as an operator $T\colon (K_0 \cap X_0)^\theta (K_0 \cap X_1)^{1-\theta} \to (K_1 \cap Y_0)^\theta (K_1 \cap Y_1)^{1-\theta}$.

Proof. Take $x_0 \in K_0 \cap X_0$, $x_1 \in K_0 \cap X_1$ and construct the element ${x_0}^\theta \cdot {x_1}^{1-\theta} \in (K \cap X_0)^\theta (K \cap X_1)^{1-\theta}$. The numerical identity

$$a^\theta \cdot b^{1-\theta} = \inf_{\varepsilon>0} \{\varepsilon\theta a + \varepsilon^{-\frac{\theta}{1-\theta}}(1-\theta)b\}, \tag{69}$$

valid for all $a > 0$, $b > 0$, implies the inequality

$$\begin{aligned} T({x_0}^\theta \cdot {x_1}^{1-\theta})(t) &\le T(\varepsilon\theta x_0(t) + \varepsilon^{-\frac{\theta}{1-\theta}}(1-\theta)x_1(t)) \\ &\le \varepsilon\theta T x_0(t) + \varepsilon^{-\frac{\theta}{1-\theta}}(1-\theta)Tx_1(t). \end{aligned} \tag{70}$$

Minimizing the right-hand side in (70) over $\varepsilon > 0$ for each fixed t and taking (69) into account, we arrive at (68).

The boundedness of T in the required sense,

$$T\colon (K_0 \cap X_0)^\theta (K_0 \cap X_1)^{1-\theta} \to (K_1 \cap Y_0)^\theta (K_1 \cap Y_1)^{1-\theta},$$

is immediate from (68). □

Lemma 3. *Fix $\theta \in (0,1)$. In the spaces $L^1_{w_0}$, $L^\infty_{w_1}$, and $L^1_{u_0}$, consider the cones $K(\downarrow) \cap L^1_{w_0}$, $K(\downarrow) \cap L^\infty_{w_1}$, and $K(\downarrow) \cap L^1_{u_0}$ for which we have a continuous embedding*

$$(K(\downarrow) \cap L^1_{w_0})^\theta (K(\downarrow) \cap L^\infty_{w_1})^{1-\theta} \subseteq (K(\downarrow) \cap L^1_{u_0})^\theta (K(\downarrow) \cap L^\infty_{w_1})^{1-\theta}.$$

Then we have a continuous embedding

$$K(\downarrow) \cap L^1_{w_0} \subseteq K(\downarrow) \cap L^1_{u_0}.$$

Proof. Lemma 8 in the last section shows that for every z of unit norm in $(K(\downarrow) \cap L^1_{w_0})^\theta$ $(K(\downarrow) \cap L^\infty_{w_1})^{1-\theta}$ we have

$$z(t) \le x_0^\theta(t) \cdot \Big(\frac{1}{\widetilde{w}(t)}\Big)^{1-\theta}(t) \quad (t \in R_+) \tag{71}$$

with $x_0 \in K(\downarrow) \cap L^1_{w_0}$ and $\|x_0 \mid L^1_{w_0}\| = 1$.

Since $\widetilde{w}$ is nonzero a.e., the claim follows from (71). □

Now everything is ready to prove an analog of Theorem 10 with conditions (63) and (64) replaced by (65) and (66).

Theorem 12. *Fix $p \in (1,\infty)$ and a weight function v satisfying* (10), (11). *Define a new function $\bar{v}$ by* (12) *and put $\theta = 1/p$.*

Suppose we are given operators $T, T' \in \mathrm{Sub}(+)$, where $T \in \mathrm{Sub}(+)$ acts boundedly in the couple

$$T\colon K(\downarrow) \cap L^p_v \to L^p_v.$$

Then there exist functions w_0, w_1 such that

$$w_0^\theta(t) \cdot w_1^{1-\theta}(t) \equiv v(t),$$

and T acts boundedly in the following couples:

$$T\colon K(\downarrow) \cap L^1_{w_0} \to L^1_{w_0}, \quad T\colon K(\downarrow) \cap L^\infty_{w_1} \to L^\infty_{w_1}.$$

Proof. We introduce a new operator T_1 by the formula

$$T_1 x(t) = Tx(t) + x(t)$$

and apply Theorem 10 to it, obtaining functions v_0, v_1, w_0, w_1 for which we have

$$v_0^\theta(t) \cdot v_1^{1-\theta}(t) \equiv \bar{v}(t), \quad w_0^\theta(t) \cdot w_1^{1-\theta}(t) \equiv v(t) \tag{72}$$

(here $\bar{v}$ is defined by (12)).

The operator $T_1 Q$ acts boundedly in the following couples:

$$TQ + Q\colon L^1_{v_0} \to L^1_{w_0}, \quad TQ + Q\colon L^\infty_{v_1} \to L^\infty_{w_1}.$$

We introduce a function u_0 by the formula

$$\|\chi_{(0,t)}u_0 \mid L^1\| \cdot \left\|\chi_{(t,\infty)}\frac{1}{v_0} \mid L^\infty\right\| \equiv 1 \tag{73}$$

and consider the space $L^1_{u_0}$. By (73) and Theorem 8, the pair $((L^1_{v_0})_+, Q)$ generates the cone $K(\downarrow) \cap L^1_{u_0}$.

Next, we introduce a function u_1 by the formula

$$\frac{1}{u_1(t)} \equiv \int_t^\infty \frac{1}{v_1(s)}\, ds \tag{74}$$

and consider the space $L^\infty_{u_1}$. By (74) and Theorem 9, the pair $((L^\infty_{v_1})_+, Q)$ generates the cone $K(\downarrow) \cap L^\infty_{u_1}$.

Since Q acts boundedly in the couples

$$Q\colon L^1_{v_0} \to L^1_{w_0}, \quad Q\colon L^\infty_{v_1} \to L^\infty_{w_1}$$

and the cones $K(\downarrow) \cap L^1_{u_0}$, $K(\downarrow) \cap L^\infty_{u_1}$ are generated by the pairs $((L^1_{v_0})_+, Q)$ and $((L^\infty_{v_1})_+, Q)$, respectively, we arrive at the continuous embeddings

$$K(\downarrow) \cap L^1_{u_0} \subseteq K(\downarrow) \cap L^1_{w_0}, \quad K(\downarrow) \cap L^\infty_{u_1} \subseteq K(\downarrow) \cap L^\infty_{w_1}. \tag{75}$$

On the other hand, since Q is positive, inequality (68) in Theorem 11 shows that for every $x_0 \in L^1_{v_0}$, $x_1 \in L^\infty_{v_1}$ we have

$$Q(x_0^\theta \cdot x_1^{1-\theta})(t) \le (Qx_0(t))^\theta \cdot (Qx_1(t))^{1-\theta} \tag{76}$$

almost everywhere. By Theorem 8, the pair $((L^p_v)_+, Q)$ generates the cone $K(\downarrow) \cap L^p_v$. Therefore, (76) yields the continuous embedding

$$K(\downarrow) \cap L^p_v \subseteq (K(\downarrow) \cap L^1_{u_0})^\theta (K(\downarrow) \cap L^\infty_{u_1})^{1-\theta}. \tag{77}$$

At the same time, formula (72), Remark 1, and the definitions yield the continuous embedding

$$(K(\downarrow) \cap L^1_{w_0})^\theta (K(\downarrow) \cap L^\infty_{w_1})^{1-\theta} \subseteq K(\downarrow) \cap L^p_v. \tag{78}$$

Consequently, by (77) and (78) we obtain

$$(K(\downarrow) \cap L^1_{w_0})^\theta (K(\downarrow) \cap L^\infty_{w_1})^{1-\theta} \subseteq K(\downarrow) \cap L^p_v \subseteq (K(\downarrow) \cap L^1_{u_0})^\theta (K(\downarrow) \cap L^\infty_{u_1})^{1-\theta}.$$

This implies the continuous embedding

$$(K(\downarrow) \cap L^1_{w_0})^\theta (K(\downarrow) \cap L^\infty_{w_1})^{1-\theta} \subseteq (K(\downarrow) \cap L^1_{u_0})^\theta (K(\downarrow) \cap L^\infty_{u_1})^{1-\theta}.$$

Together with the second embedding in (75), this yields the continuity of the embeddings

$$\begin{aligned}(K(\downarrow) \cap L^1_{w_0})^\theta (K(\downarrow) \cap L^\infty_{u_1})^{1-\theta} &\subseteq (K(\downarrow) \cap L^1_{w_0})^\theta (K(\downarrow) \cap L^\infty_{w_1})^{1-\theta} \\ &\subseteq (K(\downarrow) \cap L^1_{u_0})^\theta (K(\downarrow) \cap L^\infty_{u_1})^{1-\theta}.\end{aligned}$$

Thus,

$$(K(\downarrow) \cap L^1_{w_0})^\theta (K(\downarrow) \cap L^\infty_{u_1})^{1-\theta} \subseteq (K(\downarrow) \cap L^1_{u_0})^\theta (K(\downarrow) \cap L^\infty_{u_1})^{1-\theta}. \tag{79}$$

By (79), using Lemma 3, we obtain the embedding

$$K(\downarrow) \cap L^1_{w_0} \subseteq K(\downarrow) \cap L^1_{u_0}.$$

Comparing this with the first embedding in (75), we see that, up to equivalent norms, we have

$$K(\downarrow) \cap L^1_{w_0} = K(\downarrow) \cap L^1_{u_0}. \tag{80}$$

Thus, (73) and (80) imply the existence of a constant $c > 0$ such that for every $t > 0$ we have

$$c < \|\chi_{(0,t)} w_0 \mid L^1\| \cdot \left\| \chi_{(t,\infty)} \frac{1}{v_0} \mid L^\infty \right\| < \frac{1}{c}.$$

Combining (12) and (72) with the last relation, we see that

$$\begin{aligned}
1 &= \Big(\int_0^t (w_0^\theta(s) \cdot w_1^{1-\theta}(s))^{1/\theta}\, ds \Big)^\theta \cdot \Big(\int_t^\infty \Big(\frac{1}{v_0^\theta(s) \cdot v_1^{1-\theta}(s)} \Big)^{1/(1-\theta)} ds \Big)^{1-\theta} \\
&= \Big(\int_0^t w_0(s) \cdot w_1^{(1-\theta)/\theta}(s)\, ds \Big)^\theta \cdot \Big(\int_t^\infty \Big(\frac{1}{v_0(s)} \Big)^{\theta/(1-\theta)} \Big(\frac{1}{v_1(s)} \Big) ds \Big)^{1-\theta} \\
&\le \sup_{s \le t} w_1^{1-\theta}(s) \cdot \Big(\int_0^t w_0(s)\, ds \Big)^\theta \cdot \sup_{s \ge t} \frac{1}{v_0^\theta(s)} \cdot \Big(\int_t^\infty \frac{1}{v_1(s)}\, ds \Big)^{1-\theta} \\
&= \Big(\sup_{s \le t} w_1(s) \cdot \int_t^\infty \frac{1}{v_1(s)}\, ds \Big)^{1-\theta} \cdot \Big(\sup_{s \ge t} \frac{1}{v_0(s)} \cdot \int_0^t w_0(s)\, ds \Big)^\theta \\
&\le \Big(\frac{1}{c} \Big)^\theta \cdot \Big(\sup_{s \le t} w_1(s) \cdot \int_t^\infty \frac{1}{v_1(s)}\, ds \Big)^{1-\theta}
\end{aligned}$$

for all $t \in R_+$. This shows that the inequality

$$c^{\frac{\theta}{1-\theta}} \cdot \inf_{s \le t} \frac{1}{w_1(s)} \le \int_t^\infty \frac{1}{v_1(s)}\, ds$$

is true for all $t \in R_+$. Taking (74) and (34) into account, we can rewrite the last inequality in the following equivalent form: for all $t \in R_+$ we have

$$\frac{1}{\widetilde{w}_1(t)} \le c^{\frac{-\theta}{1-\theta}} \frac{1}{u_1(t)},$$

or, in the language of embeddings,

$$K(\downarrow) \cap L^\infty_{\widetilde{w}_1} \subseteq K(\downarrow) \cap L^\infty_{u_1}. \tag{81}$$

Theorem 5 implies the identity $K(\downarrow) \cap L^\infty_{w_1} = K(\downarrow) \cap L^\infty_{\widetilde{w}_1}$. Thus, the embedding (81) is equivalent to the embedding

$$K(\downarrow) \cap L^\infty_{w_1} \subseteq K(\downarrow) \cap L^\infty_{\tilde{u}_1}.$$

Together with the second embedding in (75), this relation shows that, up to equivalent norms, the following identity holds true:

$$K(\downarrow) \cap L^\infty_{u_1} = K(\downarrow) \cap L^\infty_{w_1}.$$

Combined with (80), this proves the theorem. □

Since the Hardy operator fits in the scope of Theorem 12, we have the following statement.

Theorem 13. *Fix* $p \in (1, \infty)$ *and consider a weight function* v *satisfying* (10) *and* (11). *Put* $\theta = 1/p$. *The Hardy operator* H *is bounded as an operator*

$$H \colon K(\downarrow) \cap L^p_v \to L^p_v$$

if and only if there exist functions w_0, w_1 *such that* a) $w_0^\theta(t) \cdot w_1^{1-\theta}(t) \equiv v(t)$ *for all* $t \in R_+$;

b) H *acts boundedly in the couples*

$$H \colon K(\downarrow) \cap L^1_{w_0} \to L^1_{w_0}, \quad H \colon K(\downarrow) \cap L^\infty_{w_1} \to L^\infty_{w_1}.$$

§5. Proofs of Theorems 1 and 7, and auxiliary lemmas

To prove Theorem 1, we need some auxiliary statements, with which we shall start.

Lemma 4. *Let X be an ideal space. Take a numerical sequence*

$$0 < \cdots < t_j < t_{j+1} < \cdots < \infty \text{ with } \lim_{j\to-\infty} t_j = 0.$$

Let the element $x = \sum_{-\infty}^{\infty} 2^{-j}\chi_{[0,t_j)}$ belong to X and satisfy the condition

$$\lim_{k\to-\infty} \Big\| \sum_{j=-\infty}^{k} 2^{-j}\chi_{[0,t_j)} \,|\, X \Big\| = 0. \tag{82}$$

Then there exists a sequence of integers $k_j : -\infty < \cdots < k_j < k_{j-1} < \cdots < k_0 < \infty$ such that

$$\sum_{i=0}^{\infty} \Big\| \sum_{j=k_{i+1}+1}^{k_i} 2^{-j}\chi_{[0,t_j)} \,|\, X \Big\| + \Big\| \sum_{j=k_0}^{\infty} 2^{-j}\chi_{[0,t_j)} \,|\, X \Big\| \le 2\|x \,|\, X\|.$$

Proof. The sequence k_i can be defined as follows. Using (82), take k_0 so as to have

$$\Big\| \sum_{-\infty}^{k_0} 2^{-j}\chi_{[0,t_j)} \,|\, X \Big\| \le 2^{-1}\|x \,|\, X\|.$$

Suppose that the numbers $k_{i-1} < k_{i-2} < \cdots < k_1 < k_0$ are constructed. Then we choose $k_i < k_{i-1}$ so as to have

$$\Big\| \sum_{-\infty}^{k_i} 2^{-j}\chi_{[0,t_j)} \,|\, X \Big\| \le 2^{-i-1}\|x \,|\, X\|.$$

The possibility of this choice follows from (82).

Since X is an ideal space, an easy calculation shows that

$$\begin{aligned}
&\sum_{i=0}^{\infty} \Big\| \sum_{j=k_{i+1}+1}^{k_i} 2^{-j}\chi_{[0,t_j)} \,|\, X \Big\| + \Big\| \sum_{j=k_0+1}^{\infty} 2^{-j}\chi_{[0,t_j)} \,|\, X \Big\| \\
&\quad \le \sum_{i=0}^{\infty} \Big\| \sum_{-\infty}^{k_i} 2^{-j}\chi_{[0,t_j)} \,|\, X \Big\| + \Big\| \sum_{j=k_0+1}^{\infty} 2^{-j}\chi_{[0,t_{j+1})} \,|\, X \Big\| \\
&\quad \le \sum_{i=0}^{\infty} 2^{-i-1}\|x \,|\, X\| + \|x \,|\, X\| \le 2\|x \,|\, X\|. \qquad \square
\end{aligned}$$

The next lemma allows us to estimate the norms of certain specific functions.

Lemma 5. *Let $p \in [1,\infty)$. Consider the space L^p_v, where the weight v satisfies (10) for every $t > 0$, i.e., the cone $K(\downarrow) \cap L^p_v$ is nondegenerate.*

Let $\{t_j\}_{j=k}^{\infty}$ be a numerical sequence such that the relations $t_{j+1} > t_j$ are fulfilled for all $j = k, k+1, \ldots$. Suppose that the element $x = \sum_{j=k}^{\infty} 2^{-j}\chi_{[0,t_j)}$ belongs to L^p_v. Then we have

$$\|x \,|\, L^p_v\|^p = 2^{(1-k)p}\|\chi_{[0,t_k)}v \,|\, L^p\|^p + \sum_{j=k}^{\infty} 2^{-(j+1)p}\|\chi_{[t_j,t_{j+1})}v \,|\, L^p\|^p, \tag{83}$$

$$
\begin{aligned}
\sum_{j=k}^{\infty} 2^{-jp} \|\chi_{[0,t_j)} v \mid L^p\|^p = \frac{1}{1-2^{-p}} (2^{-kp} \|\chi_{[0,t_k)} v \mid L^p\|^p \\
+ \sum_{j=k}^{\infty} 2^{-(j+1)p} \|\chi_{[t_j,t_{j+1})} v \mid L^p\|^p).
\end{aligned} \tag{84}
$$

Proof. First, we verify (83):

$$
\begin{aligned}
\|x \mid L^p_v\|^p &= \left\| \sum_{j=k}^{\infty} 2^{-j} \chi_{[0,t_j)} \mid L^p_v \right\|^p = \left\| 2^{-k+1} \chi_{[0,t_k)} + \sum_{j=k}^{\infty} 2^{-j} \chi_{[t_j,t_{j+1})} \mid L^p_v \right\|^p \\
&= 2^p \|2^{-k} \chi_{[0,t_k)} \mid L^p_v\|^p + \sum_{j=k}^{\infty} 2^{-pj} \|\chi_{[t_j,t_{j+1})} \mid L^p_v\|^p.
\end{aligned}
$$

Next, we prove (84). Condition (10) shows that for every t_j we have $\|\chi_{[0,t_j)} v \mid L^p\|^p < \infty$. Therefore, for every $m \geq k$ we have

$$
\|\chi_{[0,t_m)} v \mid L^p\|^p = \|\chi_{[0,t_k)} v \mid L^p\|^p + \sum_{i=k}^{m-1} \|\chi_{[t_i,t_{i+1})} v \mid L^p\|^p. \tag{85}
$$

Suppose first that the left-hand side in (84) is finite. Then, taking (85) into account, we obtain

$$
\begin{aligned}
&\sum_{j=k}^{\infty} 2^{-jp} \|\chi_{[0,t_j)} v \mid L^p\|^p \\
&= \|\chi_{[0,t_k)} v \mid L^p\|^p \sum_{j=k}^{\infty} 2^{-jp} + \sum_{i=k}^{\infty} \left(\sum_{j=i+1}^{\infty} 2^{-jp} \right) \|\chi_{[t_i,t_{i+1})} v \mid L^p\|^p \\
&= \frac{2^{-kp}}{1-2^{-p}} \|\chi_{[0,t_k)} v \mid L^p\|^p + \sum_{i=k}^{\infty} \frac{2^{-(i+1)p}}{1-2^{-p}} \|\chi_{[t_i,t_{i+1})} v \mid L^p\|^p.
\end{aligned} \tag{86}
$$

Thus, (84) is true in this case.

But if the left-hand side of (84) is infinite, we deduce that the right-hand side is also infinite because all transformations in (86) have been done for nonnegative terms. □

The next statement is a principal lemma in this paper.

Lemma 6. *Fix $p \in [1, \infty)$ and a weight function v such that* (10) *is true for all $t > 0$, i.e., the cone $K(\downarrow) \cap L^p_v$ is nondegenerate.*

We introduce a new function $\bar{v}$ by the equation

$$
\|\chi_{[0,t)} v \mid L^p\| \cdot \left\| \chi_{[t,\infty)} \frac{1}{v} \mid L^{p'} \right\| \equiv 1. \tag{87}
$$

Then if a function x from the unit ball of L^p_v has the form $x = \sum_{i=k_0}^{k_1} 2^{-i} \chi_{[0,t_i)}$ ($0 \leq t_{k_0} < \cdots < t_i < t_{i+1} < \cdots < \infty$, where k_0 is finite and k_1 may be infinite), then for every $\varepsilon > 0$ there exists a function $x_\varepsilon \in L^p{}_{\bar{v}}$ satisfying

$$
\|x_\varepsilon \mid L^p{}_{\bar{v}}\| \leq \frac{1+\varepsilon}{(2^p-1)^{1/p}} \|x \mid L^p_v\| \tag{88}
$$

and such that for all $t \in [0, \infty)$ we have

$$
(Qx_\varepsilon)(t) \geq \frac{1}{16} x(t). \tag{89}
$$

Proof. So, suppose we a given a function x in the unit ball of L^p_v and $x = \sum_{i=k_0}^{k_1} 2^{-i}\chi_{[0,t_i)}$ $(0 \le t_{k_0} < \cdots < t_i < t_{i+1} < \cdots < \infty$, k_0 is finite and $k_1$may be infinite).

It is easily seen that for every admissible $i = k_0, k_{0+1}, \ldots$ we have

$$2^{-i-1} \le x(t_i) \le 2^{-i}. \tag{90}$$

For every admissible $i \in Z$, we define a number b_i by

$$b_i = \inf\left\{ \|y \mid L^p_{\bar{v}}\| \, : \, \int_{t_i}^{\infty} y(s)\,ds \ge 2^{-i-1} \right\}. \tag{91}$$

Since $t_i < \infty$ and the weight function is finite a.e., all numbers b_i are finite. Moreover, since $p \in [1, \infty)$ and $L^p{}_{\bar{v}}$ is an ideal space, it follows that

$$\inf\left\{ \|y \mid L^p_{\bar{v}}\| \, : \, \int_{t_i}^{\infty} y(s)\,ds \ge 2^{-i-1} \right\} = \inf\left\{ \|y \mid L^p_{\bar{v}}\| \, : \, \int_{t_i}^{\infty} y(s)\,ds = 2^{-i-1} \right\}, \tag{92}$$

i.e., we may assume that we have equality in (91). By the definition of the dual space, formulas (87) and (92) yield immediately two important relations:

$$\begin{aligned} 2^{-i-1} &= \int_{t_i}^{\infty} y(s)\,ds \le \|\chi_{[t_i,\infty)} y \mid L^p_{\bar{v}}\| \cdot \|\chi_{[t_i,\infty)} \mid (L^p_{\bar{v}})'\| \\ &= \|\chi_{[t_i,\infty)} y \mid L^p_{\bar{v}}\| \cdot \left\|\chi_{[t_i,\infty)} \frac{1}{\bar{v}} \mid L^{p'}\right\|, \end{aligned} \tag{93}$$

$$b_i = \frac{2^{-i-1}}{\|\chi_{[t_i,\infty)} \frac{1}{\bar{v}} \mid L^{p'}\|} = 2^{-i-1} \cdot \|\chi_{[0,t_i)} v \mid L^p\|. \tag{94}$$

Fixing $\varepsilon > 0$, for every $i = k_0, k_0 + 1, \ldots$ we choose a nearly extremal function y_i ensuring the relations

$$\operatorname{supp} y_i \subseteq [t_i, \infty), \quad \int_{t_i}^{\infty} y_i(s)\,ds = 2^{-i-1}, \quad b_i \le \|y_i \mid L^p{}_{\bar{v}}\| \le b_i(1+\varepsilon). \tag{95}$$

The possibility of such a choice is clear.

The subsequent construction of the required function is entirely algorithmic. So, we present it in the form usual for description of algorithms. Thus, let a collection of functions $\{y_k(t)\}_{k_0}^{k_1}$ be given.

Fix $k_0 \in Z$. Put $k = k_0$, $\zeta_{k_0}(t) = y_{k_0}(t)$.

Step A. If

$$\int_{t_k}^{t_{k+1}} \zeta_k(s)\,ds \ge \frac{1}{2} \int_{t_k}^{\infty} \zeta_k(s)\,ds, \tag{96}$$

put $z_k(t) = \zeta_k(t)\chi_{[t_k,t_{k+1})}$, $k = k+1$, $\zeta_k(t) = y_k(t)$. Return to Step A.

Step B. If (96) fails, i.e., we have

$$\int_{t_k}^{t_{k+1}} \zeta_k(s)\,ds < \frac{1}{2} \int_{t_k}^{\infty} \zeta_k(s)\,ds, \tag{97}$$

then again define z_k by $z_k(t) = \zeta_k(t)\chi_{[t_k,t_{k+1})}$, remove the function y_{k+1} from the collection $\{y_k(t)\}_{k_0}^{k_1}$, put $\zeta_{k+1}(t) \equiv \zeta_k(t)$, $k = k+1$, and return to Step A.

Note that if $k_1 < \infty$, then the last step of the algorithm is done for $k = k_1 - 1$. In this case z_{k_1} should be modified. Specifically, if the last step of the algorithm is of type **B**, we define z_{k_1} by $z_{k_1}(t) = y_{k_1}(t)$, but if the last step of the algorithm is of type B, we put $z_{k_1}(t) = \zeta_{k_1-1}\chi_{[t_{k_1},\infty)}$.

First, we show that for all admissible k: $k \geq k_0$ we have

$$\int_{t_k}^{\infty} \zeta_k(s)\,ds \geq \int_{t_k}^{\infty} y_k(s)\,ds. \tag{98}$$

Indeed, for $k = k_0$ we have equality in (98). We do an induction step. If we perform Step **A** after Step **A**, then again we have equality in (98). If we perform Step **A** after Step **B**, then from (97) and the inductive hypothesis we deduce that

$$\begin{aligned}\int_{t_{k+1}}^{\infty} \zeta_{k+1}(s)\,ds &= \int_{t_{k+1}}^{\infty} \zeta_k(s)\,ds = \int_{t_k}^{\infty} \zeta_k(s)\,ds - \int_{t_k}^{t_{k+1}} \zeta_k(s)\,ds \\ &\geq \frac{1}{2}\int_{t_k}^{\infty} \zeta_k(s)\,ds \geq \frac{1}{2}\int_{t_k}^{\infty} y_k(s)\,ds = \int_{t_{k+1}}^{\infty} y_{k+1}(s)\,ds.\end{aligned}$$

Thus, (98) is proved.

The algorithm results in replacing the collection $\{y_k(t)\}_{k_0}^{k_1}$ with a new collection of functions $\{z_k(t)\}_{k_0}^{k_1}$; furthermore, the procedure implies directly that the supports of the functions in the collection $\{z_k(t)\}_{k_0}^{k_1}$ are mutually disjoint.

Now, we define the new function

$$\overline{x_{k_0}}(t) = \sum_{k_0}^{k_1} z_k(t).$$

First, we show that for every $j = k_0,\, k_0+1, \ldots$ we have

$$Q\overline{x_{k_0}}(t_j) = \int_{t_j}^{\infty} \sum_{k_0}^{k_1} z_k(s)\,ds \geq \frac{1}{4}\int_{t_j}^{\infty} y_j(s)\,ds = 2^{-j-3} \geq 2^{-3}x(t_j). \tag{99}$$

Three possibilities may occur.

a) Let $z_j(t) \equiv y_j(t)\chi_{[t_j,t_{j+1})}$, $z_{j+1}(t) \equiv y_{j+1}(t)\chi_{[t_{j+1},t_{j+2})}$, i.e., Step **A** is performed. Then (96) yields

$$Q\overline{x_{k_0}}(t_j) = \int_{t_j}^{\infty} \sum_{k_0}^{k_1} z_k(s)\,ds \geq \int_{t_j}^{t_{j+1}} z_j(s)\,ds = \int_{t_j}^{t_{j+1}} y_j(s)\,ds \geq \frac{1}{2}\int_{t_j}^{\infty} y_j(s)\,ds,$$

which proves (99) in the case in question.

b) Suppose that, starting with some $m \geq k_0$, we have

$$z_k(t)\chi_{[t_k,t_{k+1})} \equiv y_m(t)\chi_{[t_k,t_{k+1})},\ k = m, m+1, \ldots, m+l \quad (1 < l < \infty),$$

and that $z_{m+l+1}(t)$ does not coincide with $y_m(t)$ on $[t_{m+l+1}, t_{m+l+2})$. This happens if, starting with $k = m$, the algorithm walks away to Step **B** and does not change the function $\zeta_m(t)$ $(l-1)$ times, i.e., in accordance with (96) and (97), we have the relations

$$\begin{aligned}&\int_{t_j}^{t_{j+1}} y_m(s)\,ds < \frac{1}{2}\int_{t_{j+1}}^{\infty} y_m(s)\,ds \quad \text{for} \quad j = m, m+1, \ldots, m+l-1; \\ &\int_{t_{m+l}}^{t_{m+l+1}} y_m(s)\,ds \geq \frac{1}{2}\int_{t_{m+l+1}}^{\infty} y_m(s)\,ds.\end{aligned} \tag{100}$$

In this case, for $j = m,\, m+1, \ldots, m+l$ we put

$$\begin{aligned}a_j &= \int_{t_j}^{t_{j+1}} \zeta_j(s)\,ds = \int_{t_j}^{t_{j+1}} z_j(s)\,ds = \int_{t_j}^{t_{j+1}} y_m(s)\,ds, \\ d_j &= \int_{t_{j+1}}^{\infty} \zeta_j(s)\,ds = \int_{t_{j+1}}^{\infty} y_m(s)\,ds.\end{aligned} \tag{101}$$

Therefore, (100) and (101) show that for all $j = m,\ m+1, \dots, m+l$ the following inequalities hold true:

$$\begin{aligned}
Q\overline{x_{k_0}}(t_j) &= \int_{t_j}^{\infty} \sum_{k_0}^{k_1} z_k(s)\,ds \geq \int_{t_j}^{t_{m+l+1}} \sum_{i=j}^{m+l} z_j(s)\,ds \\
&= \int_{t_j}^{t_{j+m+l}} y_m(s)\,ds = a_j + a_{j+1} + \cdots + a_{m+l} \\
&= a_j + a_{j+1} + \cdots + a_{m+l-1} + \frac{1}{2}a_{m+l} + \frac{1}{2}a_{m+l} \\
&\geq a_j + a_{j+1} + \cdots + a_{m+l-1} + \frac{1}{2}a_{m+l} + \frac{1}{4}d_{m+l} \\
&\geq \frac{1}{4}\int_{t_j}^{\infty} y_m(s)\,ds = \frac{1}{4}\int_{t_j}^{\infty} \zeta_j(s)\,ds.
\end{aligned}$$

To prove (99) in the case in question, it remains to apply (98).

c) It remains to consider the case where, starting with some $m \geq k_0$, for all $k \geq m$ we have $z_k(t) \equiv y_m(t)$. This happens if, starting with $k = m$, the algorithm walks away to Step **B** and does not change the function $\zeta_m(t)$ any longer. In this case, for all $j = m$, $m+1, \dots$ we have $\zeta_m(t) \equiv \zeta_j(t) \equiv y_m(t)$. By (98), it follows that we also have

$$Q\overline{x_{k_0}}(t_j) = \int_{t_j}^{\infty} \sum_{k_0}^{k_1} z_k(s)\,ds = \int_{t_j}^{\infty} y_m(s)\,ds = \int_{t_j}^{\infty} \zeta_j(s)\,ds \geq \int_{t_j}^{\infty} y_j(s)\,ds.$$

Again, this implies (99).

Now, let $t \in (t_j, t_{j+1})$. Then the explicit form of the function x, inequalities (90) and the nonnegativity of $\overline{x_{k_0}}(t)$ imply the relation

$$Q\overline{x_{k_0}}(t) = \int_{t}^{\infty} \sum_{k_0}^{k_1} z_k(s)\,ds \geq \int_{t_{j+1}}^{\infty} \sum_{k_0}^{k_1} z_k(s)\,ds \geq 2^{-j-4} \geq \frac{1}{16}x(t_j) \geq \frac{1}{16}x(t).$$

Put $x_\varepsilon(t) \equiv \overline{x_{k_0}}(t)$. The last inequality and (99) yield (89). It remains to verify (88).

First, we prove the relation

$$\|x_\varepsilon \mid L_v^p\|^p = \sum_{j=k_0}^{k_1} \|z_j \mid L_v^p\|^p \leq \sum_{j=k_0}^{k_1} \|y_j \mid L_v^p\|^p. \tag{102}$$

The identity in (102) follows from the fact that the supports of the functions in the collection $\{z_j(t)\}_{k_0}^{k_1}$ are mutually disjoint. On the other hand, for every function y_j the algorithm either merely drops it or multiplies it by a characteristic function, i.e.,

$$\sum_{j=k_0}^{k_1} z_j = \sum_{j=k_0}^{k_1} \chi(D_j)y_j,$$

where $D_j = \varnothing$ if y_j was dropped, $D_j = [t_j,\ t_{j+1})$ if y_j was involved in the action of the algorithm only once, $D_j = [t_j,\ t_{j+2})$ if y_j was involved in the action of the algorithm twice, and so on. Therefore,

$$\sum_{j=k_0}^{k_1} \|z_j \mid L_v^p\|^p = \sum_{j=k_0}^{k_1} \|y_j\chi(D_j) \mid L_v^p\|^p \leq \sum_{j=k_0}^{k_1} \|y_j \mid L_v^p\|^p.$$

So, relations (102) are proved.

Next, taking the choice of the y_j into account, by (95) and (94) we obtain the inequality

$$\|x_\varepsilon \mid L^p_{\bar v}\|^p \le \sum_{j=k_0}^{k_1} \|y_j \mid L^p_{\bar v}\|^p \le (1+\varepsilon)^p \sum_{j=k_0}^{k_1} b_j^p \le (1+\varepsilon)^p \sum_{j=k_0}^{k_1} (2^{-j-1}\|\chi_{[0,t_j)}v \mid L^p\|)^p.$$

Together with (84), this implies

$$\|x_\varepsilon \mid L^p_{\bar v}\|^p \le \frac{2^{-p}(1+\varepsilon)^p}{1-2^{-p}}(2^{-k_0 p}\|\chi_{[0,t_{k_0})}v \mid L^p\|)^p + \sum_{j=k_0}^{k_1}(2^{-j-1}\|\chi_{[t_j,t_{j+1})}v \mid L^p\|)^p.$$

Next, using (83), we finally obtain

$$\|x_\varepsilon \mid L^p_{\bar v}\| \le \frac{(1+\varepsilon)}{(2^p-1)^{1/p}}\|x \mid L^p_v\|. \qquad \square$$

Now, everything is ready for the proof of Theorem 1. We proceed with this.

Proof. By Lemma 2, the embedding

$$QL^p{}_{\bar v} \subseteq K(\downarrow) \cap L^p{}_v$$

follows from the boundedness of Q as an operator from $L^p{}_{\bar v}$ to $L^p{}_v$ (that is how the weight $\bar v$ was chosen). Furthermore, Q takes any nonnegative function to a monotone nonincreasing function. This proves (a).

We pass to the second statement of the theorem.

Fix a function $x \in K(\downarrow) \cap L^p{}_v$ with nonzero norm. Condition (11) implies the identity

$$\lim_{t\to\infty} x(t) = 0. \tag{103}$$

Since the norm in $L^p{}_v$ is absolutely continuous, by (103) we see that there exists a strictly monotone continuous function $x_0 \in K(\downarrow) \cap L^p{}_v$ enjoying the conditions

$$x_0(t) \ge x(t) \quad (\forall t > 0); \tag{104}$$

$$\|x_0 \mid L^p{}_v\| \le 2\|x \mid L^p{}_v\|; \tag{105}$$

$$\lim_{t\to\infty} x_0(t) = 0; \tag{106}$$

$$\lim_{t\to 0} x_0(t) = \infty. \tag{107}$$

Since the continuous function $x_0(t)$ is strictly monotone, by (106), (107) we deduce that for every $i \in Z$ there exists a unique point $t_i \in (0,\infty)$ with $x_0(t_i) = 2^{-i}$. We define two new functions

$$y_{s0}(t) = \sum_{-\infty}^{\infty} \chi_{[t_i,t_{i+1})}2^{-i}, \quad y_{s1}(t) = \sum_{-\infty}^{\infty} \chi_{[0,t_{i+1})}2^{-i}.$$

Direct calculations show that for every $t > 0$ we have

$$y_{s1}(t) \equiv 2y_{s0}(t), \quad y_{s0}(t) \ge x_0(t) \ge \frac{1}{2}y_{s0}(t). \tag{108}$$

Consequently,

$$\|y_{s1} \mid L^p{}_v\| = 2\|y_{s0} \mid L^p{}_v\| \ge 2\|x_0 \mid L^p{}_v\| \ge \frac{1}{2}\|y_{s0} \mid L^p{}_v\|. \tag{109}$$

But form (105), (108), and the absolute continuity of the norm in L^p_v we deduce that

$$\begin{aligned} \lim_{i\to-\infty}\left\|\sum_{-\infty}^{i}\chi_{[0,t_{j+1})}2^{-j} \mid L^p{}_v\right\| &= \lim_{i\to-\infty} 2\left\|\sum_{-\infty}^{i}\chi_{[t_j,t_{j+1})}2^{-j} \mid L^p{}_v\right\| \\ &= 2\lim_{i\to-\infty}\|y_{s0}\chi_{[0,t_{j+1})} \mid L^p{}_v\| \le 4\lim_{i\to-\infty}\|x_0\chi_{[0,t_{j+1})} \mid L^p{}_v\| = 0. \end{aligned} \tag{110}$$

Taking (110) into account, we apply Lemma 4 to y_{s1}. This yields a sequence $\{k_i\}_0^\infty$ such that

$$\Big\| \sum_{k_0+1}^{\infty} 2^{-j} \chi_{[0,t_{j+1})} \,|\, L^p{}_v \Big\| + \sum_{i=0}^{\infty} \Big\| \sum_{k_{i+1}+1}^{k_i} 2^{-j} \chi_{[0,t_{j+1})} \,|\, L^p{}_v \Big\| \le 2\|y_{s1} \,|\, L^p{}_v\|. \tag{111}$$

We define the functions

$$z_0 = \sum_{k_0+1}^{\infty} 2^{-j} \chi_{[0,t_{j+1})}; \quad z_{i+1} = \sum_{k_{i+1}+1}^{k_i} 2^{-j} \chi_{[0,t_{j+1})} \quad (i = 0, 1, 2, \dots).$$

Fixing $\varepsilon > 0$, we apply Lemma 6 to every function $\{z_i\}_0^\infty$. This will result in a collection of functions $\{z_{i\varepsilon}\}_0^\infty$ satisfying the conditions

$$Qz_{i\varepsilon}(t) \ge \frac{1}{16} z_i(t) \quad (\forall t > 0); \tag{112}$$

$$\|z_{i\varepsilon} \,|\, L^p{}_{\bar v}\| \le \frac{1+\varepsilon}{(2^p-1)^{1/p}} \|z_i \,|\, L^p{}_v\|. \tag{113}$$

Put

$$x_\varepsilon = \sum_{i=0}^{\infty} z_{i\varepsilon}.$$

Then (104), (108), (111), and (112) show that for every $t > 0$ we have

$$Qx_\varepsilon(t) = Q\Big(\sum_{i=0}^{\infty} z_{i\varepsilon}\Big)(t) \ge \frac{1}{16} \sum_{i=0}^{\infty} z_i(t) = \frac{1}{16} y_{s1}(t) = \frac{1}{8} y_{s0}(t) \ge \frac{1}{8} x_0(t) \ge \frac{1}{8} x(t).$$

But from (105), (108), (109), and (113) we obtain

$$\begin{aligned} \|x_\varepsilon \,|\, L^p{}_{\bar v}\| &= \Big\| \sum z_{i\varepsilon} \,|\, L^p{}_{\bar v} \Big\| \le \sum \|z_{i\varepsilon} \,|\, L^p{}_{\bar v}\| \le \frac{1+\varepsilon}{(2^p-1)^{1/p}} \sum \|z_i \,|\, L^p{}_v\| \\ &\le 2\frac{1+\varepsilon}{(2^p-1)^{1/p}} \|y_{s1} \,|\, L^p{}_v\| \le 8\frac{1+\varepsilon}{(2^p-1)^{1/p}} \|x_0 \,|\, L^p{}_v\| \\ &\le 16\frac{1+\varepsilon}{(2^p-1)^{1/p}} \|x \,|\, L^p{}_v\|. \end{aligned}$$

□

We pass to the proof of Theorem 7.

Proof. By Theorem 5, the cones $K(\downarrow) \cap L_v^\infty$ and $K(\downarrow) \cap L_{\widetilde v}^\infty$ coincide.

First, we verify (a). By Lemma 2, the operator $Q\colon L_u^\infty \to L_w^\infty$ is bounded if and only if

$$\sup_{t>0} w(t) \int_t^\infty \frac{ds}{u(s)} = C_8 < \infty. \tag{114}$$

In our case, (114) has the form

$$\sup_{t>0} \widetilde v(t) \int_t^\infty \frac{ds}{\bar v(s)} = \sup_{t>0} \widetilde v(t) \int_t^\infty \Big(-\frac{d}{ds} \frac{1}{\bar v_{ac}}(s) \Big) = \sup_{t>0} \widetilde v(t) \frac{1}{\widetilde v_{ac}(t)} \le C_8. \tag{115}$$

Now, we prove (b). Let $x \in K(\downarrow) \cap L_{\widetilde v}^\infty$ and $c_0 = \sup_{t>0} x(t)\widetilde v(t)$. Put $y_\varepsilon(t) = \frac{c_0}{\bar v(t)}$. Then $\|y_\varepsilon \,|\, L_{\widetilde v}^\infty\| = c_0$, and we have

$$\begin{aligned} Qy_\varepsilon(t) &= \int_t^\infty y_\varepsilon(\tau)\, d\tau = \int_t^\infty \frac{c_0}{\bar v(\tau)}\, d\tau \\ &= c_0 \int_t^\infty \Big(-\frac{d}{ds} \frac{1}{\bar v_{ac}(s)} ds \Big) = c_0 \frac{1}{\bar v_{ac}(t)}(t) \ge \frac{c_0}{c} x(t). \end{aligned}$$

□

Now we pass to a lemma that yields a necessary and sufficient condition for the existence of an absolutely continuous function equivalent to a given weight function v.

We remark at once that, in the case of the space L_v^∞, the value of the weight function at every point is essential. Therefore, we assume for definiteness in what follows that v is continuous from the left.

Lemma 7. *Fix $p=\infty$ and a weight function $v\colon [0,\infty)\to R_+$. Next, define a function $\widetilde{v}$ as in (34). Then the existence of an absolutely continuous function $\widetilde{v}_{ac}\in K(\downarrow)\cap L_v^\infty$ such that*

$$\frac{1}{c}\widetilde{v}_{ac}(t)\le \widetilde{v}(t)\le c\widetilde{v}_{ac}(t) \tag{116}$$

for some constant C and all $t\in R_+$ is equivalent to the inequality

$$\sup_{t\in R_+}\frac{\widetilde{v}(t+0)}{\widetilde{v}(t-0)}=d<\infty. \tag{117}$$

Proof. We verify the implication (116) $\Rightarrow$ (117). Since $\widetilde{v}_{ac}$ is absolutely continuous, by (116) we obtain

$$\frac{\widetilde{v}(t+0)}{\widetilde{v}(t-0)}\le \frac{c\widetilde{v}_{ac}(t+0)}{\widetilde{v}(t-0)}\le \frac{c^2\widetilde{v}_{ac}(t+0)}{\widetilde{v}_{ac}(t-0)}=c^2, \tag{118}$$

and this proves the implication (116) $\Rightarrow$ (117).

Now, we verify the implication (117) $\Rightarrow$ (116). Since $\widetilde{v}_{ac}$ is monotone nondecreasing, the Radon–Nikodym decomposition yields the representation

$$\widetilde{v}(t)=x_1(t)+x_2(t)+x_3(t), \tag{119}$$

where $x_1,\ x_2,\ x_3\in K(\downarrow)$ are, respectively, absolutely continuous, singular, and jump functions. We construct an equivalent absolutely continuous function for each of them separately. The function x_1 is absolutely continuous itself, so nothing should be constructed for it. We have two possibilities for x_2: a) this function is positive for all $t\in R_+$, b) there exists $\tau_0\in R_+$ such that $x_2(\tau_0)=0$ and $x_2(t)>0$ for $t<\tau_0$. We consider a) first.

Put $c=x_2(1)$. For every $k\in Z$ we define $t_k\in R_+$ by $t_k=\sup\{t : x_2(t)\ge 2^k c\}$. If $0\le t_{k+1}<t_k<\infty$, then for $t\in[t_{k+1},t_k]$ we define a function x_{2ac} by the formula $x_{2ac}(t)=x_2(t_k)+(t-t_k)\frac{x_2(t_{k+1})-x_2(t_k)}{t_{k+1}-t_k}$. Then for all $t\in[t_{k+1},t_k]$ we have

$$x_{2ac}(t_k)\le x_2(t)\le x_{2ac}(t_{k+1})=2x_{2ac}(t_k)\le 2x_2(t). \tag{120}$$

If $0<t_k<t_{k-1}=\infty$, then for $t\in[t_k,\infty)$ we define a function x_{2ac} by the formula $x_{2ac}(t)=x_2(t_k)$. Then for all $t\in[t_k,\infty)$ we have

$$x_2(t)\le x_{2ac}(t_k)\le 2x_2(t). \tag{121}$$

Now, if $x_2(0)<2x_2(t_k)$, then for $t\in[0,t_k]$ we define x_{2ac} by the formula $x_{2ac}(t)=x_2(0)$. Then for $t\in[0,t_k]$ we again have (121).

Relations (120)–(121) show that in case a) we have constructed an absolutely continuous functionx_{2ac} equivalent to x_2 with constant 2.

If condition b) is fulfilled, then for all $t\in[\tau_0,\infty)$ we define x_{2ac} by the formula $x_{2ac}(t)\equiv 0$. The subsequent construction is similar to case a) treated above.

Now, we show how to construct an equivalent absolutely continuous function in $K(\downarrow)$ for x_3.

Again, there are two possibilities for x_3: a) the function x_3 is positive for all $t\in R_+$, b) there exists $\tau_0\in R_+$ such that $x_3(\tau_0)=0$ and $x_2(t)>0$ for $t<\tau_0$.

First, we consider case a). For every $k\in Z$, we define $t_k\in(\tau,\infty)$ by $t_k=\inf\{t : x_3(t)\le(d+1)^k\}$. Again by continuity, $x_3(t_k)\le(d+1)^k$ for all k.

Again, we construct a linear approximate. If $0 \le t_{k+1} < t_k < \infty$, then for $t \in [t_{k+1}, t_k]$ we define a function x_{3ac} by the formula $x_{3ac}(t) = x_3(t_k) + (t - t_k)\frac{x_3(t_{k+1}) - x_3(t_k)}{t_{k+1} - t_k}$. Then, if $x_3(t_{k+1}) \le (d+1)x_3(t_k)$, for all $t \in [t_{k+1}, t_k]$ we have

$$x_{3ac}(t_k) \le x_3(t) \le x_{3ac}(t_{k+1}) \le (d+1)x_{3ac}(t_k) \le (d+1)x_{3ac}(t). \tag{122}$$

But if $x_3(t_{k+1}) > (d+1)x_3(t_k)$, then for every $\delta > 0$ we have $x_3(t_k - \delta) > (d+1)^k$. Therefore, by (117) we obtain $x_3(t_{k+1}) \le (d+1)^{k+1} \le (d+1)x_3(t_k - \delta)$ and, consequently, $x_3(t_{k+1}) \le d(d+1)x_3(t_k)$. Thus, for all $t \in [t_{k+1}, t_k]$ we have

$$x_{3ac}(t_k) \le x_3(t) \le x_{3ac}(t_{k+1}) \le d(d+1)x_{3ac}(t_k) \le d(d+1)x_{3ac}(t). \tag{123}$$

If $0 < t_k < t_{k-1} = \infty$, then for $t \in [t_k, \infty)$ we define x_{3ac} by the formula $x_{3ac}(t) = x_3(t_k)$.

Applying (117) once again, we see that

$$x_3(t) \le x_{3ac}(t) \le d(d+1)x_{3ac}(t) \tag{124}$$

for all $t \in [t_k, \infty)$. Relations (122)–(124) show that, in case a), we have constructed an absolutely continuous function x_{3ac} equivalent to x_3 with the constant $d(d+1)$.

If b) is fulfilled, then for all $t \in [\tau_0, \infty)$ we define x_{3ac} by $x_{3ac}(t) \equiv 0$. The subsequent arguments resemble those in case a). □

In conclusion we discuss Lemma 8, required for the proof of Lemma 3.

Lemma 8. *Fix $\theta \in (0,1)$. In the spaces $L^1_{w_0}$ and $L^\infty_{w_1}$, consider the cones $K(\downarrow) \cap L^1_{w_0}$ and $K(\downarrow) \cap L^\infty_{w_1}$. Let*

$$x \in (K(\downarrow) \cap L^1_{w_0})^\theta (K(\downarrow) \cap L^\infty_{w_1})^{1-\theta}$$

satisfy

$$\|x \,|\, (K(\downarrow) \cap L^1_{w_0})^\theta (K(\downarrow) \cap L^\infty_{w_1})^{1-\theta}\| = 1. \tag{125}$$

Then there exists $x_0 \in K(\downarrow) \cap L^1_{w_0}$ with $\|x_0 \,|\, L^1_{w_0}\| = 1$ such that

$$x(t) \le x_0^\theta(t) \cdot \Big(\frac{1}{\widetilde{w}_1(t)}\Big)^{1-\theta}(t) \tag{126}$$

for all $t \in R_+$.

Proof. Theorem 5 shows that the cones $K(\downarrow) \cap L^\infty_{w_1}$ and $K(\downarrow) \cap L^\infty_{\widetilde{w}}$ coincide.

Suppose that (125) is fulfilled. This means that there exists a sequence of pairs of functions $x_{0n} \in K(\downarrow) \cap L^1_{w_0}$ with $\|x_{0n} \,|\, L^1_{w_0}\| = 1$ and $x_{1n} \in K(\downarrow) \cap L^\infty_{\widetilde{w}}$ with $\|x_{1n} \,|\, L^\infty_{\widetilde{w}}\| = 1$ such that for all $t \in R_+$ we have

$$x(t) \le \Big(1 + \frac{1}{n}\Big) \cdot x_{0n}^\theta(t) x_{1n}^{1-\theta}(t). \tag{127}$$

Since every $x \in K(\downarrow) \cap L^\infty_{\widetilde{w}_1}$ with $\|x \,|\, L^\infty_{\widetilde{w}_1}\| = 1$ satisfies $x(t) \le \frac{1}{\widetilde{w}(t)}$ for all $t \in R_+$, we can rewrite (127) in the form

$$x(t) \le \Big[\Big(1 + \frac{1}{n}\Big)^{1/\theta} \cdot x_{0n}(t)\Big]^\theta \cdot \Big(\frac{1}{\widetilde{w}_1(t)}\Big)^{1-\theta}. \tag{128}$$

We define a function y at all points $t \in R_+$ by the formula $y(t) = \inf_n \big(1 + \frac{1}{n}\big)^{1/\theta} \cdot x_{0n}(t)$. Then $y \in K(\downarrow) \cap L^1_{w_0}$, and the pointwise inequality $y(t) \le \big(1 + \frac{1}{n}\big)^{1/\theta} \cdot x_{0n}(t)$ shows that $\|y \,|\, L^1_{w_0}\| \le \big(1 + \frac{1}{n}\big)^{1/\theta}$ for every $n \in N$. This means that $\|y \,|\, L^1_{w_0}\| \le 1$. In (128), we pass to the infimum over n for every $t \in R_+$, obtaining the inequality

$$x(t) \le y(t)^\theta \cdot \Big(\frac{1}{\widetilde{w}_1(t)}\Big)^{1-\theta} \tag{129}$$

valid for all $t \in R_+$. Now, if we suppose that $\|y \,|\, L^1_{w_0}\| = q < 1$, then, putting $q_0 = q^\theta$, we see that for all $t \in R_+$ we have

$$x(t) \le \Big(\frac{y(t)}{q} q_0\Big)^{\theta} \cdot \Big(\frac{q_0}{\widetilde{w}_1(t)}\Big)^{1-\theta};$$

moreover, $\frac{q_0}{q} y \in K(\downarrow) \cap L^1_{w_0}$ with $\big\|\frac{q_0}{q} y \,|\, L^1_{w_0}\big\| = q_0$, and also $q_0 \frac{1}{\widetilde{w}_1} \in K(\downarrow) \cap L^\infty_{\widetilde{w}_1}$ with $\big\|q_0 \frac{1}{\widetilde{w}_1} \,|\, L^\infty_{\widetilde{w}_1}\big\| = q_0$, which contradicts (125). □

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