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# ON THE SIDON INEQUALITY FOR TRIGONOMETRIC POLYNOMIALS

A. O. RADOMSKII

*To Boris Sergeevich Kashin on his 65th birthday*

Abstract. A lower estimate is established for the uniform norm of a special type trigonometric polynomial in terms of the sum of the $L^1$-norms of its summands in the case where the sequence of frequencies splits into finitely many lacunary sequences. The result refines theorems known for lacunary sequences and generalizes a result of Kashin and Temlyakov, which in its turn generalizes the classical Sidon inequality.

## §1. Introduction

For $f \in L^p(0, 2\pi)$, put

$$\|f\|_p = \left( \int_0^{2\pi} |f(x)|^p \, dx \right)^{1/p} \quad \text{for} \quad 1 \le p < \infty,$$
$$\|f\|_\infty = \operatorname*{ess\,sup}_{[0,2\pi]} |f(x)| \quad \text{for} \quad p = \infty,$$
$$c_n(f) = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} \, dx, \quad n \in \mathbb{Z},$$
$$a_k(f) = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos kx \, dx,$$
$$b_k(f) = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin kx \, dx, \quad k = 0, 1, 2, \dots .$$

Given $2\pi$-periodic functions $f(x)$ and $g(x)$, we define $\langle f, g \rangle$ by the formula

$$\langle f, g \rangle = \int_0^{2\pi} f(x) g(x) \, dx,$$

and denote by $f * g$ the convolution

$$(f * g)(x) = \frac{1}{2\pi} \int_0^{2\pi} f(x - t) g(t) \, dt.$$

If $x$ is a real number, then $[x]$ denotes its integral part, and $\lceil x \rceil$ is the smallest integer $n$ such that $n \ge x$. Given a finite set $A$, we denote its cardinality by $|A|$. For a nonzero trigonometric polynomial $T(x)$, its exact order will be denoted by $\deg(T)$. For a real number $r \ge 0$, let $\mathrm{T}(r)$ denote the space of all real trigonometric polynomials of the form

$$t(x) = A + \sum_{k=1}^{[r]} a_k \cos kx + b_k \sin kx$$

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(by definition, the sum $\sum_{k=1}^{0}$ is put to be zero).

Let $\lambda > 1$ be a real number. We denote by $\Lambda(\lambda)$ the class of sequences $N = \{n_k\}_{k=1}^{\infty}$ of natural numbers such that $n_{k+1}/n_k \geq \lambda$, $k = 1, 2, \dots$. Let $\Lambda$ stand for the class of all lacunary sequences $N$, i.e.,

$$\Lambda = \bigcup_{\lambda>1} \Lambda(\lambda).$$

Finally, we introduce the class $\Lambda_\sigma$ of all monotone increasing sequences $N$ of natural numbers that admit splitting into finitely many lacunary sequences. Observe that if $N \in \Lambda_\sigma$, then $N$ is a monotone increasing sequence of natural numbers, and it can be split into finitely many sequences $N^{(j)} \in \Lambda(\mu)$, where $\mu > 1$ is any number prescribed beforehand.

The following theorem was proved by S. Sidon in 1927.

**Theorem A** (Sidon [1]). *Suppose that $\{n_k\}_{k=1}^{\infty}$ is a sequence of natural numbers satisfying*

$$\frac{n_{k+1}}{n_k} \geq \lambda > 1, \quad k = 1, 2, \dots$$

*If a trigonometric series*

$$\sum_{k=1}^{\infty} \alpha_k \cos n_k x + \beta_k \sin n_k x$$

*is the Fourier series of a bounded function $f(x)$, then*

$$\sum_{k=1}^{\infty} |\alpha_k| + |\beta_k| < \infty.$$

Sidon's method of proof was based on application of the Riesz products, which became an important tool in the theory of trigonometric and general orthogonal series. For the first time, these products arose in F. Riesz's paper [2]. Also in [1], Sidon observed that Theorem A remains valid in the case where $f(x)$ is only bounded from one side, i.e., if $f(x) \leq M$ or $f(x) \geq -M$. In fact, the proof of Theorem A in [1] implies the following estimate, which will be called the Sidon inequality:

$$\sum_{k=1}^{\infty} |\alpha_k| + |\beta_k| \leq C(\lambda) \|f\|_\infty,$$

where $C(\lambda) > 0$ is a constant depending only on $\lambda$. In particular, the next result is true.

**Theorem B.** *Suppose that $\{n_k\}_{k=1}^{\infty}$ is a sequence of natural numbers satisfying*

$$n_{k+1}/n_k \geq \lambda > 1, \quad k = 1, 2, \dots.$$

*Then for each real trigonometric polynomial*

$$f(x) = \sum_{k=1}^{m} \alpha_k \cos n_k x + \beta_k \sin n_k x, \quad m = 1, 2, \dots,$$

*we have*

$$\|f\|_\infty \geq c(\lambda) \sum_{k=1}^{m} (|\alpha_k| + |\beta_k|), \tag{1.1}$$

*with a constant $c(\lambda) > 0$ depending only on $\lambda$.*

A direction of refinement of Theorem A was related to relaxing its assumptions concerning lacunarity. In [3], Sidon himself carried his theorem over to the case where $N = \{n_k\}_{k=1}^{\infty}$ can be split into finitely many lacunary sequences, i.e., $N \in \Lambda_\sigma$. The further development in this issue was done in the papers [4–7] and others.

In 1998, Kashin and Temlyakov [8, 9] started the study of another direction of refining the Sidon theorem. In connection with estimates for the entropy numbers of some classes of functions of low smoothness, they explored the question about the possible generalizations of the Sidon inequality (1.1) where $\alpha_k \cos n_k x$ is replaced with $p_k(x) \cos n_k x$, $p_k(x)$ being a trigonometric polynomial.

**Theorem C** (Kashin and Temlyakov [8, 9]). *For any trigonometric polynomial of the form*

$$f(x) = \sum_{k=l+1}^{2l} p_k(x) \cos 4^k x,$$

*where $p_k \in \mathrm{T}(2^l)$, $k = l+1, \ldots, 2l$, $l = 1, 2, \ldots,$ we have*

$$\|f\|_\infty \geq c \sum_{k=l+1}^{2l} \|p_k\|_1$$

*with an absolute constant $c > 0$.*

The further investigation of these issues was continued by present author in [10–12]. The best result is as follows.

**Theorem D** (Radomskii [12, Theorem 1.1]). *Suppose that $\{n_k\}_{k=1}^{\infty}$ is a sequence of natural numbers satisfying $n_{k+1}/n_k \geq \lambda > 1$, $k = 1, 2, \ldots$. Then for any trigonometric polynomial*

$$f(x) = \sum_{k=l}^{m} p_k(x) \cos n_k x,$$

*where $p_k \in \mathrm{T}(\gamma n_l)$, $\gamma = \min\left(\frac{1}{6}, \frac{\lambda-1}{3}\right)$, $k = l, \ldots, m$, $m > l$, $l = 1, 2, \ldots,$ we have*

$$\|f\|_\infty \geq c(\lambda) \sum_{k=l}^{m} \|p_k\|_1, \tag{1.2}$$

*with a constant $c(\lambda) > 0$ depending only on $\lambda$.*

We note that Theorem D not only generalizes Theorem C, carrying the result over to the case of an arbitrary lacunary sequence, but also relaxes the conditions on the degrees of the $p_k(x)$. The Sidon type inequalities (1.2) are closely related to the properties of the space of quasicontinuous functions and to the QC-norm, which was introduced by Kashin and Temlyakov in [8, 9] as follows: for $f \in L^1(0, 2\pi)$ with the Fourier series $f \sim \sum_{j=0}^{\infty} \delta_j(f, x)$, where

$$\delta_0(f) = \frac{1}{2\pi} \int_0^{2\pi} f(x)\, dx,$$

$$\delta_j(f, x) = \sum_{n=2^{j-1}}^{2^j - 1} a_n(f) \cos nx + b_n(f) \sin nx, \quad j = 1, 2, \ldots,$$

we put

$$\|f\|_{\mathrm{QC}} \equiv \int_0^1 \left\| \sum_{j=0}^{\infty} r_j(t) \delta_j(f, \cdot) \right\|_\infty dt,$$

where $\{r_j(t)\}_{j=0}^{\infty}$ is the Rademacher system, see [13, Chapter 2]). In particular, in [9] the following theorem (Theorem 2.1) was proved: for any real function $f \in L^1(0, 2\pi)$ we have the inequality

$$\|f\|_{\mathrm{QC}} \geq \frac{1}{48\pi} \sum_{j=0}^{\infty} \|\delta_j(f)\|_1. \tag{1.3}$$

The relationship between C- and QC-norms is also of interest. K. I. Oskolkov showed (see [9]) that

$$\sup_{t \in \mathrm{T}(2^m)} \frac{\|t\|_\infty}{\|t\|_{\mathrm{QC}}} \geq c_1 \sqrt{m}, \quad c_1 > 0.$$

On the other hand, from (1.3) and a result of P. G. Grigor′ev (see [14]) it follows that

$$\sup_{t \in \mathrm{T}(2^m)} \frac{\|t\|_{\mathrm{QC}}}{\|t\|_\infty} \geq c_2 \sqrt{m}, \quad c_2 > 0.$$

In [15], the present author obtained a nontrivial lower estimate for the quantity

$$\sup_{t \in L} \|t\|_{\mathrm{QC}} / \|t\|_\infty$$

for a subspace $L \subset \mathrm{T}(2^m - 1)$ satisfying certain dimensional restrictions. Another important question is about the sharpness of the conditions in Theorem D. In this direction, the following fact was obtained (see [16], [12, Theorem 2.2], and also [11], where a result of similar nature was established for the sequence $n_k = 2^k$).

**Theorem E** (Grigor'ev, Radomskii)**.** *Let $\{n_k\}_{k=1}^{\infty}$ be a sequence of natural numbers such that $n_{k+1}/n_k \geq \lambda > 1$, $k = 1, 2, \ldots$, and let $\{g(k)\}_{k=1}^{\infty}$ be a monotone nondecreasing sequence of real numbers with $1 \leq g(k) \leq n_k$, $k = 1, 2, \ldots$. Then there exist real trigonometric polynomials $p_k(x)$, $k = 1, 2, \ldots$, such that*

$$\deg(p_k) \leq \frac{n_k}{g(k)}, \quad \|p_k\|_1 \geq \frac{2\pi}{5}, \quad \|p_k\|_\infty \leq 12, \quad k = 1, 2, \ldots,$$

*and*

$$\left\| \sum_{k=1}^{m} p_k(x) \cos n_k x \right\|_\infty \leq \alpha(\lambda) + \beta\sqrt{m} + 24 \log_\lambda g(m), \quad m = 1, 2, \ldots,$$

*where $\beta > 0$ is an absolute constant and $\alpha(\lambda) > 0$ depends only on $\lambda$.*

In particular, Theorem E shows that in Theorem D with $m = 2l$ the condition $p_k \in \mathrm{T}(\gamma n_l)$ cannot be replaced with the condition $p_k \in \mathrm{T}(n_k/\lambda^{r_k})$, where $\{r_k\}_{k=1}^{\infty}$ is a monotone nondecreasing sequence of positive real numbers such that $r_k/k \to 0$ as $k \to \infty$ (in particular, it does not suffice to assume that $p_k \in \mathrm{T}(cn_k)$ with $c \in (0, 1)$).

In this paper we carry the result of Theorem D to sequences of class $\Lambda_\sigma$ and relax the conditions imposed on the degrees of the trigonometric polynomials $p_k(x)$ (Theorem 1). This refinement is achieved via application of a new method of proof, based on a refinement of the Riesz products and an estimate for the number of solutions of certain Diophantine equations. Partly, we have used some ideas from S. B. Stechkin's paper [4]. It should also be noted that we prove our results for more general sums $f$ that involve the terms $q_k(x) \sin n_k x$, besides $p_k(x) \cos n_k x$.

**Theorem 1.** *Let $\varepsilon \in (0, 1)$ and $B \geq 1$ be real numbers, and let $N = \{n_k\}_{k=1}^{\infty}$ be a monotone increasing sequence of natural numbers such that $N$ can be split into $d$*

*sequences* $N^{(j)} \in \Lambda(\lceil 7\sqrt{B}\rceil)$, $j = 1, \dots, d$. *Then for each trigonometric polynomial of the form*

$$f(x) = \sum_{k=l}^{m} p_k(x) \cos n_k x + q_k(x) \sin n_k x,$$

*where* $p_k, q_k \in \mathrm{T}(r_k)$, $k = l, \dots, m$,

$$r_l = \min\left(\frac{n_{l+1} - n_l}{2(1+\varepsilon)}, \frac{n_l}{1+\varepsilon}\right),$$

$$r_k = \min\left(\frac{n_k - n_{k-1}}{2(1+\varepsilon)}, \frac{n_{k+1} - n_k}{2(1+\varepsilon)}, Bn_l\right), \quad k = l+1, \dots, m-1,$$

$$r_m = \min\left(\frac{n_m - n_{m-1}}{2(1+\varepsilon)}, Bn_l\right),$$

$m > l$, $l = 1, 2, \dots$, *we have the inequality*

$$\|f\|_\infty \geq \frac{c}{d^2 \cdot \ln^2(1 + 1/\varepsilon)} \sum_{k=l}^{m} \|p_k\|_1 + \|q_k\|_1,$$

*where* $c > 0$ *is an absolute constant.*

## §2. Proof of Theorem 1

We start with a lemma, which refines a construction used by Kashin and Temlyakov in [9].

**Lemma 1.** *Suppose* $\varepsilon \in (0, 1)$, $n \in \mathbb{N}$, *and* $p(x)$ *is a real trigonometric polynomial of the form*

$$p(x) = A + \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx, \quad |a_n| + |b_n| > 0.$$

*Then there exists a nonzero real trigonometric polynomial* $g(x)$ *such that*

i) $\deg(g) < (1+\varepsilon)n$;

ii) $\|g\|_\infty \leq 30 \ln(1 + 1/\varepsilon) =: c_0(\varepsilon)$;

iii) $\int_0^{2\pi} p(x) g(x)\, dx = \|p\|_1$.

*Proof of Lemma* 1. Let $s > n$ be a natural number. Consider the function

$$V_{n,s}(x) = \frac{1}{s-n} \sum_{j=n}^{s-1} \sum_{\nu=-j}^{j} e^{i\nu x}$$

(the de la Valleé-Poussin kernel). Then $V_{n,s}(x)$ is a real trigonometric polynomial with $\deg(V_{n,s}) = s-1$ and $c_k(V_{n,s}) = 1$ for $|k| \leq n$. It can be shown (see, e.g., [17, Chapter 1]) that

$$\|V_{n,s}\|_1 \leq 86 \ln\left(1 + \frac{s}{s-n}\right).$$

We put

$$V(x) = V_{n,s}(x), \quad s = \lceil (1+\varepsilon)n \rceil.$$

Then $V(x)$ is a real trigonometric polynomial with $\deg(V) = \lceil (1+\varepsilon)n\rceil - 1$, $c_k(V) = 1$ for $|k| \leq n$, and

$$\|V\|_1 \leq 86 \ln\left(1 + \frac{\lceil (1+\varepsilon)n\rceil}{\lceil (1+\varepsilon)n\rceil - n}\right).$$

Since

$$\frac{\lceil (1+\varepsilon)n\rceil}{\lceil (1+\varepsilon)n\rceil - n} \leq \frac{(1+\varepsilon)n + 1}{(1+\varepsilon)n - n} \leq \frac{(1+\varepsilon)n + n}{\varepsilon n} = 1 + \frac{2}{\varepsilon},$$

we have

$$\|V\|_1 \le 86\ln 2 + 86\ln\Big(1+\frac{1}{\varepsilon}\Big) \le 172\ln\Big(1+\frac{1}{\varepsilon}\Big).$$

Put

$$g(x) = \operatorname{sgn}(p) * V = \frac{1}{2\pi}\int_0^{2\pi} \operatorname{sgn}(p(x-t))V(t)\,dt.$$

Then $g(x)$ is a real continuous $2\pi$-periodic function (see, e.g., [18, vol. I]). Since $c_k(g) = c_k(\operatorname{sgn}(p)) \cdot c_k(V)$, it follows that $c_k(g) = 0$ for $|k| > \deg(V)$, whence

$$a_k(g) = \frac{1}{\pi}\int_0^{2\pi} g(x)\cos kx\,dx = c_k(g) + c_{-k}(g) = 0,$$
$$b_k(g) = \frac{1}{\pi}\int_0^{2\pi} g(x)\sin kx\,dx = i(c_k(g) - c_{-k}(g)) = 0, \quad k > \deg(V).$$

Therefore,

$$g(x) = \frac{a_0(g)}{2} + \sum_{k=1}^{\lceil(1+\varepsilon)n\rceil-1} a_k(g)\cos kx + b_k(g)\sin kx, \quad x\in\mathbb{R}. \tag{2.1}$$

For any $x$ we have

$$|g(x)| \le \frac{1}{2\pi}\int_0^{2\pi}|V(t)|\,dt < 30\ln\Big(1+\frac{1}{\varepsilon}\Big).$$

Applying the Parseval identity and using the properties of convolution, we see that

$$\langle p, g\rangle = 2\pi\sum_{k=-n}^{n} c_k(p)c_{-k}(g) = 2\pi\sum_{k=-n}^{n} c_k(p)c_{-k}(\operatorname{sgn}(p))c_{-k}(V)$$
$$= 2\pi\sum_{k=-n}^{n} c_k(p)c_{-k}(\operatorname{sgn}(p)) = \langle p, \operatorname{sgn}(p)\rangle = \|p\|_1.$$

Observe that $g(x)$ is a nonzero trigonometric polynomial. Indeed, suppose $g(x)\equiv 0$; then $0 = \langle p, g\rangle = \|p\|_1$, so that $p(x) = 0$ a.e. on $[0, 2\pi]$. Since $p(x)$ is continuous, we have $p(x) = 0$ for all $x\in[0,2\pi]$. Consequently, $2\pi A = \int_0^{2\pi} p(x)\,dx = 0$, $\pi a_k = \int_0^{2\pi} p(x)\cos kx\,dx = 0$, and $\pi b_k = \int_0^{2\pi} p(x)\sin kx\,dx = 0$, $k = 1,\dots,n$. Since $|a_n|+|b_n| > 0$ by assumption, we arrive at a contradiction. Finally, (2.1) implies that $\deg(g) \le \lceil(1+\varepsilon)n\rceil - 1 < (1+\varepsilon)n$. Lemma 1 is proved. □

We continue the proof of Theorem 1. Suppose that the trigonometric polynomial $f(x)$ has the form

$$f(x) = \sum_{k=l}^{m} p_k(x)\cos n_k x + q_k(x)\sin n_k x,$$

where $p_k, q_k \in \mathrm{T}(r_k)$, $k = l,\dots,m$,

$$r_l = \min\left(\frac{n_{l+1}-n_l}{2(1+\varepsilon)}, \frac{n_l}{1+\varepsilon}\right),$$
$$r_k = \min\left(\frac{n_k - n_{k-1}}{2(1+\varepsilon)}, \frac{n_{k+1}-n_k}{2(1+\varepsilon)}, Bn_l\right), \quad k = l+1,\dots,m-1,$$
$$r_m = \min\left(\frac{n_m - n_{m-1}}{2(1+\varepsilon)}, Bn_l\right) \quad (m > l).$$

It it easy to show that $r_k \le n_k/(1+\varepsilon)$, $k = l,\dots,m$, and

$$(1+\varepsilon)(r_s + r_k) \le n_k - n_s, \quad l \le s < k \le m. \tag{2.2}$$

Let $k \in \{l, \dots, m\}$. If $p_k(x) \equiv b$, $b \in \mathbb{R}$, then we put

$$g_k(x) = \begin{cases} 1 & \text{if } \; b \geq 0, \\ -1 & \text{if } \; b < 0. \end{cases}$$

Then $g_k(x)$ is a nonzero real trigonometric polynomial with $\deg(g_k) = 0 < (1+\varepsilon)r_k$ and we have

$$\|g_k\|_\infty = 1 < 30 \ln 2 \leq 30 \ln\Big(1 + \frac{1}{\varepsilon}\Big),$$

$\langle p_k, g_k \rangle = 2\pi|b| = \|p_k\|_1$. If $p_k(x)$ is a nonconstant trigonometric polynomial, then for the role of $g_k(x)$ we take the trigonometric polynomial constructed in Lemma 1 and corresponding to $p_k(x)$ and $\varepsilon$. Then $g_k(x)$ will be a nonzero real trigonometric polynomial with $\deg(g_k) < (1+\varepsilon)\deg(p_k) \leq (1+\varepsilon)r_k$,

$$\|g_k\|_\infty \leq 30 \ln\Big(1 + \frac{1}{\varepsilon}\Big) = c_0(\varepsilon)$$

and $\langle p_k, g_k \rangle = \|p_k\|_1$. The trigonometric polynomial $\widetilde{g}_k(x)$ for $q_k(x)$ is constructed similarly. Put $\tau_k(x) = g_k(x)/c_0(\varepsilon)$ and $\widetilde{\tau}_k(x) = \widetilde{g}_k(x)/c_0(\varepsilon)$. Then $\tau_k(x)$ and $\widetilde{\tau}_k(x)$ are nonzero trigonometric polynomials such that $\max(\deg(\tau_k), \deg(\widetilde{\tau}_k)) < (1+\varepsilon)r_k$, $\max(\|\tau_k\|_\infty, \|\widetilde{\tau}_k\|_\infty) \leq 1$, and

$$\langle p_k, \tau_k \rangle = \frac{\|p_k\|_1}{c_0(\varepsilon)}, \quad \langle q_k, \widetilde{\tau}_k \rangle = \frac{\|q_k\|_1}{c_0(\varepsilon)}, \quad k = l, \dots, m. \tag{2.3}$$

We set

$$\alpha = \frac{1}{864 c_0(\varepsilon) d}. \tag{2.4}$$

The sequence $N = \{n_k\}_{k=1}^\infty$ is split into $d$ sequences $N^{(j)} \in \Lambda(\lceil 7\sqrt{B} \rceil)$, $j = 1, \dots, d$, i.e., for any $j \in \{1, \dots, d\}$ the sequence $N^{(j)}$ is equal to $\{n_k\}_{k \in E_j}$ (in the order given within $E_j$), where $E_j$ is a set of integers $i_1 < i_2 < \dots < i_s < \dots$, $n_{i_{s+1}}/n_{i_s} \geq \lceil 7\sqrt{B} \rceil$, $s = 1, 2, \dots$, and

$$\mathbb{N} = \coprod_{j=1}^d E_j.$$

Let $\Phi = \{j \in \{1, \dots, d\}$: the set $\{l \leq k \leq m : k \in E_j\}$ is nonempty$\}$. For $j \in \Phi$, we define the Riesz product

$$J^{(j)}(x) = \frac{1}{\alpha} \prod_{l \leq k \leq m \,:\, k \in E_j} (1 + \alpha\, \tau_k(x) \cos n_k x + \alpha\, \widetilde{\tau}_k(x) \sin n_k x) = \frac{1}{\alpha} + \omega_1(x) + \omega_2(x), \tag{2.5}$$

where

$$\omega_1(x) = \sum_{l \leq k \leq m \,:\, k \in E_j} \tau_k(x) \cos n_k x + \widetilde{\tau}_k(x) \sin n_k x,$$

$$\omega_2(x) = J^{(j)}(x) - \frac{1}{\alpha} - \omega_1(x).$$

Consider the case where $|\{l \leq k \leq m : k \in E_j\}| \geq 2$. Clearly, $J^{(j)}(x) \geq 0$ for any $x$. We estimate $\langle f, J^{(j)} \rangle$ from below. Since either $p_k(x) \equiv 0$, or $\deg(p_k) < n_k$ (and similarly for $q_k(x)$), we have $\langle f, 1/\alpha \rangle = 0$. Let $s \in \{l, \dots, m\}$ be such that $s \in E_j$, and let $k \in \{l, \dots, m\}$. Suppose $k \neq s$. We shall show that

$$\begin{aligned} &\int_0^{2\pi} p_k(x) \cos n_k x (\tau_s(x) \cos n_s x + \widetilde{\tau}_s(x) \sin n_s x)\, dx \\ &\quad = \int_0^{2\pi} p_k(x) \tau_s(x) \cos n_k x \cos n_s x\, dx + \int_0^{2\pi} p_k(x) \widetilde{\tau}_s(x) \cos n_k x \sin n_s x\, dx = 0. \end{aligned} \tag{2.6}$$

We assume that $k > s$ (the case where $k < s$ is treated similarly). If $p_k(x) \equiv 0$, then (2.6) is obvious. Let $p_k(x) \equiv b$, $b \neq 0$. Since $\deg(\tau_s), \deg(\widetilde{\tau}_s) < (1+\varepsilon)r_s < (1+\varepsilon)(r_s + r_k) \leq n_k - n_s$ (see (2.2)), it follows that (2.6) is true. Now, let $p_k(x)$ be nonconstant. Since $\deg(p_k\tau_s)$, $\deg(p_k\widetilde{\tau}_s) < r_k + (1+\varepsilon)r_s < (1+\varepsilon)(r_s+r_k) \leq n_k - n_s$ (see (2.2)), we conclude that (2.6) is proved. Similar arguments show that

$$\int_0^{2\pi} q_k(x)\sin n_k x(\tau_s(x)\cos n_s x + \widetilde{\tau}_s(x)\sin n_s x)\,dx = 0. \tag{2.7}$$

Suppose that $k = s$. We prove that

$$I = \int_0^{2\pi} p_s(x)\cos n_s x(\tau_s(x)\cos n_s x + \widetilde{\tau}_s(x)\sin n_s x)\,dx = \frac{\|p_s\|_1}{2c_0(\varepsilon)}. \tag{2.8}$$

If $p_s(x) \equiv 0$, then (2.8) is obvious. Let $p_s(x) \equiv b$, $b \neq 0$. Then $\tau_s(x) = 1/c_0(\varepsilon)$ if $b > 0$, and $\tau_s(x) = -1/c_0(\varepsilon)$ if $b < 0$. Thus,

$$I = \frac{\pi|b|}{c_0(\varepsilon)} + \frac{b}{2}\int_0^{2\pi} \widetilde{\tau}_s(x)\sin 2n_s x\,dx = \frac{\|p_s\|_1}{2c_0(\varepsilon)} + \frac{b}{2}\,I_0.$$

Since $\deg(\widetilde{\tau}_s) < (1+\varepsilon)r_s \leq n_s < 2n_s$, we see that $I_0 = 0$ and (2.8) is fulfilled. Now, let $p_s(x)$ be nonconstnat. Then (see (2.3))

$$\begin{aligned} I = \frac{1}{2}\int_0^{2\pi} p_s(x)\tau_s(x)\,dx &+ \frac{1}{2}\int_0^{2\pi} p_s(x)\tau_s(x)\cos 2n_s x\,dx \\ &+ \frac{1}{2}\int_0^{2\pi} p_s(x)\widetilde{\tau}_s(x)\sin 2n_s x\,dx = \frac{\|p_s\|_1}{2c_0(\varepsilon)} + \frac{1}{2}I_1 + \frac{1}{2}I_2. \end{aligned}$$

Since $\deg(p_s\tau_s)$, $\deg(p_s\widetilde{\tau}_s) < r_s + (1+\varepsilon)r_s \leq (2+\varepsilon)n_s/(1+\varepsilon) < 2n_s$, it follows that $I_1 = 0$ and $I_2 = 0$, so that (2.8) is proved. Similarly,

$$\int_0^{2\pi} q_s(x)\sin n_s x(\tau_s(x)\cos n_s x + \widetilde{\tau}_s(x)\sin n_s x)\,dx = \frac{\|q_s\|_1}{2c_0(\varepsilon)}. \tag{2.9}$$

Relations (2.6)–(2.9) imply

$$\int_0^{2\pi} f(x)(\tau_s(x)\cos n_s x + \widetilde{\tau}_s(x)\sin n_s x)\,dx = \frac{1}{2c_0(\varepsilon)}(\|p_s\|_1 + \|q_s\|_1).$$

Consequently,

$$\langle f, \omega_1\rangle = \frac{1}{2c_0(\varepsilon)}\sum_{l\leq k\leq m:\, k\in E_j} \|p_k\|_1 + \|q_k\|_1. \tag{2.10}$$

We shall prove that

$$|\langle f, \omega_2\rangle| \leq 144\alpha\sum_{k=l}^{m} \|p_k\|_1 + \|q_k\|_1. \tag{2.11}$$

Let $k \in \{l, \dots, m\}$. It will be shown that

$$|\langle p_k(x)\cos n_k x, \omega_2\rangle| \leq 144\alpha\|p_k\|_1. \tag{2.12}$$

If $p_k(x) \equiv 0$, then (2.12) is true. In what follows we assume that $p_k \neq 0$. Let $z_1 < z_2 < \dots < z_L$ be all elements of the sequence $N^{(j)}$ that lie in the segment $[n_l, n_m]$. We have

$$\omega_2(x) = \frac{1}{\alpha}\sum_{s=2}^{L}\alpha^s \sum_{l\leq k_1<\dots<k_s\leq m:\, k_\nu\in E_j} \sum\nolimits^{(1)} a_{k_1}(x)\cdot\dots\cdot a_{k_s}(x), \tag{2.13}$$

where the sum $\sum^{(1)}$ is over all $a_{k_i}(x)$ equal to $\tau_{k_i}(x)\cos n_{k_i}x$ or to $\widetilde{\tau}_{k_i}(x)\sin n_{k_i}x$ (in total, in the sum $\sum^{(1)}$ we have $2^s$ summands). Let $\psi_n(x)$ denote the function $\tau_n(x)$

or $\widetilde{\tau}_n(x)$, and let $\varphi(n\,|\,x)$ stand for the function $\cos nx$ or $\sin nx$, assuming that for different $n$ the function $\varphi(n\,|\,x)$ may differ, i.e., they may be either sine or cosine with the corresponding frequencies. Let $s \in \{2, \dots, L\}$, and let $k_1, \dots, k_s$ be natural numbers such that $l \le k_1 < \dots < k_s \le m$ and $k_\nu \in E_j$, $\nu = 1, \dots, s$. We have

$$a_{k_1}(x) \cdot \dots \cdot a_{k_s}(x) = \psi_{k_1}(x) \cdot \dots \cdot \psi_{k_s}(x) \cdot \varphi(n_{k_1}\,|\,x) \cdot \dots \cdot \varphi(n_{k_s}\,|\,x)$$

(here, if $\varphi(n_{k_i}\,|\,x) = \cos n_{k_i}x$, then $\psi_{k_i}(x) = \tau_{k_i}(x)$, and if $\varphi(n_{k_i}\,|\,x) = \sin n_{k_i}x$, then $\psi_{k_i}(x) = \widetilde{\tau}_{k_i}(x)$). Using the well-known trigonometric formulas, we get the identity

$$a_{k_1}(x) \cdot \dots \cdot a_{k_s}(x) = \psi_{k_1}(x) \cdot \dots \cdot \psi_{k_s}(x) \sum\nolimits^{(2)} \pm \frac{1}{2^{s-1}} \varphi(n_{k_s} \pm n_{k_{s-1}} \pm \dots \pm n_{k_1}\,|\,x),$$

where the sum $\sum^{(2)}$ is over all collections of signs $\pm$ in the linear expression $n_{k_s} \pm n_{k_{s-1}} \pm \dots \pm n_{k_1}$ (in total, in $\sum^{(2)}$ we have $2^{s-1}$ summands). Each term $\varphi(n_{k_s} \pm n_{k_{s-1}} \pm \dots \pm n_{k_1}\,|\,x)$ is supplied with a uniquely determined sign $+$ or $-$. Observe that, since $N^{(j)} \in \Lambda(\lceil 7\sqrt{B}\rceil)$, we have $n_{k_s} \pm n_{k_{s-1}} \pm \dots \pm n_{k_1} > 0$ for any collection of signs. Let $i$ be an integer. We denote by $M(i\,|\,n_{k_1}, \dots, n_{k_s})$ the set of all collections of signs $(\varepsilon_1, \dots, \varepsilon_{s-1})$, $\varepsilon_\nu \in \{-1, 1\}$, $1 \le \nu \le s-1$, such that

$$n_{k_s} + \varepsilon_{s-1} n_{k_{s-1}} + \dots + \varepsilon_1 n_{k_1} = i.$$

Let $P(i\,|\,n_{k_1}, \dots, n_{k_s}) = |M(i|\,n_{k_1}, \dots, n_{k_s})|$ be the number of elements in the set

$$M(i\,|\,n_{k_1}, \dots, n_{k_s}).$$

It is easily seen that $P(i\,|\,n_{k_1}, \dots, n_{k_s})$ is equal to 0 or 1.

We have

$$\begin{aligned}&\langle p_k(x) \cos n_k x, a_{k_1}(x) \cdot \dots \cdot a_{k_s}(x)\rangle \\ &\quad = \sum\nolimits^{(2)} \frac{1}{2^{s-1}} \int_0^{2\pi} p_k(x) \cos n_k x\, \psi_{k_1}(x) \dots \psi_{k_s}(x) \cdot (\pm\varphi(n_{k_s} \pm n_{k_{s-1}} \pm \dots \pm n_{k_1}\,|\,x))\, dx.\end{aligned}$$

Since

$$\deg(p_k \psi_{k_1} \cdot \dots \cdot \psi_{k_s}) < r_k + (1+\varepsilon)(r_{k_1} + \dots + r_{k_s}) \le Bn_l + s(1+\varepsilon)Bn_l \le 3\lceil B\rceil sn_l,$$

it follows that

$$\begin{aligned}&\langle p_k(x) \cos n_k x, a_{k_1}(x) \cdot \dots \cdot a_{k_s}(x)\rangle \\ &\quad = \sum\nolimits^{(3)} \frac{1}{2^{s-1}} \int_0^{2\pi} p_k(x) \cos n_k x\, \psi_{k_1}(x) \dots \psi_{k_s}(x) \cdot (\pm\varphi(n_{k_s} \pm n_{k_{s-1}} \pm \dots \pm n_{k_1}\,|\,x))\, dx,\end{aligned}$$

where the sum $\sum^{(3)}$ is only taken over all distinct collections of signs $\pm$ such that

$$n_k - 3\lceil B\rceil sn_l \le n_{k_s} \pm n_{k_{s-1}} \pm \dots \pm n_{k_1} \le n_k + 3\lceil B\rceil sn_l.$$

Let $\Pi$ denote the set of such collections of signs. Clearly, we have

$$\Pi = \bigsqcup_{n_k - 3\lceil B\rceil sn_l \le i \le n_k + 3\lceil B\rceil sn_l} M(i\,|\,n_{k_1}, \dots, n_{k_s}).$$

Consequently,

$$|\Pi| = \sum_{n_k - 3\lceil B\rceil sn_l \le i \le n_k + 3\lceil B\rceil sn_l} P(i|\,n_{k_1}, \dots, n_{k_s}).$$

Since

$$\left| \int_0^{2\pi} p_k(x) \cos n_k x \psi_{k_1}(x) \cdot \dots \cdot \psi_{k_s}(x) \cdot (\pm\varphi(n_{k_s} \pm n_{k_{s-1}} \pm \dots \pm n_{k_1}\,|\,x))\, dx \right| \le \|p_k\|_1,$$

we obtain the inequality

$$\left|\langle p_k(x)\cos n_k x, a_{k_1}(x)\cdot\dots\cdot a_{k_s}(x)\rangle\right| \le \frac{1}{2^{s-1}}\,\|p_k\|_1\cdot \sum_{n_k-3\lceil B\rceil sn_l\le i\le n_k+3\lceil B\rceil sn_l} P(i|\,n_{k_1},\dots,n_{k_s}).$$

This implies (see (2.13)) that

$$\begin{aligned}
&\left|\langle p_k(x)\cos n_k x, \omega_2\rangle\right|\\
&\le \frac{1}{\alpha}\sum_{s=2}^{L}\alpha^s \sum_{l\le k_1<\dots<k_s\le m\colon k_\nu\in E_j} 2^s\cdot\frac{1}{2^{s-1}}\,\|p_k\|_1 \sum_{n_k-3\lceil B\rceil sn_l\le i\le n_k+3\lceil B\rceil sn_l} P(i|\,n_{k_1},\dots,n_{k_s})\\
&= 2\,\|p_k\|_1\sum_{s=2}^{L}\alpha^{s-1}\sum_{l\le k_1<\dots<k_s\le m\colon k_\nu\in E_j}\ \sum_{n_k-3\lceil B\rceil sn_l\le i\le n_k+3\lceil B\rceil sn_l} P(i|\,n_{k_1},\dots,n_{k_s}).
\end{aligned}\tag{2.14}$$

Let $s\in\{2,\dots,L\}$. Put

$$S=\sum_{l\le k_1<\dots<k_s\le m\colon k_\nu\in E_j}\ \sum_{n_k-3\lceil B\rceil sn_l\le i\le n_k+3\lceil B\rceil sn_l} P(i|\,n_{k_1},\dots,n_{k_s}).$$

We want to show that

$$S\le 5^s+1.\tag{2.15}$$

If $S$ is 0, 1, or 2, then (2.15) is obvious. In what follows we assume that $S\ge 3$. Let $\Omega$ be the set of all distinct vectors of the form $(\eta_1,\dots,\eta_t,0,\dots,0)\in\mathbb{R}^L$ such that $\eta_\nu\in\{-2,-1,0,1,2\}$, $\nu=1,\dots,t$, $\eta_t\ne 0$, $1\le t\le L$, and

$$\sum_{\nu=1}^{t}\eta_\nu z_\nu=i,\quad 0\le i\le 6\lceil B\rceil sn_l.$$

We claim that for any $v\in\Omega$ we have $t\le s$. Suppose the contrary. Then there exists a vector $(\eta_1,\dots,\eta_t,0,\dots,0)\in\Omega$ such that $\eta_\nu\in\{-2,-1,0,1,2\}$, $\nu=1,\dots,t$, $\eta_t\ne0$, and $t\ge s+1$. Let $\eta_t\in\{1,2\}$. Since

$$|\eta_{t-1}z_{t-1}+\dots+\eta_1 z_1|\le 2(z_1+\dots+z_{t-1})<\frac12\,z_t,$$

we have

$$\begin{aligned}
\eta_t z_t+\dots+\eta_1 z_1 &\ge z_t+\eta_{t-1}z_{t-1}+\dots+\eta_1 z_1>\frac12\,z_t\\
&\ge\frac12\,z_{s+1}\ge\frac12\left(\lceil 7\sqrt{B}\rceil\right)^s z_1\ge\frac12\left(\lceil 7\sqrt{B}\rceil\right)^s n_l>6\lceil B\rceil sn_l.
\end{aligned}$$

If $\eta_t\in\{-1,-2\}$, then

$$\eta_t z_t+\dots+\eta_1 z_1\le -z_t+\eta_{t-1}z_{t-1}+\dots+\eta_1 z_1<-\frac12 z_t<0.$$

We arrive at a contradiction. Consequently, indeed, for any $v\in\Omega$ we have $t\le s$. Let $G$ denote the set of all vectors of the form $(\eta_1,\dots,\eta_s,0,\dots,0)\in\mathbb{R}^L$ such that $\eta_\nu\in\{-2,-1,0,1,2\}$, $1\le\nu\le s$. Clearly,

$$|\Omega|\le|G|=5^s.\tag{2.16}$$

We enumerate (in some order) all collections $u_1,\dots,u_R$ of natural numbers $(k_1,\dots,k_s)$ such that $l\le k_1<\dots<k_s\le m$ and $k_\nu\in E_j$, $\nu=1,\dots,s$. We describe an algorithm of writing out certain $L$-dimensional vectors with coordinates 0, 1 and $-1$. We start with $i=n_k-3\lceil B\rceil sn_l$ and $t=1$. Let $i\in\{n_k-3\lceil B\rceil sn_l,\dots,n_k+3\lceil B\rceil sn_l\}$, $t\in\{1,\dots,R\}$, $u_t=(k_1,\dots,k_s)$. If the set $M(i|\,n_{k_1},\dots,n_{k_s})$ is empty, then we replace $t$ with $t+1$. If $M(i|\,n_{k_1},\dots,n_{k_s})$ is not empty, then this set containes a single element $(\varepsilon_1,\dots,\varepsilon_{s-1})$.

Let $n_{k_\nu} = z_{i_\nu}$, $\nu = 1,\dots,s$. Obviously, $1 \le i_1 < \dots < i_s \le L$. Consider the vector $v = (b_1,\dots,b_L)$, where

$$b_{i_s} = 1, \quad b_{i_\nu} = \varepsilon_\nu, \quad \nu = 1,\dots,s-1,$$

and the other coordinates are equal to 0. After that we replace $t$ with $t+1$. If $t = R$, $u_R = (k_1^*,\dots,k_s^*)$, then:

1) if $M(i|\, n_{k_1^*},\dots,n_{k_s^*})$ is empty, then we replace $i$ with $i+1$ and put $t = 1$;

2) if $M(i|\, n_{k_1^*},\dots,n_{k_s^*})$ is not empty, we write out the vector dictated by the above rule, replace $i$ with $i+1$ and put $t = 1$.

Finally, if $i = n_k + 3\lceil B\rceil sn_l$, $t = R$, $u_R = (k_1^*,\dots,k_s^*)$, then:

1) if $M(i|\, n_{k_1^*},\dots,n_{k_s^*})$ is empty, we finish the algorithm;

2) if $M(i|\, n_{k_1^*},\dots,n_{k_s^*})$ is not empty, we write out the vector in accordance with the above rule and finish the algorithm.

In the course of our algorithm, for each $i \in \{n_k - 3\lceil B\rceil sn_l,\dots,n_k + 3\lceil B\rceil sn_l\}$ and each collection of natural numbers $(k_1,\dots,k_s)$ such that $l \le k_1 < \dots < k_s \le m$ and $k_\nu \in E_j$, $\nu = 1,\dots,s$, we construct $P(i|\, n_{k_1},\dots,n_{k_s})$ vectors. Consequently, in total we have $S$ $L$-dimensional vectors. Each of these vectors has exactly $s$ nonzero coordinates equal to $\pm 1$, and the last nonzero coordinate is equal to 1. We enumerate these vectors in the order of obtaining them in the course of the algorithm. Suppose a vector $v_\delta = (b_1,\dots,b_L)$, $\delta \in \{1,\dots,S\}$ was obtained for $i$ and $(k_1,\dots,k_s)$. Then, by the construction itself of the vectors in questions (see, in particular, the definition of the set $M(i|\, n_{k_1},\dots,n_{k_s})$) we have $\sum_{\nu=1}^L b_\nu z_\nu = i$. Observe also that, in accordance with our construction, if a vector $v_\delta = (b_1,\dots,b_L)$ was obtained starting with $i$ and $(k_1,\dots,k_s)$, and $v_\gamma = (c_1,\dots,c_L)$ stemmed from $\widetilde{i}$ and $(\widetilde{k}_1,\dots,\widetilde{k}_s)$, where $\delta,\gamma \in \{1,\dots,S\}$, $\delta < \gamma$, then $\sum_{\nu=1}^L b_\nu z_\nu = i$, $\sum_{\nu=1}^L c_\nu z_\nu = \widetilde{i}$, and

$$n_k - 3\lceil B\rceil sn_l \le i \le \widetilde{i} \le n_k + 3\lceil B\rceil sn_l. \tag{2.17}$$

We show that all the resulting vectors are distinct. Suppose the contrary. Then there exist equal vectors $v_\delta$ and $v_\gamma$, where $\delta,\gamma \in \{1,\dots,S\}$, $\delta < \gamma$. Suppose $v_\delta$ was obtained from $i$ and $(k_1,\dots,k_s)$ and $v_\gamma$ from $\widetilde{i}$ and $(\widetilde{k}_1,\dots,\widetilde{k}_s)$ ($i,\widetilde{i} \in \{n_k - 3\lceil B\rceil sn_l,\dots,n_k + 3\lceil B\rceil sn_l\}$, the collections of natural numbers $(k_1,\dots,k_s)$ and $(\widetilde{k}_1,\dots,\widetilde{k}_s)$ are such that $l \le k_1 < \dots < k_s \le m$, and $l \le \widetilde{k}_1 < \dots < \widetilde{k}_s \le m$, $k_\nu,\widetilde{k}_\nu \in E_j$, $\nu = 1,\dots,s$, and $M(i|\, n_{k_1},\dots,n_{k_s}) \ne \varnothing$, $M(i|\, n_{\widetilde{k}_1},\dots,n_{\widetilde{k}_s}) \ne \varnothing$). We show that $i = \widetilde{i}$ and $k_\nu = \widetilde{k}_\nu$, $\nu = 1,\dots,s$. Let $v_\delta = (b_1,\dots,b_L) = v_\gamma$. Then $\sum_{\nu=1}^L b_\nu z_\nu = i$ and $\sum_{\nu=1}^L b_\nu z_\nu = \widetilde{i}$. Hence, $i = \widetilde{i}$. By using induction, it is not hard to prove the following claim: for a natural number $t$, if $b_1 < b_2 < \dots < b_t$ and $c_1 < c_2 < \dots < c_t$ are sequences of real numbers with $(b_1,\dots,b_t) \ne (c_1,\dots,c_t)$, then there exists $1 \le \nu \le t$ such that $b_\nu \ne c_\beta$, $\beta = 1,\dots,t$. If we assume that $(k_1,\dots,k_s) \ne (\widetilde{k}_1,\dots,\widetilde{k}_s)$, then we can find $1 \le \nu \le s$ such that $k_\nu \ne \widetilde{k}_t$, $t = 1,\dots,s$. Let $n_{k_\nu} = z_{i_\nu}$. Since the sequence $\{n_k\}_{k=1}^\infty$ is monotone increasing, our construction of vectors shows that the $i_\nu$th coordinate of $v_\delta$ is 1 or $-1$, while the $i_\nu$th coordinate of $v_\gamma$ is 0, which contradicts the fact that $v_\delta = v_\gamma$. Thus, $i = \widetilde{i}$ and $(k_1,\dots,k_s) = (\widetilde{k}_1,\dots,\widetilde{k}_s)$. The vector $v_\delta$ was constructed starting with $i$ and $(k_1,\dots,k_s)$. After that, in the course of the algorithm, we passed to other collections $(k_1',\dots,k_s')$ and, then, to other, larger, $i'$. Therefore, the vector $v_\gamma$ could not stem from $i$ and $(k_1,\dots,k_s)$, a contradiction. Consequently, all the resulting vectors are distinct.

Now we prove inequality (2.15). We subtract the first vector from the $j$th one ($2 \le j \le S$), keeping the first vector unchanged. As a result, we get $S$ vectors, and those with numbers $2 \le j \le S$ will have the form

$$(\eta_1,\dots,\eta_t,0,\dots,0) \in \mathbb{R}^L, \tag{2.18}$$

where $\eta_\nu \in \{-2,-1,0,1,2\}$, $1 \le \nu \le t$, $\eta_t \ne 0$, $1 \le t \le L$, and $\sum_{\nu=1}^t \eta_\nu z_\nu = i$, $0 \le i \le 6\lceil B\rceil s n_l$. We explain why $\eta_t \ne 0$. Indeed, should the zero vector occur, we would get two equal vectors after adding the first vector to it, so that two equal vectors would occur among $S$ initial vectors, which is impossible, as we saw above. We have $\sum_{\nu=1}^t \eta_\nu z_\nu = i$, $0 \le i \le 6\lceil B\rceil s n_l$, by (2.17). All $S-1$ vectors (2.18) are distinct. Indeed, should two of them be equal, then, after adding the first vector to them, we would get two equal vectors among the initial ones, which is impossible. Recalling (2.16), we see that

$$S - 1 \le 5^s,$$

which proves (2.15). Since $s \ge 2$, finally we get $S < 6^s$. Plugging this estimate in (2.14), we obtain the inequality

$$\left|\langle p_k(x)\cos n_k x, \omega_2(x)\rangle\right| \le 12\,\|p_k\|_1 \sum_{s=2}^{L}(6\alpha)^{s-1} < 12\,\|p_k\|_1 \frac{6\alpha}{1-6\alpha} \le 144\alpha\,\|p_k\|_1,$$

and (2.12) is proved (we have also used the fact that $\alpha < 1/12$, see (2.4)). Similar arguments show that

$$\left|\langle q_k(x)\sin n_k x, \omega_2(x)\rangle\right| \le 144\alpha\,\|q_k\|_1, \quad l \le k \le m,$$

and

$$|\langle \omega_2, 1\rangle| \le 2\pi \cdot 144\alpha \le \frac{2\pi}{\alpha}. \tag{2.19}$$

Consequently,

$$|\langle f, \omega_2\rangle| \le 144\alpha \sum_{k=l}^{m} \|p_k\|_1 + \|q_k\|_1$$

which proves (2.11). Using (2.10), we get

$$\langle f, J^{(j)}\rangle \ge \frac{1}{2c_0(\varepsilon)}\left(\sum_{l\le k\le m:\, k\in E_j} \|p_k\|_1 + \|q_k\|_1\right) - 144\alpha \sum_{k=l}^{m} \|p_k\|_1 + \|q_k\|_1.$$

It is easily seen that $\langle \omega_1, 1\rangle = 0$. Since the function $J^{(j)}$ is nonnegative, we can use (2.19) to show (see also (2.5)) that

$$\|J^{(j)}\|_1 = \langle J^{(j)}, 1\rangle = \frac{2\pi}{\alpha} + \langle \omega_2, 1\rangle \le \frac{4\pi}{\alpha}.$$

Consequently,

$$\langle f, J^{(j)}\rangle \le \|f\|_\infty \cdot \|J^{(j)}\|_1 \le \frac{4\pi}{\alpha}\,\|f\|_\infty.$$

As a result, we have

$$\frac{4\pi}{\alpha}\,\|f\|_\infty \ge \frac{1}{2c_0(\varepsilon)}\left(\sum_{l\le k\le m:\, k\in E_j} \|p_k\|_1 + \|q_k\|_1\right) - 144\alpha \sum_{k=l}^{m} \|p_k\|_1 + \|q_k\|_1. \tag{2.20}$$

If $j \in \Phi$ is such that $|\{l \le k \le m : \ k \in E_j\}| = 1$, then $J^{(j)}(x) = (1/\alpha) + \omega_1(x)$, and then

$$\frac{1}{2c_0(\varepsilon)} \sum_{l\le k\le m:\, k\in E_j} \|p_k\|_1 + \|q_k\|_1 = \langle f, J^{(j)}\rangle \le \|f\|_\infty \|J^{(j)}\|_1 = \frac{2\pi}{\alpha}\,\|f\|_\infty.$$

Therefore, in this case inequality (2.20) is also fulfilled. We see that (2.20) is true for any $j \in \Phi$. Summing these inequalities over all $j \in \Phi$ and using the fact that $|\Phi| \le d$, see also (2.4), we get the estimate

$$\frac{4\pi d}{\alpha}\|f\|_\infty \geq \frac{1}{2c_0(\varepsilon)}\left(\sum_{k=l}^{m}\|p_k\|_1+\|q_k\|_1\right) - 144\alpha d\sum_{k=l}^{m}\|p_k\|_1+\|q_k\|_1$$
$$= \frac{1}{3c_0(\varepsilon)}\sum_{k=l}^{m}\|p_k\|_1+\|q_k\|_1.$$

Recalling the explicit form of $c_0(\varepsilon)$, finally we obtain

$$\|f\|_\infty \geq \frac{c}{d^2\cdot\ln^2(1+1/\varepsilon)}\sum_{k=l}^{m}\|p_k\|_1+\|q_k\|_1,$$

where $c>0$ is an absolute constant and Theorem 1 is proved.

In [19], an inequality similar to the generalized Sidon inequality was obtained for discrete orthonormal system of a special form, a particular case of which coincides with the Walsh system.

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Steklov Mathematical Institute, Russian Academy of Sciences, Gubkin str. 8, Moscow 119991, Russia

*Email address*: artrad@list.ru

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